Chromodynamic Lensing and Single Spin Asymmetries

or: GPDs ⇒ distributions of partons in impact parameter space

spin dependence ⇒ ↓ spin asymmetries

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(brief) Motivation

**DIS** $\underset{opt.\,theorem}{\rightarrow}$ forward Compton amplitude $\underset{Bj-limit}{\rightarrow} q(x)$

\[ q(x) = \int \frac{dx^{-}}{2\pi} \langle p | q \left( -\frac{x^{-}}{2}, 0_{\perp} \right) \gamma^{+} q \left( \frac{x^{-}}{2}, 0_{\perp} \right) | p \rangle \, e^{ix^{-}x^{P^{+}}} \]

- Light-cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}} (x^{0} \pm x^{1})$
- $q(x) =$ light-cone momentum distribution of quarks in the target; $x =$ (light-cone) momentum fraction
- no information about position of partons!
(brief) Motivation

- generalization to $p' \neq p \Rightarrow \text{Generalized Parton Distributions}$

$$GPD(x, \xi, t) \equiv \int \frac{dx^-}{2\pi} \langle p' | q \left( -\frac{x^-}{2}, 0_\perp \right) \gamma^+ q \left( \frac{x^-}{2}, 0_\perp \right) | p \rangle e^{ix^-xP^+}$$

with $\Delta = p - p'$, $t = \Delta^2$, and $\xi(p^+ + p'^+) = -2\Delta^+$.

- can be probed e.g. in Deeply Virtual Compton Scattering (DVCS) (HERMES, JLab@12GeV, eRHIC, COMPASS, ...)


$$\langle J_q \rangle = \frac{1}{2} \int_0^1 dx \ x \left[ H_q(x, 0, 0) + E_q(x, 0, 0) \right]$$

**DVCS** ⇔ **GPDs** ⇔ $\vec{J}_q$

- But: what other “physical information” about the nucleon can we obtain by measuring/calculating GPDs?
Probabilistic interpretation of GPDs as Fourier transforms of impact parameter dependent PDFs

\[ H(x, 0, -\Delta^2_\perp) \xrightarrow{FT} q(x, b_\perp) \]
\[ \tilde{H}(x, 0, -\Delta^2_\perp) \xrightarrow{FT} \Delta q(x, b_\perp) \]

\[ E(x, 0, -\Delta^2_\perp) \xrightarrow{\perp} \text{distortion of PDFs when the target is transversely polarized} \]

Chromodynamik lensing and \( \perp \) single-spin asymmetries (SSA)

transverse distortion of PDFs + final state interactions \( \Rightarrow \) \( \perp \) SSA in \( \gamma N \xrightarrow{} \pi + X \)

Summary
\[
\int \frac{dx^-}{2\pi} e^{ix^-\cdot p^+} x \left< p' \left| \bar{q} \left(-\frac{x^-}{2}\right) \gamma^+ q \left(\frac{x^-}{2}\right) \right| p \right> = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p)
\]

\[
\int \frac{dx^-}{2\pi} e^{ix^-\cdot p^+} x \left< p' \left| \bar{q} \left(-\frac{x^-}{2}\right) \gamma^+ \gamma_5 q \left(\frac{x^-}{2}\right) \right| p \right> = \tilde{H}(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ \gamma_5 u(p) + \tilde{E}(x, \xi, \Delta^2) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2M} u(p)
\]

where $\Delta = p - p'$ is the momentum transfer and $\xi$ measures the longitudinal momentum transfer on the target $\Delta^+ = \xi (p^+ + p'^+)$.
\[
\int \frac{dx}{2\pi} e^{ix\cdot \bar{p}^+ x} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) \\
+ E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p)
\]

- \( x \) = mean long. momentum fraction carried by active quark
- \( \xi \) = longitudinal \((p^+)\) momentum transfer
- In general no probabilistic interpretation since initial and final state not the same
- Instead: interpretation as transition amplitude
- \( \int dx H(x, \xi, \Delta^2) = F_1(\Delta^2) \) and \( \int dx E(x, \xi, \Delta^2) = F_2(\Delta^2) \)
- \( \Leftrightarrow \) GPDs provide a decomposition of form factor w.r.t. the momentum fraction (in IMF) carried by the active quark
- Actually \( GPD = GPD(x, \xi, \Delta^2, q^2) \), but will not discuss \( q^2 \) dependence of GPDs today!
What is Physics of GPDs?

