A Proof of the Gluon Reggeization in the NLO

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Plan of the talk

- Introduction
- What does the term "gluon Reggeization" mean
- The gluon Reggeization and the BFKL approach
- Bootstrap of the gluon Reggeization
  - Bootstrap relation for elastic amplitudes
  - "Soft" bootstrap conditions
  - "Strong" bootstrap
  - Bootstrap relations for inelastic amplitudes
  - Bootstrap conditions for production vertices
- A way of the proof of the gluon Reggeization
- Summary and outlook

Introduction

In the limit of large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$ (Regge limit) the most appropriate approach for the description of the the scattering amplitudes is given by the theory of the complex angular momenta (Gribov-Regge theory). One of the remarkable properties of QCD is the Reggeization of its elementary particles. Contrary to QED, where the electron does Reggeize in perturbation theory,
M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, F. Zachariasen, 1964,
but the photon remains elementary,
S. Mandelstam, 1964,
in QCD the gluon does Reggeize
M. T. Grisaru, H. J. Schnitzer, 1973,
L.N. Lipatov, 1976,
V. F., E.A. Kuraev, L.N.Lipatov, 1975,
E.A.Kuraev, L.N. Lipatov, V.F., 1976,
Ya.Ya. Balitskii, L.N. Lipatov,1978,
Ya.Ya.Balitskii, L.N. Lipatov, V.F., 1979,

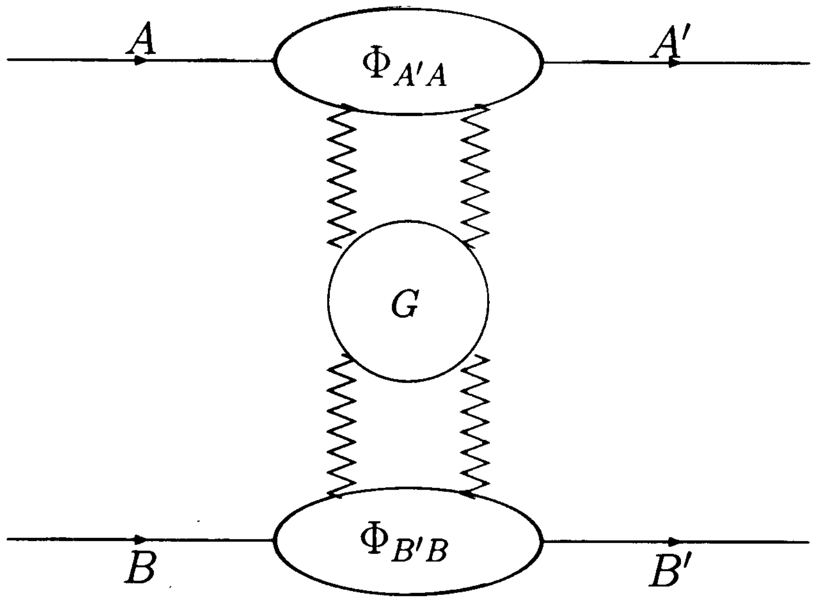
as well as the quark
V.F., V.E. Sherman, 1976,
V.F., R. Fiore, 2001,
A.V. Bogdan, V.Del Duca, V.F., E.W.Glover, 2002,
M.I. Kotsky, L.N. Lipatov, A. Principe, M.I. Vyazovsky, 2002.

The property of the Reggeization is very important for high energy QCD. BFKL (Balitskii-Fadin-Kuraev-Lipatov) approach to the description of the high energy QCD processes is based on the gluon Reggeization. The BFKL equation for resummation of leading logarithmic radiative correction to scattering amplitudes of processes with gluon exchanges in the t-channel was obtained assuming the gluon Reggeization. The Pomeron, determining high energy behaviour of cross sections, and the Odderon, responsible for the difference of particle and antiparticle cross sections, appears in QCD as a compound state of two and three Reggeized gluons respectively. Colorless objects constructed from Reggeized quarks and antiquarks should be relevant to phenomenological Reggeon trajectories successfully used for the description of processes with exchange of quantum numbers.

In the BFKL approach the amplitude for the process

$$A + B \longrightarrow A' + B'$$

at large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$, $s \gg |t|$, can be represented by the picture



and may be symbolically written as the convolution

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$$

where the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe the transitions $A \to A'$ and $B \to B'$ due to scattering on the Reggeized gluons, while G is the Green's function for the two interacting Reggeized gluons. All dependence on properties of particles A, A' $(B,\ B')$ is contained in the impact factors $\Phi_{A'A}$ $(\Phi_{B'B})$, which are energy independent, so that dependence on energy is determined by the universal (process independent) Green's function G. This representation is valid both in the leading logarithmic approximation (LLA), when only the leading terms $(\alpha_S \ln s)^n$ are resummed, and in the next-to-leading approximation (NLA), when the $\alpha_S(\alpha_S \ln s)^n$ terms are also resummed, not only for forward scattering with $t = 0$, but for the non-forward case as well.

In the case of the forward scattering ($A' = A, B' = B$, q=0) with the help of the optical theorem we obtain:

$$\sigma_{AB}(s) = \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \int \frac{d^2 q_A}{2\pi \vec{q}_A^{\,2}} \int \frac{d^2 q_B}{2\pi \vec{q}_B^{\,2}} \left(\frac{s}{s_0}\right)^{\omega} \Phi_A(\vec{q}_A) G_\omega(\vec{q}_A, -\vec{q}_B) \Phi_B(\vec{q}_B),$$

where the vector sign is used for vector components transverse to the initial momenta p_A, p_B and s_0 is a certain energy scale.

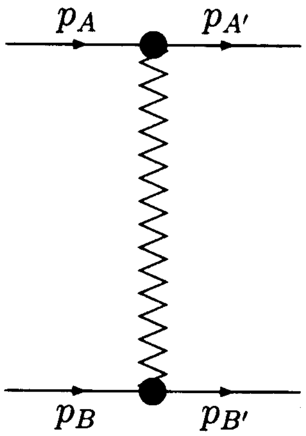
What does the term "gluon Reggeization" mean

It was claimed already that the BFKL approach is based on the gluon Reggeization. But one needs to know precisely what does it mean. The notion "Reggeization" of an elementary particle with spin s in perturbation theory was introduced for denotation of disappearance, due to radiative corrections, of non analytic in J terms $(\delta_{J,s})$ in the complex angular momentum J plane, which appear in the Born approximation from Feynman diagrams with exchange of the particle with spin s. We use this notion in more strong, but transparent sense. Talking about the gluon Reggeization in QCD we mean not only the existence of the Reggeon with gluon quantum numbers, negative signature and trajectory

$$j\,(t) = 1 + \omega(t)$$

passing through 1 at $t = 0$. We mean also that in each order of perturbation theory this Reggeon gives the leading contribution to the amplitudes of the processes at large relative energies of the participating particles and fixed (i.e. not increasing with s) momentum transfers.

