A novel strong coupling expansion of the QCD Hamiltonian[1]

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- H.-P. Pavel, $SU(2)$ *Yang-Mills quantum mechanics of spatially constant fields*, Phys. Lett. B **648** (2007) 97-106.
- H.-P. Pavel, *Expansion of the Yang-Mills Hamiltonian in spatial derivatives and glueball spectrum*, Phys. Lett. B **685** (2010) 353-364.

$SU(2)$ Yang-Mills theory (define : $E_i^a \equiv F_{i0}^a$, $B_i^a \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}^a$)

$$\mathcal{S}[A] := -\frac{1}{4}\int d^4x\ F_{\mu\nu}^a F^{a\mu\nu}\ , \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c\ , \quad a = 1,2,3\ ,$$

is invariant under the $SU(2)$ gauge transformations $U[\omega(x)] \equiv \exp(i\omega_a\tau_a/2)$

$$A_{a\mu}^\omega(x)\tau_a/2 = U[\omega(x)]\left(A_{a\mu}(x)\tau_a/2 + \frac{i}{g}\partial_\mu\right)U^{-1}[\omega(x)]\ ,$$

Quantum Theory: exploit the time dependence of the gauge transformations to put

$$\mathrm{Weyl\ Gauge:}\quad \mathrm{A_{a0}(x) = 0}\ , \quad \mathrm{a = 1,2,3}\ , \qquad \rightarrow \quad \mathrm{E_{ai}(x) = -\dot{A}_{ai}(x)}$$

and impose can. com. rel. on the spatial fields using the Schrödinger representation

$$[\Pi_{ai}(x), A_{bj}(y)] = i\delta_{ab}\delta_{ij}\delta^3(x-y) \quad \longrightarrow \quad \Pi_{ai}(x) = -E_{ai}(x) = -i\frac{\delta}{\delta A_{ai}(x)}\ .$$

The physical states Ψ have to satisfy: $\left(B_{ai}(A) = \epsilon_{ijk}\left(\partial_j A_{ak} + \frac{1}{2}g\epsilon_{abc}A_{bj}A_{ck}\right)\right)$

$$\mathrm{Schr\ddot{o}d.\ equ.:}\qquad H\Psi = E\Psi\ , \qquad H = \int d^3x \frac{1}{2}\sum_{a,i}\left[\left(\frac{\delta}{\delta A_{ai}(x)}\right)^2 + B_{ai}^2(A(x))\right]$$

$$\mathrm{Gauss\ laws:}\qquad G_a(x)\Psi = 0\ , \qquad G_a(x) = -i\left(\delta_{ac}\partial_i + g\epsilon_{abc}A_{bi}(x)\right)\frac{\delta}{\delta A_{ci}(x)}$$

$G_a(x)$ generators of the residual time-independent gauge transformations,

$$[G_a(x), H] = 0\ , \qquad [G_a(x), G_b(y)] = ig\delta^3(x-y)\epsilon_{abc}G_c(x)\ ,$$

matrix elements: $\langle\Phi_1|\mathcal{O}|\Phi_2\rangle = \int\prod_x\prod_{ik} dA_{ik}(x)\ \Phi_1^*\mathcal{O}\Phi_2$.

The "Symmetric Gauge"

Point transformation to the new set of adapted coordinates, the 3 $q_j \quad (j = 1, 2, 3)$ and the 6 elements $S_{ik} = S_{ki} \quad (i, k = 1, 2, 3)$ of the pos. definite symmetric 3×3 matrix S

$$A_{ai}\left(q, S\right) = O_{ak}\left(q\right) S_{ki} - \frac{1}{2g}\epsilon_{abc}\left(O\left(q\right)\partial_i O^T\left(q\right)\right)_{bc} ,$$

where $O(q)$ is an orthog. 3×3 matrix parametrized by the q_i. It corresponds to the gauge choice

$$\chi_i(A) = \epsilon_{ijk} A_{jk} = 0 \quad (\text{"symmetric gauge"}).$$

Symmetric gauge exists: one can prove that at least for strong coupling any time-independent gauge field can be carried over uniquely into the symmetric gauge.

$$\rightarrow \qquad G_a \Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i}\Phi = 0 \quad (\text{Abelianisation})$$