Definition of GPDs resembles that of form factors

\[
\langle p' \mid \hat{O} \mid p \rangle = H(x, \xi, \Delta^2) \bar{u}(p')\gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p)
\]

with \( \hat{O} \equiv \int \frac{dx^-}{2\pi} e^{ix^- \bar{p}^+ x} \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \)

\( \leftrightarrow \) relation between PDFs and GPDs similar to relation between a charge and a form factor

\( \leftrightarrow \) If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs?
<table>
<thead>
<tr>
<th>operator</th>
<th>forward matrix elem.</th>
<th>off-forward matrix elem.</th>
<th>position space</th>
</tr>
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<td>$Q$</td>
<td>$F(t)$</td>
<td>$\rho(\vec{r})$</td>
</tr>
<tr>
<td>$\int \frac{dx^- e^{ixp^+ x^-}}{4\pi} \bar{q}\left(\frac{-x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right)$</td>
<td>$q(x)$</td>
<td>$H(x, \xi, t)$</td>
<td>?</td>
</tr>
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## Form Factors vs. GPDs

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<td>$q(x,\vec{b}_{\perp})$</td>
</tr>
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$q(x,\vec{b}_{\perp}) = \text{impact parameter dependent PDF}$
Impact parameter dependent PDFs

define state that is localized in $\perp$ position:

$$|p^+, R_\perp = 0, \lambda\rangle \equiv N \int d^2 p_\perp |p^+, p_\perp, \lambda\rangle$$

Note: $\perp$ boosts in IMF form Galilean subgroup $\Rightarrow$ this state has

$$R_\perp \equiv \frac{1}{P^+} \int dx^- d^2 x_\perp x_\perp T^{++}(x) = 0_\perp$$

(parton interpretation: $R_\perp = \sum_i x_i b_\perp, i$)

cf.: working in CM frame in nonrel. physics ($\rightarrow$ Soper’s thesis)

define impact parameter dependent PDF

$$q(x, b_\perp) \equiv \int \frac{dx^-}{4\pi} \langle p^+, 0_\perp | \bar{q} \left( -\frac{x^-}{2}, b_\perp \right) \gamma^+ q \left( \frac{x^-}{2}, b_\perp \right) |p^+, 0_\perp \rangle e^{ixp^+x^-}$$
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[
q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | q(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} \\
= |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p_\perp' \int dx^- \langle p^+, p_\perp' | q(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-}
\]
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+ x^-} \]

\[ = |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+ x^-} \]

\[ = |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{q}(-\frac{x^-}{2}, 0_\perp) \gamma^+ q(\frac{x^-}{2}, 0_\perp) | p^+, p_\perp \rangle e^{ixp^+ x^-} \times e^{ib_\perp \cdot (p_\perp - p'_\perp)} \]
Impact parameter dependent PDFs

use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} \]

\[ = |N|^2 \int d^2p_\perp \int d^2p_\perp' \int dx^- \langle p^+, p_\perp' | \bar{q}(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-} \]

\[ \times e^{ib_\perp \cdot (p_\perp - p_\perp')} \]

\[ = |N|^2 \int d^2p_\perp \int d^2p_\perp' H \left( x, 0, -(p_\perp' - p_\perp)^2 \right) e^{ib_\perp \cdot (p_\perp - p_\perp')} \]

\[ \rightarrow q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{ib_\perp \cdot \Delta_\perp} \]
Impact parameter dependent PDFs

\[ q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta^2_\perp) e^{ib_\perp \cdot \Delta_\perp} \]

\((\Delta_\perp = p_\perp - p'_\perp, \xi = 0)\)

- \(q(x, b_\perp)\) has physical interpretation of a density

\[ q(x, b_\perp) \geq 0 \quad \text{for} \quad x > 0 \]

\[ q(x, b_\perp) \leq 0 \quad \text{for} \quad x < 0 \]
GPDs allow simultaneous determination of longitudinal momentum and transverse position of partons

\[ q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{ib_\perp \cdot \Delta_\perp} \]

\( q(x, b_\perp) \) has interpretation as density (positivity constraints!)

\[ q(x, b_\perp) \sim \langle p^+, 0_\perp | b^+(xp^+, b_\perp) b(xp^+, b_\perp) | p^+, 0_\perp \rangle \]
\[ = | b(xp^+, b_\perp) | p^+, 0_\perp \rangle |^2 \geq 0 \]

→ positivity constraint on models
Discussion:  \( GPD \leftrightarrow q(x, b_{\perp}) \)

- Nonrelativistically such a result not surprising! Absence of relativistic corrections to identification \( H(x, 0, -\Delta_{\perp}^2) \overset{FT}{\longleftrightarrow} q(x, b_{\perp}) \) due to Galilean subgroup in IMF.

