Black Holes and Nilpotent Orbits

Pietro Frè
University of Torino

JINR – Dubna
December 16th 2011

Based on common work with Aleksander S. Sorin & Mario Trigiante

A well defined mathematical problem

The goal is just to find and classify all spherical symmetric solutions of Supergravity with a static metric of Black Hole type:

The solution of this problem is found by reformulating it into the context of a very rich mathematical framework which involves:

1. The Geometry of COSET MANIFOLDS
2. The theory of **Liouville Integrable systems** constructed on Borel-type subalgebras of SEMISIMPLE LIE ALGEBRAS
3. The addressing of a very topical issue in conyemporary ADVANCED LIE ALGEBRA THEORY namely:
    1. ***THE CLASSIFICATION OF ORBITS OF NILPOTENT OPERATORS***

The N=2 Supergravity Theory

$$\mathcal{L}^{(4)} = \sqrt{\det g}\Big[-2R[g] - \frac{1}{6}\partial_{\hat{\mu}}\phi^a\partial^{\hat{\mu}}\phi^b h_{ab}(\phi) + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\hat{\mu}\hat{\nu}} F^{\Sigma|\hat{\mu}\hat{\nu}}\Big] + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\hat{\mu}\hat{\nu}} F^{\Sigma}_{\hat{\rho}\hat{\sigma}} \epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$$

We have gravity and **n** vector multiplets

2 n scalars yielding n complex scalars z^i

and n+1 vector fields $\mathbf{A}^{\Lambda}$

The matrix $N_{\Lambda\Sigma}$ encodes together with the metric h_{ab} Special Geometry

The main point

1) space-like p-branes as the cosmic billiards, or

2) time-like p-branes as several rotational invariant black-holes in $D = 4$ and more general solitonic branes in diverse dimensions

reduce to geodesic equations on coset manifolds of the type

$$\mathcal{M} = \frac{\mathrm{U}}{\mathrm{H}} \quad or \quad \mathcal{M}^{\star} = \frac{\mathrm{U}}{\mathrm{H}^{\star}} \simeq \exp\left[\mathrm{Solv}_{\mathcal{M}}\right]$$

Dimensional Reduction to D=3

THE C-MAP

D=4 SUGRA with SK_n ⟶ D=3 σ-model on Q_{4n+4}

$$ds^2_Q = \frac{1}{4}\left[dU^2 + g_{i\bar{\jmath}}\, dz^i\, d\bar{z}^{\bar{\jmath}} + e^{-2U}\,(da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z})^2 \mp 2\, e^{-U}\, d\mathbf{Z}^T\, \mathcal{M}_4(z,\bar{z})\, d\mathbf{Z}\right]$$

Space red. / Time red.
Cosmol. / Black Holes

$$\underbrace{\{U, a\}}_{2} \bigcup \underbrace{\{z^i\}}_{2n} \bigcup \underbrace{\mathbf{Z} = \{Z^\Lambda, Z_\Sigma\}}_{2n+2}$$ 4n + 4 coordinates

Gravity **scalars** **From vector fields**

$$\mathcal{M}_4 = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \\ \hline \mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \end{array}\right)$$

SUGRA BH.s = one-dimensional Lagrangian model

Evolution parameter $\tau \sim \frac{1}{r}$ $\dot{f} \equiv \frac{d}{d\tau} f$

$$\mathcal{L} = \dot{U}^2 + h_{rs}\,\dot{\phi}^r\,\dot{\phi}^s + e^{-2U}\,(\dot{a} + \mathbf{Z}^T \mathbb{C} \dot{\mathbf{Z}})^2 + 2\,e^{-U}\,\dot{\mathbf{Z}}^T\,\mathcal{M}_4\,\dot{\mathbf{Z}}$$

$$\mathcal{L} = \begin{cases} v^2 & > & 0 \\ v^2 & = & 0 \\ -\,v^2 & < & 0 \end{cases}$$

Time-like geodesic = non-extremal Black Hole

Null-like geodesic = extremal Black Hole

Space-like geodesic = naked singularity

A Lagrangian model can always be turned into a Hamiltonian one by means of standard procedures.