Physical quantum Hamiltonian of $SU(2)$ YM theory in symmetric gauge

The correctly ordered physical quantum Hamiltonian (Christ and Lee 1980) in the symmetric gauge in terms of the physical variables $S_{ik}(\mathbf{x})$ and the corresponding canonically conjugate momenta $P_{ik}(\mathbf{x}) \equiv -i\delta/\delta S_{ik}(\mathbf{x})$ reads

$$\begin{aligned} H(S,P) \;=\; & \frac{1}{2}\mathcal{J}^{-1}\int d^3\mathbf{x}\; P_{ai}\;\mathcal{J} P_{ai} + \frac{1}{2}\int d^3\mathbf{x}\,(B_{ai}(S))^2 \\ & -\mathcal{J}^{-1}\int d^3\mathbf{x}\int d^3\mathbf{y}\Big\{\Big(D_i(S)_{ma}P_{im}\Big)(\mathbf{x})\mathcal{J} \\ & \langle \mathbf{x}\; a|{}^*D^{-2}(S)|\mathbf{y}\; b\rangle\Big(D_j(S)_{bn}P_{nj}\Big)(\mathbf{y})\Big\} \end{aligned} \quad (1)$$

with the covariant derivative $D_i(S)_{kl} \equiv \delta_{kl}\partial_i - g\epsilon_{klm}S_{mi}$,
the FP operator

$$^*D_{kl}(S) \equiv \epsilon_{kmi}D_i(S)_{ml} = \epsilon_{kli}\partial_i - g\gamma_{kl}(S)\ , \qquad \gamma_{kl}(S) \equiv S_{kl} - \delta_{kl}\mathrm{tr}S \quad (2)$$

and the Jacobian $\mathcal{J} \equiv \det|{}^*D|$
The matrix element of a physical operator O is given by

$$\langle\Psi'|O|\Psi\rangle \;\propto \int\prod_{\mathbf{x}}\Big[dS(\mathbf{x})\Big]\mathcal{J}\,\Psi'^*[S]O\Psi[S]\ . \quad (3)$$

The inverse of the FP operator can be expanded in the number of spatial derivatives

$$\begin{aligned} \langle \mathbf{x}\; k|{}^*D^{-1}(S)|\mathbf{y}\; l\rangle \;\equiv\; & {}^*D^{-1}_{kl}(S)^{(\mathbf{x})}[\delta^3(\mathbf{x}-\mathbf{y})] \\ =\; & -\frac{1}{g}\gamma^{-1}_{kl}(S)(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + \frac{1}{g^2}\gamma^{-1}_{ka}(\mathbf{x})\epsilon_{abc}\partial^{(\mathbf{x})}_c\Big[\gamma^{-1}_{bl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\Big] \\ & -\frac{1}{g^3}\gamma^{-1}_{ka}(\mathbf{x})\epsilon_{abc}\partial^{(\mathbf{x})}_c\Big[\gamma^{-1}_{bi}(\mathbf{x})\epsilon_{ijk}\partial^{(\mathbf{x})}_k\Big[\gamma^{-1}_{jl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\Big]\Big] + \ldots\ . \end{aligned}$$

Derivative Expansion (1)

In order to perform a consistent expansion, also the nonlocality in the Jacobian $\mathcal{J}$ has to be taken into account. The Jacobian $\mathcal{J}$ factorizes

$$\mathcal{J} = \mathcal{J}_0 \widetilde{\mathcal{J}} \tag{4}$$

with the local

$$\mathcal{J}_0 \equiv \det|\gamma| = \prod_{\mathbf{x}} \det|\gamma(\mathbf{x})| \ , \quad \det|\gamma(\mathbf{x})| = \prod_{i<j} (\phi_i(\mathbf{x}) + \phi_j(\mathbf{x})) \tag{5}$$

and the nonlocal

$$\widetilde{\mathcal{J}} \equiv \det|{}^*D| / \det|\gamma| \tag{6}$$

Now I include the nonlocal part of the measure into the wavefunctional

$$\widetilde{\Psi}(S) := \widetilde{\mathcal{J}}^{-1/2} \Psi(S) \tag{7}$$

leading to the corresponding transformed Hamiltonian

$$\widetilde{H} := \widetilde{\mathcal{J}}^{1/2} H \widetilde{\mathcal{J}}^{-1/2} \tag{8}$$

being Hermitean with respect to the local measure $\mathcal{J}_0$ on the cost of extra terms V_{measure} from the nonlocal factor $\widetilde{\mathcal{J}}$ of the original measure $\mathcal{J}$

$$\widetilde{H}(S,P) \ = \ \ H(S,P)\Big|_{J \to J_0} + V_{\mathrm{measure}}(S) \tag{9}$$

The matrix element of a physical operator O is given by

$$\langle \Psi' | O | \Psi \rangle \ \propto \int \prod_{\mathbf{x}} \left[dS(\mathbf{x}) \prod_{i<j} (\phi_i(\mathbf{x}) + \phi_j(\mathbf{x})) \right] \Psi'^*[S] O \Psi[S] \ . \tag{10}$$

