Report on implementation of the MK-model for resonance single-pion production into GENIE

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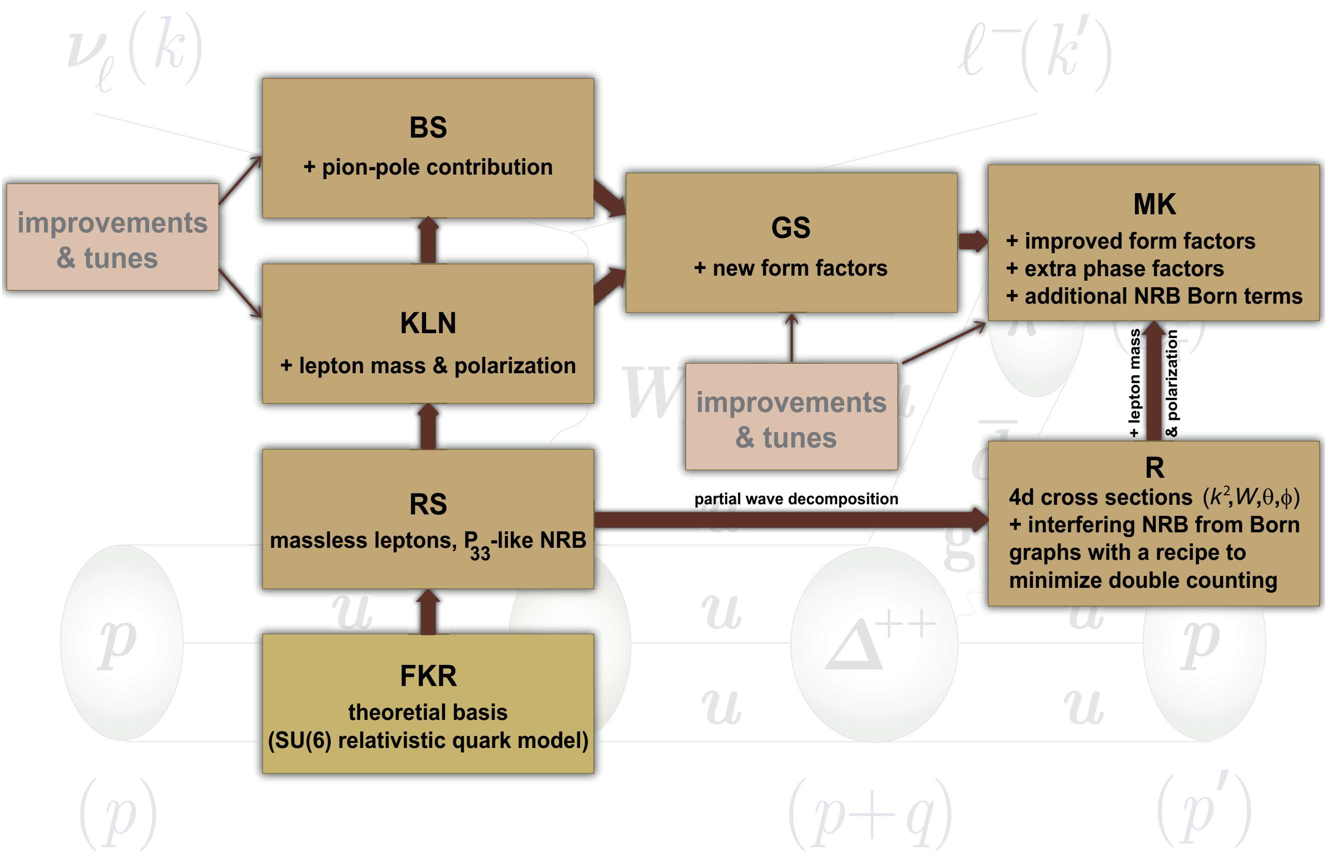
Abstract

In this report, we intend to set out the questions and problems which have arisen during the implementation of the MK-model [1–4] in GENIE Generator.

The report consists of two parts. In the first part, we list yet unfixed bugs and mistakes, because they require thorough study for their correction. In the second part, we list resolved problems. The bugs and mistakes in this part have been fixed either after intensive discussions with the author of the model[1] or if the bugs are trivial.