- \( b_{\perp} \) distribution measured w.r.t. \( R_{\perp}^{CM} = \sum_i x_i r_{i,\perp} \)
  \( \leftarrow \) width of the \( b_{\perp} \) distribution should go to zero as \( x \to 1 \), since the active quark becomes the \( \perp \) center of momentum in that limit!
  \( \leftarrow H(x, 0, t) \) must become \( t \)-indep. as \( x \to 1 \).
  (recently confirmed in LGT calcs. by J.W. Negele et al.)

- very similar results for impact parameter dependent polarized quark distributions (nucleon longitudinally polarized)

\[
\Delta q(x, b_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} \tilde{H}(x, 0, -\Delta_{\perp}^2) e^{i b_{\perp} \cdot \Delta_{\perp}}
\]

- inequality: \( |\Delta q(x, b_{\perp})| \leq |q(x, b_{\perp})| \)
Use intuition about nucleon structure in position space to make predictions for GPDs:

- **large** \( x \): quarks from **localized** valence ‘core’,
- **small** \( x \): contributions from **larger** ‘meson cloud’

\[ H(x, 0, t) \] as \( x \) decreases

- **small** \( x \), expect transverse size to increase
The physics of $E(x, 0, -\Delta^2_\perp)$

- So far: only unpolarized (or long. polarized) nucleon

In general, use ($\Delta^+ = 0$)

$$
\int \frac{dx^- e^{ip^+ x^- x}}{4\pi} \left< P+\Delta, \uparrow \left| \bar{q} \left( \frac{-x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| P, \uparrow \right> = H(x,0,-\Delta^2_\perp)
$$

$$
\int \frac{dx^- e^{ip^+ x^- x}}{4\pi} \left< P+\Delta, \uparrow \left| \bar{q} \left( \frac{-x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| P, \downarrow \right> = -\frac{\Delta_x - i\Delta_y}{2M} E(x,0,-\Delta^2_\perp).
$$

- Consider nucleon polarized in $x$ direction (in IMF)

$|X\rangle \equiv |p^+, \mathbf{R}_\perp = 0_\perp, \uparrow\rangle + |p^+, \mathbf{R}_\perp = 0_\perp, \downarrow\rangle$.

$\hookrightarrow$ unpolarized quark distribution for this state:

$$
q_X(x, b_\perp) = q(x, b_\perp) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2\Delta_\perp}{(2\pi)^2} E(x, 0, -\Delta^2_\perp) e^{ib_\perp \cdot \Delta_\perp}
$$
The physics of $E(x, 0, -\Delta^2_{\perp})$

- $q_X(x, b_{\perp}) \geq 0$ (for $x > 0$) $\Rightarrow$

$$q(x, b_{\perp}) \geq \left| \frac{1}{2M} \nabla_{b_{\perp}} \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} E(x, 0, -\Delta^2_{\perp}) e^{ib_{\perp} \cdot \Delta_{\perp}} \right|$$

- Actually, stronger (“Soffer-type”) inequality exists (Pobylitsa):

$$|q(x, b_{\perp})|^2 \geq |\Delta q(x, b_{\perp})|^2 + \left| \frac{1}{2M} \nabla_{b_{\perp}} \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} E(x, 0, -\Delta^2_{\perp}) e^{ib_{\perp} \cdot \Delta_{\perp}} \right|^2$$
The physics of $E(x, 0, -\Delta^2_\perp)$

- $q_X(x, b_\perp)$ in transversely polarized nucleon is transversely distorted compared to longitudinally polarized nucleons!

- **Mean displacement of flavor $q$ ($\perp$ flavor dipole moment)**

$$d_y^q \equiv \int dx \int d^2 b_\perp q_X(x, b_\perp)b_y = \frac{1}{2M} \int dx E_q(x, 0, 0) = \frac{k^p_q}{2M}$$

with $k^{p/u,d}_q \equiv F^u/d_2(0) = O(1 - 2)$ \implies $d_y^q = O(0.2 \text{ fm})$}

- CM for flavor $q$ shifted relative to CM for whole proton by

$$\int dx \int d^2 b_\perp q_X(x, b_\perp)xb_y = \frac{1}{2M} \int dx x E_q(x, 0, 0)$$