Derivative Expansion (2): Zeroth order Hamiltonian

To zeroth order we obtain physical Yang-Mills Hamiltonian

$$H_0 = \int d^3\mathbf{x} \sum_{m,n} \left[-\frac{1}{2}\left(\frac{\delta}{\delta S_{mn}}\right)^2 - \frac{\delta^3(\mathbf{0})}{2}\left[\gamma^{-1}_{mn}(S) - \delta_{mn}\mathrm{tr}(\gamma^{-1}(S))\right]\frac{\delta}{\delta S_{mn}} \right.$$

$$\left. -\frac{1}{2}\gamma^{-2}_{mn}(S) J^{\mathrm{spin}}_m J^{\mathrm{spin}}_n + \frac{g^2}{2}\left(\mathrm{tr}^2 S^2 - \mathrm{tr} S^4\right)\right], \qquad J^{\mathrm{spin}}_m \equiv 2\epsilon_{mij} S_{is}\frac{\delta}{\delta S_{sj}}$$

Diagonalize the positive definite symmetric 3×3 matrix field S

$$S = R^T(\alpha,\beta,\gamma)\ \mathrm{diag}\ (\phi_1,\phi_2,\phi_3)\ R(\alpha,\beta,\gamma)\ ,$$

with the $SO(3)$ matrix R parametrized by the three Euler angles $\chi = (\alpha,\beta,\gamma)$.

$$\text{Jacobian}\quad \mathcal{J} \propto \prod_{\mathrm{i}\neq\mathrm{j}} |\phi_{\mathrm{i}} - \phi_{\mathrm{j}}| \quad \rightarrow \quad 0 < \phi_1 < \phi_2 < \phi_3 \quad \text{(principle orbits)}\ .$$

In terms of the principal-axes variables, the zeroth order Hamiltonian reads

$$H_0 = \frac{1}{2}\int d^3\mathbf{x} \sum_{ijk}^{\mathrm{cyclic}} \left[-\frac{\partial^2}{\partial\phi_i^2} - \frac{2\delta^3(\mathbf{0})}{\phi_i^2 - \phi_j^2}\left(\phi_i\frac{\partial}{\partial\phi_i} - \phi_j\frac{\partial}{\partial\phi_j}\right) + \xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} + g^2\phi_j^2\phi_k^2 \right]$$

with the intrinsic spin angular momenta $\xi_i(\mathbf{x})$, $\quad J^{\mathrm{spin}}_i(\mathbf{x}) = -R_{ij}(\chi(\mathbf{x}))\xi_j(\mathbf{x})$,

$$[J^{\mathrm{spin}}_i(\mathbf{x}), \xi_j(\mathbf{y})] = 0\ , \qquad [\xi_i(\mathbf{x}),\xi_j(\mathbf{y})] = -i\epsilon_{ijk}\delta^3(\mathbf{x}-\mathbf{y})\xi_k(\mathbf{x})\ .$$

The matrix elements become

$$\langle\Phi_1|\mathcal{O}|\Phi_2\rangle = \prod_x \int d\alpha(x)\sin\beta d\beta(x) d\gamma(x) \int_{0<\phi_1<\phi_2<\phi_3} d\phi_1(x)d\phi_2(x)d\phi_3(x)(\phi_1^2-\phi_2^2)(\phi_2^2-\phi_3^2)(\phi_3^2-\phi_1^2)\Phi_1^*\mathcal{O}\Phi_2$$

Derivative expansion (3): 1st and 2nd order Hamiltonian (magnetic part)

The first order magnetic part reads

$$V^{(\partial)}_{\text{magn}} = -\frac{g}{2}\int d^3\mathbf{x}\sqrt{10}\, C^{1\gamma}_{2A\ 2B}\,(\phi_1\phi_2)^{(2)}_A\, i\partial_\gamma\,(\phi_3)^{(2)}_B \tag{11}$$

and the second order magnetic part

$$V^{(\partial\partial)}_{\text{magn}} = -\frac{1}{3}\int d^3\mathbf{x}\Big[(\phi_3)^{(0)}\,\Delta\,(\phi_3)^{(0)} + (\phi_3)^{(2)}_A\,\Delta\,(\phi_3)^{(2)}_A - \sqrt{2}\,(\phi_3)^{(0)}\,\Delta^{(2)}_A\,(\phi_3)^{(2)}_A$$
$$-\frac{1}{2}\sqrt{\frac{7}{2}}\,C^{2C}_{2A\ 2B}\,(\phi_3)^{(2)}_A\,\Delta^{(2)}_B\,(\phi_3)^{(2)}_C\Big] \tag{12}$$