SO BLACK-HOLE PROBLEM = DYNAMICAL SYSTEM

FOR SK_n = symmetric coset space THIS DYNAMICAL SYSTEM is LIOUVILLE INTEGRABLE, always!

When homogeneous symmetric manifolds

$$\frac{\mathsf{U}_{\mathsf{D}=4}}{\mathsf{H}_{\mathsf{D}=4}} \to \frac{\mathsf{U}_{\mathsf{D}=3}}{\mathsf{H}_{\mathsf{D}=4}}$$

C-MAP

$$\mathsf{U}_{\mathsf{D}=3} \supset \mathsf{U}_{\mathsf{D}=4}$$

General Form of the Lie algebra decomposition

$$\mathsf{adj}(\mathbb{U}_{D=3}) = \mathsf{adj}(\mathbb{U}_{D=4}) \oplus \mathsf{adj}(\mathsf{SL}(2,\mathbb{R})_{\mathsf{E}}) \oplus W_{(2,\mathbf{W})}$$

$$\begin{aligned}
[T^a, T^b] &= f^{ab}{}_c\, T^c \\
[L^x, L^y] &= f^{xy}{}_z\, L^z, \\
[T^a, W^{iM}] &= (\Lambda^a)^M_N\, W^{iN}, \\
[L^x, W^{iM}] &= (\lambda^x)^i{}_j\, W^{jM}, \\
[W^{iM}, W^{jN}] &= \epsilon^{ij}\, (K_a)^{MN}\, T^a + \mathbb{C}^{MN}\, k^{ij}_x\, L^x
\end{aligned}$$

Relation between $\frac{U}{H}$ and $\frac{U}{H^\star}$

One just changes the sign of the scalars coming from $W_{(2,R)}$ part in:

$$\mathrm{adj}(G_{D=3}) = \mathrm{adj}(G_{D=4}) \oplus \mathrm{adj}(SL(2,\mathbb{R})) \oplus W_{(2,R)}$$

where R is a **symplectic** representation of $G_{D=4}$

Examples

$$\frac{E_{8(8)}}{SO(16)} \to \frac{E_{8(8)}}{SO^\star(16)}$$
$$\frac{SO(4,4)}{SO(4)\times SO(4))} \to \frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$$
$$\frac{G_{(2,2)}}{SU(2)\times SU(2)} \to \frac{G_{(2,2)}}{SU(1,1)\times SU(1,1)}$$

The simplest example $G_{2(2)}$

One vector multiplet

$$\mathrm{adj}\,\left[\mathfrak{g}_{2(2)}\right] = (\mathrm{adj}\,[\mathfrak{sl}(2,\mathbb{R})_E]\ \mathbf{1}) \oplus (\mathbf{1}\,,\,\mathrm{adj}\,[\mathfrak{sl}(2,\mathbb{R})]) \oplus (\mathbf{2}\,,\,\mathbf{4})$$

$$g_{z\bar{z}} dz\, d\bar{z} = \frac{3}{4}\frac{1}{(\mathrm{Im} z)^2}\partial^\mu z\, \partial_\mu \bar{z}$$ **Poincaré metric**

$$\Omega(z) = \begin{pmatrix} -\sqrt{3}z^2 \\ z^3 \\ \sqrt{3}z \\ 1 \end{pmatrix}$$ **Symplectic section**

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z) = \begin{pmatrix} -\frac{3z+\bar{z}}{2z\bar{z}} & -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} \\ -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} & -\frac{z+3\bar{z}}{2z\bar{z}^3} \end{pmatrix}$$ **Matrix $N_{\Lambda\Sigma}$**

OXIDATION 1

The metric

$$ds^2_{(4)} = -e^{U(\tau)}\,(dt + A_{KK})^2 + e^{-U(\tau)}\left[e^{4A(\tau)}\,d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

where $A_{KK} = 2\,\mathbf{n}\cos\theta\,d\varphi$

Taub-NUT charge

$$\underbrace{\left[e^{-2U}\left(\dot a + Z^\Lambda \dot Z_\Lambda - Z_\Sigma \dot Z^\Sigma\right)\right]}_{\mathbf{n} = \text{Taub NUT charge}}$$

