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Hidden symmetry, separation of variables and interbasis expansions in the two-dimensional hydrogen atom

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Abstract. Expansions for each fundamental basis of the hydrogen atom over two others are found and an additional integral of motion corresponding to an elliptic basis is determined. Representations of the elliptic basis as a superposition of polar and parabolic states are obtained. Certain interesting limiting cases are investigated.

Introduction

Until recently the systems with hidden symmetry have been analysed mainly by two methods (Fock 1935, Bargmann 1936), the first consisting of reformulating the Schrödinger equation and rewriting it in such a form that hidden symmetry becomes obvious, and the second of constructing by a classical analogy the integrals of motion, which turn out to be generators of the group of hidden symmetry.

In this paper we should like to pay attention to a strong relationship between hidden symmetry and the separability of variables in the Schrödinger equation. The discovery of this relationship has recently initiated an intensive application of the method of separation of variables to the equations of mathematical physics and provided many new results in this branch of mathematics (Miller 1977). In the framework of the method of separation of variables the eigenvalues of the generators acquire the meaning of the separation constants and the eigenfunctions, the solutions (the fundamental bases) of the Schrödinger equation in different coordinates are common for the Hamiltonian and each generator of the group of hidden symmetry.

In the theory of systems with hidden symmetry one often deals with the matrix elements of the operators with respect to one basis of the system. Usually, these operators are linear combinations of the degrees of generators of the group of hidden symmetry. Sometimes, such matrix elements can be calculated only in the case when one knows the expansions of the basis, corresponding to the matrix element, over each fundamental basis. The aforesaid determines the role of the theory of interbasis expansions both in pure mathematics (Miller 1977) and physical applications (Komarov *et al* 1976, Malkin and Man'ko 1979).

This paper is devoted to interbasis expansions for a two-dimensional hydrogen atom thoroughly studied by Zaslow and Zandler (1965) and by Cisneras and McIntosh (1968).

In the first section we present the information on fundamental bases of the two-dimensional hydrogen atom in the discrete spectrum region. Section 2 is devoted to

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the calculation of the coefficients defining mutual expansions between polar and two parabolic bases. In the third section the method of separation of variables is used to find the integral of motion specific of the elliptic basis. In the fourth section this integral of motion is used to construct the elliptic basis, first as a superposition of polar bases and then as a superposition of parabolic ones. The fifth section is devoted to the study of the polar $R\to 0$ and parabolic $R\to\infty$ limits of the elliptic basis. Much of the material is supplied in tabular form, in the appendix.

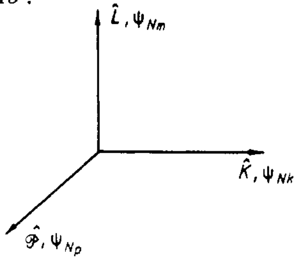
1. Fundamental bases of the two-dimensional hydrogen atom

Information on the group of hidden O(3) symmetry of the two-dimensional hydrogen atom in the discrete spectrum region can be represented in table A1. Hereafter atomic units $\hbar = e = m = 1$ are used. The generators of this group will be denoted by $\hat{L}$, $\hat{\mathscr{P}}$ and $\hat{K}$. The general eigenfunctions of the Hamiltonian

$$\hat{H}\psi = \left\{-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{(x^2+y^2)^{1/2}}\right]\psi = E_N\psi$$

$$E_N = -\tfrac{1}{2}\omega^2 = -2/(2N+1)^2$$

and of each generator $\hat{L}$, $\hat{\mathscr{P}}$ and $\hat{K}$, i.e. the fundamental bases of the two-dimensional hydrogen atom, have the meaning of the Schrödinger equation solutions obtained by the method of separation of variables in the polar and two parabolic coordinates at right angles to each other (Englefield 1972). All the fundamental bases at given N are orthonormalised over the second quantum index. The generators $\hat{L}$, $\hat{\mathscr{P}}$ and $\hat{K}$ commute with the Hamiltonian $\hat{H}$ and satisfy the commutation relations $\{\hat{\mathscr{P}}, \hat{K}\} = \mathrm{i}\hat{L}$, $\{\hat{L}, \hat{\mathscr{P}}\} = \mathrm{i}\hat{K}$ and $\{\hat{K}, \hat{L}\} = \mathrm{i}\hat{\mathscr{P}}$.



2. Expansions between the fundamental bases

From the commutation relations between $\hat{L}$, $\hat{\mathscr{P}}$ and $\hat{K}$ and the meaning of the fundamental basis it follows that the expansions between these bases can be interpreted as rotations through ninety degrees in the relevant coordinate planes. As the rotations are planar, the expansion coefficients should coincide with an accuracy up to the phase factor with the Wigner d-function of the right angle (table A2). The first column and upper row of this table represent the fundamental bases of the two-dimensional hydrogen atom, and the remaining cells represent the coefficients of interbasis expansions, by which we mean the expansions of the bases of the left column over the bases of the upper row. These expansions imply the summation over quantum numbers p, k and m, respectively. We start from the expansion of the first parabolic basis over the polar one

