# Effect of the relativistic spin rotation on two-particle spin composition

R. Lednicky, V. L. Lyuboshitz, V. V. Lyuboshitz

(Joint Institute for Nuclear Research, Dubna, Russia)

X International Workshop on High Energy Spin Physics — SPIN-03

Dubna, September 16-20, 2003

## 1 Essence of the relativistic spin rotation.

In the modern relativistic theory of reactions the S-matrix is usually parametrized on the basis of the formalism of inhomogeneous Lorentz group. This is similar to the nonrelativistic theory but includes the essential modification – the relativistic spin rotation. In so doing, the particle spin state in the frame K is set by the Lorentz transformation along the direction of the particle velocity from the frame K to the particle rest frame. Thus, the spin direction depends on the frame, from which the Lorentz transformation to the rest frame is performed. This is the essence of the effect of the relativistic spin rotation at the transition from one frame to another. The relativistic spin rotation is a purely kinematic effect conditioned by the setting of the spin of a particle in its rest frame and by the noncommutativity of successive Lorentz transformations along noncolinear directions.

E. P. Wigner, Ann. Math 40, 149 (1939)
Yu. M. Shirokov, Zh. Eksp. Teor. Fiz. 33, p. 861, 1196, 1208 (1957)
L. L. Foldy, Phys. Rev. 102, 568 (1956)
H. P. Stapp, Phys. Rev. 103, 425 (1956)
Chou Kuang-chao, M. I. Shirokov, Zh. Exp. Teor. Fiz. 34, 1230 (1958) [Sov. Phys. JETP 7, 851 (1958)]
V. I. Ritus, Zh. Eksp. Teor. Fiz. 40, 352 (1961) [Sov. Phys. JETP 13, 240 (1961)]

The spin state of a system of two free particles with the momenta $\vec{p_1}$ and $\vec{p_2}$ in an arbitrary frame $K$ => described by the two-particle spin density matrix $\rho^{(1,2)}_{m_1 m_1'; m_2 m_2'}$.

$m_1, m_1'$ => spin projections of the first particle in its rest frame;

$m_2, m_2'$ => spin projections of the second particle in its rest frame

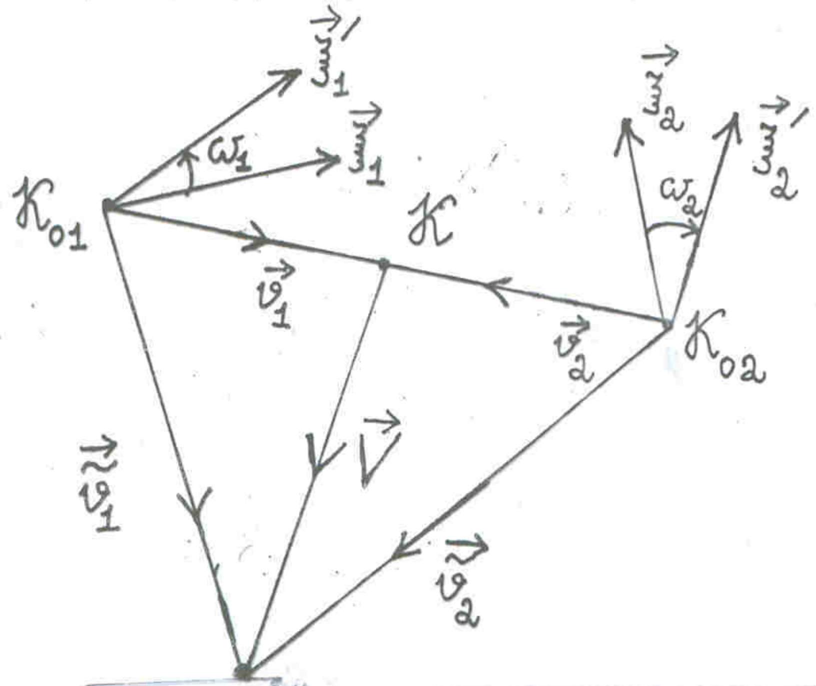
The frame $K$, rest frames → parallel respective spatial axes

Projections → onto the common coordinate axis $z$

[R. Lednicky, V.V. Lyuboshitz, V.L. Lyuboshitz, Yad. Fiz. 66, 1007 (2003) [Phys. At. Nucl. 66, 975 (2003)]

In the case of two free particles with a nonzero relative momentum, at the transition from the frame $K$ to another frame $K'$ with the parallel respective axes the angles of the relativistic spin rotation are different, when the particle velocities do not coincide and, besides, they are not colinear with the vector of relative velocity of the frames $K$ and $K'$. That leads to the frame dependence of the fractions of states with the different values of total spin at the addition of the spins set in the rest frames of two particles.