The electromagnetic charges

$$\mathcal{Q}^M = \sqrt{2}\left[e^{-U}\mathcal{M}_4\,\dot Z - \mathbf{n}\,\mathbb{C}\,Z\right]^M = \begin{pmatrix} p^\Lambda \\ e_\Sigma \end{pmatrix}$$

From the σ-model viewpoint all these first integrals of the motion

$$e^{2A(\tau)} = \begin{cases} \dfrac{v^2}{\sinh^2(v\tau)} & \text{if } v^2 > 0 \\ \dfrac{1}{\tau^2} & \text{if } v^2 = 0 \end{cases}$$

Extremality parameter

OXIDATION 2

The electromagnetic field-strenghts

$$F^{\Lambda} = 2\, p^{\Lambda} \sin\theta\, d\theta \wedge d\varphi + \dot{Z}^{\Lambda} d\tau \wedge (dt + 2\mathbf{n} \cos\theta\, d\varphi)$$

U, a, $\phi \sim$ z, Z^A *parameterize in the G/H case the coset representative*

$$\mathbb{L}(\Phi) = \exp\left[-a\, L_{+}^{E}\right] \exp\left[\sqrt{2}\, Z^{M}\, \mathcal{W}_{M}\right] \mathbb{L}_{4}(\phi) \exp\left[U\, L_{0}^{E}\right]$$

gen. in **(2,W)**

Coset repres. in D=4

Element of Borel $(U_{D=3})$

Ehlers SL(2,R)

From coset rep. to Lax equation

$$\Sigma(\tau) \equiv \mathbb{L}^{-1}(\tau)\frac{d}{d\tau}\mathbb{L}(\tau)$$ **From coset representative**

$$\begin{aligned} \Sigma(\tau) &= L(\tau) \oplus W(\tau) \\ W(\tau) \in \mathbb{H}^\star &\Rightarrow \eta W^T(\tau) + W(\tau)\eta = 0 \\ L(\tau) \in \mathbb{K} &\Rightarrow \eta L^T(\tau) - L(\tau)\eta = 0 \end{aligned}$$ **decomposition**

$$W(\tau) = L_>(\tau) - L_<(\tau)$$ **R-matrix**

$$\frac{d}{d\tau}L(\tau) = [W(\tau)\,,\,L(\tau)]$$ **Lax equation**

Integration algorithm

Initial conditions $$L_0 = L(0) \quad , \quad \mathbb{L}_0 = \mathbb{L}(0)$$

Building block $$\mathcal{C}(\tau) := \exp\left[-2\,\tau\,L_0\right]$$

$$\mathfrak{D}_i(\mathcal{C}) := \mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i}(\tau) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i}(\tau) \end{pmatrix}, \quad \mathfrak{D}_0(\tau) := 1\,.$$

$$\left(\mathbb{L}(\tau)^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathcal{C})\mathfrak{D}_{i-1}(\mathcal{C})}}\mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{i,j} \end{pmatrix}$$

Found by Fre & Sorin 2009 - 2010

Key property of integration algorithm

$$L(\tau) = \mathcal{Q}(\mathcal{C})\, L_0\, (\mathcal{Q}(\mathcal{C}))^{-1}$$

$$\mathcal{Q}(\mathcal{C}) \in \mathsf{H}^{\star}$$

Hence all LAX evolutions occur within distinct orbits of H*

Fundamental Problem: classification of ORBITS

The role of H*

$$\mathsf{U}_{D=3} \supset \begin{cases} \mathsf{H} \\ \mathsf{and} \\ \mathsf{H}^{\star} \end{cases}$$

H: **Max. comp. subgroup** — **COSMOL.**

H*: **Different real form of H** — **BLACK HOLES**

In our simple $G_{2(2)}$ model

$$\mathbb{H}^{\star} = \mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$$

The method of standard triplets

The basic theorem proved by mathematicians is that any nilpotent element of a Lie algebra $X \in \mathfrak{g}$ can be regarded as belonging to a triplet of elements $\{x, y, h\}$ satisfying the standard commutation relations of the $\mathfrak{sl}(2)$ Lie algebra, namely:

$$[h\,,\,x] = x \quad ; \quad [h\,,\,y] = -y \quad ; \quad [x\,,\,y] = 2\,h$$

Hence the classification of nilpotent orbits is just the classification of embeddings of an $\mathfrak{sl}(2)$ Lie algebra in the ambient one, modulo conjugation by the full group $\mathrm{G}_{\mathbb{R}}$ or by one of its subgroups. In our case the relevant subgroup is $\mathrm{H}^{\star} \subset \mathrm{G}_{\mathbb{R}}$.