This is a living document which is being constantly updated. Any comments, suggestions and corrections are welcome.

In the current version of this report, we discuss the latest version of the Monoo code in which she got out of the phase factors earlier belonging to the vector and axial parts of each resonance amplitude and were the subject of fine-tuning. From our point of view, this simplification (rejecting the phase factors) is a very right step.



[1] In sections of this part, we point the version of the erratum or code where they were fixed.

Contents

Chapter 1

Still unfixed problems, bugs and mistakes

1.1 Ambiguity in calculation of signs of the amplitudes

In this section, we show that one will get a contradiction in calculation of the helicity amplitudes if one try to calculate these by two different ways: directly and by using some known symmetry relations. First, we recall the general formulas.

1.1.1 General relations

From ((3.34)– [2])[1] it follows that

$$\widetilde{F}^{\lambda_k(p)}_{\lambda_2,\lambda_1}(\theta,\phi) = \frac{1}{2M}\langle N\pi,\lambda_2|e^\alpha_p J^V_\alpha|N,\lambda_1\rangle, \quad \widetilde{G}^{\lambda_k(p)}_{\lambda_2,\lambda_1}(\theta,\phi) = \frac{1}{2M}\langle N\pi,\lambda_2|e^\alpha_p J^A_\alpha|N,\lambda_1\rangle, \tag{1.1}$$

where e_p is the gauge boson polarization, $p = \{L, R, +, -\}$, $\lambda_k(L) = -1$, $\lambda_k(R) = 1$, and $\lambda_k(\pm) = 0$. We will omit argument p in λ_k when it is clear from context. So, from hereon λ_k denotes either the symbols $L, R, +, -$ or a relevant number, as the context admits. Let's now express the matrix element of single pion production as product of two matrix elements: resonance production and resonance decay to nucleon-pion pair (similar to Eq. ((4.18)– [2])).

$$\langle N\pi,\lambda_2|e^\alpha J_\alpha|N,\lambda_1\rangle = \langle N\pi,\lambda_2|R,\lambda\rangle\langle R,\lambda|e^\alpha J_\alpha|N,\lambda_1\rangle, \tag{1.2}$$

where for CC processes $J_\alpha = J^V_\alpha - J^A_\alpha$. First, let's find the matrix element for the resonance production. It can be found from Eq. ((3.17)– [2]) that the convolution of lepton current with helicity λ and hadronic current is

$$e^\alpha_\lambda J_\alpha = \left(C_{L_\lambda} e^\alpha_L + C_{R_\lambda} e^\alpha_R + C_\lambda e^\alpha_\lambda\right) J_\alpha, \tag{1.3}$$

where $\lambda = -(+)$ denotes left(right) lepton helicity, and e^α_L, e^α_R, e^α_+ and e^α_- are defined by Eq. (1.28).

Using Eqs. ((4.5, 4.6)– [2]), the matrix element of resonance production can be written as

$$\begin{aligned}\langle R,\lambda|e^\alpha_\lambda J_\alpha|N,\lambda_1\rangle &= 2M_R\langle R,\lambda|(C_{L_\lambda} e^\alpha_L + C_{R_\lambda} e^\alpha_R + C_\lambda e^\alpha_\lambda)F_\alpha|N,\lambda_1\rangle \\ &= 2M_R\langle R,\lambda|C_{L_\lambda} F_- + C_{R_\lambda} F_+ + C_\lambda F^{(\lambda)}_0|N,\lambda_1\rangle,\end{aligned} \tag{1.4}$$

[1] Here and further, we denote by ((A)–[B]) the formula (A) from Ref. [B].

where

$$F_- = e^\alpha_L F_\alpha = +\frac{1}{\sqrt{2}}(F_x - iF_y), \quad F_0^{(-)} = e^\alpha_- F_\alpha = \frac{1}{\sqrt{Q^2}}\left(\mathcal{Q}^\star_- F_0 + \nu^\star_- F_z\right),$$

$$F_+ = e^\alpha_R F_\alpha = -\frac{2}{\sqrt{2}}(F_x + iF_y), \quad F_0^{(+)} = e^\alpha_+ F_\alpha = \frac{1}{\sqrt{Q^2}}\left(\mathcal{Q}^\star_+ F_0 + \nu^\star_+ F_z\right),$$

at that $(F_-)^\dagger = -F_+$, $(F_0^{(-)})^\dagger = F_0^{(-)}$ and $(F_0^{(+)})^\dagger = F_0^{(+)}$. The resonance production amplitudes in the RS model [5] are expressed as

$$f_{\pm|2s_z|} = \langle N, s_z \pm 1|F_\pm|R, s_z\rangle, \quad f_{0\pm} = \langle N, s_z \pm \frac{1}{2}|F_0^{(\pm)}|R, s_z \pm \frac{1}{2}\rangle. \tag{1.5}$$

