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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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SCALAR PARTICLE SCATTERING AMPLITUDE ANALYZED ON THE BASIS OF THE BETHE-SALPETER EQUATION

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Изучение амплитуды рассеяния скалярных частиц на основе уравнения Бете-Солпитера

С помощью операторного метода, использующего модифицированную теорию возмущений в экспоненте, решается уравнение Бете-Солпитера для двухчастичной функции Грина скалярных частиц в случае обобщенного локального ядра. Получено аналитическое представление для функции Грина в первом порядке указанной теории. Найдена амплитуда рассеяния двух скалярных частиц в обобщенном лестничном приближении. В пределе больших S для амплитуды получено представление эйконального типа.

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Scalar Particle Scattering Amplitude Analyzed on the Basis of the Bethe-Salpeter Equation

The Bethe-Salpeter equation for the two-particle Green function is solved in the case of the generalized local kernel with the help of the operator method using the modified perturbation theory in exponent. The scalar particle scattering amplitude in the ladder approximation is found and its asymptotic in the region of high energies and fixed momentum transfers is investigated. The eikonal representation of the scattering amplitude with the phase corresponding to superposition of the Yukawa quasipotentials is obtained.

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Dubna, 1971

§1. Introduction

As is well known the problems of the quantized field theory may be reduced to finding the Green functions of the interacting particles. For this purpose the functional integration methods are widely used [1,2,3]. The calculation of the functional integrals however may be performed only in the case of the Gauss type functions.

On this account it is of considerable interest to find the two-particle Green function directly from the Bethe-Salpeter equation using the operator method and modified perturbation theory[3]. Such an attempt was undertaken in the paper[4]. Here the Bethe-Salpeter equation is solved with the help of the mentioned methods in the case of the scalar particles and the generalized local kernel.

In the second paragraph the Green function's representation through coefficient functions $T_n(x,\kappa,\nu)$ is found and the equations for the latter are written out.

In the third paragraph the Green function in the first order of the modified perturbation theory is calculated. It must be noted that this approximation is nothing else than the so-called " $k_i k_j = 0$ approximation" introduced in the papers[2,3,5]

In the fourth paragraph the scattering amplitude is derived in the above mentioned approximation, which corresponds to the sum of the ladder-graphs with the propagators in which the terms of the $K_i K_j$ -type are rejected.

The fifth paragraph is devoted to the investigation of the asymptotic behaviour of the found scattering amplitude in the region $S \to \infty$, t - fixed. The eikonal type representation for the ladder approximation is obtained and examples of BS equation's kernels corresponding to the smooth quasipotentials are given.

Note that recently the straight-line paths approximation was suggested for the investigation of the asymptotic of the sums of the ladder-type graphs with crossings /6,7,8/. The essence of the method is following: in the scattering process the nucleons conserve their "individuality" because of the "softness" of the virtual mesons. In the case of scalar particle exchange an assumption of "softness" appears to be essential for the correct reproduction of the asymptotics of mentioned diagrams with the help of the eikonal formula /13/.

On the other hand the condition of "softness" does not change the asymptotic of the pure ladder-type graphs /11/.

§2. The Operator Method of Solving the Bethe-Salpeter Equation

On account of the translation invariance the Green function in the momentum space depends only upon three independent variables p , q and E , which are connected with the external momenta by the following relations:

$$p = \frac{q_1 - p_1}{2}, \quad q = \frac{q_2 - p_2}{2}, \quad E = p_1 + q_1 .$$

where p_1 and q_1 are the momenta of the incoming particles, p_2 and q_2 - the momenta of the outgoing ones.

In this case the BS equation is

$$[(p+\frac{E}{2})^2-m^2+i\varepsilon][(p-\frac{E}{2})^2-m^2+i\varepsilon]G(p,q/E)=$$
$$=\delta^{(4)}(p-q)+\frac{g^2}{(2\pi)^4 i}\int d^4p' K(p,p'/E)\ G(p',q/E). \tag{1}$$

We shall consider only the kernels $K(p,p'/E)$, which depend upon the difference $p-p'$. Writing the equation (1) in the following form

$$[(p^2+\frac{E^2}{4}-m^2)^2-(pE)^2+2i\varepsilon(p^2+\frac{E^2}{4}-m^2)]G(p,q/E)=$$
$$=\delta^{(4)}(p-q)+\frac{g^2}{(2\pi)^4 i}\int d^4p' K(p-p'/E)G(p',q/E) \tag{2}$$

and performing the Fourier transformation

$$G(p,q/E)=\int d^4x\,d^4y\,e^{-ipx+iqy}G(x,y/E), \tag{3}$$

$$K(p-p'/E)=\int d^4x\,e^{-i(p-p')x}K(x/E) , \tag{4}$$

we can rewrite our equation in the coordinate space

$$[(-\Box+\frac{E^2}{4}-m^2)^2+(E^\mu\partial_\mu)^2+2i\varepsilon(-\Box+\frac{E^2}{4}-m^2)]G(x,y/E)=$$
$$=\frac{1}{(2\pi)^4}\delta^{(4)}(x-y)-i\lambda K(x/E)\ G(x,y/E). \tag{5}$$