$$\psi_{Np}(u, v) = \sum_{m=-N}^{N} V_{pm}^{N}\psi_{Nm}(r, \varphi). \tag{2.1}$$

It is obvious that the overlapping integral between these bases is very complicated to calculate. The following method is more effective. At large r the bases are considerably simplified and contain r in the same degree. Therefore, within this limit the dependences on r are reduced in both parts of the expansion (2.1), and the orthonormalisation condition of the functions $e^{im\varphi}$ leads to

$$V_{pm}^{N}=\frac{(-1)^{N-m}}{2\pi}\left(\frac{(N+m)!(N-m)!}{(N+p)!(N-p)!}\right)^{1/2}$$
$$\times\int_0^{2\pi}(1+\cos\varphi)^{\frac{1}{2}(n-p)}(1-\cos\varphi)^{\frac{1}{2}(N-p)}\,e^{im\varphi}\,d\varphi.$$

By calculating this integral, comparing the answer for V_{pm}^{N} with the known formula from angular momentum theory (Varshalovich *et al* 1975):

$$d_{\mu,\mu'}^{J}\left(\frac{\pi}{2}\right)=\frac{(-1)^{\mu-\mu'}}{2^{J}}\left(\frac{(J+\mu)!(J-\mu)!}{(J+\mu')!(J-\mu')!}\right)^{1/2}\sum_k(-1)^k\binom{J+\mu'}{k}\binom{J-\mu'}{k+\mu-\mu'},$$

and taking into account the symmetry relation

$$d_{\mu,\mu'}^{J}(\beta)=(-1)^{\mu-\mu'}d_{\mu',\mu}^{J}(\beta),$$

we come to the conclusion that the expansion (2.1) has the form

$$\psi_{Np}(u,v)=(-i)^{N-p}\sum_{m=-N}^{N}d_{p,m}^{N}(\tfrac{1}{2}\pi)\psi_{Nm}(r,\varphi),\tag{2.2}$$

which is in agreement with the data of table A2. The Wigner functions are tabulated completely, and therefore there is no problem in using the expansion (2.2) at certain values of quantum numbers N and p. Now we turn to the expansion of the second parabolic basis over the first one

$$\psi_{Nk}(\bar{u},\bar{v})=\sum_{p=-N}^{N}T_{kp}^{N}\psi_{Np}(u,v).$$

The parabolic coordinates of the first and second type are related by transformation of rotation by angle $\pi/4$

$$\bar{u}=(u+v)/\sqrt{2},\qquad \bar{v}=(u-v)/\sqrt{2}.$$

It has been shown in a paper by Pogosyan *et al* (1981) that under the rotation of the system of coordinates by an angle α, the product of the Hermite polynomials is transformed according to the rule

$$\frac{H_{N+k}(x\cos\alpha-y\sin\alpha)H_{N-k}(x\sin\alpha+y\cos\alpha)}{[(N+k)!(N-k)!]^{1/2}}$$
$$=\sum_{p=-N}^{N}d_{k,p}^{N}(2\alpha)\frac{H_{N+p}(x)H_{N-p}(y)}{[(N+p)!(N-p)!]^{1/2}}.$$

Using this rule, we immediately have

$$\psi_{Nk}(\bar{u},\bar{v})=\sum_{p=-N}^{N}d_{k,p}^{N}(\tfrac{1}{2}\pi)\psi_{Np}(u,v).\tag{2.3}$$

The expansion of the second parabolic basis over the polar one can be obtained by the summation theory (Varshalovich *et al* 1975):

$$\sum_{\mu''=-J}^{J}d_{\mu'',\mu}^{J}(\tfrac{1}{2}\pi)d_{\mu'',\mu}^{J}(\tfrac{1}{2}\pi)\,\bar{e}^{i\frac{1}{2}\pi\mu''}=(-1)^{\mu'}(-i)^{\mu+\mu'}d_{\mu',\mu}^{J}(\tfrac{1}{2}\pi)$$

and has the form

$$\psi_{Nk}(\bar{u}, \bar{v}) = (-\mathrm{i})^{N+k} \sum_{m=-N}^{N} (-\mathrm{i})^m d^N_{k,m}(\tfrac{1}{2}\pi)\psi_{Nm}(r, \varphi). \tag{2.4}$$

The expansion coefficients inverse to the expansions (2.2)-(2.3), coincide with those given in table A2. This fact is a consequence of the orthonormalisation of the fundamental basis leading to the orthonormalisation of the expansion coefficients.