## 2 Application to the case of two-particle spin states




$K_{01}$ → the rest frame of the first particle

$K_{02}$ → the rest frame of the second particle

$K$ → the c.m.s. of the particle pair

$K'$ → the laboratory frame

The frames $K_{01}$ and $K$; $K_{02}$ and $K$; $K$ and $K'$ → with the parallel respective axes

$\vec{p}_1 = \vec{k}$, $\vec{p}_2 = -\vec{k}$ → momenta of the first and second particles, respectively, in the c.m.s; $M_1, M_2$ – masses of the particles.

$\vec{v}_1 = c\vec{k}/\sqrt{\vec{k}^2 + M_1^2c^2}$ → velocity of the first particle in the c.m.s

$\vec{v}_2 = -c\vec{k}/\sqrt{\vec{k}^2 + M_2^2c^2}$ → velocity of the second particle in the c.m.s.

$\vec{V}$ → velocity of the c.m.s of the particle pair in the laboratory frame

$\tilde{\vec{v}}_1$ → velocity of the first particle in the laboratory frame (result of relativistic addition of velocities $\vec{v}_1$ and $\vec{V}$)

$\tilde{\vec{v}}_2$ → velocity of the second particle in the laboratory frame (result of relativistic addition of velocities $\vec{v}_2$ and $\vec{V}$).

$\vec{\xi}_1, \vec{\xi}_2$ → spin polarizations of the first and second particles in the c.m.s; $\vec{\xi}_1', \vec{\xi}_2'$ → the same in the laboratory frame

$\omega_1, \omega_2$ → angles of the relativistic spin rotation for the first and second particles, respectively, at the Lorentz transformation from the c.m.s. of the particle pair to the laboratory frame.

Opposite directions of spin rotation for the first and second particles (around the axis $z$ parallel to the vector $[\vec{k}\vec{V}]$:

$\omega_1 > 0$, $\omega_2 < 0$. $|\vec{\zeta}_1| = |\vec{\zeta}_1'|$; $\zeta_{1z} = \zeta_{1z}'$; $|\vec{\zeta}_2| = |\vec{\zeta}_2'|$; $\zeta_{2z} = \zeta_{2z}'$

Expressions for the rotation angles $\omega_1, \omega_2$:

$$\sin\omega_1 = \gamma\gamma_1 \frac{v_1 v}{c^2} \sin\Theta \frac{1+\gamma+\gamma_1+\tilde{\gamma}_1}{(1+\gamma)(1+\gamma_1)(1+\tilde{\gamma}_1)};$$

Stapp formula

[H.P. Stapp, Phys. Rev. 103, 425 (1956)]

$$\sin\omega_2 = -\gamma\gamma_2 \frac{v_2 v}{c^2} \sin\Theta \frac{1+\gamma+\gamma_2+\tilde{\gamma}_2}{(1+\gamma)(1+\gamma_2)(1+\tilde{\gamma}_2)};$$

$$\cos\omega_1 = 1 - \frac{(\gamma-1)(\gamma_1-1)}{1+\tilde{\gamma}_1}\sin^2\Theta;$$

$$\cos\omega_2 = 1 - \frac{(\gamma-1)(\gamma_2-1)}{1+\tilde{\gamma}_2}\sin^2\Theta;$$

$\Theta \rightarrow$ angle between the vectors $\vec{k}$ and $\vec{V}$ $(0 \leq \Theta \leq \pi)$.

