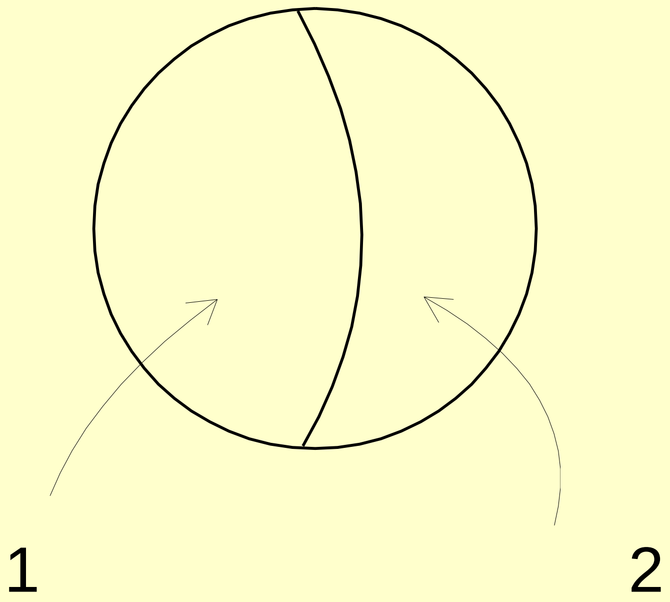
Holographic Entanglement Renyi Entropy

Dmitri V. Fursaev
Dubna University &
Bogoliubov Laboratory
JINR

Round Table Italia-Russia @ Dubna
17.12.2011
"Black Holes in Mathematics and Physics"

Quantum entanglement

quantum mechanics:
states of subsystems may not be described independently
= states are entangled



importance:
studying correlations of different systems (especially at strong couplings), critical phenomena and etc

Entropy as a measure of entanglement

$$\rho_1 = \mathrm{Tr}_2\,\rho \quad - \quad \text{reduced density matrix}$$

$$S_1 = -\mathrm{Tr}_1\,\rho_1 \ln \rho_1 \quad - \quad \text{entanglement entropy}$$

$$S_1^{(n)} = \frac{\mathrm{Tr}_1\,\rho_1^{\;n}}{1-n} \quad - \quad \text{entanglement Renyi entropy}$$

$$n = 2, 3, 4, \ldots. \qquad S_1^{(n)} \to S_1 \quad , \quad n \to 1$$

Computation of the reduced density matrix and entanglement entropy is a difficult problem, in general

entanglement has to do with quantum gravity:

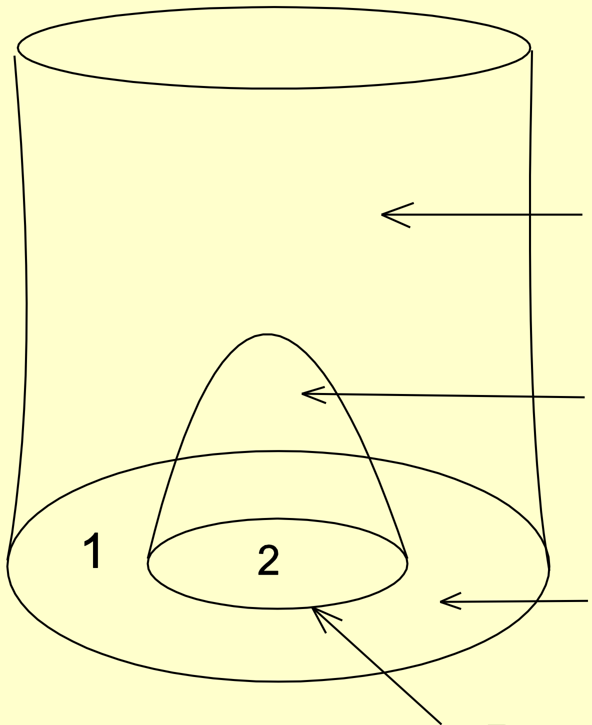
- possible source of the entropy of a black hole (states inside and outside the horizon);