Angular momenta i.e. α-labels

Embeddings of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are characterized by the branching law of any representation of $\mathfrak{g}$ into irreducible representations of $\mathfrak{h}$. In the case of the $\mathfrak{sl}(2) \sim \mathfrak{so}(1,2)$ algebra the branching law is expressed by listing the angular momenta $\{j_1, j_2, \ldots j_n\}$ of the irreducible blocks into which the fundamental representations decomposes.

$$\sum_{i=1}^{n} (2j_i + 1) = N$$

The representations $j_1, j_2, \ldots\ldots, j_3$ are called the α-labels

The classification algorithm

$$\mathbb{U} = \mathbb{H} \oplus \mathbb{K}$$

For nilpotent $\mathbb{K}$ elements we choose the central element h in the Cartan subalgebra $\mathcal{C} \subset \mathbb{H}^{\star}$. The Weyl group $\mathcal{W}$ is the symmetry group of the root system Δ. If $\mathcal{C} \subset \mathrm{H}^{\star}$

$$\Delta = \Delta_H \bigoplus \Delta_K$$

$\Delta_{H,K}$ contains the roots represented in $\mathbb{H}^{\star}$, respectively $\mathbb{K}$.

$\boxed{\mathcal{W}_H \subset \mathcal{W}}$ is the subgroup which respects the splitting

The Hweyl subgroup

Given

Cartan element h corresponding to a partition $\{j_1, j_2, \ldots j_n\}$, we consider its Weyl orbit and we split this Weyl orbit into m suborbits corresponding to the m cosets:

$$\frac{\mathcal{W}}{\mathcal{W}_H} \quad ; \quad m \equiv \frac{|\mathcal{W}|}{|\mathcal{W}_H|}$$

Each Weyl suborbit corresponds to an H*-orbit of the neutral elements h in the standard triples. We just have to separate those triples whose x and y elements lie in $\mathbb{K}$ from those whose x and y elements lie in $\mathbb{H}^\star$. By construction if the x and y elements of one triple lie in $\mathbb{K}$, the same is true for all the other triples in the same $\mathcal{W}_H$ orbit. Weyl transformations outside $\mathcal{W}_H$ mix instead $\mathbb{K}$-triples with $\mathbb{H}^\star$ ones.

Given h one can impose the commutation relations:

$$\begin{array}{rcl} [h\,,\,x] & = & x \\ [x\,,\,x^T] & = & 2\,h \end{array}$$

as a set of algebraic equations for x. Typically these equations admit more than one solution[2].

β-labels

When continuous parameters are left over in the solutions space, signaling the existence of a continuous part in the $\mathcal{S}_h$ stabilizer, the direct construction of $\mathcal{S}_h$ orbits is more involved and time consuming. An alternative method, however, is available to distribute the obtained solutions into distinct orbits which is based on invariants. Let us define the non-compact operator:

$$X_c \equiv \mathrm{i}\,(x - x^T)$$

and consider its adjoint action on the maximal compact subalgebra $\mathbb{H} \subset \mathbb{U}$ which, by construction, has the same dimension as $\mathbb{H}^\star$. We name β-labels the spectrum of eigenvalues of that adjoint matrix[3]:

$$\beta - \text{label} = \text{Spectrum}\left[\text{adj}_{\mathbb{H}}\,(X_c)\right]$$

Since the spectrum is an invariant property with respect to conjugation, x-solutions that have different β-labels belong to different $\mathrm{H}^\star$ orbits necessarily. Actually they even belong to different orbits with respect to the full group U. In fact there exists a one-to-one correspondence between nilpotent U orbits in $\mathbb{U}$ and β-labels, which directly follows from the celebrated Kostant-Sekiguchi theorem So we arrange the different solutions of

$$\left[x\,,\,x^T\right] = 2\,h$$

into orbits by grouping them according to their β-labels

γ-labels

The set of possible β-labels at fixed choice of the partition $\{j_1, j_2, \ldots j_n\}$ is predetermined since it corresponds to the set of γ-labels [31]. Let us define these latter. Given the central element h of the triple, we consider its adjoint action on the subalgebra $\mathbb{H}^\star$ and we set:

$$\gamma - \text{label} = \text{Spectrum}\left[\text{adj}_{\mathbb{H}^\star}(h)\right]$$