Lorentz-factors:

$$\gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}; \quad \gamma_2 = \left(1 - \frac{v_2^2}{c^2}\right)^{-1/2}; \quad \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2};$$

$$\tilde{\gamma}_1 = \left(1 - \frac{\tilde{v}_1^2}{c^2}\right)^{-1/2} = \gamma_1\gamma\left(1 + \frac{v_1 V}{c^2}\cos\Theta\right);$$

$$\tilde{\gamma}_2 = \left(1 - \frac{\tilde{v}_2^2}{c^2}\right)^{-1/2} = \gamma_2\gamma\left(1 - \frac{v_2 V}{c^2}\cos\Theta\right).$$

The case of particles with equal masses:

$\vec{v}_1 = -\vec{v}_2$; $\gamma_1 = \gamma_2$ (but $\tilde{\gamma}_1 \neq \tilde{\gamma}_2$, when $V\cos\Theta \neq 0$)

In the case of colinearity of the velocity vectors $\vec{v}_1(\vec{v}_2)$ and $\vec{V}$ (i.e. $\Theta=0$ or $\Theta=\pi$) $\Rightarrow$ both the rotation angles $\omega_1, \omega_2$ turn to zero: $\omega_1=\omega_2\equiv 0$.

Nonrelativistic velocities in the c.m.s of the pair $\Rightarrow$

$$\omega_1 \approx \frac{\gamma}{\gamma+1}\frac{v_1 V}{c^2}\sin\Theta;\quad \omega_2 \approx \frac{\gamma}{\gamma+1}\frac{v_2 V}{c^2}\sin\Theta;\quad \omega_1 \ll 1, |\omega_2| \ll 1$$

Ultrarelativistic limit $(\gamma_1\to\infty, \gamma_2\to\infty)$;

$$\tilde{\gamma}_1/\gamma_1 \to \gamma\left(1+\frac{V}{c}\cos\Theta\right),\quad \tilde{\gamma}_2/\gamma_2 \to \gamma\left(1-\frac{V}{c}\cos\Theta\right) \Rightarrow$$

$$\sin\omega_1 \approx \frac{V}{c}\sin\Theta\,\frac{1+\gamma\left(1+\frac{V}{c}\cos\Theta\right)}{(1+\gamma)\left(1+\frac{V}{c}\cos\Theta\right)};\quad \sin\omega_2 \approx -\frac{V}{c}\sin\Theta\,\frac{1+\gamma\left(1-\frac{V}{c}\cos\Theta\right)}{(1+\gamma)\left(1-\frac{V}{c}\cos\Theta\right)}$$

$$\cos\omega_1 \approx 1-\frac{\gamma-1}{\gamma\left(1+\frac{V}{c}\cos\Theta\right)}\sin^2\Theta;\quad \cos\omega_2 \approx 1-\frac{\gamma-1}{\gamma\left(1-\frac{V}{c}\cos\Theta\right)}\sin^2\Theta$$

(the angles of the relativistic spin rotation become equal to the corresponding angles of momentum rotation (the aberration angles) at the transition from the c.m.s. of the particle pair to the laboratory frame). Exact equalities $\to$ for massless particles (photons, neutrinos)

Transformation of the two-particle spin density matrix at the transition from the c.m.s. of the particle pair to the laboratory frame, taking into account the relativistic spin rotation:

$$\hat{\rho}'^{(1,2)} = \hat{D}^{(1)}(\omega_1)\otimes\hat{D}^{(2)}(\omega_2)\,\hat{\rho}^{(1,2)}\,\hat{D}^{+(1)}(\omega_1)\,D^{+(2)}(\omega_2).$$

Here: $\hat{D}^{(1)}(\omega_1) = \exp(i\omega_1 \hat{\vec{j}}_1 \vec{n})$, $\hat{D}^{(2)}(\omega_2) = \exp(i\omega_2 \hat{\vec{j}}_2 \vec{n})$

$\Rightarrow$ matrices of space rotation;

$\vec{n}$ – unit vector along the direction of the vector $[\vec{k}\vec{V}]$;

$\hat{\vec{j}}_1, \hat{\vec{j}}_2 \rightarrow$ vector spin operators of the first and second particles

## 3. Spin-½ particles. Transformations of components of the correlation tensor

Structure of the two-particle spin density matrix in the c.m.s. of the pair.

$$\hat{\rho}^{(1,2)} = \frac{1}{4}\Big[\hat{I}^{(1)} \otimes \hat{I}^{(2)} + (\vec{\sigma}^{(1)}\vec{P}_1) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes (\vec{\sigma}^{(2)}\vec{P}_2) + \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik}\, \hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\Big]$$

$\hat{I} \rightarrow$ two-row unit matrix

$\vec{\sigma} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\} \rightarrow$ the vector Pauli operator;

$\vec{P}_1 = \langle \vec{\sigma}^{(1)} \rangle \rightarrow$ polarization vector of the first particle

$\vec{P}_2 = \langle \vec{\sigma}^{(2)} \rangle \rightarrow$ polarization vector of the second particle

$T_{ik} = \langle \hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)} \rangle \rightarrow$ components of the correlation tensor (in general, $T_{ik} \neq P_{1i} P_{2k}$, equality $\rightarrow$ for independent particles only).

