Italian-Russian Dubna Round Table on Black Holes in Mathematics and Physics, Dubna

Black holes, hyperbolical geometry and dark matter

A.Yu. Kamenshchik

University of Bologna and INFN, Bologna
L.D. Landau Institute for Theoretical Physics, Moscow

December 15-18, 2011

Based on:

M. Gaudin, V. Gorini, A. Kamenshchik, U. Moschella and V. Pasquier,
Gravity of a static massless scalar field and a limiting Schwarzschild-like geometry,
International Journal of Modern Physics D 15 (2006) 1387.

L. Rizzi, S. L. Cacciatori, V. Gorini, A. Kamenshchik and O.F. Piattella,
Dark matter effects in vacuum spacetime,
Physical Review D 82 (2010) 027301.

Content

1. Introduction.
2. Static spherically and hyperbolically symmetric solutions with massless scalar field.
3. Kantowski-Sachs cosmologies and the duality between cosmological and static solutions.
4. Empty hyperbolic space and its properties - pseudo-Schwarzschild geometry.
5. Dark matter effects in vacuum spacetime.
6. Concluding remarks.

Introduction

- We have rediscovered the static speherically symmetric solution of the Einstein equations in the presence of massless scalar field.

1. I.Z. Fisher, Journal of Experimantal and Theoretical Physics (1948)
2. O. Bergman and R. Leipnik (1957)
3. H.A. Buchdahl (1959)
4. A.I. Jannis, E.T. Newman and J. Winicour (1968)
5. M. Wyman (1981)
6. A.G. Agnese and M. La Camera (1985)
7. B.C. Xanthopoulos and T. Zannias (1989)

- This solution can be easily obtained by using the duality between static and cosmological solutions (Kantowski-Sachs cosmologies).
- This duality includes the exchange between the radial and temporal coordinates and between the spherical and the hyperbolical symmetries.
- In the limiting case of the absence of the scalar field we obtain the Schwarzschild and pseudo-Schwarzschild geometries.
- The pseudo-Schwarzschild geometry has rather particular properties. In particular, it can mimic the presence of dark matter.

Static spherically and hyperbolically symmetric solutions with massless scalar field.

$$ds^2 = b^2(r)dt^2 - a^2(r)(dr^2 + d\theta^2 + \sin^2\theta d\varphi^2).$$

$$G_t^t = \frac{a'(r)^2 + a(r)^2 - 2\,a(r)\,a''(r)}{a(r)^4} = \varepsilon,$$

$$G_r^r = \frac{a(r)^2\,b(r) - b(r)\,a'(r)^2 - 2\,a(r)\,a'(r)\,b'(r)}{a(r)^4\,b(r)} = -\varepsilon,$$

$$G_\theta^\theta = G_\phi^\phi = \frac{b(r)\,a'(r)^2 - a(r)\,b(r)\,a''(r) - a(r)^2\,b''(r)}{a(r)^4\,b(r)} = -\varepsilon$$

$$\varepsilon = \frac{4\pi\phi'^2}{a^2}.$$

It is convenient to introduce the notations:

$$A \equiv \frac{a'}{a}, \qquad B \equiv \frac{b'}{b}.$$

Then the Einstein equations are

$$A' + A^2 + AB - 1 = 0,$$
$$A' + A^2 - B' - B^2 - 1 = 0.$$

$$A = -\frac{B'}{B} - B,$$

and B^{-1} satisfies the equation for an upside-down harmonic oscillator:

$$\left(\frac{1}{B}\right)'' - \frac{1}{B} = 0.$$

$$ds^2 = \left(\tanh\frac{r}{2}\right)^{2\gamma} dt^2 - \frac{a_0^2 \sinh^2 r}{\left(\tanh\frac{r}{2}\right)^{2\gamma}}(dr^2 + d\theta^2 + \sin^2\theta d\varphi^2),$$

$$\phi'^2 = \frac{\varepsilon a^2}{4\pi} = \frac{(1-\gamma^2)}{4\pi\sinh^2 r}.$$

The values $\gamma = \pm 1$ correspond to the limiting cases of empty spacetimes.