\text{not surprising to find that second moment of $E_q$ is related to angular momentum carried by flavor $q$}
Comparison of a non-rotating sphere that moves in $z$ direction with a sphere that spins at the same time around the $z$ axis and a sphere that spins around the $x$ axis. When the sphere spins around the $x$ axis, the rotation changes the distribution of momenta in the $z$ direction (adds/subtracts to velocity for $y > 0$ and $y < 0$ respectively). For the nucleon, the resulting modification of the (unpolarized) momentum distribution is described by $E(x, 0, -\Delta^2_\perp)$. 
simple model for \( E_q(x, 0, -\Delta^2) \)

For simplicity, make ansatz where \( E_q \propto H_q \)

\[
E_u(x, 0, -\Delta^2) = \frac{\kappa_u^p}{2} H_u(x, 0, -\Delta^2) \\
E_d(x, 0, -\Delta^2) = \kappa_d^p H_d(x, 0, -\Delta^2)
\]

with

\[
\kappa_u^p = 2\kappa_p + \kappa_n = 1.673 \\
\kappa^p_d = 2\kappa_n + \kappa_p = -2.033.
\]

Satisfies: \( \int dx E_q(x, 0, 0) = \kappa^p_q \)

Model too simple but illustrates that anticipated distortion is very significant since \( \kappa_u \) and \( \kappa_d \) known to be large!
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Single Spin Asymmetry (Sivers)

example: $\gamma p \rightarrow \pi X$ (Breit frame)

What is the sign/magnitude of the left-right asymmetry?

$\perp$ asymmetry of outgoing $\pi$ resulting from both Sivers and Collins effect

Sivers: asymmetry of $\pi$ due to asymmetry of $\perp$ momentum of outgoing quark $\langle k_\perp \rangle \sim \int dx \int d^2k_\perp f(x, k_\perp) k_\perp$ with

$$f(x, k_\perp) \propto \int \frac{d\xi - d^2\xi_\perp}{(2\pi)^3} e^{ip \cdot \xi} \langle P, S | \bar{q}(0) U_{[0, \infty]} \gamma^+ U_{[\infty, \xi]} q(\xi) | P, S \rangle |_{\xi^+=0}. $$

with $U_{[0, \infty]} = P \exp \left( i g \int_0^\infty d\eta^- A^+(\eta) \right)$
Modulo gauge links this yields ... (Mankiewicz et al., Sterman, Boer et al.,..)

\[
\langle \mathbf{k}_\perp \rangle \sim \left\langle P, S \left| \bar{q}(0) \gamma^+ \int_0^\infty \frac{d\eta^-}{2\pi} G^{+\perp}(\eta)q(\xi) \right| P, S \right\rangle
\]

physical (semi-classical) interpretation:

- net transverse momentum is result of averaging over the transverse force from spectators on active quark

\[
\int_0^\infty \frac{d\eta^-}{2\pi} G^{+\perp}(\eta) \text{ is } \perp \text{ impulse due to FSI}
\]

- What is sign/magnitude of this result?
connection with \(⊥\) distortion of PDFs

- example: \(\gamma p \rightarrow \pi X\) (Breit frame)

\[ \vec{p}_{\gamma} \]

\[ \vec{p}_N \]

\(\pi^+\)

- \(u, d\) distributions in \(⊥\) polarized proton have left-right asymmetry in \(⊥\) position space (T-even!); sign determined by \(\kappa_u\) & \(\kappa_d\)

- attractive FSI deflects active quark towards the center of momentum

\(\leftrightarrow\) FSI converts left-right position space asymmetry of leading quark into right-left asymmetry in momentum

- compare: convex lens that is illuminated asymmetrically

\(\leftrightarrow\) semi-classical picture for recent results by Brodsky et al.

- natural explanation for correlation between sign of \(\kappa_q\) and sign of Sivers contribution to SSA that has been seen in some models (Brodsky at al., Feng,..)
other predictions:
Other topics

- QCD evolution
- extrapolating to $\xi = 0$
DVCS allows probing GPDS

\[ \int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle \]

GPDS resemble both PDFs and form factors: defined through matrix elements of light-cone correlator, but \( \Delta \equiv p' - p \neq 0 \).

t-dependence of GPDS at \( \xi = 0 \) (purely \( \perp \) momentum transfer) \( \Rightarrow \) Fourier transform of impact parameter dependent PDFs \( q(x, b_\perp) \)

knowledge of GPDS for \( \xi = 0 \) provides novel information about nonperturbative parton structure of nucleons: distribution of partons in \( \perp \) plane

\[
q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{ib_\perp \cdot \Delta_\perp} \\
\Delta q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta_\perp^2) e^{ib_\perp \cdot \Delta_\perp}
\]

\( q(x, b_\perp), \Delta q(x, b_\perp) \) have probabilistic interpretation, e.g. \( q(x, b_\perp) > 0 \) for \( x > 0 \)
$\frac{\Delta_{\perp}}{2M} E(x, 0, -\Delta_{\perp}^2)$ describes how the momentum distribution of unpolarized partons in the $\perp$ plane gets transversely distorted when the nucleon is polarized in $\perp$ direction.