V. L. Lyuboshitz, M. I. Podgoretsky, Yad. Fiz 60, 45 (1997)

R. Lednicky, V. L. Lyuboshitz, Phys. Lett. B508, 146 (2001)

Matrices of space rotations for the first and second particles:

$$\hat{D}^{(1)}(\omega_1) = \cos\frac{\omega_1}{2} + i\vec{\sigma}^{(1)}\vec{n}\sin\frac{\omega_1}{2}$$

$$\hat{D}^{(2)}(\omega_2) = \cos\frac{\omega_2}{2} + i\vec{\sigma}^{(2)}\vec{n}\sin\frac{\omega_2}{2}$$

$\left(\vec{n} = \dfrac{[\vec{k}\vec{V}]}{|[\vec{k}\vec{V}]|} \rightarrow \text{unit vector}\right)$

$\Rightarrow$ transformations of the polarization parameters for two spin-½ particles at the transition from the c.m.s. of the particle pair to the laboratory frame, taking into account the relativistic spin rotation by the angles $\omega_1$, $\omega_2$, are as follows:

a) Components of polarization vectors in the laboratory frame.

$$P'_{1x} = P_{1x}\cos\omega_1 - P_{1y}\sin\omega_1; \quad P'_{2x} = P_{2x}\cos\omega_2 - P_{2y}\sin\omega_2;$$

$$P'_{1y} = P_{1y}\cos\omega_1 + P_{1x}\sin\omega_1; \quad P'_{2y} = P_{2y}\cos\omega_2 + P_{2x}\sin\omega_2$$

$$P'_{1z} = P_{1z}, \quad P'_{2z} = P_{2z}$$

b) Diagonal components of the correlation tensor $T_{ik}$ in the laboratory frame:

$$T'_{xx} = (T_{xx}\cos\omega_1 - T_{yx}\sin\omega_1)\cos\omega_2 - (T_{xy}\cos\omega_1 - T_{yy}\sin\omega_1)\sin\omega_2$$

$$T'_{yy} = (T_{yy}\cos\omega_1 + T_{xy}\sin\omega_1)\cos\omega_2 + (T_{yx}\cos\omega_1 + T_{xx}\sin\omega_1)\sin\omega_2$$

$$T'_{zz} = T_{zz}$$

(Here and further $z \parallel [\vec{k}\vec{V}]$, axes $x, y$ → in the plane perpendicular to the vector $[\vec{k}\vec{V}]$)

c) Nondiagonal components of the correlation tensor $T_{ik}$ in the laboratory frame.

$$T'_{xy} = \left(T_{xy}\cos\omega_1 - T_{yy}\sin\omega_1\right)\cos\omega_2 + \left(T_{xx}\cos\omega_1 - T_{yx}\sin\omega_1\right)\sin\omega_2$$

$$T'_{yx} = \left(T_{yx}\cos\omega_1 + T_{xx}\sin\omega_1\right)\cos\omega_2 - \left(T_{yy}\cos\omega_1 + T_{xy}\sin\omega_1\right)\sin\omega_2$$

$$T'_{xz} = T_{xz}\cos\omega_1 - T_{yz}\sin\omega_1; \quad T'_{zx} = T_{zx}\cos\omega_2 - T_{zy}\sin\omega_2;$$

$$T'_{yz} = T_{yz}\cos\omega_1 + T_{xz}\sin\omega_1; \quad T'_{zy} = T_{zy}\cos\omega_2 + T_{zx}\sin\omega_2$$

d) Transformation of the „trace" of the correlation tensor $T_{ik}$:

$T = T_{xx} + T_{yy} + T_{zz}$ (in the c.m.s.). In the laboratory frame:

$$T' = T'_{xx} + T'_{yy} + T'_{zz} = \left(T_{xx} + T_{yy}\right)\cos(\omega_1 - \omega_2) + \left(T_{xy} - T_{yx}\right)\sin(\omega_1 - \omega_2) + T_{zz}.$$

For the symmetric tensor $T_{ik}$: $\boxed{T' = T - 2\left(T_{xx} + T_{yy}\right)\sin^2\frac{\omega_1 - \omega_2}{2}}$.