Let us explain this in more details. Consider the elastic scattering process $A + B \rightarrow A' + B'$
in Regge kinematical region:
$s \simeq -u \rightarrow \infty$, t fixed (i.e. not growing with s) For this amplitude the term gluon Reggeization used by us means that the elastic scattering amplitude with the gluon quantum numbers in the t-channel and negative signature (*i.e.* odd under $s \leftrightarrow u$ exchange) has the Regge form



$$\left(\mathcal{A}_8^-\right)_{AB}^{A'B'} = \Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)}\right]\Gamma^c_{B'B}\ ;$$

$$j(t) = 1 + \omega(t)\ ; j(0) = 1\ ,$$

$\omega(t)$ – Reggeized gluon trajectory ,

$$\Gamma^c_{A'A} = g\langle A'|T^c|A\rangle\Gamma_{A'A}\ ,$$

T^c -colour group generators in fundamental (quarks) or adjoint (gluons)representation.

With this assumption the vertices $\Gamma^{(0)}_{A'A}$ and the gluon Regge trajectory $\omega^{(1)}$ can be easily calculated in the leading order (LO). The result is:

$$\Gamma^{(0)}_{A'A} = \delta_{\lambda_{A'}\lambda_A} \ , \quad \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{D-1}} \frac{N}{2} \int \frac{d^{D-2}k_\perp}{k_\perp^2 (q-k)_\perp^2} = -\bar{g}^2 (\vec{q}^{\,2})^\epsilon \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} \ ,$$

where Γ is the Euler gamma-function and

$$\bar{g}^2 = \frac{g^2 N \Gamma(1-\epsilon)}{(4\pi)^{D/2}} \ , \qquad t = q^2 \simeq q_\perp^2 \ ;$$

We keep the space-time dimension $D = 4 + 2\epsilon \neq 4$ to regularize infrared divergencies and use the the Sudakov decomposition of momenta:

$$p = \beta p_1 + \alpha p_2 + p_\perp \ ' \qquad p_\perp^2 = -\vec{p}^{\,2} \ ;$$

(p_1, p_2) - light-cone basis of the initial particle momenta plane

$$p_A = p_1 + \frac{m_A^2}{s} p_2 \ , \qquad p_B = p_2 + \frac{m_B^2}{s} p_1 \ , \qquad 2 p_1 \cdot p_2 = s \ .$$

Of course, neither the calculation, nor the results are not so simple in the next-to-leading order (NLO). In particular, s-cannel helicity is not conserved in this approximation, so that

$$\Gamma^{(1)}_{A'A} = \delta_{\lambda_{A'}\lambda_A} \Gamma^{(+)}_A(t) + \delta_{\lambda_{A'},-\lambda_A} \Gamma^{(-)}_A(t) \ .$$

The vertices $\Gamma^{(\pm)}_A$ for quarks and gluons were calculated at arbitrary D.

V. F., L.N. Lipatov, 1993; V. F., R. Fiore, 1992;
V. F., R. Fiore, A. Quartarolo; 1994; V. F., R. Fiore, M.I. Kotsky, 1995

The two-loop contribution $\omega^{(2)}(t)$ to the gluon trajectory was obtained at arbitrary D in terms of integrals over transverse momenta.

V. F., 1995; V. F., R. Fiore, M.I. Kotsky, 1995; V. F., R. Fiore, A. Quartarolo, 1996; V. F., R. Fiore, M.I. Kotsky, 1996

The integral can be expressed in terms of elementary functions only for $\epsilon \to 0$:

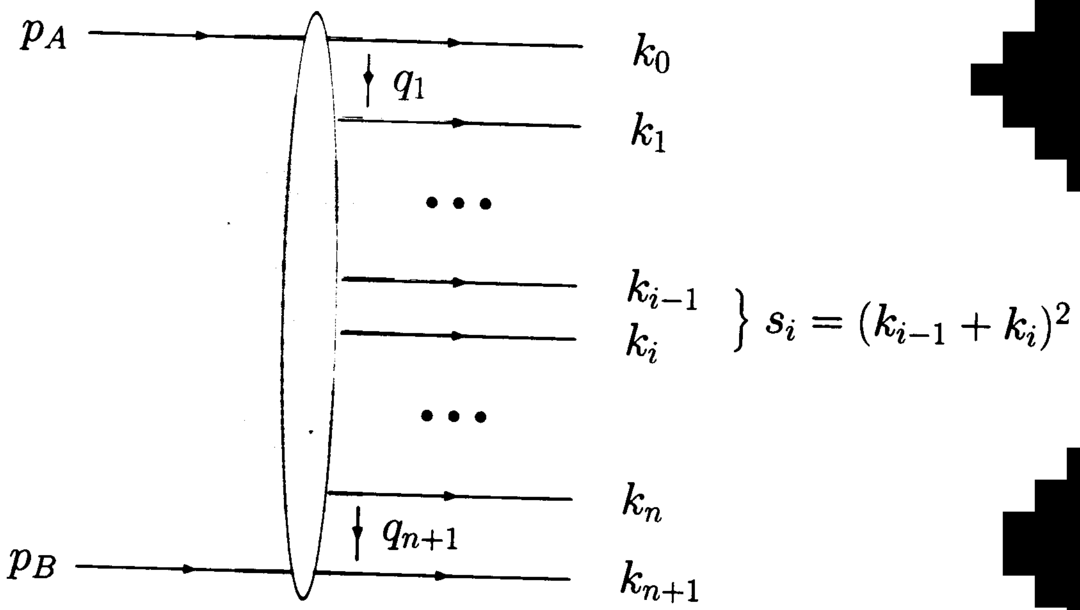
$$\omega^{(2)}(t) \simeq \left(\frac{\bar{g}^2 \left(\vec{q}^{\,2}\right)^\epsilon}{\epsilon} \right)^2 \left[\frac{11}{3} + \left(2\psi'(1) - \frac{67}{9} \right) \epsilon + \left(\frac{404}{27} + \psi''(1) - \frac{22}{3}\psi'(1) \right) \epsilon^2 \right] \ ,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