3. Elliptic integral of motion

The necessary information on the elliptic basis is collected in table A3. The elliptic basis (Mardoyan *et al* 1984) $\psi^{(\pm)}_{Nq}(\xi, \eta; R) = f^{(\pm)}_{Nq}(\xi; R) g^{(\pm)}_{Nq}(\eta; R)$ is an eigenfunction of the Hamiltonian $\hat{H}$ and of an additional elliptic integral of motion $\hat{\Lambda}$ obtained by the method of separation of variables. The eigenvalues $\lambda^{(\pm)}_q(R)$ of the operator $\hat{\Lambda}$ have the meaning of separation constants in the elliptic coordinates. The elliptic basis is divided into two sub-bases $\psi^{(+)}_{Nq}(\xi, \eta; R)$ and $\psi^{(-)}_{Nq}(\xi, \eta; R)$, the first being even and the second odd with respect to the change $\eta \to -\eta$. Such a division is possible due to the invariance of the operators $\hat{H}$ and $\hat{\Lambda}$ relative to the inversion with respect to the variable η. A positive integer q numbers in ascending order the values of the separation constants $\lambda^{(\pm)}_q(R)$ and determines the number of zeros of the functions $g^{(\pm)}_{Nq}(\eta; R)$, the so-called angular elliptic functions. In the sub-basis which is even with respect to η, $0 \leq q \leq N$, and in the odd sub-basis $1 \leq q \leq N$. The number of zeros of the radial elliptic function $f^{(\pm)}_{Nq}(\xi; R)$ is given by the difference $N - q$. The total number of zeros of the elliptic basis at a given discrete value of the energy E_N is N. The elliptic parameter R may change within the limits $0 \leq R \leq \infty$. Now we shall show how to determine the form of the elliptic integral of motion $\hat{\Lambda}$ using the method of separation of variables. The process of separation of variables in the elliptic coordinates generates the separation constant Q and leads to a pair of ordinary differential equations

$$(\mathrm{d}^2/\mathrm{d}\xi^2 + R\cosh\xi - \tfrac{1}{4}\omega^2 R^2 \cosh^2\xi)\psi_1(\xi) = -Q\psi_1(\xi)$$
$$(\mathrm{d}^2/\mathrm{d}\eta^2 - R\cos\eta + \tfrac{1}{4}\omega^2 R^2 \cos^2\eta)\psi_2(\eta) = Q\psi_2(\eta).$$

Eliminating the energy parameter ω^2, we get the operator

$$\hat{Q} = \frac{1}{\cosh^2\xi - \cos^2\eta}\left(\cos^2\eta\frac{\partial^2}{\partial\xi^2} + \cosh^2\xi\frac{\partial^2}{\partial\eta^2}\right) - \frac{R\cosh\xi\cos\eta}{\cosh\xi + \cos\eta} \tag{3.1}$$

the eigenvalue of which is the constant Q, and the eigenfunction is the solution of the equation $\hat{H}\psi = E_N\psi$. Since in the limits $R \to 0$ and $R \to \infty$ the elliptic system of coordinates, presented in table A3, transforms into the polar and first parabolic systems, it is clear *a priori* that the operator $\hat{Q}$ should be a linear combination of operators $\hat{L}^2$ and $\hat{\mathscr{P}}$ which in these limits transforms into $\hat{L}^2$ and $\hat{\mathscr{P}}$, respectively. The choice of $\hat{L}^2$ instead of $\hat{L}$ is due to the fact that as $R \to 0$, i.e. $\eta \to \varphi$, the parity with respect to the inversion $\eta \to -\eta$ is conserved. To determine the weight factors and the free constant in the aforementioned linear combination, we turn in expression (3.1) to the Cartesian coordinates and compare the obtained result with the form of the operators $\hat{L}^2$, $\hat{\mathscr{P}}$, and $\hat{H}$ written in terms of the Cartesian coordinates also. After some tedious calculations we get

$$\hat{Q} = -\hat{L}^2 - \omega R\hat{\mathscr{P}} - \tfrac{1}{2}R^2\hat{H}.$$

It follows from this formula that the elliptic bases $\psi^{(+)}_{Nq}(\xi, \eta; R)$ and $\psi^{(-)}_{Nq}(\xi, \eta; R)$ are eigenfunctions of the operator $\hat{\Lambda}$ represented in table A3, which correspond to eigenvalues

$$\lambda^{(\pm)}_q(R) = Q^{(\pm)}_q - \tfrac{1}{4}\omega^2 k^2.$$

In the subsequent formulae we shall use the constants $\lambda^{(\pm)}_q(R)$ rather than $Q^{(\pm)}_q(R)$, and therefore, in describing table A3 we have called them the elliptic separation constants.

4. Elliptic basis

Let us consider the superposition of polar bases

$$\psi^{(\pm)}_{Nq}(\xi, \eta; R) = \sum_{m=-N}^{N} W^{(\pm)}_{Nqm}(R)\psi_{Nm}(r, \varphi) \tag{4.1}$$

for which by definition

$$\hat{\Lambda}\psi^{(\pm)}_{Nq}(\xi, \eta; R) = \lambda^{(\pm)}_q(R)\psi^{(\pm)}_{Nq}(\xi, \eta; R). \tag{4.2}$$

As $\eta \to -\eta$ the polar angle φ changes sign, and therefore, taking account of the symmetry properties of the elliptic sub-bases $\psi^{(+)}_{Nq}(\xi, \eta; R)$ and $\psi^{(-)}_{Nq}(\xi, \eta; R)$ with respect to the inversion $\eta \to -\eta$, we get from (4.1) the following conditions:

$$W^{(\pm)}_{Nq,-m}(R) = \pm W^{(\pm)}_{Nqm}(R). \tag{4.3}$$

Now we substitute (4.1) into (4.2), multiply the resulting equation by $\psi^*_{Nm'}(r, \varphi)$ and integrate over the two-dimensional volume. Then, from the orthonormalisation of the polar basis there follow two systems of homogeneous equations, which should be satisfied by the expansion coefficients (4.1)

$$\sum_{m=-N}^{N} \left(\mathcal{P}_{m'm} + \frac{\lambda^{(\pm)}_q(k) + m'^2}{\omega R}\delta_{mm'} \right) W^{(\pm)}_{Nqm}(R) = 0. \tag{4.4}$$