Here we have introduced $\lambda = g^2$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

To solve the equation (5) we shall use the method of the Fock's fifth parameter

$$G(x,y/E) = i\int_0^\infty d\nu\, \Phi(\nu)\,, \tag{6}$$

where $\Phi(\nu)$ is determined by the symbolic relation

$$\Phi(\nu) = \exp\{-i\nu[(-\Box + \frac{E^2}{4} - m^2)^2 + (E^\mu\partial_\mu)^2 + i\lambda K(x/E) + 2i\varepsilon(-\Box + \frac{E^2}{4} - m^2)]\}\frac{\delta^{(4)}(x-y)}{(2\pi)^4}$$

and satisfies the following differential equation with the boundary conditions

$$i\frac{\partial\Phi(\nu)}{\partial\nu} = [(-\Box + \frac{E^2}{4} - m^2)^2 + (E^\mu\partial_\mu)^2 + i\lambda K(x/E) + 2i\varepsilon(-\Box + \frac{E^2}{4} - m^2)]\,\Phi(\nu)\,, \tag{7}$$

$$\Phi(\nu=0) = \frac{1}{(2\pi)^4}\,\delta^{(4)}(x-y)\,.$$

We shall seek $\Phi(\nu)$ in the following form

$$\Phi(\nu) = \frac{1}{(2\pi)^8}\int d^4k\, \exp\, i\{T(x,k,\nu) + k(x-y) + [(kE)^2 - (k^2 + \frac{E^2}{4} - m^2)^2 - 2i\varepsilon(k^2 + \frac{E^2}{4} - m^2)]\nu\}\,. \tag{8}$$

It is obvious that the function $T(x,\kappa,\nu)$ satisfies the boundary condition

$$T(x,\kappa,\nu=0)=0. \tag{9}$$

Calculations, as simple as they are cumbrous, result in the following equation for $T(x,\kappa,\nu)$

$$\begin{aligned}\frac{\partial T}{\partial\nu} &= -i\Box^2 T + 2(\partial_\mu\partial_\nu T)^2 + 4(\partial^\mu T)(\Box\partial_\mu T) + (\Box T)^2 + \\ &+ 4\kappa^\mu(\Box\partial_\mu T) + 4i(\partial^\mu T)(\partial^\nu T)(\partial_\mu\partial_\nu T) + i(4\kappa^\mu\kappa^\nu - \\ &- E^\mu E^\nu)(\partial_\mu\partial_\nu T) + 8i\kappa^\mu(\partial^\nu T)(\partial_\mu\partial_\nu T) + 2i(\Box T)(\partial_\mu T)^2 + \\ &+ 2i\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)(\Box T) + 4i\kappa^\mu(\partial_\mu T)(\Box T) - (\partial_\mu T)^4 - \\ &- (4\kappa^\mu\kappa^\nu - E^\mu E^\nu)(\partial_\mu T)(\partial_\nu T) - 2\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)(\partial_\mu T)^2 - \\ &- 4\kappa^\mu(\partial_\mu T)(\partial_\nu T)^2 - 4\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)\kappa^\mu(\partial_\mu T) + \\ &+ 2(E\kappa)E^\mu(\partial_\mu T) - i\lambda K(x/E).\end{aligned} \tag{10}$$

Now we seek the solution of the equation (10) in the form of the power series

$$T(x,\kappa,\nu) = \sum_{n=1}^{\infty}\lambda^n T_n(x,\kappa,\nu). \tag{11}$$

For the coefficient function T_1 we obtain the following equation

$$\begin{aligned}\frac{\partial T_1}{\partial\nu} &= -i\Box^2 T_1 + 4\kappa^\mu(\Box\partial_\mu T_1) + i(4\kappa^\mu\kappa^\nu - E^\mu E^\nu)\cdot \\ &(\partial_\mu\partial_\nu T_1) + 2i\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)(\Box T_1) - 4\left(\kappa^2 + \frac{E^2}{4} - \right. \\ &\left. - m^2 + i\varepsilon\right)\kappa^\mu(\partial_\mu T_1) + 2(E\kappa)E^\mu(\partial_\mu T_1) - iK(x/E)\end{aligned} \tag{12}$$

and

$$T_1(x,\kappa,\nu=0)=0. \tag{13}$$

It is not difficult to derive an equation for the arbitrary coefficient function $T_n(x, \kappa, \nu)$. Keeping in mind the possible future applications we shall write down the equation for $T_2(x,\kappa,\nu)$