The spin-0 fields are

$$(\phi_3)^{(0)} := \frac{1}{\sqrt{3}}\,(\phi_1+\phi_2+\phi_3)$$

and the spin-2 fields can be written in terms of D-functions as

$$(\phi_3)^{(2)}_A := \sqrt{\frac{2}{3}}\left[\left(\phi_3-\frac{1}{2}\,(\phi_1+\phi_2)\right)D^{(2)}_{A0}(\chi)+\frac{\sqrt{3}}{2}\,(\phi_1-\phi_2)\,D^{(2)}_{A2+}(\chi)\right]$$

The second order differential operators are the spin-0 Laplace operator

$$\Delta \equiv \partial_x^2+\partial_y^2+\partial_z^2$$

as well as the corresponding spin-2 operators

$$\Delta^{(2)}_0 \equiv \partial_z^2-\frac{1}{2}\,(\partial_x^2+\partial_y^2) \qquad \Delta^{(2)}_{2+} \equiv \frac{\sqrt{3}}{2}\,(\partial_x^2-\partial_y^2)$$
$$\Delta^{(2)}_{2-} \equiv \sqrt{3}\,\partial_x\partial_y \qquad \Delta^{(2)}_{1+} \equiv \sqrt{3}\,\partial_y\partial_z \qquad \Delta^{(2)}_{1-} \equiv \sqrt{3}\,\partial_z\partial_x$$

Coarse graining (1): Equivalence to an expansion in $\lambda = g^{-2/3}$

Set an ultraviolet cutoff a by introducing an infinite spatial lattice of granulas $G(\mathbf{n},a)$, here cubes of length a, situated at sites $\mathbf{x} = a\mathbf{n}$ ($\mathbf{n} = (n_1, n_2, n_3) \in Z^3$), and considering the averaged variables

$$\phi(\mathbf{n}) := \frac{1}{a^3} \int_{G(\mathbf{n},a)} d\mathbf{x}\ \phi(\mathbf{x})$$

(where in particular $\delta(\mathbf{0}) \to 1/a^3$), and the discretised spatial derivatives (s=1,2,3),

$$\partial_s \phi(\mathbf{n}) := \lim_{N\to\infty} \sum_{n=1}^{N} w_N(n) \frac{1}{2na} \left(\phi(\mathbf{n}+n\mathbf{e}_s) - \phi(\mathbf{n}-n\mathbf{e}_s)\right) \tag{13}$$

$$\partial_s^2 \phi(\mathbf{n}) := \lim_{N\to\infty} \sum_{n=1}^{N} w_N(n) \frac{1}{(na)^2} \Big(\phi(\mathbf{n}+n\mathbf{e}_s) + \phi(\mathbf{n}-n\mathbf{e}_s) - 2\phi(\mathbf{n})\Big) \tag{14}$$

with the unit lattice vectors $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$ and the distrib.

$$w_N(n) := 2\frac{(-1)^{n+1}(N!)^2}{(N-n)!(N+n)!}\ , \quad 1 \le n \le N\ , \quad \sum_{n=1}^{N} w_N(n) = 1\ . \tag{15}$$

Apply the rescaling transformation

$$\phi_i = \frac{g^{-1/3}}{a}\phi_i' \qquad \pi_i = \frac{g^{1/3}}{a^2}\pi_i' \qquad \xi_i = \frac{1}{a^3}\xi_i' \tag{16}$$

I obtain the expansion of the Hamiltonian in $\lambda = g^{-2/3}$

$$H = \frac{g^{2/3}}{a}\left[\mathcal{H}_0 + \lambda \sum_\alpha \mathcal{V}_\alpha^{(\partial)} + \lambda^2 \left(\sum_\beta \mathcal{V}_\beta^{(\Delta)} + \sum_\gamma \mathcal{V}_\gamma^{(\partial\partial\neq\Delta)}\right) + \mathcal{O}(\lambda^3)\right]\ , \tag{17}$$

Coarse graining (2): The "free" Hamiltonian

The "free" Hamiltonian

$$\mathcal{H}_0 \quad = \quad \sum_{\mathbf{n}} \mathcal{H}_0^{QM}(\mathbf{n}) \ , \tag{18}$$

is the sum of the Hamiltonians of $SU(2)$-Yang-Mills quantum mechanics of constant fields in each box,

$$\mathcal{H}_0^{QM} \quad = \quad \frac{1}{2}\sum_{ijk}^{\text{cyclic}} \left[\pi_i^2 - \frac{2i}{\phi_j^2 - \phi_k^2}(\phi_j\pi_j - \phi_k\pi_k) + \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + \phi_j^2\phi_k^2\right] \tag{19}$$