V. F., M. Kotsky, 1996; J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998;
V.Del Duca, E.W.N. Glover, 2001

Amplitudes for production of particles have complicated analytical structure. It is not simple even in the multi-Regge kinematics (MRK)

$$A + B \ \rightarrow \ \tilde{A} + \tilde{B} + n$$



$$k_i = \beta_i p_1 + \alpha_i p_2 + k_{i\perp} \ , \qquad s\alpha_i\beta_i = k_i^2 - k_{i\perp}^2 = k_i^2 + \vec{k}_i^2 \qquad k_0 = p_{\tilde{A}} \ , \quad k_{n+1} = p$$

$$1 \approx \alpha_{n+1} \gg \alpha_n \gg \alpha_{n-1} \ldots . \gg \alpha_0$$

$$1 \approx \beta_0 \gg \beta_1 \gg \beta_2 \ldots . \gg \beta_{n+1}$$

$$s \ \gg \ s_{ij} = (k_i + k_j)^2 \ \gg \ |k_{i\perp}^2|$$

In this regime

$$q_i = p_A - \sum_{j=0}^{i-1} k_j = -(p_B - \sum_{l=i}^{n+1} k_l) \simeq \beta_i p_1 - \alpha_{i-1} p_2 - \sum_{j=0}^{i-1} k_{j\perp} \ ,$$

$$t_i = q_i^2 \simeq q_{i\perp}^2 = -\vec{q}_i^{\,2}$$

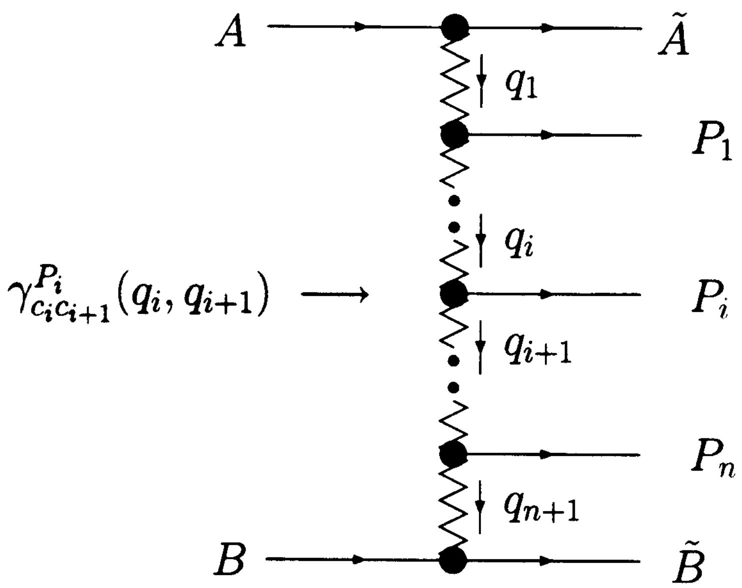
Quasi multi-Regge kinematics (QMRK): instead of one of the particles in the MRK there are a couple of particle with limited invariant mass.

The real parts of the production amplitudes in the MRK and QMRK have a simple factorized form

$$\Re\, \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s\, \Gamma_{\tilde{A}A}^{c_1} \left(\prod_{i=1}^{n} \gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1}) \left(\frac{s_i}{s_R}\right)^{\omega(t_i)} \frac{1}{t_i} \right) \frac{1}{t_{n+1}} \left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})} \Gamma_{\tilde{B}B}^{c_{n+1}}$$

s_R - energy scale, irrelevant in the LLA

$\gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1})$ - (non-local) effective vertex for production of the particles P_i in the Reggeon-Reggeon collisions



LLA: $\{P_i\}$ states of a single gluon

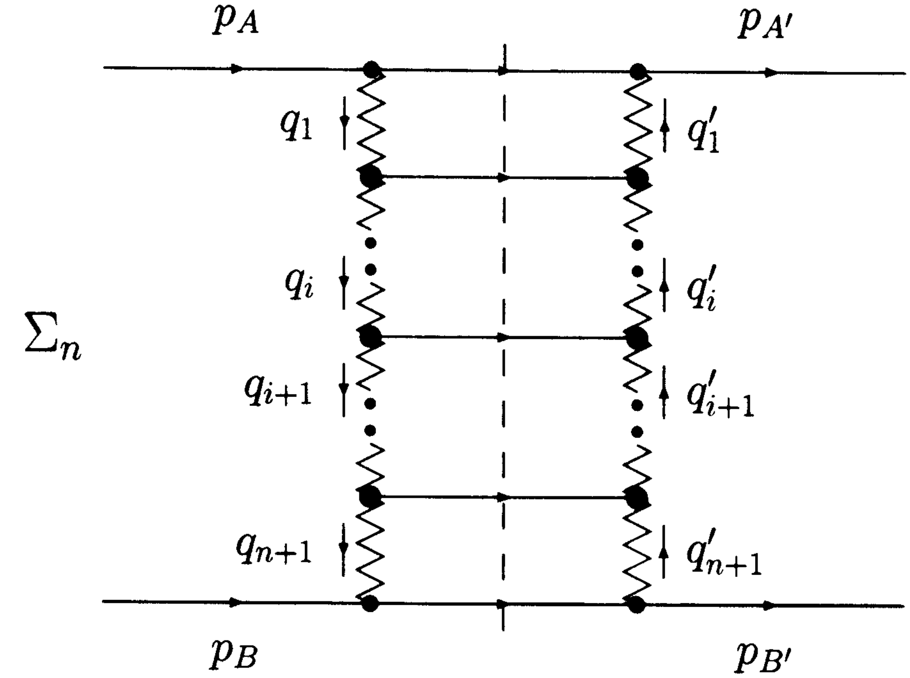
$$\gamma_{c_i c_{i+1}}^{G_i}(q_i, q_{i+1}) = g T_{c_i c_{i+1}}^{d_i} e_\mu^*(k_i) C^\mu(q_i, q_{i+1})$$

$$C(q_{i+1}, q_i) = -q_i - q_{i+1} + \left(\frac{q_i^2}{(k_i\, p_1)} + 2\frac{(k_i\, p_2)}{(p_1\, p_2)}\right) p_1 - \left(\frac{q_{i+1}^2}{(k_i\, p_2)} + 2\frac{(k_i\, p_1)}{(p_1\, p_2)}\right) p_2$$