$$\frac{\partial T_2}{\partial \nu} = -i\Box^2 T_2 + 4\kappa^\mu(\Box\partial_\mu T_2) + i(4\kappa^\mu\kappa^\nu - E^\mu E^\nu)(\partial_\mu\partial_\nu T_2) + 2i\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)(\Box T_2) - 4\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)\kappa^\mu(\partial_\mu T_2) + 2(E\kappa)E^\mu(\partial_\mu T_2) - iS(x,\kappa,\nu) \tag{14}$$

and

$$T_2(x,\kappa,\nu=0) = 0, \tag{15}$$

where

$$S(x,\kappa,\nu) = i\Big\{2(\partial_\mu\partial_\nu T_1)^2 + 4(\partial^\mu T_1)(\Box\partial_\mu T_1) + (\Box T_1)^2 + 8i\kappa^\mu(\partial^\nu T_1)(\partial_\mu\partial_\nu T_1) + 4i\kappa^\mu(\partial_\mu T_1)(\Box T_1) - (4\kappa^\mu\kappa^\nu - E^\mu E^\nu)(\partial_\mu T_1)(\partial_\nu T_1) - 2\left(\kappa^2 + \frac{E^2}{4} - m^2 + i\varepsilon\right)(\partial_\mu T_1)^2\Big\}. \tag{16}$$

Let us note that the equations (12) and (14) differ only in functions $K(x/E)$ and $S(x,\kappa,\nu)$.

§3. The Green Function of the Bethe-Salpeter Equation

At first we shall solve the equations (12) and (14). Passing to the momentum space

$$T_2(x,\kappa,\nu) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}\, \check{T}_2(p,\nu),$$

$$S(x,\kappa,\nu) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}\, \check{S}(p,\nu),$$

we obtain the equation

$$\frac{\partial \check{T}_2}{\partial \nu} = -i\left\{\left[\rho^2+2(\kappa\rho)\right]^2+2\left[\rho^2+2(\kappa\rho)\right]\left(\kappa^2+\frac{E^2}{4}-m^2+i\varepsilon\right)-\right.$$

$$\left.-(E,\rho+\kappa)^2+(E\kappa)^2\right\}\check{T}_2-i\check{S}(\rho,\nu), \tag{17}$$

from which

$$\check{T}_2(\rho,\nu) = -i\int_0^\nu d\nu'\,\check{S}(\rho,\nu-\nu')\exp\left\{-i\nu'\left\{\left[\rho^2+2(\kappa\rho)\right]^2+\right.\right.$$

$$\left.\left.+2\left(\kappa^2+\frac{E^2}{4}-m^2+i\varepsilon\right)\left[\rho^2+2(\kappa\rho)\right]-(E,\rho+\kappa)^2+(E\kappa)^2\right\}\right\}. \tag{18}$$

If we substitute $\check{S}(\rho,\nu-\nu')$ for $K(\rho/E)$ and take into account that the kernel K does not depend upon ν, (18) turns into the solution for the function $\check{T}_1(\rho,\nu)$

$$\check{T}_1(\rho,\nu) = -i\int_0^\nu d\nu'\,K(\rho/E)\exp\left\{-i\nu'\left\{\left[\rho^2+2(\kappa\rho)\right]^2+\right.\right.$$

$$\left.\left.+2\left(\kappa^2+\frac{E^2}{4}-m^2+i\varepsilon\right)\left[\rho^2+2(\kappa\rho)\right]-(E,\rho+\kappa)^2+(E\kappa)^2\right\}\right\}. \tag{19}$$

Let us denote

$$\mathcal{E}(\kappa,\rho) = \left[\rho^2+2(\kappa\rho)\right]^2+$$

$$+2\left(\kappa^2+\frac{E^2}{4}-m^2\right)\left[\rho^2+2(\kappa\rho)\right]-(E,\rho+\kappa)^2+(E\kappa)^2$$

and

$$I(\rho,\kappa) = 2\varepsilon\left[\rho^2+2(\kappa\rho)\right].$$

Then (19) takes the form

$$\check{T}_1(p,\nu) = K(p,E)\,\frac{e^{-i\nu[R(p,k)+i\,I(p,k)]}-1}{R(p,k)+i\,I(p,k)}\,, \qquad (20)$$

and we obtain

$$T_1(x,k,\nu) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx} K(p/E)\,\frac{e^{-i\nu[R(p,k)+iI(p,k)]}-1}{R(p,k)+i\,I(p,k)}\,. \qquad (21)$$

Consider $T_1(x,k,\nu)$ as a function of the complex variable $\nu = \nu_1 + i\,\nu_2$. Integral (21) converges if

$$Re\,(-i\nu_1+\nu_2)\,[\,R(p,k)+i\,I(p,k)] < 0$$

for $p \to \infty$.