Example: angular correlation between the momenta of two protons in the rest frames of two $\Lambda$-hyperons, decaying into the channel $\Lambda \to p + \pi^-$ with space parity nonconservation:

$$\boxed{dW = \frac{1}{2}\left(1 + \frac{\alpha^2}{3} T\cos\varphi\right) d(\cos\varphi)}$$

($\alpha = 0.642 \to$ factor of P-odd asymmetry)

G. Alexander, H. J. Lipkin, Phys. Lett. B352, 162 (1995)
R. Lednicky, V. L. Lyuboshitz Phys. Lett. B508, 146 (2001)
R. Lednicky, V. V. Lyuboshitz, V. L. Lyuboshitz, Yad. Fiz. 66, 1007 (2003)

Here $T \rightarrow$ the „trace" of the correlation tensor of the $\Lambda\Lambda$-pair in its c.m.s.

This structure of the angular correlation $dW$ holds for the transition to the rest frames of two $\Lambda$-particles from an arbitrary frame. For the laboratory frame $\rightarrow T$ is replaced by $T'$.

## 4 Fractions of the singlet and triplet states.

The „trace" of the correlation tensor of the system of two spin-$1/2$ particles is the linear combination of the relative fractions of the singlet (total spin $S=0$) and triplet ($S=1$) states:

$$\boxed{T = \langle \vec{\sigma}^{(1)} \otimes \vec{\sigma}^{(2)} \rangle = \rho_t - 3\rho_s \quad (\rho_t + \rho_s = 1)}$$

[R. Lednicky, V.L. Lyuboshitz. Phys. Lett. B508, 146 (2001)]

a) The pure singlet state of the particle pair in its c.m.s.

$\rho_s = 1;\ \rho_t = 0;\ T_{ik} = -\delta_{ik},\ T = -3.$

Transformation to the laboratory frame gives:

$T' = -3 + 4\sin^2\frac{\omega_1 - \omega_2}{2} \rightarrow$ „trace" of the correlation tensor

$\Rightarrow$ the relative fractions of the singlet and triplet states in the laboratory frame are as follows:

$$\boxed{\rho_s' = \cos^2\frac{\omega_1 - \omega_2}{2},\quad \rho_t' = \sin^2\frac{\omega_1 - \omega_2}{2}}$$

($\omega_1, \omega_2 \rightarrow$ angles of relativistic spin rotation)

Two-particle singlet state:

$$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|+\tfrac{1}{2}\rangle_z^{(1)}|-\tfrac{1}{2}\rangle_z^{(2)} - |-\tfrac{1}{2}\rangle_z^{(1)}|+\tfrac{1}{2}\rangle_z^{(2)}\right)$$

Triplet state with the zero projection of total spin onto the spin rotation axis $z$:

$$|\Psi_{10}\rangle = \frac{1}{\sqrt{2}}\left(|+\tfrac{1}{2}\rangle_z^{(1)}|-\tfrac{1}{2}\rangle_z^{(2)} + |-\tfrac{1}{2}\rangle_z^{(1)}|+\tfrac{1}{2}\rangle_z^{(2)}\right).$$

In accordance with the expressions for matrices of space rotations, the singlet state $|\Psi_{00}\rangle$ in the c.m.s. of the pair is transformed into the following superposition of the singlet and triplet states in the laboratory frame:

$$|\Psi_0'\rangle = \frac{1}{\sqrt{2}}\left(\exp\left(i\frac{\omega_1-\omega_2}{2}\right)|+\tfrac{1}{2}\rangle_z^{(1)}|-\tfrac{1}{2}\rangle_z^{(2)} - \exp\left(-i\frac{\omega_1-\omega_2}{2}\right)|-\tfrac{1}{2}\rangle_z^{(1)}|+\tfrac{1}{2}\rangle_z^{(2)}\right) =$$

$$= \cos\left(\frac{\omega_1-\omega_2}{2}\right)|\Psi_{00}\rangle + i\sin\left(\frac{\omega_1-\omega_2}{2}\right)|\Psi_{10}\rangle.$$

b) The pure triplet state $|\Psi_{10}\rangle$ in the c.m.s. of the particle pair.