We can choose the spin quantization axis in such a way that $\lambda_1 = -s_{1z}$, $\lambda_2 = -s_{2z}$, and $\lambda = s_{Rz}$. Therefore,

$$\begin{aligned}
\langle R, \lambda|e^\alpha_L F_\alpha|N_1, \lambda_1 = -(\lambda+1)\rangle &= \langle R, \lambda|F_-|N_1, \lambda_1 = -(\lambda+1)\rangle \\
&= \langle N_1, \lambda_1 = -(\lambda+1)|(F_-)^\dagger|R, \lambda\rangle^* \\
&= \langle N_1, \lambda_1 = -(\lambda+1)| - F_+|R, \lambda\rangle^* \\
&= -\langle N_1, s_{1z} = (\lambda+1)|F_+|R, s_{Rz} = \lambda\rangle^* \\
&= -f^*_{+|2\lambda|} = -f_{+|2\lambda|} \\
\langle R, \lambda|e^\alpha_R F_\alpha|N_1, \lambda_1 = -(\lambda-1)\rangle &= \langle R, \lambda|F_+|N_1, \lambda_1 = -(\lambda-1)\rangle \\
&= \langle N_1, \lambda_1 = -(\lambda-1)|(F_+)^\dagger|R, \lambda\rangle^* \\
&= \langle N_1, \lambda_1 = -(\lambda-1)| - F_-|R, \lambda\rangle^* \\
&= -\langle N_1, s_{1z} = (\lambda-1)|F_-|R, s_{Rz} = \lambda\rangle^* \\
&= -f^*_{-|2\lambda|} = -f_{-|2\lambda|},
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
\langle R, \lambda = \pm\frac{1}{2}|e^\alpha_- F_\alpha|N_1, \lambda_1 = \mp\frac{1}{2}\rangle &= \langle R, \lambda = \pm\frac{1}{2}|F_0^{(-)}|N_1, \lambda_1 = \mp\frac{1}{2}\rangle \\
&= \langle N_1, \lambda_1 = \mp\frac{1}{2}|(F_0^{(-)})^\dagger|R, \lambda = \pm\frac{1}{2}\rangle^* \\
&= \langle N_1, \lambda_1 = \mp\frac{1}{2}|F_0^{(-)}|R, \lambda = \pm\frac{1}{2}\rangle^* \\
&= \langle N_1, s_{1z} = \pm\frac{1}{2}|F_0^{(-)}|R, s_{Rz} = \pm\frac{1}{2}\rangle^* \\
&= (f_{0\pm}^{(-)})^* = f_{0\pm}^{(-)}, \\
\langle R, \lambda = \pm\frac{1}{2}|e^\alpha_+ F_\alpha|N_1, \lambda_1 = \mp\frac{1}{2}\rangle &= \langle R, \lambda = \pm\frac{1}{2}|F_0^{(+)}|N_1, \lambda_1 = \mp\frac{1}{2}\rangle \\
&= \langle N_1, \lambda_1 = \mp\frac{1}{2}|(F_0^{(+)})^\dagger|R, \lambda = \pm\frac{1}{2}\rangle^* \\
&= \langle N_1, \lambda_1 = \mp\frac{1}{2}|F_0^{(+)}|R, \lambda = \pm\frac{1}{2}\rangle^* \\
&= \langle N_1, s_{1z} = \pm\frac{1}{2}|F_0^{(+)}|R, s_{Rz} = \pm\frac{1}{2}\rangle^* \\
&= (f_{0\pm}^{(+)})^* = f_{0\pm}^{(+)},
\end{aligned} \tag{1.7}$$

where we use the property that RS helicity amplitudes $f_{\pm|2s_z|}$ and $f_{0\pm}$ are real values (see Table II of Ref. [5]). The resonance decay matrix element postulated by Minoo is given by Eq ((4.20)– [2]):

$$\langle N\pi, \lambda_2 | R, \lambda\rangle = \sigma^D C^j_{N\pi}\sqrt{\chi_E}\kappa C^I_{N\pi} f_{\text{BW}}, \tag{1.8}$$

where the meaning of all symbols can be found in Ref. [2].