Since $R(p,k) = p^4 + O(p^3)$ and $I(p,k) = O(p^2)$ when $p \to \infty$, $T_1(x,k,\nu)$ is defined by the formula (21) in the lower half-plane $\nu_2 < 0$ of the complex variable ν . To obtain the function $T_1(x,k,\nu)$ one must perform an analytic continuation to the real axes. Hence we have

$$T_1(x,k,\nu-i\delta) = \frac{1}{(2\pi)^4}\int d^4p\, K(p/E)\, e^{ipx}\,\frac{e^{-i(\nu-i\delta)[R+iI]}-1}{R(p,k)+iI(p,k)}\,.$$

Replacing p by $p_1 = \dfrac{p}{1-i\alpha}$, where $\alpha > 0$ is a small quantity, we obtain

$$T_1(x,\kappa,\nu-i\delta)=\frac{(1-i\alpha)^4}{(2\pi)^4}\int_\gamma d^4\rho_1\, K(\rho_1(1-i\alpha)/E)\,e^{i(1-i\alpha)\rho_1 x}\cdot$$

$$\frac{e^{-i(\nu-i\delta)[R(\rho_1(1-i\alpha))+iI(\rho_1(1-i\alpha))]}-1}{R(\rho_1(1-i\alpha))+iI(\rho_1(1-i\alpha))}\,.$$

Since the expression in square brackets takes the form $\rho_1^4[(1-\alpha^2)^2-4\alpha^2-4i\alpha(1-\alpha^2)]$ when ρ_1 tends to ∞ and we have

$$Re(-i\nu-\delta)[(1-\alpha^2)^2-4\alpha^2-4i\alpha(1-\alpha^2)]=$$
$$=-\delta[1-6\alpha^2+\alpha^4]-\nu\alpha(1-\alpha^2)<0\,,$$

when $Im\,\rho_1^4=0$ and δ is arbitrarily small, we can deformate the integration path γ to the real axis and put $\delta=0$.

Finally we have

$$T_1(x,\kappa,\nu)=\frac{(1-i\alpha)^4}{(2\pi)^4}\int d^4\rho\, K(\rho(1-i\alpha)/E)\,e^{i(1-i\alpha)\rho x}\cdot$$
$$\frac{e^{-i\nu[R(\rho(1-i\alpha),\kappa]+iI(\rho(1-i\alpha),\kappa)]}-1}{R(\rho(1-i\alpha),\kappa)+iI(\rho(1-i\alpha),\kappa)}\,. \qquad (22)$$

In the formula (22) α is infinitely small. We can put $\alpha=0$ everywhere except for the term which makes the integral converge. Thus

$$T_1(x,\kappa,\nu)=\frac{1}{(2\pi)^4 i}\int d^4\rho\int_0^\nu d\nu'\,K(\rho/E)\,e^{i\rho x-\alpha\rho^4}$$
$$\exp\{-i\nu'\{[\rho^2+2(\kappa\rho)]^2+2[\rho^2+2(\kappa\rho)](\kappa^2+\frac{E^2}{4}-$$
$$-m^2+i\varepsilon)-(E,\rho+\kappa)^2+(E\kappa)^2\}\}\,. \qquad (23)$$

In the first order of the modified perturbation theory the function $T(x,\kappa,\nu)$ in the formula (8) is replaced by

$\lambda T_1(x,k,\nu)$ defined by the formula (23). Then using the equations (6) and (3) we find in this approximation the Green function of the BS equation

$$G_1(p,q/E)=\frac{i}{(2\pi)^4}\int d^4x\, e^{-i(p-q)x}\cdot$$
$$\cdot\int_0^\infty d\nu\, e^{-i\nu[(q+\frac{E}{2})^2-m^2+i\varepsilon][(q-\frac{E}{2})^2-m^2+i\varepsilon]}\cdot$$
$$\exp\{\frac{\lambda}{(2\pi)^4}\int d^4\rho\int_0^\nu d\nu' K(\rho/E)e^{i\rho x-\alpha\rho^4}\exp\{-i\nu'[[(\rho+ \tag{24}$$
$$+q+\frac{E}{2})^2-m^2+i\varepsilon][(\rho+q-\frac{E}{2})^2-m^2+i\varepsilon]-[(q+\frac{E}{2})^2-m^2+i\varepsilon]\cdot$$
$$\cdot[(q-\frac{E}{2})^2-m^2+i\varepsilon]]\}\}.$$

The derived formula defines the Green function $G_1(p,q/E)$ only for such p and q when all the integrals converge. For example, with the interaction taken off $\lambda\to 0$ we obtain from (24)

$$G_{1f}(p,q/E)=\frac{i}{(2\pi)^4}\int d^4x\, e^{-i(p-q)x}\frac{e^{-i\nu[(q+\frac{E}{2})^2-m^2+i\varepsilon][(q-\frac{E}{2})^2-m^2+i\varepsilon]}-1}{-i[(q+\frac{E}{2})^2-m^2+i\varepsilon][(q-\frac{E}{2})^2-m^2+i\varepsilon]},$$

where $\nu\to\infty$. The integrals converge (or, what is the same, there exists a limit) when

$$Im[(q+\frac{E}{2})^2-m^2+i\varepsilon][(q-\frac{E}{2})^2-m^2+i\varepsilon]<0$$

or $q^2+\frac{E^2}{4}<m^2$.