- d=4 supersymmetric BH’s are equivalent to 2, 3,… qubit systems

- entanglement entropy allows a *holographic interpretation* for CFT’s with AdS duals

Holographic Formula for Entanglement Entropy (n=1)

Ryu and Takayanagi,
hep-th/0603001, 0605073



AdS_{d+1} (bulk space)

$\tilde{B}$ minimal (least area) surface in the bulk

4d space-time manifold (asymptotic boundary of AdS)

B separating surface

entropy of entanglement

$$S = \frac{\tilde{A}}{4G^{(d+1)}}$$

is measured in terms of the area of $\tilde{B}$

$G^{(d+1)}$ is the gravity coupling in AdS

Holographic formula enables one to compute entanglement entropy in strongly correlated systems with the help of geometrical methods (the Plateau problem);

Ryu-Takayanagi formula passes several non-trivial tests:

- in 2D and 4D CFT's (at weak coupling);
- for different quantum states;
- for different shapes and topologies of the separating surface in boundary CFT

Is it possible to find a holographic description of entanglement Renyi entropy?

Plan:

- new result: Renyi entropies in 2D and 4D CFT’s (at weak couplings);

- Difficulties with a holographic description Renyi entropies in CFT’s and a (possible) wayout;

Entanglement Renyi Entropy in CFT’s at weak coupling

1st step: representation in terms of a 'partition function'

$$\rho = e^{-H/T} / \mathrm{Tr}\, e^{-H/T} \; - \text{thermal density matrix}$$

$$Z^{(n)}(T) \equiv \mathrm{Tr}_1 \left(\mathrm{Tr}_2\, e^{-H/T}\right)^n \; - \text{a partition function}$$

$$Z^{(1)}(T) = Z(T)$$

$$Z(\beta, T) \equiv Z^{(n)}(T) \quad , \quad \beta = 2\pi n$$

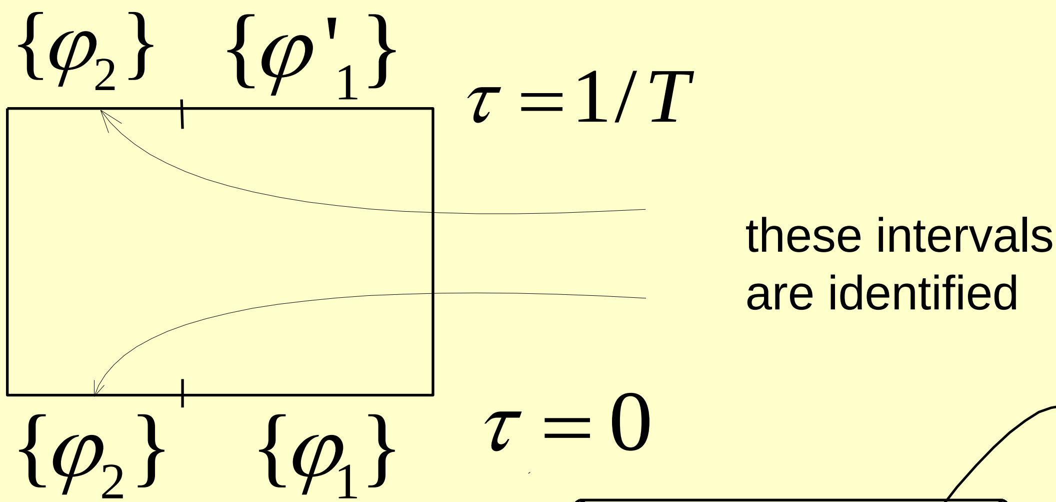
$$\beta - \text{"inverse temperature"}$$

$$S_1(T) = -\lim_{\beta\to 1}\left(\beta\partial_\beta - 1\right)\ln Z(\beta, T)$$