The standard (Schwarzschild-like) form of the metric is also of interest. In the limit $\gamma = \pm 1$ we obtain

$$ds^2 = \left(1 - \frac{2a_0}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{2a_0}{R}} - R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

Static hyperbolic metric

$$ds^2 = b^2(r)dt^2 - a^2(r)(dr^2 + d\chi^2 + \sinh^2\chi d\varphi^2).$$

In this case $\frac{1}{B}$ behaves as an harmonic oscillator:

$$\left(\frac{1}{B}\right)'' + \frac{1}{B} = 0.$$

Kantowski-Sachs cosmologies and the duality between cosmological and static solutions.

Consider the hyperbolic metric. By interchanging the time and the radial variables

$$t \leftrightarrow r$$

it becomes

$$ds^2 = -a^2(t)dt^2 + b^2(t)dr^2 - a^2(t)(d\chi^2 + \sinh^2\chi d\varphi^2).$$

To remedy for the incorrect signs in front of dr^2 and dt^2 we make the replacement

$$g_{\alpha\beta} \rightarrow -g_{\alpha\beta}$$

and get

$$ds^2 = a^2(t)dt^2 - b^2(t)dr^2 + a^2(t)(d\chi^2 + \sinh^2\chi d\varphi^2).$$

Now the last two terms of the metric have wrong signs. To correct them, we make one more replacement:

$$\chi \to i\theta,$$

which finally produces the metric

$$ds^2 = a^2(t)dt^2 - b^2(t)dr^2 - a^2(t)(d\theta^2 + \sin^2\theta d\varphi^2).$$

This set of transformations also determines a map of the components of the Einstein tensor into those of a Kantowski-Sachs spherical universe filled with the time-dependent massless scalar field $\phi(t)$.
A similar set of transformations performs the transition from the static spherically symmetric metric to the cosmological hyperbolic Kantowski-Sachs metric.

The non-triviality of these transformations consists in the fact that they not only exchange the radial and time variables among themselves, but also substitute the spherical symmetry by the hyperbolical one and vice versa.

Empty hyperbolic space and its properties - pseudo-Schwarzschild geometry.

In the pseudo-Schwarzschild form the metric is

$$ds^2 = \left(\frac{2a_0}{\rho} - 1\right) dt^2 - \frac{d\rho^2}{\left(\frac{2a_0}{\rho} - 1\right)} - \rho^2(d\chi^2 + \sinh^2\chi d\varphi^2).$$

This metric was written by B. Harrison in 1959 and was called the degenerate solution III-9 but its properties were not discussed.

The pseudo-Schwarzschild metric has a horizon at $\rho = 2a_0$, analogously to the Schwarzschild horizon. The singularity occurs at $\rho = 0$ and is also analogous to the Schwarzschild singularity. The main difference lies in the fact that this metric is defined at $\rho < 2a_0$, i.e. inside the horizon, in contrast to the Schwarzschild metric.

In the vicinity of singularity

The velocity of a massive particle in the vicinity of the singularity vanishes and the latter cannot reach the singularity. There is no obstruction for the light rays (massless particles) which fall to the singularity $\rho = 0$.

One can construct Kruskal-type coordinates, analogous to those describing the Schwarzschild manifold.