In this region

$$G_{1f}(p,q/E)=\frac{\delta^{(4)}(p-q)}{[(q+\frac{E}{2})^2-m^2+i\varepsilon][(q-\frac{E}{2})^2-m^2+i\varepsilon]}.$$

Now the derived expression can be analytically continued to the whole real axis and the usual two-particle Green function in the absence of interaction can be obtained.

§4. The Scattering Amplitude

It is known that to find a scattering amplitude with the help of Green function (24) it is necessary to pick out four pole terms, corresponding to the free ends and then to proceed to the limit

$$\tilde{f}(p_1, p_2; q_1, q_2) = \lim_{p_i^2, q_i^2 \to m^2} (p_1^2 - m^2)(q_1^2 - m^2) \cdot$$

$$(p_2^2 - m^2)(q_2^2 - m^2) i G(p, q/E) .$$

Let us write the expression for the Green function (24) using new variables p_1 , q_1 , p_2 and q_2

$$G_1 = \frac{i}{(2\pi)^4} \int d^4x \, e^{i(p_1 - p_2)x} \int_0^\infty d\nu \, e^{-i\nu[p_2^2 - m^2 + i\varepsilon][q_2^2 - m^2 + i\varepsilon]} \, e^{i\lambda T_1(\nu)}.$$

Note that when $\nu \to \infty$ and $p_1^2, q_1^2 \to m^2$

$$T_1(x, q, \nu) \to T_1(\infty) =$$

$$= \frac{1}{(2\pi)^4 i} \int d^4 p \, e^{ipx - \alpha p^4} \frac{K(p/E)}{i[(p+q_2)^2 - m^2 + i\varepsilon][(p-p_2)^2 - m^2 + i\varepsilon]} .$$

We can put $\alpha = 0$ since the written integral converges, for example for the kernels $K(p)$ decreasing when $p \to \infty$.

Denote

$$\varphi(x) = -\frac{1}{2(2\pi)^4}\int d^4k\, e^{ikx} K(k/E)\, \mathcal{D}^c(k+q_2)\, \mathcal{D}^c(k-p_2). \qquad (25)$$

Using the following formula

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} \int_0^\infty d\nu\, e^{-i\nu(\alpha+i\varepsilon)(\beta+i\varepsilon)}\, e^{i\lambda T_1(\nu)} = \frac{1}{i}\,\frac{e^{i\lambda T_1(\infty)}}{(\alpha+i\varepsilon)(\beta+i\varepsilon)}, \qquad (26)$$

which is fair for $T_1(\infty) < \infty$ and $\alpha+\beta < 0$, we obtain

$$\tilde{f}(p_1,p_2,q_1,q_2) = \lim_{p_1^2, q_1^2 \to m^2} (p_1^2 - m^2)(q_1^2 - m^2)\,\cdot$$

$$\cdot\,\frac{i}{(2\pi)^4}\int d^4x\, e^{i(p_1-p_2)x}\, \exp[2i\lambda\varphi(x)].$$

Subtracting $\tilde{f}(\lambda=0)$, we shall find the formula for the scattering amplitude, in which the two pole terms are already picked out and cancelled.

$$f(p_1,q_1,p_2,q_2) = \lim_{p_1^2, q_1^2 \to m^2} (p_1^2 - m^2)(q_1^2 - m^2)\,\cdot \qquad (27)$$

$$\frac{2i^2}{(2\pi)^4}\int d^4x\, e^{i(p_1-p_2)x}\, e^{i\lambda\varphi(x)} \sin\lambda\varphi(x).$$

Note that $\varphi(x)$ does not contain momenta p_1 and q_1, and the expression $e^{i(p_1-p_2)x}\varphi(x)$ has the necessary pole singularities, which will be shown further. Hence (27) can be rewritten in the following form

$$f(p_1,q_1,p_2,q_2)=\frac{2i^2}{(2\pi)^4}\int d^4x\, e^{i\lambda\varphi(x)}\frac{\sin\lambda\varphi(x)}{\varphi(x)}\tilde{\varphi}(x), \tag{28}$$

where

$$\tilde{\varphi}(x)=\lim_{p_1^2,q_1^2\to m^2}(p_1^2-m^2)(q_1^2-m^2)e^{i(p_1-p_2)x}\varphi(x). \tag{29}$$

Using the definition of the function $\varphi(x)$ (see (25)), we obtain

$$e^{i(p_1-p_2)x}\varphi(x)=$$

$$=-\frac{1}{2(2\pi)^4}\int d^4k\, e^{i(k+p_1-p_2)x}K(k/E)\mathcal{D}^c(k+q_2)\mathcal{D}^c(k-p_2)=$$

$$=-\frac{1}{2(2\pi)^4}\int d^4\rho\, e^{i\rho x}K(\rho-p_1+p_2/E)\mathcal{D}^c(\rho+q_1)\mathcal{D}^c(\rho-p_1)=$$

$$=-\frac{1}{2(2\pi)^4}\int d^4\rho\, e^{i\rho x}\frac{K(\rho-p_1+p_2/E)}{(\rho^2-2\rho p_1+p_1^2-m^2+i\varepsilon)(\rho^2+2\rho q_1+q_1^2-m^2+i\varepsilon)}.$$

Let us denote $\alpha=\rho^2-2\rho p_1$, $\beta=\rho^2+2\rho q_1$ and $z=p_1^2-m^2+i\varepsilon=q_1^2-m^2+i\varepsilon$.