The relation between the standard helicity amplitudes and helicity amplitudes defined by Eq. (1.1) is given by Eq. ((3.67)– [2]):

$$F_{\mu,\lambda}(\theta,\phi) = e^{i[\lambda_1\pi+\lambda_2(\pi+2\phi)]}\widetilde{F}^{\lambda_k}_{\lambda_2,\lambda_1}(\theta,\phi), \quad G_{\mu,\lambda}(\theta,\phi) = e^{i[\lambda_1\pi+\lambda_2(\pi+2\phi)]}\widetilde{G}^{\lambda_k}_{\lambda_2,\lambda_1}(\theta,\phi), \tag{1.9}$$

where λ is spin of resonance, $\lambda = \lambda_k - \lambda_1$, $\mu = -\lambda_2$, and $\lambda \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$, $\mu \in \{-\frac{1}{2}, \frac{1}{2}\}$.

The multipole expansion of $F_{\mu,\lambda}(\theta,\phi)$:

$$\begin{aligned} F_{\mu,\lambda}(\theta,\phi) &= \sum_j \begin{Bmatrix} F^j_{\mu,\lambda}, & \text{if } \lambda_k \text{ is L or R} \\ F^{0j}_{\mu,\lambda}, & \text{if } \lambda_k \text{ is + or –} \end{Bmatrix}(2j+1)d^j_{\lambda,\mu}(\theta)e^{i(\lambda-\mu)\phi}, \\ G_{\mu,\lambda}(\theta,\phi) &= \sum_j \begin{Bmatrix} G^j_{\mu,\lambda}, & \text{if } \lambda_k \text{ is L or R} \\ G^{0j}_{\mu,\lambda}, & \text{if } \lambda_k \text{ is + or –} \end{Bmatrix}(2j+1)d^j_{\lambda,\mu}(\theta)e^{i(\lambda-\mu)\phi}. \end{aligned} \tag{1.10}$$

By comparing Eqs. (1.9) and (1.10) one obtains:

$$\begin{aligned} \widetilde{F}^{\lambda_k}_{\lambda_2,\lambda_1}(\theta,\phi) &= (-1)^{(\lambda_1+\lambda_2)}\sum_j \begin{Bmatrix} F^j_{\mu,\lambda}, & \text{if } \lambda_k \text{ is L or R} \\ F^{0j}_{\mu,\lambda}, & \text{if } \lambda_k \text{ is + or –} \end{Bmatrix}(2j+1)d^j_{\lambda,\mu}(\theta)e^{i(\lambda_k-\lambda_1-\lambda_2)\phi}, \\ \widetilde{G}^{\lambda_k}_{\lambda_2,\lambda_1}(\theta,\phi) &= (-1)^{(\lambda_1+\lambda_2)}\sum_j \begin{Bmatrix} G^j_{\mu,\lambda}, & \text{if } \lambda_k \text{ is L or R} \\ G^{0j}_{\mu,\lambda}, & \text{if } \lambda_k \text{ is + or –} \end{Bmatrix}(2j+1)d^j_{\lambda,\mu}(\theta)e^{i(\lambda_k-\lambda_1-\lambda_2)\phi}. \end{aligned} \tag{1.11}$$

where the values $F^j_{\mu,\lambda}$, $F^{0j}_{\mu,\lambda}$, $G^j_{\mu,\lambda}$, $G^{0j}_{\mu,\lambda}$ are the coefficients of multipole expansion. The amplitudes $F^j_{\mu,\lambda}$ are defined similarly to Eq. ((30)– [6]) (up to a common factor). The superscript "0“ in Eqs. (1.10) and (1.11) hereinafter is omitted when it does not lead to misunderstanding. The standard helicity amplitudes possess the following symmetry properties:

$$F_{-\mu,-\lambda}(\theta,\phi) = -e^{i(\lambda-\mu)(\pi-2\phi)}F_{\mu,\lambda}(\theta,\phi). \tag{1.12}$$