$$ds^2 = f^2(u,v)(dv^2 - du^2) - \rho^2(d\chi^2 + \sinh^2\chi d\varphi^2),$$

$$f^2(u,v) = \frac{32a_0^3}{\rho}\exp\left(-\frac{\rho}{2a_0}\right).$$

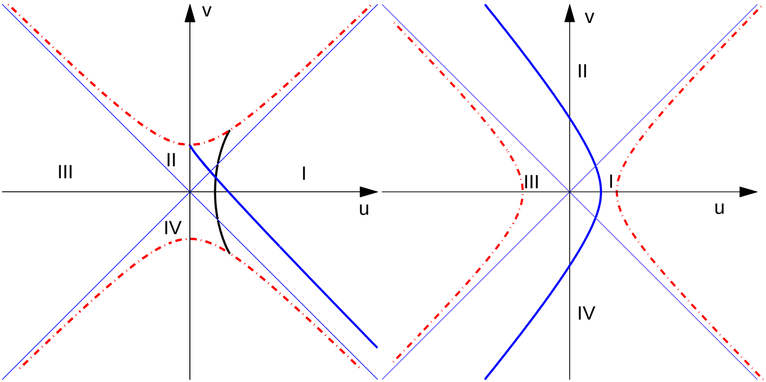
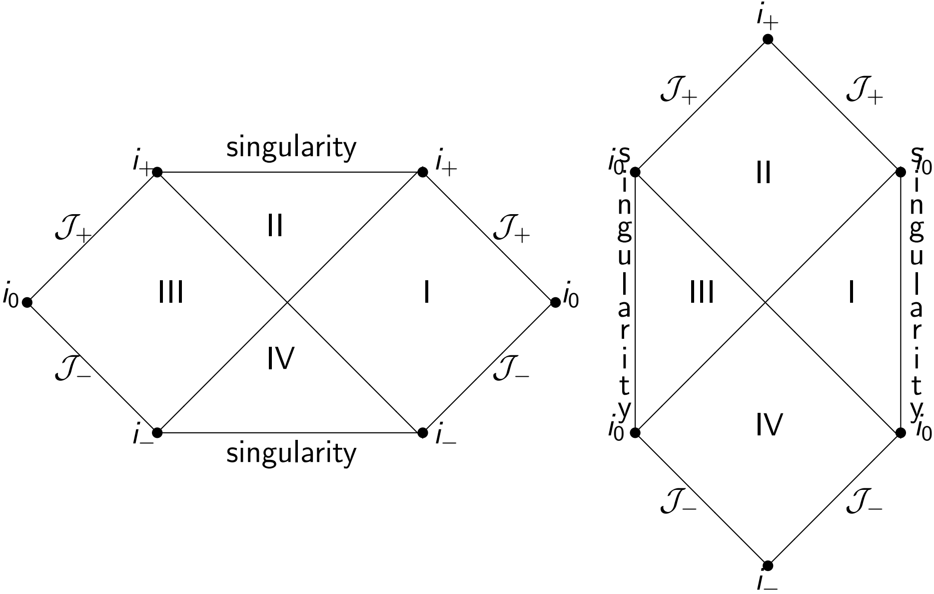


Figure: The Kruskal diagrams for the Schwarzschild (on the left) and pseudo-Schwarzschild (on the right) manifolds.

The Kruskal diagram for the pseudo-Schwarzschild manifold is rotated by $\pi/2$ with respect to the Kruskal diagram for the Schwarzschild manifold. In contrast to the Schwarzschild singularity which is spacelike, the pseudo-Schwarzschild singularity is timelike.

In the Schwarzschild case one has two types of timelike radial geodesics: those whose motion is finite and those whose motion is infinite.

In the pseudo-Schwarzschild manifold there exists only one type of timelike geodesics: a test particle travels along the geodesics from spatial infinity, cross the horizon, reaches the turning point, (where the distance of the particle from the singularity attains its minimum value), cross again the horizon and travels to spatial infinity.



Penrose diagrams

Motion outside of the horizon

$$ds^2 = \frac{d\tilde{t}^2}{1 - \frac{2a_0}{\tilde{t}}} - d\tilde{\rho}^2 \left(1 - \frac{2a_0}{\tilde{t}}\right) - \tilde{t}^2 (d\chi^2 + \sinh^2 \chi d\varphi^2).$$

This metric is non-stationary and when $\tilde{t} \to \infty$ it becomes the Minkowski metric. It asymptotically tends to the Minkowski metric, written in a rather particular way: it represents a direct product of the line $\tilde{\rho}$ times the (2+1) - dimensional Milne manifold.