$\rho_s = 0,\ \rho_t = 1;\ T_{zz} = -1;\ T_{xx} = T_{yy} = 1;\ T = 1.$

Transformations lead to the results:

$T' = 1 - 4\sin^2\frac{\omega_1-\omega_2}{2}$ → the trace of the correlation tensor in the laboratory frame ⟹

the relative fractions of the singlet and triplet states in the laboratory frame:

$$\boxed{\rho_s' = \sin^2\frac{\omega_1-\omega_2}{2}; \quad \rho_t' = \cos^2\frac{\omega_1-\omega_2}{2}.}$$

The state $|\Psi_{10}\rangle$ is transformed, at the transition to the laboratory frame, into the following superposition of the singlet and triplet states:

$$|\Psi_1'\rangle = \cos\left(\frac{\omega_1-\omega_2}{2}\right)|\Psi_{10}\rangle + i\sin\left(\frac{\omega_1-\omega_2}{2}\right)|\Psi_{00}\rangle$$

c) The unpolarized triplet in the c.m.s. of the particle pair:

$\rho_s = 0$, $\rho_t = 1$; $T_{ik} = (1/3)\,\delta_{ik}$; $T = 1$.

According to the transformations; in the laboratory frame;

$T' = 1 - \frac{4}{3}\sin^2\frac{\omega_1-\omega_2}{2} \rightarrow$ „trace" of the correlation tensor.

$\rho_s' = \frac{1}{3}\sin^2\frac{\omega_1-\omega_2}{2}$; $\rho_t' = 1 - \frac{1}{3}\sin^2\frac{\omega_1-\omega_2}{2} \rightarrow$ relative fractions of spin states.

---

The factor of spin mixing:

$R = \sin^2\frac{\omega_1-\omega_2}{2}$. For the case of two particles with equal masses ($\vec{v}_1 = -\vec{v}_2$, $v_1 = |\vec{v}_1| = |\vec{v}_2|$, $\gamma_1 = \gamma_2$)

$$\Rightarrow R = \left(\frac{v_1 v}{c^2}\right)^2 \sin^2\theta \left[\left(\frac{1}{\gamma} + \frac{1}{\gamma_1}\right)^2 + \left(\frac{v_1 v}{c^2}\right)^2 \sin^2\theta\right]^{-1}$$

(here $\vec{v}_1$ – velocity in the c.m.s. of the particle pair, $\gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}$ → Lorentz-factor in the c.m.s, $\vec{v}$ → velocity of the c.m.s. of the pair with respect to the laboratory frame, $v = |\vec{v}|$, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$; $\Theta$ → angle between $\vec{v}_1$ and $\vec{v}$).

Maximum of $R$ ⇒ corresponds to $\Theta = \pi/2$.

In the ultrarelativistic case, under the conditions $\gamma_1 \gg 1, \gamma \gg 1$ and $\sin\Theta \gg \max\left(\frac{1}{\gamma}, \frac{1}{\gamma_1}\right)$, $R \to 1$. Thus, in this case the singlet state in the c.m.s. of the pair becomes the triplet state with the zero projection of total spin onto the spin rotation axis $z$ in the laboratory frame, and the triplet state with the zero projection of total spin in the c.m.s. → the singlet state in the laboratory frame.

## 5 Conclusions.

1) The effect of the relativistic spin rotation for a system of two free particles is investigated, setting the spin states of the particles in the corresponding particle rest frames. The transformations of the correlation tensor components at the transition from the c.m.s. of two particles to the laboratory frame are considered. It is shown that the angles of the relativistic spin rotation for two particles are generally different except for the case when the vectors of the c.m.s. particle velocities are colinear with the velocity vector of c.m.s. in the laboratory frame.

2) It is ascertained that the effect of the relativistic spin rotation leads to the dependence of the two-particle spin composition (in the particular case of spin-1/2 particles → the singlet and triplet fractions) on the concrete frame in which the relativistic two-particle system with a nonzero vector of relative velocity is analyzed.