$$S_1^{(n)}(T) = \frac{2\pi\, Z(\beta, T) - \beta Z(T)}{2\pi - \beta}$$

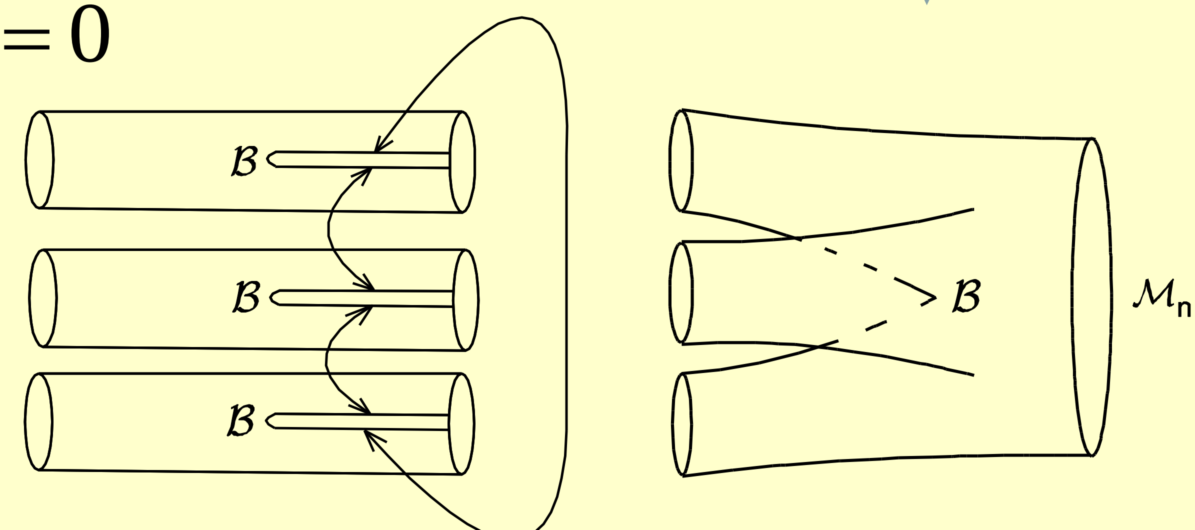
2[d] step: relation of a 'partition function' to an effective action on a 'curved space'

$$W(\beta,T) = -\ln Z(\beta,T) - \text{effective action}$$



$$\mathrm{Tr}_1 \rho_1^{\ 3}$$

one glues n
(n=3) copies



a 'curved space' with conical singularity at the separating point (surface)

3[d] step: use results of spectral geometry

$$W=\frac{1}{2}\sum_k \eta_k \ln\det L_k \quad , \quad \eta_k=\pm 1$$

$L_k -$ Laplace operators of different spin fields on M_n

$$W=\sum_{p=0}^{d-1}\Lambda^{d-p}\,\frac{A_p}{d-p}-A_d\ln(\Lambda/\mu)+\ldots$$ for dimension d even,

$$A_p=\sum_k \eta_k\, A_{k,p}$$, where $A_{k,p}$: $\mathrm{Tr}\, e^{-tL_k}=\sum_{p=0}^{\infty} t^{\frac{p-d}{2}} A_{k,p}+\ldots;$

$\Lambda -$ is a UV cutoff; μ is a physical scale (mass, inverse syze etc)

an example: a scalar Laplacian $L_0=-\nabla^2$:

$$A_0=O(n)\,,\ A_2=\frac{1}{24\pi}\int_{M_n} R+\frac{1}{12\gamma_n}(\gamma_n{}^2-1)\int_B \quad , \quad \gamma_n=n^{-1}$$

There are non-trivial contributions from conical singularities located at the 'separating' surface B

computations

$$S = \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s_p}{d-p} + s_d \ln(\Lambda/\mu) + \ldots,$$

$$S^{(n)} = \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s^{(n)}{}_p}{d-p} + s^{(n)}{}_d \ln(\Lambda/\mu) + \ldots, -$$ Renyi entropies

$$s_p \equiv -\lim_{n\to 1}(n\partial_n - 1)A_p(n) \quad , \quad s^{(n)}{}_p \equiv \frac{nA_p(1) - A_p(n)}{n-1}$$

$$s_0 = s^{(n)}{}_0 = 0 \quad , \; s_{2k+1} = s^{(n)}{}_{2k+1} = 0 -$$ (if boundaries are absent)