The geodesic equation for massive particles can be reduced to the following form:

$$\left(\frac{d\tilde{\rho}}{d\tilde{t}}\right)^2 = \frac{v^2 g_{tt}^3}{1 - v^2 + v^2 g_{tt}},$$

where v is the asymptotic value of the velocity $d\tilde{\rho}/d\tilde{t}$ when $\tilde{t} \to \infty$.

Dark matter effects in vacuum spacetime.

Changing the sign of the mass in the formula for the metric of the hyperbolical spacetime we obtain

$$ds^2 = \frac{dr^2}{1 + \frac{2m}{r}} - \left(1 + \frac{2m}{r}\right) d\tau^2 - r^2(d\chi^2 + \sinh^2 \chi d\varphi^2).$$

Introducing new coordinates

$$t = r \cosh \chi, \ \rho = r \sinh \chi,$$

one obtains

$$ds^2 = \frac{r^3 - 2m\rho^2}{r^2(r + 2m)} dt^2 + \frac{4mt\rho}{r^2(r + 2m)} dt d\rho - \frac{r^3 + 2mt^2}{r^2(r + 2m)} d\rho^2 - \left(1 + \frac{2m}{r}\right) dz^2 - \rho^2 d\phi^2 \ ,$$

At large values of $t/2m$ this metric becomes

$$ds^2 \sim dt^2 - dz^2 - (d\rho^2 + \rho^2 d\phi^2).$$

An observer becomes Minkowskian if he waits long enough ($t \to \infty$).

The markers (t, z, ρ, ϕ) can be interpreted as cylindrical coordinates for an Minkowskian observer.

Particle dynamics.

In a Newtonian approximation one has

$$\frac{d^2\rho}{dt^2} \sim -\Gamma^{\rho}_{tt} \sim -\frac{m}{t^3}\rho,$$

where the Christoffel symbol is expanded at the leading order in $2m/t$.

A test mass in planar motion feels the same forse as if it were immersed in a cylindrical and homogeneous distribution of matter which extends throughout the whole space with a time depending density

$$\delta(t) = \frac{mc^2}{G}\frac{1}{2\pi(ct)^3}.$$

The observer concludes that some form of matter is homogeneously spread in the portion of the space he is living in, even though no matter is present at all.

Moreover, this matter density is decreasing with the third power of time, hence he also concludes that this distribution is dynamic, as if it were linearly expanding in every direction. Similar conclusions hold from the study of the trajectories of light rays.

Concluding remarks.

The empty space limits for spherical and hyperbolical Kantowski-Sachs cosmological solutions are

$$ds^2 = 4a_0^2 \cos^4 \frac{t}{2} dt^2 - \tan^2 \frac{t}{2} dr^2 - 4a_0^2 \cos^4 \frac{t}{2} (d\theta^2 + \sin^2 \theta d\varphi^2),$$

$$ds^2 = 4a_0^2 \cosh^4 \frac{t}{2} dt^2 - \tanh^2 \frac{t}{2} dr^2 - 4a_0^2 \cosh^4 \frac{t}{2} (d\chi^2 + \sinh^2 \chi d\varphi^2).$$

The metrics of the empty Kantowski-Sachs universes, written in this form do not describe complete manifolds.

Making the transformation

$$\tilde{t} = 2a_0 \cos^2 \frac{t}{2}$$

one sees that the first metric describes the internal part of the Schwarzschild world. Completing the manifold we construct the external part of the Schwarzschild world, which is static.

Similarly, making the transformation

$$\tilde{t} = 2a_0 \cosh^2 \frac{t}{2}$$

we get the metric, describing the external part of the pseudo-Schwarzschild while its completion gives the static pseudo-Schwarzschild geometry below the horizon.

Since our toy model has several bizarre features, we do not claim it provides a realistic answer to the origin of dark matter. It merely serves as an indication that dark matter effects may not necessarily be due to extra particles: in the absence of corroborating signals one should be open to investigating alternative possibilities.