Here $\mathcal{P}_{m'm}$ denotes the generator $\hat{\mathcal{P}}$ matrix element with respect to the polar bases, i.e.

$$\mathcal{P}_{m'm} = \int \psi^*_{Nm'}(r, \varphi)\hat{\mathcal{P}}\psi_{Nm}(r, \varphi)\, \mathrm{d}v. \tag{4.5}$$

This matrix element can be calculated by two methods depending on whether the operator $\mathcal{P}$ is written in the polar coordinates or whether the polar bases are expanded over the parabolic ones. Let us first write the operator in the polar coordinates

$$\hat{\mathcal{P}} = \frac{1}{\omega}\left[\cos\varphi\left(1 + \frac{1}{r}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{2}\frac{\partial}{\partial r} \right) + \sin\varphi\left(\frac{\partial}{\partial r} - \frac{1}{2r} \right)\frac{\partial}{\partial\varphi} \right].$$

Upon substituting this operator into the expression for the matrix element $\mathcal{P}_{m'm}$ and integrating over the polar angle, we get the formula

$$\mathcal{P}_{m'm} = (1/2\omega)(\delta_{m',m+1}t_{m+1,m} + \delta_{m',m-1}t_{m,m-1})$$

in which the value of $t_{m,m-1}$ is obtained from $t_{m+1,m}$ by changing $m \to m-1$, and $t_{m+1,m}$ has the form

$$t_{m+1,m} = \int_0^\infty rR^{(r)}_{N,m+1}\{1 + (m + \tfrac{1}{2})[(\mathrm{d}/\mathrm{d}r) - (m/r)]\}R^{(r)}_{Nm}\, \mathrm{d}r.$$

A direct calculation of this integral is very tedious and it is easier to use the following method. We write down two Schrödinger equations, one with the quantum number m and the other with m', multiply the first equation by $R_{Nm'}(r)$ and the second by $R^{(r)}_{Nm}$, subtract one from the other, and integrate over $\mathrm{d}r$. Hence, one can easily establish the identity

$$\int_0^\infty rR^{(r)}_{Nm'}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{m'^2-m^2-1}{2r}\right)R^{(r)}_{Nm}\,\mathrm{d}r=0.$$

It follows from this identity that the contribution to the integral $t_{m+1,m}$ comes only from the first term in brackets, so that

$$t_{m+1,m}=\int_0^\infty rR^{(r)}_{N,m+1}R^{(r)}_{Nm}\,dr=-\omega[(N-m)(N+m+1)]^{1/2}$$

and for the matrix element $\mathscr{P}_{m'm}$ we have

$$\mathscr{P}_{m'm}=-\tfrac{1}{2}[(N-m)(N+m+1)]^{1/2}\,\delta_{m,m'-1}-\tfrac{1}{2}[(N+m)(N-m+1)]^{1/2}\,\delta_{m,m'+1}. \tag{4.6}$$

Following now the second method, substituting into (4.5) the expansion of the polar basis over the first parabolic basis, one can immediately get

$$\mathscr{P}_{m'm}=\sum_{p=-N}^{N} p\,d^{N}_{p,m}(\tfrac{1}{2}\pi)\,d^{N}_{p,m'}(\tfrac{1}{2}\pi).$$

The sum on the right is calculated using the recurrence relation known from the angular momentum theory (Varshalovich *et al* 1975):

$$-\mu d^{J}_{\mu,\mu'}(\tfrac{1}{2}\pi)=\tfrac{1}{2}[(J+\mu')(J-\mu'+1)]^{1/2}d^{J}_{\mu,\mu'-1}(\tfrac{1}{2}\pi)+\tfrac{1}{2}[(J-\mu')(J+\mu'+1)]^{1/2}d^{J}_{\mu,\mu'+1}(\tfrac{1}{2}\pi) \tag{4.7}$$

and the orthonormalisation condition

$$\sum_{\mu''=-J}^{J} d^{J}_{\mu'',\mu}(\beta)\,d^{J}_{\mu'',\mu'}(\beta)=\delta_{\mu\mu'}.$$