(Attractive) final state interaction converts $\perp$ position space asymmetry into $\perp$ momentum space asymmetry

$\leftrightarrow$ simple physical explanation for sign of Sivers asymmetry

Similar mechanism also applicable to many other semi-inclusive events, such as transverse polarizations in hyperon production.


extrapolating to $\xi = 0$

- **bad news:** $\xi = 0$ not directly accessible in DVCS since long. momentum transfer necessary to convert virtual $\gamma$ into real $\gamma$

- **good news:** moments of GPDs have simple $\xi$-dependence (polynomials in $\xi$)
  $\leftrightarrow$ should be possible to extrapolate!

even moments of $H(x, \xi, t)$:

$$
H_n(\xi, t) \equiv \int_{-1}^{1} dx x^{n-1} H(x, \xi, t) = \sum_{i=0}^{\left[ \frac{n-1}{2} \right]} A_{n,2i}(t) \xi^{2i} + C_n(t)
$$

$$
= A_{n,0}(t) + A_{n,2}(t) \xi^2 + \ldots + A_{n,n-2}(t) \xi^{n-2} + C_n(t) \xi^n,
$$
i.e. for example

\[ \int_{-1}^{1} dxxH(x, \xi, t) = A_{2,0}(t) + C_2(t)\xi^2. \]

- For \( n^{th} \) moment, need \( \frac{n}{2} + 1 \) measurements of \( H_n(\xi, t) \) for same \( t \) but different \( \xi \) to determine \( A_{n,2i}(t) \).
- GPDs @ \( \xi = 0 \) obtained from \( H_n(\xi = 0, t) = A_{n,0}(t) \)
- similar procedure exists for moments of \( \tilde{H} \)
QCD evolution

So far ignored! Can be easily included because:

- For $t \ll Q^2$, leading order evolution $t$-independent
- For $\xi = 0$ evolution kernel for GPDs same as DGLAP evolution kernel

likewise:

- impact parameter dependent PDFs evolve such that different $b_\perp$ do not mix (as long as $\perp$ spatial resolution much smaller than $Q^2$)
above results consistent with QCD evolution:

\[
H(x, 0, -\Delta^2_\perp, Q^2) = \int d^2b_\perp q(x, b_\perp, Q^2)e^{ib_\perp \Delta_\perp} \\
\tilde{H}(x, 0, -\Delta^2_\perp, Q^2) = \int d^2b_\perp \Delta q(x, b_\perp, Q^2)e^{ib_\perp \Delta_\perp}
\]

where QCD evolution of \( H, \tilde{H}, q, \Delta q \) is described by DGLAP and is independent on both \( b_\perp \) and \( \Delta^2_\perp \), provided one does not look at scales in \( b_\perp \) that are smaller than \( 1/Q \).
suppression of crossed diagrams

Flow of the large momentum $q$ through typical diagrams contributing to the forward Compton amplitude. a) ‘handbag’ diagrams; b) ‘cat’s ears’ diagram. Diagram b) is suppressed at large $q$ due to the presence of additional propagators.
define state that is localized in position space (center of mass frame)

\[ | \vec{R} = \vec{0} \rangle \equiv \mathcal{N} \int d^3 \vec{p} | \vec{p} \rangle \]

define charge distribution (for this localized state)

\[ \rho(\vec{r}) \equiv \langle \vec{R} = \vec{0} | j^0(\vec{r}) | \vec{R} = \vec{0} \rangle \]
use translational invariance to relate to same matrix element that appears in def. of form factor

\[ \rho(\vec{r}) \equiv \left\langle \vec{R} = \vec{0} \left| j^0(\vec{r}) \right| \vec{R} = \vec{0} \right\rangle \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{r}) | \vec{p} \rangle \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle e^{i \vec{r} \cdot (\vec{p} - \vec{p}')} \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' F \left( - (\vec{p}' - \vec{p})^2 \right) e^{i \vec{r} \cdot (\vec{p} - \vec{p}')} \]

\[ \rho(\vec{r}) = \int \frac{d^3 \vec{\Delta}}{(2\pi)^3} F(-\vec{\Delta}^2) e^{i \vec{r} \cdot \vec{\Delta}} \]
express quark-bilinear in twist-2 GPD in terms of light-cone ‘good’ component $q_+(+) = \frac{1}{2} \gamma^- \gamma^+ q$