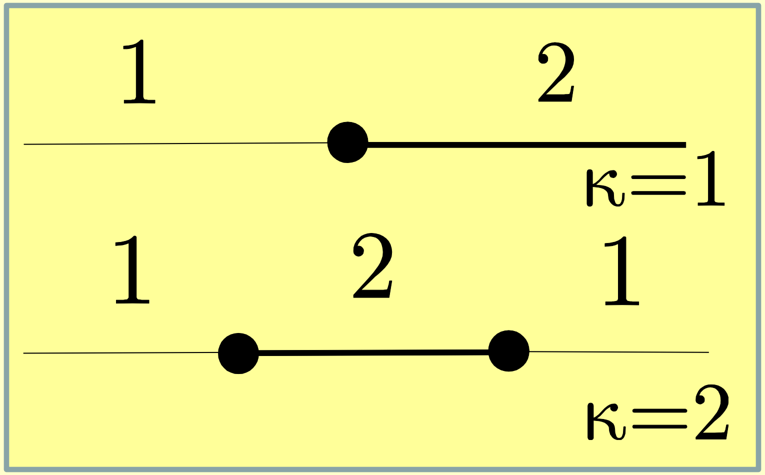
2D CFT: "c" massless scalars and spinors

$$W = \frac{a}{2}\ln\det\nabla^2 - b\ln\det\gamma^\mu\nabla_\mu$$

$c = a + b$ – CFT central charge

$$s_2 = \frac{c}{6}k \quad , \quad s^{(n)}{}_2 = \frac{c}{12}k(1+\gamma_n)$$

$$S = \frac{c}{6}k\ln(L/\varepsilon)\,,$$



$S^{(n)} = \frac{c}{12}(1+\gamma_n)k\ln(L/\varepsilon),$ – Renyi entropy

$\varepsilon \equiv \Lambda^{-1}\,,\quad L$ – a typical syze of the system,

the result holds for a system on an interval devided into 2 or 3 parts

$k = 1, 2$- the number of separating points (which yield conical singularities)

4D N=4 super SU(N) Yang-Mills theory at weak coup.

6 scalar multiplets, 4 multiplets of Weyl spinors, 1 multiplet of gluon fields

$$S^{(n)} = \frac{1}{2}\Lambda^2 s^{(n)}{}_2 + s^{(n)}{}_4 \ln(\Lambda/\mu) + \ldots$$

$$s^{(n)}{}_2 = \frac{d(N)}{4\pi}\gamma_n A(B) - \text{area of the separating surface } B$$

Conformal invariance

$$s^{(n)}{}_4 = d(N)(a(\gamma_n)F_a + c(\gamma_n)F_c + b(\gamma_n)F_b)$$

$$F_a = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x\, R(B) \quad , \qquad a(\gamma_n) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 7\gamma_n + 15),$$

$$F_c = -\frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x\, C_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho \ , \quad c(\gamma_n) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 3\gamma_n + 3)$$

$$F_b = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x \left(\frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i) - \mathrm{Tr}(k_i k_i) \right) , \ b(\gamma_n) = ?$$

$R(B)$ – scalar curvature of B, n_i^μ – a pair of unit orthogonal normals to B,

$C_{\mu\nu\lambda\rho}$ – Weyl tensor of M at B, $(k_i)_{\mu\nu}$ – extrinsic curvatures of B

F_a, F_b, F_c – invariant with respect to the Weyl transformations $g_{\mu\nu}{}'(x) = e^{2\omega(x)} g_{\mu\nu}(x)$

$\lim_{n\to 1} b(\gamma_n) = 1,$ ('holographic' arguments by S.N. Solodukhin, arXiv:0802.3117)