The low energy spectrum and eigenstates of $\mathcal{H}_0^{QM}$ at each site $x_{\mathbf{n}}$ appearing in (18),

$$\mathcal{H}_0^{QM}(\mathbf{n})|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} = \epsilon_i^{(S)}(\mathbf{n})|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} \ , \tag{20}$$

characterised by the quantum numbers of spin S, M, are known with high accuracy. Due to the positivity of the range $(0 < \phi_1 < \phi_2 < \phi_3)$, all states should satisfy

$$\text{either} \quad \partial_{\phi_1}\Phi(\phi)|_{\phi_1=0} = 0 \quad ((+)\ \text{b.c.}) \qquad \text{or} \quad \Phi(\phi)|_{\phi_1=0} = 0 \quad ((-)\ \text{b.c.}), \tag{21}$$

in accordance with the invariance of the Hamiltonian $\mathcal{H}_0$ under parity transformation $\phi \to -\phi$. Although the potential term $\frac{1}{2}g^2\left(\phi_1^2\phi_2^2 + \phi_2^2\phi_3^2 + \phi_1^2\phi_3^2\right)$ has the classical zero energy minima ("valleys") $\phi_1 = \phi_2 = 0 \ ; \quad \phi_3 \ \text{arbitrary}$, the spectrum of the quantum Hamiltonian is purely discrete. The matrix elements are

$$\langle\Phi_1|\mathcal{O}|\Phi_2\rangle = \int d\alpha \sin\beta d\beta d\gamma \int_{0<\phi_1<\phi_2<\phi_3} d\phi_1 d\phi_2 d\phi_3 (\phi_1^2 - \phi_2^2)(\phi_2^2 - \phi_3^2)(\phi_3^2 - \phi_1^2)\Phi_1^*\mathcal{O}\Phi_2 \ .$$

The lowest energies for the $(+)$ and $(-)$ b.c. are

$$\epsilon_0^+ = 4.1167 \ , \quad \epsilon_0^- = 8.7867 \ . \tag{22}$$

The energies relative to ϵ_0

$$\mu_i^{(S)+} := \epsilon_i^{(S)+} - \epsilon_0^+ \ , \quad \mu_i^{(S)-} := \epsilon_i^{(S)-} - \epsilon_0^- \ , \tag{23}$$

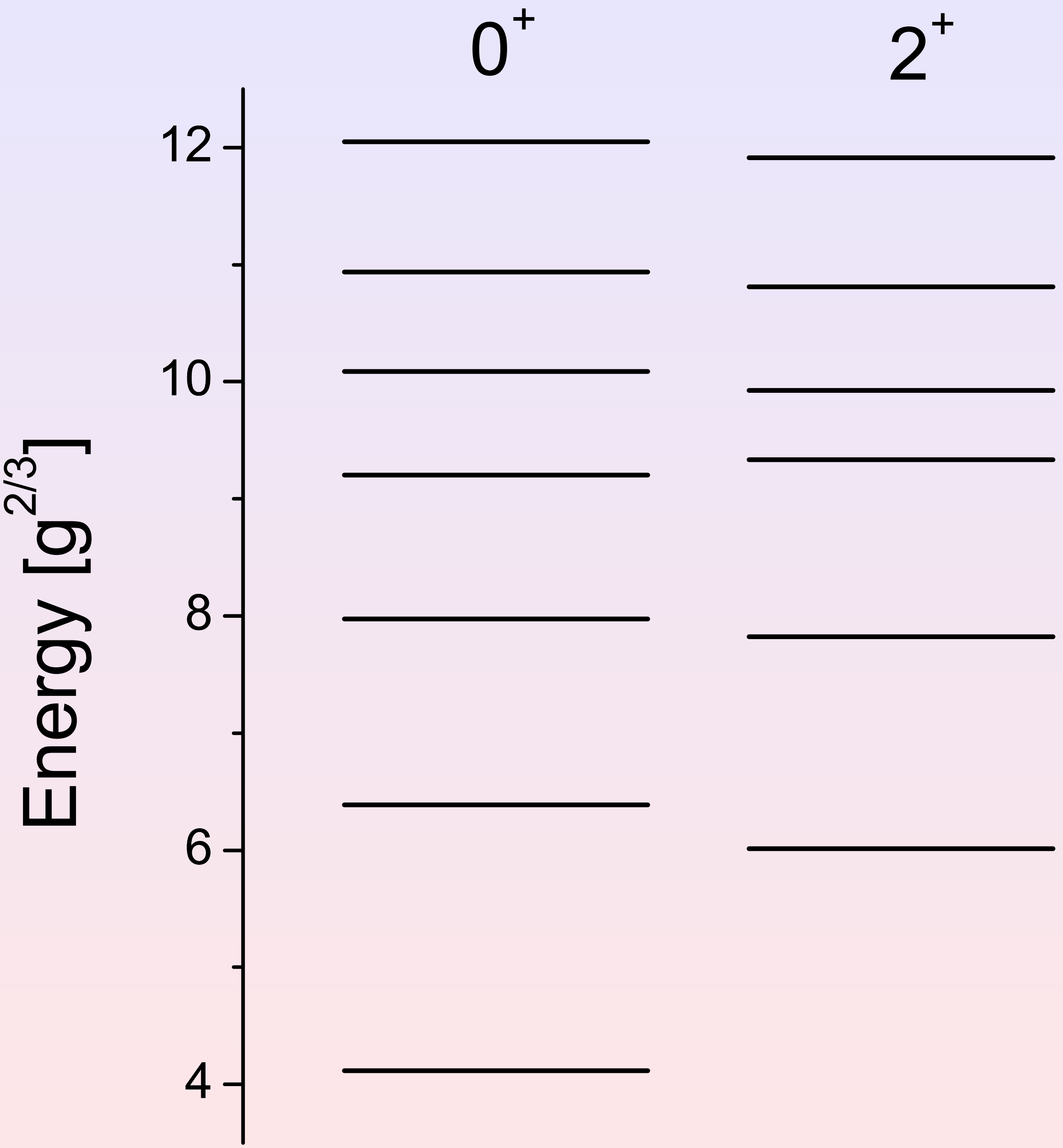
for the lowest spin-0,2,3 and 4 states are summarized in Table 1a and 1b. Spin-1 states are absent for both cases. The underlined values correspond to stable excitations below threshold