$$q'\gamma^+ q = q'_+(+)\gamma^+ q(+) = \sqrt{2} q'_(+) q(+) .$$

expand $q(+) \text{ in terms of canonical raising and lowering operators}$

$$q_+(x^-, x_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2k_\perp}{2\pi} \sum_s \left[ u_+(k, s) b_s(k^+, k_\perp) e^{ikx} + v_+(k, s) d_s^\dagger(k^+, k_\perp) e^{ikx} \right] ,$$
density interpretation of \( q(x, b_\perp) \)

with usual (canonical) equal light-cone time \( x^+ \) anti-commutation relations, e.g.

\[
\{ b_r(k^+, k_\perp), b_s^\dagger(q^+, q_\perp) \} = \delta(k^+ - q^+)\delta(k_\perp - q_\perp)\delta_{rs}
\]

and the normalization of the spinors is such that

\[
\bar{u}_{(+)}(p, r)\gamma^+ u_{(+)}(p, s) = 2p^+\delta_{rs}.
\]

Note: \( \bar{u}_{(+)}(p', r)\gamma^+ u_{(+)}(p, s) = 2p^+\delta_{rs} \) for \( p^+ = p'^+ \), one finds for \( x > 0 \)

\[
q(x, b_\perp) = \mathcal{N}' \sum_s \int \frac{d^2 k_\perp}{2\pi} \int \frac{d^2 k'_\perp}{2\pi} \langle p^+, 0_\perp | b_s^\dagger(x p^+, k'_\perp) b_s(x p^+, k_\perp) | p^+, 0_\perp \rangle \\
\times e^{ib_\perp \cdot (k_\perp - k'_\perp)}.
\]
Switch to mixed representation:

**momentum** in longitudinal direction

**position** in transverse direction

\[ \tilde{b}_s(k^+, x_\perp) \equiv \int \frac{d^2k_\perp}{2\pi} b_s(k^+, k_\perp) e^{ik_\perp \cdot x_\perp} \]

\[ q(x, b_\perp) = \sum_s \langle p^+, 0_\perp | \tilde{b}_s^\dagger(xp^+, b_\perp) \tilde{b}_s(xp^+, b_\perp) | p^+, 0_\perp \rangle . \]

\[ = \sum_s \left| \tilde{b}_s(xp^+, b_\perp) | p^+, 0_\perp \rangle \right|^2 \]

\[ \geq 0. \]
Boosts in nonrelativistic QM

\[ \vec{x}' = \vec{x} + \vec{v}t \quad t' = t \]

purely kinematical (quantization surface \( t = 0 \) inv.)

**1.** boosting wavefunctions very simple

\[ q_{\vec{v}}(\vec{p}_1, \vec{p}_2) = q_{\vec{0}}(\vec{p}_1 - m_1 \vec{v}, \vec{p}_2 - m_2 \vec{v}). \]

**2.** dynamics of center of mass

\[ \vec{R} = \sum_i x_i \vec{r}_i \quad \text{with} \quad x_i \equiv \frac{m_i}{M} \]

decouples from the internal dynamics
Relativistic Boosts

\[ t' = \gamma \left( t + \frac{v}{c^2} z \right), \quad z' = \gamma (z + vt) \quad x'_\perp = x_\perp \]

generators satisfy Poincaré algebra:

\[
\begin{align*}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
[M^{\mu\nu}, M^{\rho\lambda}] &= i \left( g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho} \right)
\end{align*}
\]

rotations: \( M_{ij} = \varepsilon_{ijk} J_k \), boosts: \( M_{i0} = K_i \).
introduce generator of \( \perp \) ‘boosts’:

\[
B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}}
\]

Poincaré algebra \( \Rightarrow \) commutation relations:

\[
[J_3, B_k] = i\varepsilon_{kl}B_l \quad [P_k, B_l] = -i\delta_{kl}P^+
\]

\[
[P^-, B_k] = -iP_k \quad [P^+, B_k] = 0
\]

with \( k, l \in \{x, y\} \), \( \varepsilon_{xy} = -\varepsilon_{yx} = 1 \), and \( \varepsilon_{xx} = \varepsilon_{yy} = 0 \).
Together with $[J_z, P_k] = i\varepsilon_{kl} P_l$, as well as

$$
\begin{align*}
[P^-, P_k] &= [P^-, P^+] = [P^-, J_z] = 0 \\
[P^+, P_k] &= [P^+, B_k] = [P^+, J_z] = 0.
\end{align*}
$$

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

\begin{align*}
P^- &\quad \rightarrow \quad \text{Hamiltonian} \\
P_\perp &\quad \rightarrow \quad \text{momentum in the plane} \\
P^+ &\quad \rightarrow \quad \text{mass} \\
L_z &\quad \rightarrow \quad \text{rotations around } z\text{-axis} \\
B_\perp &\quad \rightarrow \quad \text{generator of boosts in the plane},
\end{align*}

back to discussion
Consequences

- many results from NRQM carry over to $\perp$ boosts in IMF, e.g.