We note that the main contribution to the limit (29) is given by the integrals over the regions near zero, so α and β may be considered arbitrarily small. The limit (29) is equal to the coefficient C_{-2} in the Loran expansion of the function $e^{i(p_1-p_2)x}\varphi(x)$ in z . As is known, this coefficient is given by the formula

$$C_{-2} = \frac{1}{2\pi i} \oint_{\gamma} dz\, e^{i(p_1-p_2)x} \varphi(x)\, z =$$

$$= -\frac{1}{2(2\pi)^4} \int d^4\rho\, e^{i\rho x} K(\rho - p_1 + p_2/E) \frac{1}{2\pi i} \oint_{\gamma} \frac{dz \cdot z}{(\alpha+z)(\beta+z)},$$

where α and β are so small that they fall into the contour γ. Then the contour integral is equal to 1 and we obtain for C_{-2}

$$C_{-2} = -\frac{1}{2(2\pi)^4} \int d^4\rho\, e^{i\rho x} K(\rho - p_1 + p_2/E).$$

Denoting

$$\varphi_0(x) = -\frac{1}{2(2\pi)^4} \int d^4k\, e^{ikx} K(k/E), \qquad (30)$$

we can write

$$\tilde{\varphi}(x) = e^{i(p_1-p_2)x} \varphi_0(x). \qquad (31)$$

In the formula (31) all the momenta are already taken on the mass shell.

Substituting (31) in (28) we obtain the final expression for the scattering amplitude

$$f(p_1, q_1, p_2, q_2) = \frac{2i^2}{(2\pi)^4} \int d^4x\, e^{i(p_1-p_2)x} e^{i\lambda\varphi(x)} \sin\lambda\varphi(x) \frac{\varphi_0(x)}{\varphi(x)}, \qquad (32)$$

where the functions $\varphi(x)$ and $\varphi_0(x)$ are defined by the formulae (25) and (30).

Expanding the amplitude (32) in powers of the coupling constant $\lambda = g^2$, it is not difficult to make sure that we have obtained a consistent expression corresponding to the sum of the ladder graphs in the $K_i K_j = 0$ approximation with the generalized propagator $\mathcal{K}(\kappa/E)$. It may be of certain interest the further analysis of the formula (32) with various kernels $\mathcal{K}(p-p'/E)$ of the BS equation, and also the account of corrections to the considered approximation with the help of the coefficient function $T_2(x, \kappa, \nu)$, defined by the equations (16), (18).

§5. The Eikonal Representation of the Scattering Amplitude

In the present paragraph we consider the asymptotic behaviour of the scattering amplitude defined by the formula (32) in case of the fixed momentum transfers t and high energies $s \to \infty$. The essence of the used method is following. We expand the expression (32) in powers of the coupling constant λ and find the discontinuity Δf_n of the amplitude in the n-th order of perturbation theory in case of $s \to \infty$. Then the complete amplitude is restored with the help of dispersion relations and the summing of series $\sum \lambda^n f_n$ is performed.

Thus expanding in a power series $f = \sum_{n=1}^{\infty} \lambda^n f_n$, we obtain

$$f_n = \frac{(2i\lambda)^n i}{n!\,(2\pi)^4} \int d^4x\, e^{i(p_1-p_2)x} \varphi_0(x)\, \varphi^{n-1}(x) =$$

$$= \frac{(2i\lambda)^n i}{n!\,(2\pi)^{4n}} \left(-\frac{1}{2}\right)^n \int d^4\rho_1 \cdots d^4\rho_{n-1}\, \mathcal{K}(p_2 - p_1 - \rho_1 - \cdots - \rho_{n-1}) \cdot$$

$$\cdot \prod_{j=1}^{n-1} K(\rho_j)\, \mathcal{D}^c(\rho_j + q_2)\, \mathcal{D}^c(\rho_j - p_2).$$

The discontinuity of the amplitude f_n follows from this expression

$$\Delta f_n = \frac{i\lambda^n (2\pi i)^{n-1}}{n!(2\pi)^{3n+1} i}\int d^4\rho_1 \cdots d^4\rho_{n-1} K(p_2 - p_1 - \rho_1 - \cdots - \rho_{n-1}) \cdot \prod_{j=1}^{n-1} K(\rho_j)\,\delta(\rho_j^2 + 2\rho_j q_2)\,\delta(\rho_j^2 - 2\rho_j p_2). \tag{33}$$