$$\mu_{\text{th}}^+ = 3.796 \ \ (= 2\mu_1^{(2)+}) \ , \quad \mu_{\text{th}}^- = 5.089 \ \ (= 2\mu_1^{(2)-}) \ , \tag{24}$$

for decay into two spin-2 excitations $\mu_1^{(2)}$ (lightest in the spectrum).

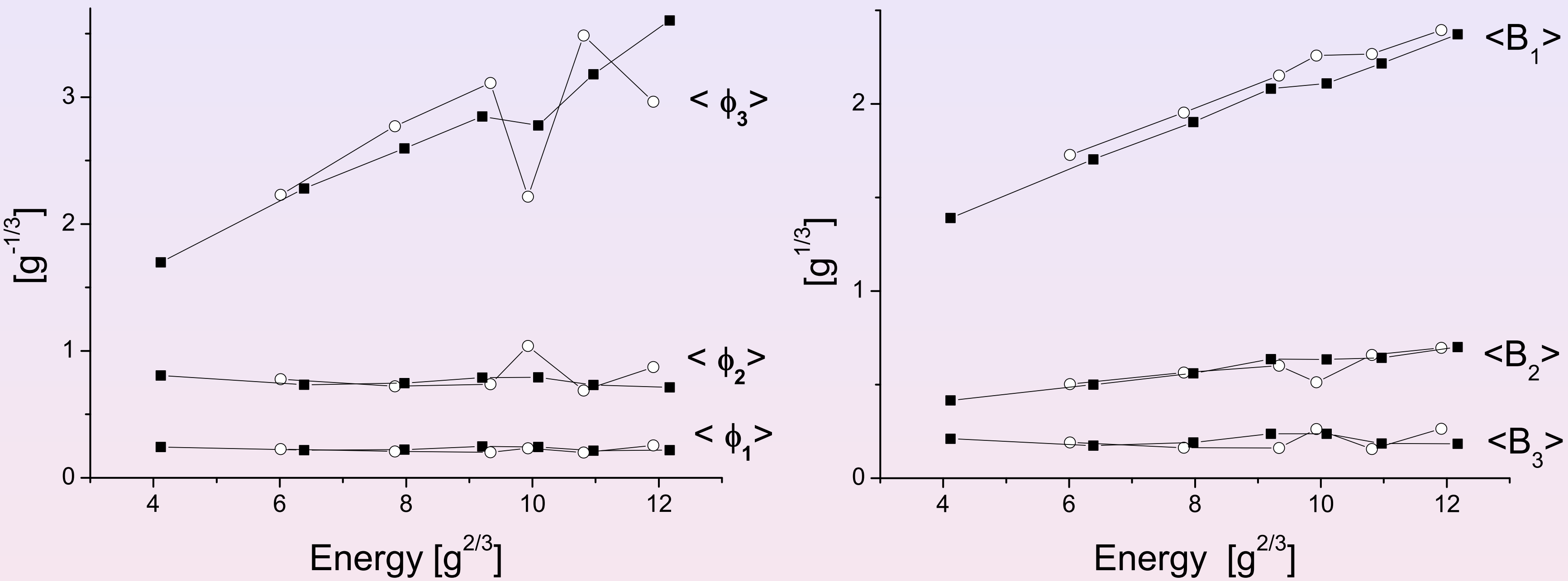
$\mu_i^{(S)+}$	$S=0$	$S=2$	$S=3$	$S=4$	$\mu_i^{(S)-}$	$S=0$	$S=2$	$S=3$	$S=4$
$i=1$	$\underline{2.270}$	$\underline{1.898}$	8.009	$\underline{3.61}$	$i=1$	$\underline{3.268}$	$\underline{2.545}$	9.250	$\underline{4.93}$
$i=2$	3.857	$\underline{3.704}$	10.815	5.23	$i=2$	5.233	5.212	12.78	7.37
$i=3$	5.09	5.22	13.1	6.9	$i=3$	6.803	6.612	15.38	9.6