This simple calculation gives the result obtained above (4.6). Formula (4.6) and the system of equations (4.1) result in the trinomial recurrence relations:

$$\begin{aligned}\{[\lambda^{(\pm)}_q(R)+m^2/\omega R]\}\,&W^{(\pm)}_{Nqm}(R)\\ &=\tfrac{1}{2}[(N-m)(N+m+1)]^{1/2}\,W^{(\pm)}_{Nqm+1}(R)\\ &\quad+\tfrac{1}{2}[(N+m)(N-m+1)]^{1/2}\,W^{(\pm)}_{Nqm-1}(R)\end{aligned} \tag{4.8}$$

$$\sum_{m=-N}^{N} W^{(\pm)*}_{Nqm}(R)\,W^{(\pm)}_{Nqm}(R)=1. \tag{4.9}$$

The result obtained together with the normalisation condition is a starting point in the programme of calculating the expansion coefficients (4.1). Some expressions for $W^{(\pm)}_{Nqm}(R)$ are given in tables A4 and A5. The latter also represent the equations which are used to determine the eigenvalues of the separation constants. The method described above may be applied to obtain the exact expansion of the elliptic basis in terms of the parabolic basis

$$\psi^{(\pm)}_{Nq}(\xi,\eta;R)=\sum_{p=-N}^{N} u^{(\pm)}_{Nqp}(R)\,\psi_{Np}(u,v). \tag{4.10}$$

Here, as well as in the expansion (4.1), the LHS is not assumed to be known *a priori*, and the constructive idea consists in that $\psi_{Nq}^{(\pm)}(\xi, \eta; R)$ satisfies equation (4.2). In this way the calculations lead to the trinomial recurrence relation

$$[\lambda_q^{(\pm)}(R)+\omega pR+\tfrac{1}{2}(N^2+N-p^2)]u_{Nqp}^{(\pm)}(R)$$
$$=\tfrac{1}{4}[(N-p)(N-p-1)(N+p+1)(N+p+2)]^{1/2}u_{Nqp+2}^{(\pm)}(R)$$
$$+\tfrac{1}{4}[(N+p)(N+p-1)(N-p+1)(N-p+2)]^{1/2}u_{Nqp-2}^{(\pm)}(R) \qquad (4.11)$$

which should be solved with the normalisation conditions

$$\sum_{p=-N}^{N} u_{Nqp}^{(\pm)*}(R)u_{Nqp}^{(\pm)}(R)=1. \qquad (4.12)$$

Substituting into (4.1) the expansion of the polar basis over the first parabolic one and having compared the result obtained with (4.10), we have

$$u_{Nqp}^{(\pm)}(R)=(\mathrm{i})^{N-p}\sum_{m=-N}^{N} d_{p,m}^{N}(\tfrac{1}{2}\pi)\,W_{Nqm}^{(\pm)}(R). \qquad (4.13)$$

The values of the coefficients $u_{Nqp}^{(\pm)}(R)$, collected in tables A6 and A7, are calculated on the basis of this formula. Note that the Wigner d-function has been chosen with the same phases as in the monograph by Varshalovich *et al* (1975). At the known $W_{Nqm}^{(\pm)}(R)$, formula (4.13) is more convenient for calculating the coefficients $u_{Nqp}^{(\pm)}(R)$ than the recurrence relation (4.10).

5. Limits $R\to 0$ and $R\to\infty$ in the elliptic basis

In view of the complexity of the elliptic basis, it is expedient to consider separately the limiting transitions in the formula obtained as $R\to 0$ and $R\to\infty$. In the elliptic coordinates we have chosen the Coulomb field centre as the origin of the coordinates. Therefore, in the limits $R\to 0$ and $R\to\infty$ (the proton and electron coordinates are thought to be fixed) the elliptic coordinates turn into the polar and first parabolic ones. It is obvious that the elliptic sub-basis does not change its parity in the course of such limiting transitions. This means that as $R\to 0$ the transition may proceed to the polar sub-bases with a given parity with respect to the inversion $\varphi\to-\varphi$. In these sub-bases the exponent $e^{im\varphi}$ is substituted by the cosine and sine of $m\varphi$; in this case $0\leqslant m\leqslant N$ for an even basis and $1\leqslant m\leqslant N$ for an odd basis, so that the multiplicity of degeneracy of the energy spectrum would be equal, as usual, to $2N+1$. The parabolic sub-bases with a given parity have the form $f_{N+p}^{(\pm)}(u)g_{N-p}^{(\pm)}(v)$. In this case for an even basis the quantum numbers N and p have equal parity and for an odd basis different parity, so that in the first case $p=-N, -N+2, \ldots, N-2, N$ (in all, $N+1$ values), and in the second case $p=-N+1, -N+3, \ldots, N-3, N-1$ (in all, N values). The possibility of transformation of the elliptic basis into the polar and parabolic ones follows from an explicit form of the operator $\hat{\Lambda}$. Indeed, as $R\to 0$ and $R\to\infty$ this operator tends to $-\hat{L}^2$ and $-\omega R\hat{\mathscr{P}}$, respectively. Hence, it is clear that the behaviour of the elliptic separation constants $\lambda_q^{(\pm)}(R)$ as $R\to 0$ and $R\to\infty$ is determined by table A8. Under these limiting transitions the quantum number q conserves the meaning of a quantity providing the number of zeros of the angular elliptic function. Now we consider the behaviour of the coefficients $W_{Nqm}^{(+)}(R)$ and $W_{Nqm}^{(-)}(R)$ in the limits under consideration.