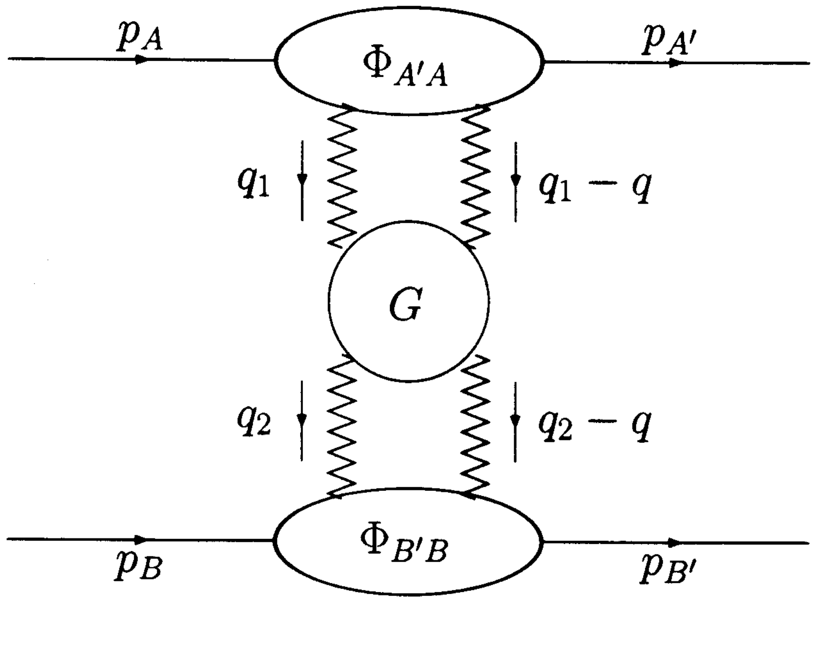
L.N. Lipatov, 1976; V. F., E.A. Kuraev, L.N. Lipatov, 1975

The gluon Reggeization and the BFKL approach

Unitarity relations:



That gives



$\Phi_{P'P}^{(\mathcal{R},\nu)}$ – impact factors in the t-channel color state $(\mathcal{R},\nu)$

$G_\omega^{(\mathcal{R})}$ – Mellin transform of the Green's functions for Reggeon-Reggeon scattering in the t-channel color representation $\mathcal{R}$

Colour decomposition:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\mathcal{R}} (\mathcal{A}_{\mathcal{R}})_{AB}^{A'B'}$$

$$\text{Im}_s (A_{\mathcal{R}})_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2} \int \frac{d^{D-2}q_2}{\vec{q}_2^{\,2}(\vec{q}_2-\vec{q})^2} \sum_{\nu} \Phi_{A'A}^{(\mathcal{R},\nu)}(\vec{q}_1;\vec{q};s_0)$$
$$\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0}\right)^{\omega} G_\omega^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q}) \right] \Phi_{B'B}^{(\mathcal{R},\nu)}(-\vec{q}_2;-\vec{q};s_0)$$

The Green's functions obey the equation

$$\omega G_\omega^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q}) = \vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2 \delta^{(D-2)}(\vec{q}_1-\vec{q}_2)$$
$$+ \int \frac{d^{D-2}q_r}{\vec{q}_r^{\,2}(\vec{q}_r-\vec{q})^2} \mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_r;\vec{q})\, G_\omega^{(\mathcal{R})}(\vec{q}_r,\vec{q}_2;\vec{q})$$

$$\mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2;\vec{q}) = \left[\omega\left(-\vec{q}_1^{\,2}\right) + \omega\left(-(\vec{q}_1-\vec{q})^2\right)\right] \delta^{(D-2)}(\vec{q}_1-\vec{q}_2) + \mathcal{K}_r^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2;\vec{q})$$

Kernel: "virtual" part "real" part

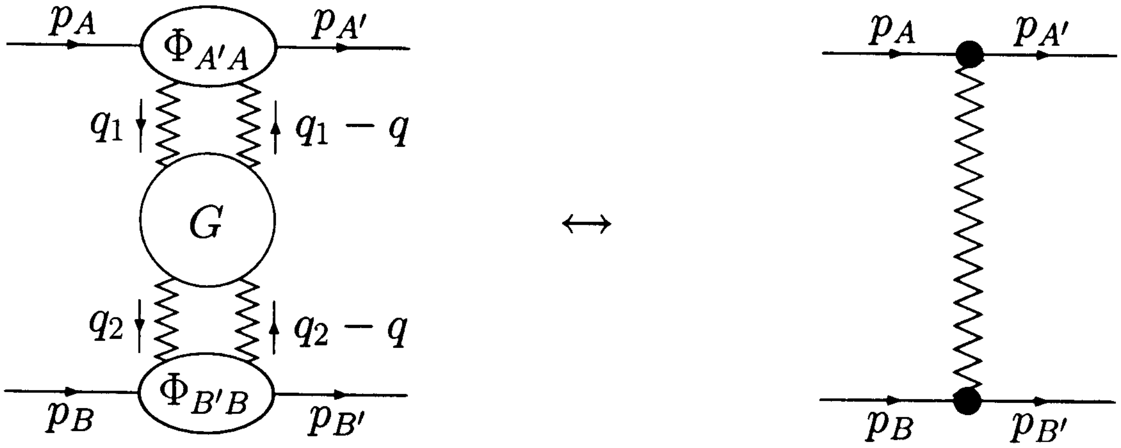
V. F., E.A. Kuraev, L.N. Lipatov, 1975

Impact factors and kernel are defined without any ambiguity in terms of the gluon Regge trajectory and vertices of Reggeon interactions

- The NLO kernel is known completely in the forward case ($t = 0$) for the all color representations
- In the non-forward case, the $\mathcal{K}_{RRGG}^{(B)}$ contribution is missing for the singlet color representation
- The NLA kernel is known completely in the non-forward case for the octet color representation

Bootstrap of the gluon Reggeization

In the case of Gluon quantum numbers in the t-channel the representation for the elastic scattering process $A + B \longrightarrow A' + B'$ derived from s-channel unitarity, must reproduce the representation with one Reggeized gluon exchange in the t-channel.



$$\mathbf{Im}_s(A_{8^-})_{AB}^{A'B'} \quad = \quad \Gamma_{A'A}^{c\,(\mathrm{Born})} \left(\frac{s}{|t|}\right)^{1+\omega^{(1)}(t)} \pi\omega^{(1)}(t)\Gamma_{A'A}^{c\,(\mathrm{Born})}$$

It leads to the following conditions: [V.F., R. Fiore (1999)]

First bootstrap condition (on the NLA kernel)

$$\frac{g^2Nt}{2\,(2\pi)^{D-1}}\int\frac{d^{D-2}q_1}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\int\frac{d^{D-2}q_2}{\vec{q}_2^{\,2}\vec{q}_2^{\,\prime\,2}}\mathcal{K}^{(8)(1)}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right) = \omega^{(1)}\left(t\right)\omega^{(2)}\left(t\right)$$

Second bootstrap condition (on the NLA impact factors)

$$\frac{ig\sqrt{N}t}{(2\pi)^{D-1}}\int\frac{d^{D-2}q_1}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\Phi_{A'A}^{(8,a)(1)}(\vec{q}_1,\vec{q};s_0) = \Gamma_{A'A}^{a(1)}\,\omega^{(1)}(t)$$
$$+\frac{1}{2}\,\Gamma_{A'A}^{a(B)}\left[\omega^{(2)}(t) + \left(\omega^{(1)}(t)\right)^2\ln\left(\frac{s_0}{\vec{q}^{\,2}}\right)\right]$$

In the NLA, the check of the bootstrap was very important

- as a (partial) check of correctness of the NLO BFKL calculations
- since a formal proof of the gluon Reggeization to all orders of perturbation theory does not exist in this approximation.