Energy specrum for 0^+ and 2^+



The energy of the first 2^+ state is lower than the first excited 0^+ state!

$\langle\phi_i\rangle$ and $\langle B_i\rangle \equiv \langle g\phi_j\phi_k\rangle$ expectation values for $0^+$and 2^+



$\langle\phi_3\rangle$ is raising with increasing excitation, whereas $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ are practically constant. $\langle B_3\rangle$ is practically constant with increasing excitation, whereas $\langle B_1\rangle$ and $\langle B_2\rangle$ are raising.

Coarse graining (3): Free many particle states

1. Free many particle states (completely decoupled granulas):

The eigenstates of $H_0 = \sum_{\mathbf{n}} H_0^{QM}(\mathbf{n})$ are "free" many particle states (completely decoupled granulas), the "free" vacuum

$$|0\rangle \equiv \bigotimes_{\mathbf{n}} |\Phi_0\rangle_{\mathbf{n}} \;\rightarrow\; E_{\text{vac}}^{\text{free}} = \mathcal{N}\epsilon_0 \frac{g^{2/3}}{a}$$

the "free" one-particle excited states,

$$|S, M, i, \mathbf{k}\rangle \;\equiv\; \sum_{\mathbf{n}} e^{i\mathbf{k}.\mathbf{x}_{\mathbf{n}}} \left[|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m}\neq\mathbf{n}} |\Phi_0\rangle_{\mathbf{m}} \right]$$

$$\rightarrow\; E_i^{(S)\text{free}}(k) \;=\; \mu_i^{(S)} \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}}$$

the "free" two-particle excited states,

$$|S_1, M_1, i_1, \mathbf{n}_1; S_2, M_2, i_2, \mathbf{n}_2\rangle \;\equiv\; |\Phi_{i_1,M_1}^{(S_1)}\rangle_{\mathbf{n}_1} \otimes |\Phi_{i_2,M_2}^{(S_2)}\rangle_{\mathbf{n}_2} \left[\bigotimes_{\mathbf{m}\neq\mathbf{n}_1,\mathbf{n}_2} |\Phi_0\rangle_{\mathbf{m}} \right]$$

$$\rightarrow\; E_{i_1,i_2}^{(S_1,S_2)\text{free}} = \;=\; (\mu_{i_1}^{(S_1)} + \mu_{i_2}^{(S_2)}) \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}}$$

and so on .

2. Include interactions V^{Δ} and V^{∂} using 1st and 2nd order perturbation theory:

$$E_n^{\text{int}} = \widetilde{E}_n + \langle\widetilde{n}|V^{\Delta}|\widetilde{n}\rangle + \sum_{n'} \frac{|\langle\widetilde{n}|V^{\partial}|\widetilde{n}'\rangle|^2}{\widetilde{E}_n - \widetilde{E}_{n'}}$$

Coarse graining (4)

$$\begin{aligned}
\langle \phi \Delta \phi \rangle &\rightarrow \langle \phi(\mathbf{x}) \frac{1}{4a^2} \sum_{i=1}^{3} [\phi(\mathbf{x} + 2\mathbf{e}_i a) + \phi(\mathbf{x} - 2\mathbf{e}_i a) - 2\phi(\mathbf{x})] \rangle \\
&= \frac{1}{4a^2} \sum_{i=1}^{3} [\langle \phi(\mathbf{x}) \rangle \langle \phi(\mathbf{x} + 2\mathbf{e}_i a) \rangle + \langle \phi(\mathbf{x}) \rangle \langle \phi(\mathbf{x} - 2\mathbf{e}_i a) \rangle - 2 \langle \phi(\mathbf{x}) \phi(\mathbf{x}) \rangle] \\
&= \langle \phi \rangle \Delta \langle \phi \rangle - \frac{3}{2a^2} \left(\langle \phi(\mathbf{x})^2 \rangle - \langle \phi(\mathbf{x}) \rangle^2 \right)
\end{aligned}$$