From (4.1) it follows that

$$W^{(\pm)}_{Nqm}(R) = \int \psi^{*}_{Nm}(r, \varphi)\psi^{(\pm)}_{Nq}(\xi, \eta; R)\, \mathrm{d}v \tag{5.1}$$

and therefore the dependence on R of the coefficients is determined only by the quantum numbers N and q. This means that as $R \to 0$ and $R \to \infty$ all the coefficients $W^{(\pm)}_{Nqm}(R)$ in the trinomial recurrence relation (4.8) have the same order with respect to R. Now we multiply (4.8) by ωR and make R tend to zero. It follows from table A8 that in this limit the expansion coefficients (4.1) equal zero at $m \neq q$. The values of $W^{(\pm)}_{Nqm}(R)$ different from zero are then determined from the normalisation condition (4.9) and equal

$$W^{(\pm)}_{Nqm}(0) = \tfrac{1}{2}\,\delta_{qm}, \qquad 1 \leq m \leq N$$

$$W^{(\pm)}_{Nq0}(0) = \delta_{q0}.$$

Table A8 also shows that within the limit $R \to \infty$, (4.8) turns into two relations

$$-(2q-N)\,W^{(+)}_{Nqm}(\infty) = \tfrac{1}{2}[(N-m)(N+m+1)]^{1/2}\,W^{(+)}_{Nqm+1}(\infty) + \tfrac{1}{2}[(N+m)(N-m+1)]^{1/2}\,W^{(+)}_{Nqm-1}(\infty)$$

$$-(2q-N-1)\,W^{(-)}_{Nqm}(\infty) = \tfrac{1}{2}[(N-m)(N+m+1)]^{1/2}\,W^{(-)}_{Nqm+1}(\infty) + \tfrac{1}{2}[(N+m)(N-m+1)]^{1/2}\,W^{(-)}_{Nqm-1}(\infty).$$

The comparison of these formulae with the recurrence relations (4.7) convinces us that the limiting values of the coefficients $W^{(\pm)}_{Nqm}(\infty)$ coincide with the Wigner d-function of the right angle up to the phase factor, which may depend on quantum numbers N and q alone. To provide the transformation of the expansion (4.1) as $R \to \infty$ into (2.2), it suffices to choose the phase factors as follows:

$$W^{(+)}_{Nqm}(\infty) = (-\mathrm{i})^{N-(2q-N)}\, d^{N}_{2q-N,m}(\tfrac{1}{2}\pi)$$

$$W^{(-)}_{Nqm}(\infty) = (-\mathrm{i})^{N-(2q-N-1)}\, d^{N}_{2q-N-1,m}(\tfrac{1}{2}\pi).$$

Then, using formula (4.13), it is obvious that within the limits $R \to 0$ and $R \to \infty$ the expansion (4.10) turns into the expansion of the polar basis over the first parabolic one and into the identical transformation respectively.

6. Conclusion

A complete analysis of the two-dimensional hydrogen atom should also include the case $E > 0$, when O(2, 1) becomes the group of hidden symmetry. This problem is very complicated and we shall undertake it in the near future.

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Appendix

Table A1. Generators of the group of hidden O(3) symmetry and the fundamental bases.

$x = r\cos\varphi$	$0 \leqslant r < \infty$	$\hat{H}\psi_{Nm} = E_N\psi_{Nm}$	$\psi_{Nm} = \left(\frac{\pi\omega^3}{\pi}\right)^{1/2}\left(\frac{(N+\lvert m\rvert)!}{(N-\lvert m\rvert)!}\right)^{1/2}(2\omega r)^{\lvert m\rvert}/(2\lvert m\rvert)!\, \mathrm{e}^{-\omega r}F(-N+\lvert m\rvert; 2\lvert m\rvert+1; 2\omega r)\,\mathrm{e}^{\mathrm{i}m\varphi}$
$y = r\sin\varphi$	$0 \leqslant \varphi \leqslant 2\pi$	$\hat{L}\psi_{Nm} = m\psi_{Nm}$	$\hat{L} = -\mathrm{i}\,\partial/\partial\varphi$; $-N \leqslant m \leqslant N$ and integer, $\omega = (-2E_N)^{1/2}$
$x = \frac{u^2 - v^2}{2}$	$0 \leqslant u < \infty$	$\hat{H}\psi_{Np} = E_N\psi_{Np}$	$\psi_{Np} = \left(\frac{\omega^3}{\pi}\right)^{1/2}\frac{\mathrm{e}^{-\frac{1}{2}\omega(u^2+v^2)}}{[2^{2N}(N+p)!(N-p)!]^{1/2}}H_{N+p}(\sqrt{\omega}u)H_{N-p}(\sqrt{\omega}v)$, ($H$ are Hermite polynomials)
$y = uv$	$-\infty < v < \infty$	$\hat{\mathscr{P}}\psi_{Np} = p\psi_{Np}$	$\hat{\mathscr{P}} = \frac{1}{2\omega}\frac{1}{u^2+v^2}\left(u^2\frac{\partial^2}{\partial v^2} - v^2\frac{\partial^2}{\partial u^2} + 2(u^2 - v^2)\right)$
$x = \bar{u}\bar{v}$	$0 \leqslant \bar{u} < \infty$	$\hat{H}\psi_{Nk} = E_N\psi_{Nk}$	ψ_{Nk} and $\hat{K}$ are obtained from ψ_{Np} and $\hat{\mathscr{P}}$ by the change of $p \to k$, $u \to \bar{u}$, $v \to \bar{v}$, p and $k \in [-N, N]$ and integers
$y = \frac{\bar{u}^2 - \bar{v}^2}{2}$	$-\infty < \bar{v} < \infty$	$\hat{K}\psi_{Nk} = k\psi_{Nk}$	

Table A2. Coefficients of the fundamental interbasis expansions.