It is not difficult to verify that in the c.m. system with the axis z along $\vec{p}_2$ we have in the case of $S \to \infty$ the following expressions

$$p_2 \simeq \left\{ \frac{\sqrt{S}}{2}, 0, 0, \frac{\sqrt{S}}{2} \right\}$$

$$q_2 \simeq \left\{ \frac{\sqrt{S}}{2}, 0, 0, -\frac{\sqrt{S}}{2} \right\}$$

$$p_1 - p_2 \simeq \left\{ 0, \vec{\Delta}_\perp, 0 \right\}.$$

In this case it is easy to integrate over ρ_{j0} and ρ_{jz} in (33), which brings us to the expression (34) for the discontinuity Δf_n

$$\Delta f_n \simeq \frac{i\lambda^n (2\pi i)^{n-1}}{n!(2\pi)^{3n+1} i\, 2^{n-1}} \frac{1}{S^{n-1}} \int d^2\vec{\rho}_{1\perp} \cdots d^2\vec{\rho}_{n-1\perp} \cdot \sum_{m=0}^{n-1} (-1)^{n-m-1} C_{n-1}^m K(p_2 - p_1 - \rho_1^{(\alpha)} - \cdots - \rho_m^{(\alpha)} - \rho_{m+1}^{(\beta)} - \cdots - \rho_{n-1}^{(\beta)}) \prod_{j=1}^{m} K(\rho_j^{(\alpha)}) \prod_{j=m+1}^{n-1} K(\rho_j^{(\beta)}), \tag{34}$$

where

$$\rho_j^{(\alpha)} = \left\{ 0, \vec{\rho}_{j\perp}, \sqrt{S} \right\}$$

$$\rho_j^{(\beta)} = \left\{ 0, \vec{\rho}_{j\perp}, \frac{\vec{\rho}_{j\perp}^{\,2}}{\sqrt{S}} \right\}. \tag{35}$$

It is clear that in the case of the kernel $K(k)\sim\frac{1}{k^{2+\varepsilon}}$ when $k\to\infty$ the main contribution to the asymptotic of the Δf_n gives the first term $m=0$ in the formula (34).

Thus when $s\to\infty$ we have

$$\Delta f_n \simeq \frac{i\lambda^n(-2\pi i)^{n-1}}{n!(2\pi)^{3n+1}i\,2^{n-1}}\,\frac{1}{s^{n-1}}\int d^2\vec{\rho}_{1\perp}\cdots d^2\vec{\rho}_{n-1\perp}\cdot \tag{36}$$

$$\cdot\tilde{K}(-\vec{\Delta}_\perp-\vec{\rho}_{1\perp}-\cdots-\vec{\rho}_{n-1\perp})\prod_{j=1}^{n-1}\tilde{K}(\vec{\rho}_{j\perp}),$$

where the following notation is adopted

$$\tilde{K}(\vec{\rho}_\perp)=K(\rho)\qquad\text{when}\qquad \rho=\{0,\vec{\rho}_\perp,0\}. \tag{37}$$

Defining the function

$$\Phi_0(\vec{x}_\perp)=\frac{1}{(2\pi)^2}\int d^2\vec{\rho}_\perp\, e^{-i\vec{\rho}_\perp\vec{x}_\perp}\,\tilde{K}(\vec{\rho}_\perp), \tag{38}$$

we can rewrite the expression (36) in the form

$$\Delta f_n\simeq\frac{C_n}{s^{n-1}},\qquad s\to\infty,$$

where

$$C_n=\frac{i\lambda^n(-2\pi i)^{n-1}}{n!(2\pi)^{n+3}2^{n-1}i}\int d^2\vec{x}_\perp\, e^{-i\vec{\Delta}_\perp\vec{x}_\perp}\left[\Phi_0(\vec{x}_\perp)\right]^n. \tag{39}$$

Note that if the kernel $K(k)$ is given by the representation

$$K(k)=\int_0^\infty\frac{\rho(\varkappa)\,d\varkappa}{k^2-\varkappa^2+i\varepsilon},$$

we have

$$\Phi_0(\vec{x}_\perp) = -\frac{1}{2\pi}\int_0^\infty d\varkappa\, \rho(\varkappa)\, K_0(\varkappa|\vec{x}_\perp|), \qquad (40)$$

where K_0 is the well known McDonald function.