The first bootstrap condition has been verified

- at arbitrary space-time dimension, for the part concerning the quark contribution to the kernel (in massless QCD)
  V.F., R. Fiore, A. Papa, (1999)

Due to simplicity of this part of the kernel:

$$\mathcal{K}_r^{Q\bar{Q}}\left(\vec{q}_1, \vec{q}_2; \vec{q}\,\right) = \frac{g^4 n_f N}{24(2\pi)^5}\times$$

$$\left\{-\frac{1}{\pi^\epsilon}\frac{6}{(4\pi)^{2\epsilon}}\Gamma(-\epsilon)\frac{[\Gamma(2+\epsilon)]^2}{\Gamma(4+2\epsilon)}\left[\frac{(\vec{k}^2)^\epsilon}{\vec{k}^2}\left(\vec{q}_1^{\,2}\vec{q}_2^{\,\prime 2}+\vec{q}_2^{\,2}\vec{q}_1^{\,\prime 2}\right)+\vec{q}^{\,2}\left((\vec{q}^{\,2})^\epsilon-(\vec{q}_1^{\,2})^\epsilon-(\vec{q}_2^{\,2})^\epsilon\right)\right.\right.$$

$$\left.\left.-\frac{\left(\vec{q}_1^{\,2}\vec{q}_2^{\,\prime 2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime 2}\right)}{\vec{k}^2}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\right] + \left(\vec{q}_1 \longleftrightarrow \vec{q}_1^{\,\prime},\ \vec{q}_2 \longleftrightarrow \vec{q}_2^{\,\prime}\right)\right\}$$

it was not a complicated problem.

- in the $D \to 4$ limit, for the part concerning the gluon contribution to the kernel V.F., R. Fiore, M.I. Kotsky, (2000)

The second bootstrap condition is process-dependent (it should be checked for every new impact factor which is calculated!).

It has been checked at arbitrary space-time dimension for quark and gluon impact factors in QCD with massive quarks.
V.F., R. Fiore, M.I. Kotsky, A. Papa (2000)

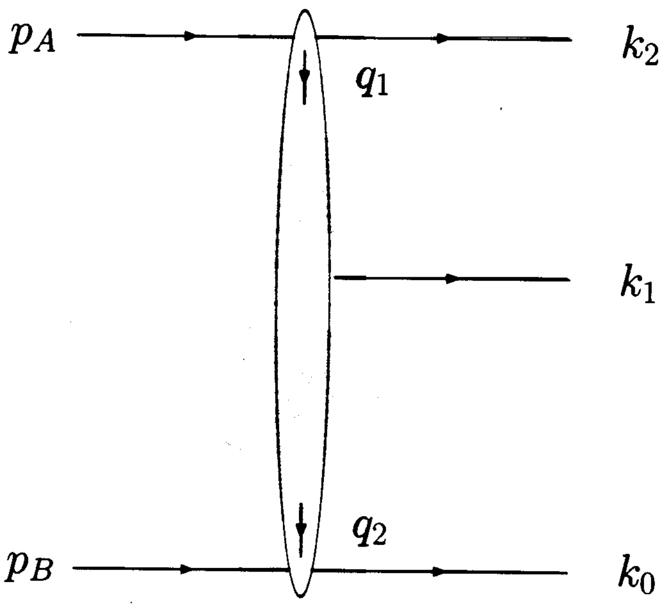
Strong bootstrap

The bootstrap conditions for elastic amplitudes give a possibility to check, but not to prove the hypothesis of the gluon Reggeization. However, it can be proved (so that the BFKL approach can obtain a firm basis), with the help of s-channel unitarity relations for production amplitudes.

Of course, we need to use analytical properties of these amplitudes. Unfortunately, we know not much about them. But, fortunately, we don't need much.

Let us consider the one-particle production amplitude

$$A + B \;\rightarrow\; A' + B' + G$$



$\} \; s_1 = (k_0 + k_1)^2$

$\} \; s_2 = (k_1 + k_2)^2$

in MRK. Note that we are interested in only negative signatutes in both t_i- channels, i.e. in the part of the amplitude which is antisymmetric with respect to any of the substitutions $s_i \rightarrow -s_i$, $i = 1, 2$. Note that due to the relation

$$\frac{s_1 s_2}{s} = \vec{k}_1^{\,2}$$

which is hold in all physical channels, this part is also antisymmetric with respect to $s \rightarrow -s$.

Dependece on $\vec{k}_1{}^2$ can be very complicated (the simplest example is $(-\vec{k}_1{}^2)^\epsilon$; it reproduces correctly real value at all negative s-is). Fortunately, it can be made irrelevant taking appropriate combinations of the discontinuities. .

An important observation is that a discontinuity of a product fg is expressed through the discontinuities of f and g:

$$f_+g_+ - f_-g_- = \frac{1}{2}(f_+ - f_-)(g_+ + g_-) + \frac{1}{2}(f_+ + f_-)(g_+ - g_-) \ .$$

The second important ovservation that in the sum of discontinuities of $F(\vec{k}_1{}^2)$ on s_1 and s (as well as the sum of the discontinuities on s_2 and s or the difference of the discontinuities on s_1 and s_2) is zero. From these two observations it follows that in such sum (or diference) we can take instead of analytic functions of $\vec{k}_1{}^2$ their real parts.