Results

Hence 1st and 2nd order perturbation theory together give the result

$$E_{\rm vac}^{+} = \mathcal{N}\frac{g^{2/3}}{a}\left[4.1167 + 29.894\lambda^2 + \mathcal{O}(\lambda^3)\right] ,$$

for the energy of the interacting glueball vacuum up to λ^2 for the $(+)$ b.c. .
First and second order perturbation theory give the results (up to λ^2)

$$E_1^{(0)+}(k) - E_{\rm vac}^{+} = \left[\, 2.270 + 13.511\lambda^2 + \mathcal{O}(\lambda^3)\right] \frac{g^{2/3}}{a} + 0.488\frac{a}{g^{2/3}}k^2 + \mathcal{O}((a^2k^2)^2) ,$$

for the energy spectrum of the interacting spin-0 glueball for the $(+)$ b.c. .
Similar results for the $(-)$ b.c.
Lorentz invariance : $E = \sqrt{M^2 + k^2} \simeq M + \frac{1}{2M}\, k^2 \quad \rightarrow \quad \tilde{c}^{(i)} = 1/[2\mu_i]$
Consider $J = L + S$ states:

$$|J=0,k\rangle \sim \alpha_1^{(0)} \sum_{\mathbf{n}} j_0(kr)\left[|\Phi_1^{(0)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m}\neq\mathbf{n}} |\Phi_0\rangle_{\mathbf{m}}\right]$$

$$+ \sum_{S,i}^{\rm stable} \alpha_i^{(S)} \sum_{\mathbf{n}} j_S(kr) \sum_M Y_{SM}(\theta,\phi)\left[|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m}\neq\mathbf{n}} |\Phi_0\rangle_{\mathbf{m}}\right] ,$$

Glueball mass and coupling constant renormalisation in the IR

Consider the physical mass

$$M = \frac{g_0^{2/3}}{a}\left[\mu + c g_0^{-4/3}\right] \ . \tag{25}$$

Demanding its independence of box size a, one obtains

$$\gamma(g_0) \equiv a\frac{d}{da}g_0(a) = \frac{3}{2}g_0\frac{\mu + c g_0^{-4/3}}{\mu - c g_0^{-4/3}}$$

which vanishes for the two cases, $g_0 = 0$ or $g_0^{4/3} = -c/\mu$. The first solution corresponds to the perturbative fixed point, and the second, if it exists $(c < 0)$, to an infrared fixed point. My result for $c_1^{(0)}/\mu_1^{(0)} = 5.95(1.34)$ suggests, that no infrared fixed points exist, in accordance with the corresponding result of Wilsonian lattice QCD[2]. Solving the above equation (25) for positive $(c > 0)$ I obtain

$$g_0^{2/3}(Ma) = \frac{Ma}{2\mu} + \sqrt{\left(\frac{Ma}{2\mu}\right)^2 - \frac{c}{\mu}}\ , \qquad a > a_c := 2\sqrt{c\mu}/M \tag{26}$$

with the physical glueball mass M. critical coupling $g_0^2|_c = 14.52\ (1.55)$ and

$$\text{for } M \sim 1.6 \text{ GeV} : \quad a_c \sim 1.4 \text{ fm } (0.9 \text{ fm}) \ .$$

Connect the behaviour of the bare coupling constant (26), obtained for boxes of large size a, with those obtained for small boxes (see Luescher 1983), to get information about the intermediate region, including the possibility of existence of phase transitions.

[2] In comparison, the $SU(2)$ result from strong coupling on the lattice (Muenster 1981): $aM = 4\log(g_0^2) + O(g_0^{-2}) \rightarrow \gamma(g_0) = \frac{1}{2}g_0\log(g_0^2) + \ldots$ does not contain infrared fixed points.

Conclusions

- Using the symmetric gauge $\epsilon_{ijk} A_{jk} = 0$, an expansion of the physical quantum Hamiltonian of $SU(2)$ Yang-Mills theory in the number of spatial derivatives can be carried out.
- This expansion is equivalent to a strong coupling expansion in $g^{-2/3}$ for large box sizes a, similar to Lueschers weak coupling expansion in $g^{2/3}$ applicable for small boxes.
- Using the very accurate results of Yang-Mills quantum mechanics obtained with the variational method , quite accurate results can in principle be obtained for the dispersion relation of glueballs. Useful alternative to lattice calculations based on the Wilson-loop.
- The present (preliminary) results for the spin-0 mass suggest that the running coupling goes to infinity for large boxes. No infrared fixed points exist.
- Problems: a.o. question of Lorentz invariance of dispersion relation.

Ongoing work:

- Extension to mass of spin 2 particles.
- Inclusion of Quarks: Glueball-Spectrum $\longrightarrow$ Hadron-, Glueball- and Hybrid-Spectrum
- Investigation of flux tubes and string tension.