Obviously all h-operators in the same $\mathcal{W}_H$-orbit have the same γ-label. Hence the set of possible γ-labels corresponding to the same partition $\{j_1, j_2, \ldots j_n\}$ contains at most as many elements as the order of lateral classes $\frac{\mathcal{W}}{\mathcal{W}_H}$. The actual number can be less when some $\mathcal{W}_H$-orbits of h-elements coincide[4]. Given the set of γ-labels pertaining to one $\{j_1, j_2, \ldots j_n\}$-partition the set of possible β-labels pertaining to the same partition is the same. We know a priori that the solutions to eq.(2.15) will distribute in groups corresponding to the available β-labels. Typically all available β-labels will be populated, yet for some partition $\{j_1, j_2, \ldots j_n\}$ and for some chosen γ-label one or more β-labels might be empty.

Final classification of orbits

The above discussion shows that by naming α-label the partition $\{j_1, j_2, \ldots j_n\}$ (branching rule of the fundamental representation of $\mathbb{U}$ with respect to the embedded $\mathfrak{sl}(2)$) the orbits can be classified and named with a triple of indices:

$$\mathcal{O}^{\alpha}_{\gamma\beta}$$

the set of $\gamma\beta$-labels available for each α-label being determined by means of the action of the Weyl group as we have thoroughly explained.

What we have described is a precise algorithm to construct triple representative of nilpotent **H*** orbits of nilpotent operators in **K**

Example of $G_{2,2}$: Partitions

- (j=3) ⟹ The largest orbit NO5
- (j=1, j=1/2, j=1/2) ⟹ The orbit NO2
- (j=1, j=1,j=0) ⟹ Splits into NO3 and NO4 orbits
- (j=1/2, j=1/2, j=0, j=0, j=0) The smallest orbit NO1

Classification of Nilpotent Orbits for: $\frac{G_{2,2}}{SL(2)\times SL(2)}$

	d_n	α – label	$\gamma\beta$ – labels	Orbits	$\mathcal{W}_H$ – classes
	7	[j=3]	$\gamma\beta_1 = \{8_1 4_1 0_1\}$	$\mathcal{O}_1^1$	$(\times, \gamma_1, \times)$
	3	[j=1]×2[j=1/2]	$\gamma\beta_1 = \{3_1 1_1 0_1\}$	$\mathcal{O}_1^2$	$(\gamma_1, \gamma_1, \times)$
	3	2[j=1]×[j=0]	$\gamma\beta_1 = \{4_1 0_2\}$ $\gamma\beta_2 = \{2_2 0_1\}$	$\begin{array}{c\|cc} & \beta_1 & \beta_2 \\ \hline \gamma_1 & \mathcal{O}_{1,1}^3 & \mathcal{O}_{1,2}^3 \\ \gamma_2 & \mathcal{O}_{2,1}^3 & \mathcal{O}_{2,2}^3 \end{array}$	$(\gamma_1, \gamma_2, \gamma_2)$
	2	2[j=1/2]×3 [j=0]	$\gamma\beta_1 = \{1_2 0_1\}$	$\mathcal{O}_1^4$	$(0, \gamma_1, \gamma_1)$

Tits Satake Theory

- To each non maximally non-compact real form **U** (non split) of a Lie algebra of rank r_1 is associated a unique subalgebra $\mathbf{U_{TS}} \subset \mathbf{U}$ which is maximally split.
- $\mathbf{U_{TS}}$ has rank $r_2 < r_1$
- The Cartan subalgebra $\mathbf{C_{TS}} \subset \mathbf{U_{TS}}$ is the non compact part of the full cartan subalgebra

Tits Satake projection

Tits-Satake (TS) projection of *special homogeneous ($\mathcal{SH}$) manifolds*:

$$\mathcal{SH} \overset{\text{Tits-Satake}}{\Longrightarrow} \mathcal{SH}_{\text{TS}}$$

1. π_{TS} is a projection: different manifolds $\mathcal{SH}_i$ have the same image $\pi_{\text{TS}}(\mathcal{SH}_i)$.