Now, if the variables s_1, s_2 and s does not enter into the combination $s_1 s_2/s = \vec{k}^{\,2}$, they can enter only as $\hat{S}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)$, with $n_1 + n_2 + n_3 = n$, where n is less or equal the order of perturbation theory and $\hat{S}$ is the operator of symmetrization with respect to exchanges

$$s_1 \leftrightarrow -s_1 \ , \quad s \leftrightarrow -s \ ,$$

and

$$s_2 \leftrightarrow -s_2 \ , \quad s \leftrightarrow -s \ .$$

Note that the terms containing products of $\ln(-s_i)\ln(s_i)$, where s_i can be s_1, s_2 or s, are forbiden, on the same ground as the terms containing $\ln(-s)\ln(s)$ are forbiden in elastic scattering amplitudes. Since in the NLO we need to keep only the first two leading total powers n, calculating imaginary part of discontinuity in anyone of variables s_1, s_2 or s we can take only real parts of logarithms of other variables. It means that

$$\Re\left[\frac{1}{-2\pi i}disc_{s_i}\left(\hat{\mathcal{S}}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)\right)\right]$$

$$=\frac{1}{2}\frac{\partial}{\partial\ln(s_i)}\Re\left[\hat{\mathcal{S}}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)\right]\ ,$$

where s_i can be s_1, s_2 or s and the partial derivative is taken at fixed $s_j \neq s_i$.

Therefore we have, for example

$$\Re\left[\frac{1}{-2\pi i s}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\left(\frac{\partial}{\partial\ln(s_1)}+\frac{\partial}{\partial\ln(s)}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'GB'}\right]\ ,$$

where in the right part the first derivative is taken at fixed s_2 and s and the second at fixed s_1 and s_2. Using that

$$\left(\frac{\partial}{\partial\ln(s_1)}+\frac{\partial}{\partial\ln(s)}\right)f(s_1,s_2,s)=\frac{\partial}{\partial\ln(s_1)}f(s_1,s_2,\frac{s_1,s_2}{\vec{k}_1^{\,2}})\ ,$$

we come to

$$\Re\left[\frac{1}{-2\pi i s}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\frac{\partial}{\partial\ln(s_1)}\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'GB'}\right]\ ,$$

where in the right part the amplitude is considered as function of s_1, s_2 and $\vec{k}_1^{\,2}$.

The requirement of the Reggeized form of the amplitude in the right part gives us the bootstrap relation:

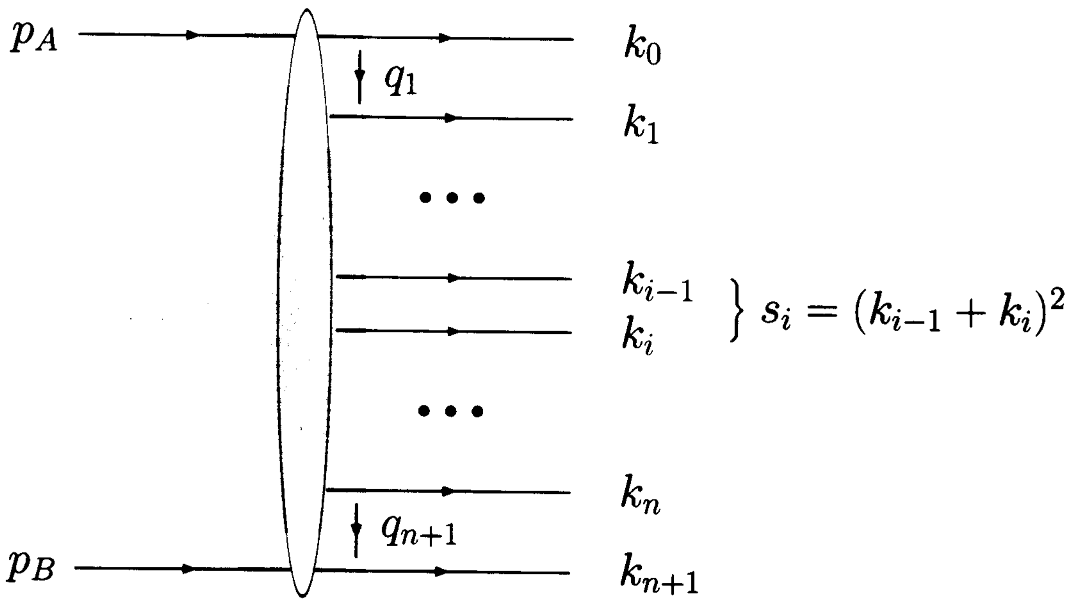
$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\omega(t_2)\Re\,\mathcal{A}_{AB}^{A'GB'}\ .$$

In the same way we obtain

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_2}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\omega(t_2)\Re\,\mathcal{A}_{AB}^{A'GB'}\ .$$

The bootstrap relations for amplitudes of multi-particle production can be derived in the same way. Denoting $\mathcal{A}_{AB}^{A'B'+n}$ the amplitude of the production of n particles in the MRK in the process

$$A + B \;\rightarrow\; A' + B' + n,$$



we have

$$\Re\left[\frac{1}{-2\pi i s}\left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}}\right)\mathcal{A}_{AB}^{A'B'+n}\right]$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial \log s_{k,k+1}} - \frac{\partial}{\partial \log s_{k-1,k}}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'B'+n}(s_{i-1,i})\right] \;,$$

where $s_{ij} = (k_i + k_j)^2$ and in the right part the amplitudes are expressed in terms of $s_{i-1,i}$, which are considered as independent variables.

If the amplitudes have the Reggeized form,

$$\Re\mathcal{A}(s_{i_1,i}) = 2s\,\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1})\left(\frac{s_i}{s_R}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}} \,.$$

we obtain bootstrap relations.

The bootstrap relations:

$$\Re\left[\frac{1}{-2\pi i}\left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}}\right)\mathcal{A}_{AB}^{A'B'+n}\right]$$
$$= \frac{1}{2}\left(\omega(t_{k+1}) - \omega(t_k)\right)\Re\,\mathcal{A}_{AB}^{A'B'+n}$$

are much more restrictive ("strong bootstrap") than the conditions obtained from the requirement of the Reggezation in the elastic amplitudes. Their fulfilment means a proof of the Reggeization in the NLO, since the energy dependence of amplitudes can be calculated order by order in perturbation theory using the equalities

$$\Re\left[\frac{1}{-2\pi i s}\left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}}\right)\mathcal{A}_{AB}^{A'B'+n}\right]$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial\log s_{k,k+1}} - \frac{\partial}{\partial\log s_{k-1,k}}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'B'+n}(s_{i-1,i})\right] ,$$

valid in the NLO.

Indeed, the discontinuities entering in the L.H.S. of these equations in some order in the coupling constant g can be expressed with the help of the unitarity relations through the multi-particle amplitudes in lower orders in g. Fulfillment of the bootstrap relations means that the energy dependence has the Regge form.