Entanglement entropy (n=1)

$$s_4 = \lim_{n\to 1} s^{(n)}{}_4 = cF_c + aF_a + bF_b$$

$$c = \lim_{n\to 1} c(\gamma_n) = \frac{1}{4}, \quad a = \lim_{n\to 1} a(\gamma_n) = \frac{1}{4}, \quad b = \lim_{n\to 1} b(\gamma_n) = \frac{1}{4}$$

$$a = c$$

relation to the trace anomaly in $D = 4$

$$\left\langle T_\mu^\mu \right\rangle = -aE_4 - cI_4$$

$$E_4 = \frac{1}{16\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$I_4 = -\frac{1}{16\pi^2} C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$$

$$C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} + \frac{1}{2}(g_{\mu\rho}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\rho} - g_{\mu\lambda}R_{\nu\rho} - g_{\nu\rho}R_{\mu\lambda}) + \frac{R}{6}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$$

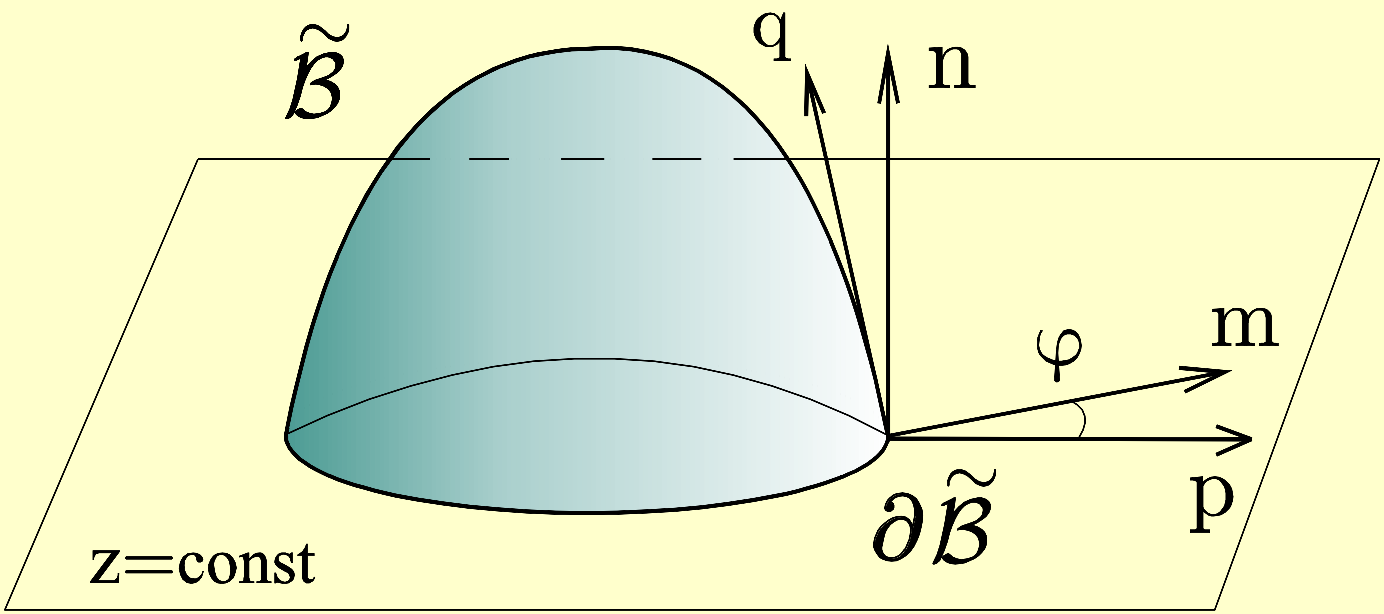
Cardy's conjecture: "charge" a decreases monotonically along RG flows

Toward a holographic description of Entanglement Renyi Entropy in CFT's

2 options

- Holographic Renyi entropy in CFT is described by RT-formula, but the background metric depends on # of replicas (Headrick 2010, Hung, Myers, Smolkin 2011)