2. π_{TS} preserves the rank of $\mathcal{G}_M$.

3. π_{TS} maps special homogeneous into special homogeneous manifolds and preserves the three classes of special manifolds (real special, special Kähler, special quaternionic)

Universality Classes

$$
\begin{array}{ccccc}
\text{Very Special real} & \overset{\mathbf{r}\text{-map}}{\Longrightarrow} & \text{Special Kähler} & \overset{\mathbf{c}\text{-map}}{\Longrightarrow} & \text{Quaternionic-Kähler} \\
\pi_{\mathrm{TS}} \Downarrow & & \pi_{\mathrm{TS}} \Downarrow & & \pi_{\mathrm{TS}} \Downarrow \\
(\text{Very Special real})_{\mathrm{TS}} & \overset{\mathbf{r}\text{-map}}{\Longrightarrow} & (\text{Special Kähler})_{\mathrm{TS}} & \overset{\mathbf{c}\text{-map}}{\Longrightarrow} & (\text{Quaternionic-Kähler})_{\mathrm{TS}}
\end{array}
$$

The main consequence of the above features is that the whole set of special homogeneous manifolds and hence of associated supergravity models is distributed into a set of *universality classes* which turns out to be composed of extremely few elements.

One example

$$\mathcal{SKO}_{2s+2} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2+2s)}{\mathrm{SO}(2)\times \mathrm{SO}(2+2s)}$$

$$\mathcal{QM}_{(4,4+2s)} = \frac{\mathrm{U}_{D=3}}{\mathrm{H}} = \frac{\mathrm{SO}(4,4+2s)}{\mathrm{SO}(4)\times \mathrm{SO}(4+2s)} \; .$$

$$\mathcal{QM}^{\star}_{(4,4+2s)} = \frac{\mathrm{U}_{D=3}}{\mathrm{H}^{\star}} = \frac{\mathrm{SO}(4,4+2s)}{\mathrm{SO}(2,2)\times \mathrm{SO}(2,2+2s)} \; .$$

Tits-Satake Projection SO(4,5)