The bootstrap relations

$$\Re\left(\sum_{l=k+1}^{n+1}\frac{disc_{s_{kl}}\mathcal{A}_{AB}^{A'B'+n}}{-2\pi i}-\sum_{l=0}^{k-1}\frac{disc_{s_{lk}}\mathcal{A}_{AB}^{A'B'+n}}{-2\pi i}-\right)$$

$$=\frac{1}{2}\left(\omega(t_{k+1})-\omega(t_k)\right)\Re\,\mathcal{A}_{AB}^{A'B'+n}$$

impose connections on the gluon trajectory, colour octet BFKL kernel and vertices of interaction of the Reggeized gluon with quarks an gluons (bootstrap conditions). To formulate these conditions it is convenient to introduce operators in transverse momentum representation. From t-channel point of view we have to consider two interacting Reggeized gluons with "coordinates" in the transverse momentum space $\vec{r}$ and $\vec{q}-\vec{r}$, where $\vec{q}$ is the total transverse momentum in the t-channel. Let us introduce $\hat{\vec{r}}$ as the operator of "coordinate" of one of the Reggeized gluons in the transverse momentum space:

$$\hat{\vec{r}}\,|\vec{q}_i\rangle=\vec{q}_i|\vec{q}_i\rangle\ .$$

The total transverse momentum $\vec{q}$ is conserved and is considered as the c-number. It is convenient to use the normalization

$$\langle\vec{q}_1|\vec{q}_2\rangle=\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^{\,2}\delta^{(D-2)}(\vec{q}_1-\vec{q}_2)\ ,$$

so that

$$\langle A|B\rangle=\langle A|\vec{k}\rangle\langle\vec{k}|B\rangle=\int\frac{d^{D-2}k}{\vec{k}^2(\vec{k}-\vec{q})^2}A(\vec{k})B(\vec{k})\ .$$

The impact factors appear in this formalism as the wave functions of the t-channel states

$$\Phi_{A'A}(\vec{q}_1,\vec{q},s_0)=\langle\vec{q}_1|A'A\rangle$$

and the BFKL kernel as the operator

$$\mathcal{K}(\vec{q}_2,\vec{q}_1,\vec{q})=\langle\vec{q}_2|\hat{\mathcal{K}}|\vec{q}_1\rangle\ .$$

In these denotations the s-channel discontinuity of the elastic amplitude can be written as

$$\frac{disc_s \mathcal{A}_{AB}^{A'B'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle \ .$$

If the bootstrap relation

$$\frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle = \frac{\omega(t)}{2} 2s\, \Gamma_{A'A} \left(\frac{s}{s_0}\right)^{\omega(t)} \Gamma_{B'B}$$

were exact, then it should be

$$\left(\hat{\mathcal{K}} - \omega(t)\right)|A'A^a\rangle = 0$$

and

$$|A'A^a\rangle = \Gamma^a_{A'A}|R_\omega\rangle \ ,$$

with an universal (process independent) eigenstate $|R_\omega\rangle$ of the kernel with the eigenvalue $\omega(t)$. In usual denotations it reads

$$\int \frac{d^{(D-2)}q_2}{\vec{q}_2^{\,2}(\vec{q}_2 - \vec{q})^2} \mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{q})\, R_\omega(\vec{q}_2, \vec{q}, s_0) = \omega(t)\, R_\omega(\vec{q}_1, \vec{q}, s_0)$$

and

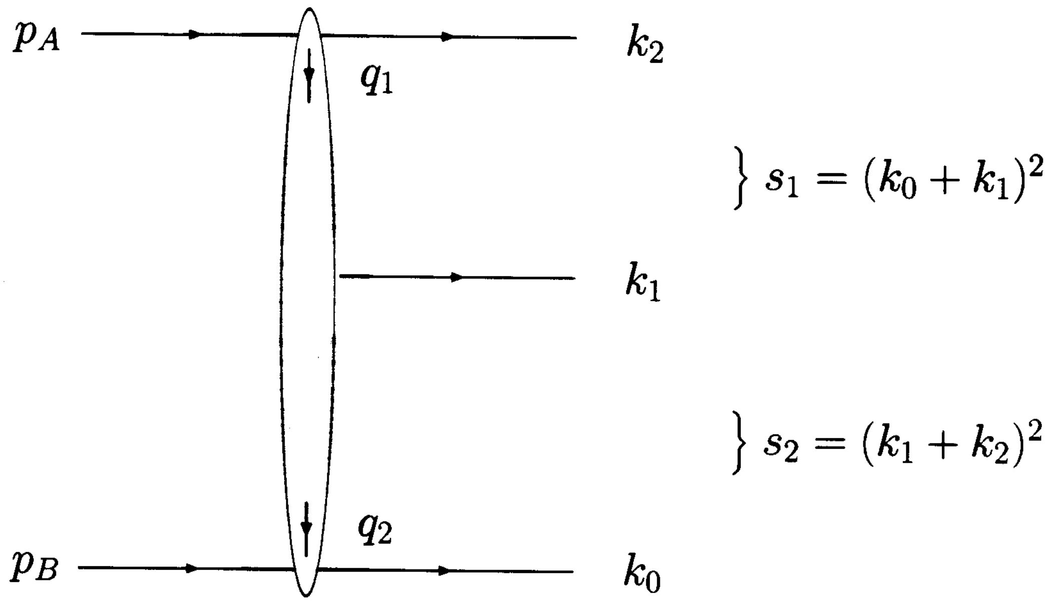
$$\Phi^a_{A'A}(\vec{q}_1, \vec{q}, s_0) = \Gamma^a_{A'A} R_\omega(\vec{q}_1, \vec{q}, s_0)$$

These conditions are known as the strong bootstrap conditions for the kernel and impact factors. They were suggested, without derivation, by
M. Braun, G.P. Vacca,1999.

In fact, since we work in the NLO, these conditions can not be derived from the bootstrap requirement for the elastic amplitudes. Nevertheless, they were checked and their fulfillment was shown for the quark and gluon impact factors
M. Braun, G.P. Vacca,1999,
V.F., R. Fiore, M.I. Kotsky, A. Papa, 2000,
for the quark contribution to the kernel
M. Braun, G.P. Vacca,1999,
V.F, R. Fiore, M.I. Kotsky, A. Papa, 2000
and, finaly, quite recently for the gluon part of the kernel
V.F, A. Papa, 2002.

It occurs, that these conditions must be satisfied for fulfillment of the bootstrap relations for inelastic amplitudes. Thus, for the gluon production amplitude

$$A + B \;\to\; A' + B' + G$$



in the MRK we obtain

$$\frac{disc_{s_2} \mathcal{A}_{AB}^{A'GB'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}} \Gamma_{A'A} \frac{1}{t_1} \left(\frac{s}{s_0}\right)^{\omega(t_1)} \langle GR| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle \; ,$$

where $\langle GR|$ is the t_2-channel state of the produced gluon G and the t_1-channel Reggeized gluon R with the wave function, which is expressed in terms of vertices of interactions of the Reggeized gluon (in fact, this wave function is the impact factor in the case when instead of on mass-shell particle we have the Reggeized gluon).