  - $\perp$ boosts kinematical

    $$ q_{\Delta}(x, k_{\perp}) = q_{0}(x, k_{\perp} - x\Delta_{\perp}) $$

    $$ q_{\Delta}(x, k_{\perp}, y, l_{\perp}) = q_{0}(x, k_{\perp} - x\Delta_{\perp}, y, l_{\perp} - y\Delta_{\perp}) $$

  - Transverse center of momentum $R_{\perp} \equiv \sum_{i} x_{i} r_{\perp,i}$ plays role similar to NR center of mass, e.g. $\int d^{2}p_{\perp} |p^{+}, p_{\perp}\rangle$ corresponds to state with $R_{\perp} = 0_{\perp}$.

back
Center of Momentum

- Field theoretic definition

\[ p^+ R_\perp \equiv \int dx^- \int d^2x_\perp T^{++}(x) x_\perp = M^{+\perp} \]

- \( M^{+\perp} = B^\perp \) generator of transverse boosts

- Parton representation:

\[ R_\perp = \sum_i x_i r_{\perp,i} \]

\( (x_i = \text{momentum fraction carried by } i^{th} \text{ parton}) \)

back
Poincaré algebra:

\[ [P^\mu, P^\nu] = 0 \]
\[ [M^{\mu\nu}, P^\rho] = i (g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu) \]
\[ [M^{\mu\nu}, M^{\rho\lambda}] = i (g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho}) \]

rotations: \( M_{ij} = \varepsilon_{ijk} J_k \),  
boosts: \( M_{i0} = K_i \).

back
introduce generator of \( \downarrow \) ‘boosts’:

\[
B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}}
\]

Poincaré algebra \( \Longrightarrow \) commutation relations:

\[
\begin{align*}
\left[J_3, B_k\right] &= i\varepsilon_{kl} B_l \\
\left[P_k, B_l\right] &= -i\delta_{kl} P^+ \\
\left[P^-, B_k\right] &= -i P_k \\
\left[P^+, B_k\right] &= 0
\end{align*}
\]

with \( k, l \in \{x, y\}, \varepsilon_{xy} = -\varepsilon_{yx} = 1, \text{ and } \varepsilon_{xx} = \varepsilon_{yy} = 0. \)
Together with \([J_z, P_k] = i \varepsilon_{kl} P_l\), as well as

\[
\begin{align*}
[P^-, P_k] &= [P^-, P^+] = [P^-, J_z] = 0 \\
[P^+, P_k] &= [P^+, B_k] = [P^+, J_z] = 0.
\end{align*}
\]

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

\[
\begin{align*}
P^- &\rightarrow \text{Hamiltonian} \\
P_{\perp} &\rightarrow \text{momentum in the plane} \\
P^+ &\rightarrow \text{mass} \\
L_z &\rightarrow \text{rotations around } z\text{-axis} \\
B_{\perp} &\rightarrow \text{generator of boosts in the plane,}
\end{align*}
\]
Consequences of Galilean subgroup

- many results from NRQM carry over to \( \perp \) boosts in IMF, e.g.
- \( \perp \) boosts kinematical

\[
\psi_{\Delta \perp}(x, k_{\perp}) = \psi_{0 \perp}(x, k_{\perp} - x \Delta_{\perp})
\]

\[
\psi_{\Delta \perp}(x, k_{\perp}, y, l_{\perp}) = \psi_{0 \perp}(x, k_{\perp} - x \Delta_{\perp}, y, l_{\perp} - y \Delta_{\perp})
\]

- Transverse center of momentum \( \mathbf{R}_{\perp} \equiv \sum_i x_i \mathbf{r}_{\perp,i} \) plays role similar to NR center of mass, e.g. \( |p^+, \mathbf{R}_{\perp} = 0_{\perp} \rangle \equiv \int d^2 p_{\perp} |p^+, \mathbf{p}_{\perp} \rangle \) corresponds to state with \( \mathbf{R}_{\perp} = 0_{\perp} \).
Proof that $B_\perp |p^+, R_\perp = 0_\perp \rangle = 0$