- The background geometry does not change but RT formula for holographic Renyi entropy should be modified (what is investigated below)



$\tilde{B}-$ is a holographic surface in the bulk;

$\partial\tilde{B}-$ belongs to conformal class of B (the surface in CFT);

tilt angle

$$\varphi = \frac{z}{2}k + \dots$$

k $-$ extrinsic curvature of B

Holographic Renyi Entropy (a suggestion)

$$S^{(n)} = \frac{1}{4G_N^{(5)}}(\gamma_n A(\tilde{B}) + 2\pi(\tilde{a}(\gamma_n)\tilde{F}_a + \tilde{c}(\gamma_n)\tilde{F}_c + \tilde{b}(\gamma_n)\tilde{F}_b) + ...$$

$A(\tilde{B})$ – volume of $\tilde{B}$;

$\tilde{F}_a, \tilde{F}_c, \tilde{F}_b$ – are local invariant functionals on $\tilde{B}$;

$$A(\tilde{B}) = \frac{1}{2z^2} A(B) + \frac{\pi}{2}(F_a + F_c + F_b) \ln\frac{\mu}{z} + ...$$

z – position of the boundary (a UV cutoff in CFT)

$$\tilde{F}_{a,b,c} = F_{a,b,c} \ln\frac{\mu}{z} + ...$$

$$\tilde{a}(\gamma_n) = a(\gamma_n) - \frac{1}{4}, \tilde{c}(\gamma_n) = c(\gamma_n) - \frac{1}{4}, \tilde{b}(\gamma_n) = b(\gamma_n) - \frac{1}{4}$$

Let $\tilde{M}$ be asymptotically solution to the 5D Einstein eqs with negative Λ

$$\tilde{R}_{MN} - \frac{1}{2}\tilde{R}\,\tilde{g}_{MN} - \frac{3}{l^2}\tilde{g}_{MN} = 0$$

Let $\tilde{B}$ be a a minimal codimension 2 hypersurface in $\tilde{M}$, ($\partial\tilde{B}$ conformal to B)

Then:

$$\tilde{F}_c = -\frac{1}{\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3x \left[\tilde{R}_{KLMN} l^K m^L l^M m^N + \frac{1}{l^2} \right]$$

$$\tilde{F}_b = -\frac{1}{2\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3x\, K_{MN} K^{MN}$$

$\tilde{R}_{KLMN}$ – Riemann tensor of $\tilde{M}$

$\tilde{\sigma}$ – metric induced on $\tilde{B}$

l, m – normal vectors of $\tilde{B}$, l – is time-like, $(l \cdot m) = 0$,

K_{MN} – extrinsic curvature tensor of $\tilde{B}$ for m^N

Non-local invariants (new property)

$$\tilde{F}_a = F_a \ln\frac{\mu}{z} + ...$$

$$F_a = 2\chi = 4 - \text{topological invariant of } B$$

$$\tilde{F}_a \equiv 2\ln\frac{A(\partial B)}{l^3}$$

$$A(\partial B) = \frac{l^3}{2z^2} A(B) + ...$$

$$\ln\frac{A(\partial B)}{l^3} = 2\ln\frac{\mu}{z} + \ln\frac{A(\partial B)}{2\mu^2} + ...$$

another option:

$$\tilde{F}_a \equiv -\ln \mu^2 \Delta_2(\partial B)$$

$$\Delta_2(\partial B) - \text{scalar Laplacian on } \partial B$$

Summary:

● new result for the entanglement Renyi entropies (ERE) in D=4 CFT's

● ERE is a local invariant functional which have a structure similar to EE -> possibility to find a holographic description of ERE

● a conjecture for holographic ERE: modification of RT formula:

- local and non-local invariant structures in the bulk;
- explicit dependence on dimensionality and the replica parameter of holographic ERE;
- the holographic surface and the background metric do not depend on the replica parameter

thank you for attention