For the s-channel discontinuity we obtain

$$\frac{disc_s \mathcal{A}_{AB}^{A'GB'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s_1}{s_0}\right)^{\hat{\mathcal{K}}} \hat{\mathcal{G}} \left(\frac{s_2}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle \ ,$$

where $\hat{\mathcal{G}}$ is the operator of the gluon production with change of total two-Reggeon state momentum from q_1 to q_2. The matrix elements of these operator are also expressed in terms of vertices of interactions of the Reggeized gluon.

The bootstrap relation

$$\Re \left[\frac{1}{-2\pi i} \left(disc_{s_2} + disc_s\right) \mathcal{A}_{AB}^{A'GB'}\right]$$

$$= \frac{\omega(t_2)}{2}\, 2s\, \Gamma^i_{A'A} \frac{1}{t_1} \left(\frac{s_1}{s_0}\right)^{\omega(t_1)} \gamma^G_{ij}(q_1, q_2)\, \frac{1}{t_2} \left(\frac{s_2}{s_0}\right)^{\omega(t_2)} \Gamma^j_{B'B}$$

leads, together with the strong bootstrap conditions for the impact factors and kernel, to new bootstrap condition, which can be written as

$$\frac{-2t_2}{(2\pi)^{D-1}} \left[t_1 \langle R_\omega|\hat{\mathcal{G}}|R_\omega\rangle + \langle RG|R_\omega\rangle\right]$$

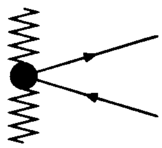
$$= \omega(t_2) \gamma^G_{ij}(q_1, q_2)$$

Fulfillment of this bootstrap condition is not checked yet.

Till now we have discussed the bootstrap relations for the scattering amplitudes in the MRK. But in the NLA in the unitarity relations not only multi-Regge, but quasi-multi-Regge kinematics as well does contribute, so that in the derivation of the NLO BFKL the multi-Regge form was assumed for production amplitudes in the QMRK as well.

If rapidities of components of the the produced couple (it can be or gg or $q\bar{q}$ pair) are far away from rapidities of colliding particles, then it is created by two Reggeized gluons, and its production is described by the vertices

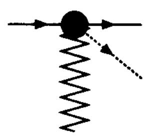
$\gamma_{c_1 c_2}^{P_1 P_2}(q_1, q_2)$



where q_1, c_1 and $-q_2, c_2$ are momenta and colour indices of the Reggeized gluons.

If the pair is produced in the region of fragmentation of particle A due to interaction of this particle with the Reggeized gluon with momentum q and colour index c, the production is described by the vertices

$\Gamma^c_{\{P_1 P_2\},A}(q)$



The produced particles can be gg or $q\bar{q}$ pair if the particle A is a gluon and qg when the particle A is a quark.

Note that because using of multi-Regge kinematics leads to lost of large logarithm, these vertices are needed only in the LO.

The QMRK vertices must satisfy the following bootstrap conditions:

$$
-if^{jbb'}\int\frac{d^{D-2}r_\perp}{r_\perp^2(q_2-r)_\perp^2}\left[\sum_G\gamma_{ib}^G(q_1,r)\Gamma^{b'}_{\{P_1P_2\},G}\right.
$$

$$
+igf^{ia'b}\frac{q_{1\perp}^2}{(q_1-r)_\perp^2}\gamma_{a'b'}^{P_1(k_1)P_2(k_2)}(q_1-r,q_2-r)
$$

$$
+\sum_{P_1'}\Gamma^{b'}_{P_1,P_1'}\gamma_{ib}^{P_1'(k_1')P_2(k_2)}(q_1,r)
$$

$$
+\sum_{P_2'}\Gamma^{b'}_{P_2,P_2'}\gamma_{ib}^{P_1(k_1)P_2'(k_2')}(q_1,r)
$$

$$
+igf^{ia'a}\frac{q_{1\perp}^2}{(q_1-k_1')_\perp^2k_{1\perp}'^{\,2}}
$$

$$
\left.\times\gamma_{ab}^{P_2(k_2)}(q_1-k_1',r)\gamma_{a'b'}^{P_1(k_1)}(k_1',q_2-r)\right]
$$

$$
=-g\frac{N_c}{2}\gamma_{ij}^{P_1(k_1)P_2(k_2)}(q_1,q_2)\int\frac{d^{D-2}r_\perp}{r_\perp^2(q_2-r)_\perp^2}\,.
$$

(note that the last term in the L.H.S. does contribute only in the case of two-gluon producion)

and

$$
\int\frac{d^{D-2}q_{1\perp}}{q_{1\perp}^2(q-q_1)_\perp^2}\frac{if^{cc_1c_1'}}{N_c}\sum_{\{i\}}\Gamma^{c_1}_{\{i\},A}(q_1)\Gamma^{c_1'}_{\{i\},P_1P_2}(q-q_1)
$$

$$
=\frac{g}{2}\Gamma^c_{P_1P_2,A}(q)\int\frac{d^{D-2}q_{1\perp}}{q_{1\perp}^2(q-q_1)_\perp^2}.
$$

These conditions are checked
V. F., M.G. Kozlov, A.V. Reznichenko, 2002
and it is shown that they are satisfied.

Summary

- The gluon Reggeization is a remarkable property of QCD. It is very important for description of high energy processes. In particular, the BFKL approach is based on this property.

- The gluon Reggeization is proved in the LLA, but still remains a hypothesis in the NLO.
- Selfconsistency of the hypothesis can be checked, and, hopefully,the gluon Reggeization can be proved in the NLO using unitarity relations.
- The selfconsistency requires fulfillment of bootstrap conditions on the gluon trajectory and vertices. Most restrictive are conditions arising from the requirement of the Reggeized form of inelastic amplitudes (strong bootstrap conditions).
- The strong bootstrap conditions for the impact factors of quarks and gluons and the BFKL kernel are proved to be satisfied.
- Now it is checked also that the conditions for the pair production in the fragmentation region, as well as for the pair production in Reggeon-Reggeon collisions are satisfied.
- The condition for the one-gluon production in Reggeon-Reggeon collisions is formulated now. Its check is under consideration.
- Fulfillment of all these conditions opens a way to prove the gluon Reggeization in the NLA.