Use

$$e^{-i\mathbf{v}_\perp \cdot \mathbf{B}_\perp} |p^+, \mathbf{p}_\perp, \lambda \rangle = |p^+, \mathbf{p}_\perp + p^+ \mathbf{v}_\perp, \lambda \rangle$$

$$\implies$$

$$e^{-i\mathbf{v}_\perp \cdot \mathbf{B}_\perp} \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle = \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle$$

$$\implies$$

$$B_\perp \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda \rangle = 0$$
Ansatz: \( H_q(x, 0, -\Delta^2_\perp) = q(x) e^{-a\Delta^2_\perp (1-x) \ln \frac{1}{x}}. \)

\[
q(x, b_\perp) = q(x) \frac{1}{4\pi a(1-x) \ln \frac{1}{x}} e^{-\frac{b^2_\perp}{4a(1-x) \ln \frac{1}{x}}}
\]
simple model for $q(x, b_\perp)$
Application: ↓ hyperon polarization

model for hyperon polarization in \( pp \rightarrow Y + X \) \((Y \in \Lambda, \Sigma, \Xi)\) at high energy:

- peripheral scattering
- \( s\bar{s} \) produced in overlap region, i.e. on “inside track”

\( \leftrightarrow \) if \( Y \) deflected to left then \( s \) produced on left side of \( Y \) (and vice versa)

\( \leftrightarrow \) if \( \kappa_s > 0 \) then intermediate state has better overlap with final state \( Y \) that has spin down (looking into the flight direction)

\( \leftrightarrow \) remarkable prediction: \( \vec{P}_Y \sim -\kappa_s \vec{p}_P \times \vec{p}_Y \).
Figure 1: \( P + P \rightarrow Y + X \) where the incoming \( P \) (from bottom) is deflected to the left during the reaction. The \( s\bar{s} \) pair is assumed to be produced in the overlap region, i.e. on the left ‘side’ of the \( Y \).
SU(3) analysis for $\kappa_s^B$ yields (assuming $|\kappa_s^p| \ll |\kappa_u^p|, |\kappa_d^p|$)

$$\kappa_s^\Lambda = \kappa^p + \kappa_s^p = 1.79 + \kappa_s^p$$
$$\kappa_s^\Sigma = \kappa^p + 2\kappa^n + \kappa_s^p = -2.03 + \kappa_s^p$$
$$\kappa_s^\Xi = 2\kappa^p + \kappa^n + \kappa_s^p = 1.67 + \kappa_s^p.$$ 

$\leftrightarrow$ expect (polarization $\mathcal{P}$ w.r.t. $\vec{p}_P \times \vec{P}_Y$)

$$\mathcal{P}_\Lambda < 0 \quad \mathcal{P}_\Sigma > 0 \quad \mathcal{P}_\Xi < 0$$

$\bullet$ exp. result:

$$0 < \mathcal{P}_{\Sigma^0} \approx \mathcal{P}_{\Sigma^-} \approx \mathcal{P}_{\Sigma^+} \approx -\mathcal{P}_\Lambda \approx -\mathcal{P}_{\Xi^0} \approx -\mathcal{P}_{\Xi^-}$$
Figure 2: Schematic view of the transverse distortion of the $s$ quark distribution (in grayscale) in the transverse plane for a transversely polarized hyperon with $\kappa_s^Y > 0$. The view is (from the rest frame) into the direction of motion (i.e. momentum into plane) for a hyperon that moves with a large momentum. In the case of spin down (a), the $s$-quarks get distorted towards the left, while the distortion is to the right for the case of spin up (b).
anomalous magnetic moment coupling in Dirac eq:

\[
\frac{i\kappa}{2M} \bar{q} \sigma^{\mu\nu} q F_{\mu\nu} = \frac{i\kappa}{2M} \left[ \bar{q} \sigma^{ij} q F_{ij} + 2 \bar{q} \sigma^{0\nu} q F_{0\nu} \right]
\]

\[\leftrightarrow \kappa \left[ \vec{\sigma} \cdot \vec{B} + (\sigma \times \vec{p}) \cdot \vec{E} \right]\]

moving spin \(\frac{1}{2}\) particle with anomalous magnetic moment has (viewed from observer at rest) transverse electric dipole moment, which is perp. to both its spin and momentum.

\[\leftrightarrow \perp \text{ distortion of } q(x, b_{\perp}) \text{ is consequence of Lorentz invariance for Dirac particle with anomalous magnetic moment.}\]