		$\psi_{Nm}(r,\varphi)$	$\psi_{Np}(u,v)$	$\psi_{Nk}(\bar{u},\bar{v})$
22034	$\psi_{Nm}(r,\varphi)$	1	$(\mathrm{i})^{N-p}d^{N}_{pm}(\pi/2)$	$(\mathrm{i})^{N+kl+m}d^{N}_{km}(\pi/2)$
22035	$\psi_{Np}(u,v)$	$(-\mathrm{i})^{N-p}d^{N}_{pm}(\pi/2)$	1	$d^{N}_{kp}(\pi/2)$
22036	$\psi_{Nk}(\bar{u},\bar{v})$	$(-\mathrm{i})^{N+k+m}d^{N}_{km}(\pi/2)$	$d^{N}_{kp}(\pi/2)$	1

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Table A3. Elliptic integral of motion and elliptic basis.

$x=\frac{R}{2}(\cosh\xi\cos\eta+1)$	$0\leqslant\xi<\infty$	$\hat{H}\psi^{(\pm)}_{Nq}=E_N\psi^{(\pm)}_{Nq}$	$\hat{H}=-\frac{2}{R^2(\cosh^2\xi-\cos^2\eta)}\times\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)+\frac{1}{R(\cosh\xi+\cos\eta)}$
$y=\frac{R}{2}\sinh\xi\sin\eta$	$0\leqslant\eta\leqslant2\pi$	$\hat{\Lambda}\psi^{(\pm)}_{Nq}=\lambda^{(\pm)}_{Nq}\psi^{(\pm)}_{Nq}$	$\hat{\Lambda}=-\hat{L}^2-\omega R\hat{\mathscr{P}}$

Table A4. Expansion coefficients of the even elliptic sub-basis over the polar one.

N	m	$W^{(+)}_{Nqm}(R)$	$\lambda^{(+)}_q$
0	0	1	$\lambda^{(+)}_q=0$
1	0	$\left(\frac{\lambda^{(+)}_q+1}{2\lambda^{(+)}_q+1}\right)^{1/2}$	$\lambda^{(+)}_q(\lambda^{(+)}_q+1)=\frac{4}{9}R^2$
1	±1	$\frac{3}{2\sqrt{2}}\frac{\lambda^{(+)}_q}{R}\left(\frac{\lambda^{(+)}_q+1}{2\lambda^{(+)}_q+1}\right)^{1/2}$	
2	0	$\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	
2	±1	$\frac{5}{2\sqrt{6}}\frac{\lambda^{(+)}_q}{R}\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	$\lambda^{(+)}_q(\lambda^{(+)}_q+1)(\lambda^{(+)}_q+4)=\frac{16R^2}{25}(\lambda^{(+)}_q+3)$
2	±2	$\frac{1}{\sqrt{6}}\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	

Table A5. Expansion coefficients of the even elliptic sub-basis over the polar one.

N	m	$W^{(-)}_{Nqm}(R)$	$\lambda^{(-)}_q$
1	±1	$\pm\frac{1}{\sqrt{2}}$	$\lambda^{(-)}_q=-1$
2	±1	$\pm\frac{1}{\sqrt{2}}\left(\frac{\lambda^{(-)}_q+4}{2\lambda^{(-)}_q+5}\right)^{1/2}$	$(\lambda^{(-)}_q+1)(\lambda^{(-)}_q+4)=\frac{4}{25}R^2$
2	±2	$\pm\frac{5}{2\sqrt{2}}\frac{\lambda^{(-)}_q+1}{R}\left(\frac{\lambda^{(-)}_q+4}{2\lambda^{(-)}_q+5}\right)^{1/2}$	
3	±1	$\pm\frac{1}{\sqrt{2}}\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	$(\lambda^{(-)}_q+1)(\lambda^{(-)}_q+4)(\lambda^{(-)}_q+9)=\frac{16R^2}{49}(\lambda^{(-)}_q+6)$
3	±2	$\pm\frac{7}{2\sqrt{5}}\frac{\lambda^{(-)}_q+1}{R}\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	
3	±3	$\pm\left(\frac{3}{10}\right)^{1/2}\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	

Table A6. Expansion coefficients of the even elliptic sub-basis over the parabolic one.