Using now the dispersion relation

$$f_n = \frac{1}{2\pi i}\int_{S_0}^\infty \frac{ds'\,\Delta f_n(s')}{s'-s} \simeq \frac{C_n}{2\pi i}\int_{S_0}^\infty \frac{ds'}{s'^{\,n-1}(s'-s)}, \quad n \geq 2 \qquad (41)$$

we obtain the leading asymptotic term

$$f_n \simeq -\frac{C_n}{2\pi i}\,\frac{\ln s}{s^{n-1}}. \qquad (42)$$

Since our formulae for the discontinuity are fair in the asymptotic region of high energies $s \to \infty$, S_0 is a big quantity and we have rejected in (42) the terms of the type $\frac{1}{S_0\, s^{n-2}}$ considering that they decrease more rapidly than the leading term. Let us also note that the formula (42) is fair when $n \geq 2$.

Taking into account the definition (39) of the quantity C_n, we can sum the power series in λ^n and we can obtain the scattering amplitude in the eikonal form.

$$f = f_{(1)} - i\sum_{n=2}^{\infty}\left(\frac{s\ln s}{2\pi^2}\right)\frac{(i\lambda/2s)^n}{(2\pi)i\,n!}\int d^2\vec{x}_\perp\, e^{-i\vec{\Delta}_\perp\vec{x}_\perp}\left[-\Phi_0(\vec{x}_\perp)\right]^n =$$

$$= f_{(1)} - \frac{s\ln s}{\pi(2\pi)^{4}}\int d^2\vec{x}_\perp\, e^{-i\vec{\Delta}_\perp\vec{x}_\perp}\left[e^{-\frac{i\lambda}{2s}\Phi_0(\vec{x}_\perp)} - 1 + \frac{i\lambda}{2s}\Phi_0(\vec{x}_\perp)\right], \qquad (43)$$

where Φ_0 is defined by the formula (38) and in the special case by the formula (40).

In the formula (43) the quantity $f_{(1)}$ is the scattering amplitude in the first order in λ and is defined by the evident expressions

$$f_{(1)} = -\frac{i^2\lambda}{(2\pi)^4} K(p_2 - p_1) = -\frac{i^2\lambda}{(2\pi)^4} \tilde{K}(-\vec{\Delta}_\perp) = -\frac{i^2\lambda}{(2\pi)^4} \int d^2\vec{x}_\perp e^{-i\vec{\Delta}_\perp \vec{x}_\perp} \Phi_0(\vec{x}_\perp). \quad (44)$$

The formula (43) is the generalization of the eikonal representation for the ladder graphs with the modified propagators. The presence of the "extra" term $\frac{\lambda}{2s\, \Phi_0(\vec{x}_\perp)}$ in square brackets is connected with the fact that $f_{(1)}$ does not depend at all on s, and the dependence of the phase is thus that $\ln s$, which appears in the all following orders, can by no means be cancelled (how it occurs if we consider the cross-diagrams/9/). Let us also note that in the case of the scalar φ^3 -theory(i.e. $\rho(\varkappa) = const\, \delta(\varkappa - \mu)$ in the formula (40)) the obtained expression (43) gives the correct asymptotic for the ladder graphs in each order (see for example/11/). This fact confirms once more the validity of the $k_i k_j = 0$ approximation for a certain class of diagrams in the considered asymptotic region.

Since the expression (43) has as its quantum-mechanical analogy the Glauber representation, we can find the corresponding interaction potential $V(s, r)$. Really,

$$\frac{\lambda}{2s} \Phi_0(\vec{x}_\perp) = \frac{1}{s} \int_{-\infty}^{+\infty} V\left(\sqrt{x_z^2 + \vec{x}_\perp^2}\right) dx_z \quad . \qquad (45)$$

In the case described by the formula (40) we obtain the following equation for the interaction potential

$$-\frac{g^2 \pi}{(2\pi)^2} \int_0^\infty \rho(\varkappa) K_0(\varkappa|\vec{x}_\perp|) d\varkappa = \int_{-\infty}^{+\infty} V\left(\sqrt{x_z^2 + \vec{x}_\perp^2}\right) dx_z \cdot (46)$$

Using the well-known representation

$V(r)$	$\rho(x)$
$\frac{const}{r^2+a^2}\left(1-e^{-\frac{2\pi r}{a}}\right)$	$\begin{cases} \cos(xa), & x < \frac{2\pi}{a} \\ 0, & x > \frac{2\pi}{a} \end{cases}$ $a>0$
$const\,\frac{1+b\sqrt{r^2+a^2}}{(r^2+a^2)^{3/2}}\,e^{-b\sqrt{r^2+a^2}}$	$\begin{cases} 0, & x < b \\ x J_0\left(a\sqrt{x^2-b^2}\right), & x > b \end{cases}$
$const\; e^{-\alpha\sqrt{r}}$	$\left(\frac{\alpha^2}{2x}-3\right)\frac{\exp(-\alpha^2/4x)}{x^{5/2}}$

In conclusion we want to note that we have mainly considered the kernels decreasing as $K \sim \frac{1}{k^{2+\varepsilon}}$ when $k \to \infty$, which is convenient for the comparison with the models of the φ^3-type and which is notable for its simplicity and clearness. However the suggested method can also be applied to the nondecreasing kernels, which correspond to the more realistic models.

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