The 37 Universal Nilpotent Orbits

N	d_n	$\alpha -$ label	$\gamma\beta -$ labels	Orbits
1	9	[j=4]	$\gamma\beta_1 = \{\pm 0_{p_s}, \pm 4_{2s+2}, \pm 8_{2s+1}, \pm 12_1\}$	$\mathcal{O}^1_1$
2	7	[j=3]×2[j=0]	$\gamma\beta_1 = \{\pm 0_{2s+p_s}, \pm 4_{2s+3}, \pm 8_1\}$ $\gamma\beta_2 = \{\pm 0_{p_s}, \pm 2_{2s}, \pm 4_3, \pm 6_{2s}, \pm 8_1\}$	β_1, β_2 γ_1: $\mathcal{O}^2_{1,1}$, $\mathcal{O}^2_{1,2}$ γ_2: $\mathcal{O}^2_{2,1}$, $\mathcal{O}^2_{2,2}$
3	5	[j=2]×2[j=1/2]	$\gamma\beta_1 = \{\pm 0_{p_s}, \pm 1_{2s+1}, \pm 3_2, \pm 4_{2s}, \pm 5_1\}$	$\mathcal{O}^3_1$
4	4	2[j=3/2]×[j=0])	$\gamma\beta_1 = \{\pm 0_{p_s}, \pm 1_{2s}, \pm 2_2, \pm 3_{2s}, \pm 4_2\}$	$\mathcal{O}^4_1$
5	3	3[j=1]	$\gamma\beta_1 = \{\pm 0_{p_s+1}, \pm 2_{4s+2}, \pm 4_1\}$	$\mathcal{O}^5_1$
6	3	[j=1]×2[j=1/2]×2[j=0]	$\gamma\beta_1 = \{\pm 0_{2s+p_s}, \pm 1_{2s+3}, \pm 3_1\}$ $\gamma\beta_2 = \{\pm 0_{p_s}, \pm 1_{2s+3}, \pm 2_{2s}, \pm 3_1\}$	β_1, β_2 γ_1: $\mathcal{O}^6_{1,1}$, $\mathcal{O}^6_{1,2}$ γ_2: $\mathcal{O}^6_{2,1}$, $\mathcal{O}^6_{2,2}$
7	3	2[j=1]×3[j=0]	$\gamma\beta_1 = \{\pm 0_{4s+p_s+3}, \pm 4_1\}$ $\gamma\beta_2 = \{\pm 0_{2s+p_s}, \pm 2_{2s+4}\}$ $\gamma\beta_3 = \{\pm 0_{p_s+3}, \pm 2_{4s}, \pm 4_1\}$	β_1, β_2, β_3 γ_1: $\mathcal{O}^7_{1,1}$, $\mathcal{O}^7_{1,2}$, $\mathcal{O}^7_{1,3}$ γ_2: $\mathcal{O}^7_{2,1}$, $\mathcal{O}^7_{2,2}$, $\mathcal{O}^7_{2,3}$ γ_3: $\mathcal{O}^7_{3,1}$, $\mathcal{O}^7_{3,2}$, $\mathcal{O}^7_{3,3}$
8	2	4[j=1/2]×[j=0]	$\gamma\beta_1 = \{\pm 0_{p_s+2}, \pm 1_{4s}, \pm 2_2\}$	$\mathcal{O}^8_1$
9	3	[j=1]×6[j=0]	$\gamma\beta_1 = \{\pm 0_{4s+p_s+2}, \pm 2_2\}$ $\gamma\beta_2 = \{\pm 0_{2s+p_s+2}, \pm 2_{2s+2}\}$	β_1, β_2 γ_1: $\mathcal{O}^6_{1,1}$, $\mathcal{O}^6_{1,2}$ γ_2: $\mathcal{O}^6_{2,1}$, $\mathcal{O}^6_{2,2}$
10	2	2[j=1/2]×5[j=0]	$\gamma\beta_1 = \{\pm 0_{2s+p_s}, \pm 1_{2s+4}\}$	$\mathcal{O}^{10}_1$
11	5	[j=2]×[j=1]×[j=0]	$\gamma\beta_1 = \{\pm 0_{p_s+1}, \pm 2_{4s}, \pm 4_3\}$ $\gamma\beta_2 = \{\pm 0_{2s+p_s+1}, \pm 4_{2s+3}\}$ $\gamma\beta_3 = \{\pm 0_{p_s}, \pm 2_{2s+3}, \pm 4_{2s}, \pm 6_1\}$	β_1, β_2, β_3 γ_1: $\mathcal{O}^{11}_{1,1}$, $\mathcal{O}^{11}_{1,2}$, $\times$ γ_2: $\mathcal{O}^{11}_{2,1}$, $\times$, $\mathcal{O}^{11}_{2,3}$ γ_3: $\times$, $\mathcal{O}^{11}_{3,2}$, $\mathcal{O}^{11}_{3,3}$
12	5	[j=2]×4[j=0]	$\gamma\beta_1 = \{\pm 0_{2s+p_s}, \pm 2_{2s+2}, \pm 4_2\}$ $\gamma\beta_2 = \{\pm 0_{2s+p_s}, \pm 2_2, \pm 4_{2s+2}\}$	β_1, β_2 γ_1: $\mathcal{O}^{12}_{1,1}$, $\mathcal{O}^{12}_{1,2}$ γ_2: $\mathcal{O}^{12}_{2,1}$, $\mathcal{O}^{12}_{2,2}$

$$p_s = s(2s-1) + 4.$$

$$\frac{\mathrm{SO}(4,4+2s)}{\mathrm{SO}(2,2)\times \mathrm{SO}(2,2+2s)}$$

Tits Satake

$$\frac{\mathrm{SO}(4,5)}{\mathrm{SO}(2,2)\times \mathrm{SO}(2,5)}$$

Спосибо за внимание

Thank you for your attention

Tiger Tiger, burning bright in the forests of the night,
Who could frame thy fearful Supersymmetry..........?