N	p	$U^{(+)}_{Nqp}(R)$	$\lambda^{(+)}_q$
0	0	1	$\lambda^{(+)}_q=0$
1	−1	$-\frac{3}{4R}\left(\lambda^{(+)}_q-\frac{2}{3}R\right)\left(\frac{2(\lambda^{(+)}_q+1)}{2\lambda^{(+)}_q+1}\right)^{1/2}$	$\lambda^{(+)}_q(\lambda^{(+)}_q+1)=\frac{4}{9}R^2$
1	1	$+\frac{3}{4R}\left(\lambda^{(+)}_q-\frac{2}{3}R\right)\left(\frac{2(\lambda^{(+)}_q+1)}{2\lambda^{(+)}_q+1}\right)^{1/2}$	
2	−2	$\frac{25}{24}\left(\frac{3}{2}\right)^{1/2}\frac{\lambda^{(+)}_q}{R^2}\left(\lambda^{(+)}_q+1+\frac{4R}{5}\right)\times\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	$\lambda^{(+)}_q(\lambda^{(+)}_q+1)(\lambda^{(+)}_q+4)=\frac{16R^2}{25}(\lambda^{(+)}_q+3)$
2	0	$\frac{2}{\lambda^{(+)}_q+4}\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	
2	2	$\frac{25}{24}\left(\frac{3}{2}\right)^{1/2}\frac{\lambda^{(+)}_q}{R^2}\left(\lambda^{(+)}_q+1-\frac{4R}{5}\right)\times\left[1+\frac{25}{12}\left(\frac{\lambda^{(+)}_q}{R}\right)^2+\frac{1}{3}\left(\frac{\lambda^{(+)}_q}{\lambda^{(+)}_q+4}\right)^2\right]^{-1/2}$	

Table A7. Expansion coefficients of the even elliptic sub-basis over the parabolic one.

N	p	$U^{(-)}_{Nqp}(R)$	$\lambda^{(-)}_q$
1	0	i	$\lambda^{(-)}_q=-1$
2	−1	$\frac{i}{\sqrt{2}}\frac{5}{2R}\left(\lambda^{(-)}_q+1+\frac{2R}{5}\right)\left(\frac{\lambda^{(-)}_q+4}{2\lambda^{(-)}_q+5}\right)^{1/2}$	$(\lambda^{(-)}_q+4)=\frac{4}{25}R^2$
2	1	$+\frac{i}{\sqrt{2}}\frac{5}{2R}\left(\lambda^{(-)}_q+1-\frac{2R}{5}\right)\left(\frac{\lambda^{(-)}_q+4}{2\lambda^{(-)}_q+5)}\right)^{1/2}$	
3	−2	$i\frac{49}{8\sqrt{5}}\frac{\lambda^{(-)}_q+1}{R^2}\left(\lambda^{(-)}_q+4+\frac{4R}{7}\right)\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	$(\lambda^{(-)}_q+1)(\lambda^{(-)}_q+4)(\lambda^{(-)}_q+9)$
3	0	$i\frac{2\sqrt{6}}{\lambda^{(-)}_q+9}\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	$=\frac{16R^2}{49}(\lambda^{(-)}_q+6)$
3	2	$i\frac{49}{8\sqrt{5}}\frac{\lambda^{(-)}_q+1}{R^2}\left(\lambda^{(-)}_q+4-\frac{4R}{7}\right)\times\left[1+\frac{49}{10}\left(\frac{\lambda^{(-)}_q+1}{R}\right)^2+\frac{3}{5}\left(\frac{\lambda^{(-)}_q+1}{\lambda^{(-)}_q+9}\right)^2\right]^{-1/2}$	

Table A8. Behaviour of the elliptic separation constants within the limits $R\to 0$ and $R\to\infty$.

$\lambda^{(+)}_q(R)\xrightarrow{R\to 0}-q^2;\ 0\leqslant q\leqslant N$	$\lambda^{(+)}_q(R)\xrightarrow{R\to\infty}-\omega R(2q-N);\ 0\leqslant q\leqslant N$
$\lambda^{(-)}_q(R)\xrightarrow{R\to 0}-q^2;\ 1\leqslant q\leqslant N$	$\lambda^{(-)}_q(R)\xrightarrow{R\to\infty}-\omega R(2q-N-1);\ 1\leqslant q\leqslant N$

References

Bargmann V 1936 *J. Physique* **99** 576

Cisneros A and McIntosh H 1968 *J. Math. Phys.* **10** 277

Englefield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley)

Fock V 1935 *J. Physique* **98** 145

Komarov I, Ponomarev L I and Slavyanov S Yu 1976 *Spheroidal and Coulomb Spheroidal Functions* (Moscow: Nauka) (in Russian)

Malkin I and Man'ko V 1979 *Dynamic Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka) (in Russian)

Mardoyan L, Pogesyan G S, Sissakian A N and Ter-Antonyan V M 1984 *Teor. Mat. Fis.* **61** 99

Miller W Jr 1977 *Symmetry and Separation of Variables* (London: Addison-Wesley)

Pogosyan G, Smorodinsky Ya A and Ter-Antonyan V M 1981 *J. Phys. A: Math. Gen.* **14** 769

Varshalovich D *et al* 1975 *Quantum Theory of Angular Momentum* (Leningrad: Nauka) (in Russian)

Zaslow B and Zandler M 1965 *Am. J. Phys.* **35** 1118