Substituting Eq. (1.10) into Eq. (1.12) we obtain

$$\begin{aligned} F_{-\mu,-\lambda}(\theta,\phi) &= -e^{i(\lambda-\mu)(\pi-2\phi)}\sum_j F^j_{\mu,\lambda}(2j+1)d^j_{\lambda,\mu}(\theta)e^{i(\lambda-\mu)\phi} \\ &= -e^{i(\lambda-\mu)\pi}\sum_j F^j_{\mu,\lambda}(2j+1)d^j_{\lambda,\mu}(\theta)e^{-i(\lambda-\mu)\phi}. \end{aligned} \tag{1.13}$$

Using the symmetry properties of the functions $d^j_{\lambda\mu}$ (see Eq. ((A1)– [7]))

$$d^j_{\lambda,\mu}(\theta) = d^j_{-\mu,-\lambda}(\theta) = (-1)^{\lambda-\mu}d^j_{\mu,\lambda}(\theta), \tag{1.14}$$

one obtains

$$d^j_{\lambda,\mu}(\theta) = d^j_{-\mu,-\lambda,}(\theta) = (-1)^{\lambda-\mu}d^j_{-\lambda,-\mu}(\theta). \tag{1.15}$$

Then Eq. (1.13) can be rewritten as

$$F_{-\mu,-\lambda}(\theta,\phi) = -\sum_j F^j_{\mu,\lambda}(2j+1)d^j_{-\lambda,-\mu}(\theta)e^{-i(\lambda-\mu)\phi}. \tag{1.16}$$

From the other hand, if one put $-\mu$ and $-\lambda$ instead of μ and λ in Eq. (1.10)

$$F_{-\mu,-\lambda}(\theta,\phi) = \sum_j F^j_{-\mu,-\lambda}(2j+1)d^j_{-\lambda,-\mu}(\theta)e^{-i(\lambda-\mu)\phi}. \tag{1.17}$$

Subtracting Eq. (1.17) from Eq. (1.16) we find:

$$0 = \sum_j (F^j_{-\mu,-\lambda} + F^j_{\mu,\lambda})(2j+1)d^j_{-\lambda,-\mu}(\theta)e^{-i(\lambda-\mu)\phi}. \tag{1.18}$$

Now let's use the orthonormality of the functions

$$\int_0^\pi d^j_{\lambda,\mu}(\theta)d^{j'}_{\lambda,\mu}(\theta)d\,\mathrm{cos}(\theta) = \frac{2}{2j+1}\delta_{jj'}. \tag{1.19}$$

Multiplying Eq. (1.17) by $d^{j'}_{\lambda,\mu}$ and integrating, we obtain

$$F^j_{-\mu,-\lambda} = -F^j_{\mu,\lambda}. \tag{1.20}$$

Studying Table II of Ref [5] one can note that the following expressions are valid:

$$\begin{aligned}
&\text{for } j = l+\frac{1}{2}: \quad f^V_{+1,+3,0+} = -f^V_{-1,-3,0-}, \quad f^A_{+1,+3,0+} = +f^A_{-1,-3,0-},\\
&\text{for } j = l-\frac{1}{2}: \quad f^V_{+1,+3,0+} = +f^V_{-1,-3,0-}, \quad f^A_{+1,+3,0+} = -f^A_{-1,-3,0-}.
\end{aligned} \tag{1.21}$$

We have prepared everything needed to make calculation by both mentioned ways. Let's consider, as an example, the calculation of $\widetilde{F}^L_{\frac{1}{2}\frac{1}{2}}(\theta,\phi)$ for which $\lambda_2 = \frac{1}{2}$, $\lambda_1 = \frac{1}{2}$, $\lambda_k = -1$, $\mu = -\frac{1}{2}$ and $\lambda = -\frac{3}{2}$.

1.1.2 Derivation by direct way

Using Eqs. (1.1)–(1.8) and (1.11) we can find the coefficients of multipole expansion for $\widetilde{F}^{-1}_{\frac{1}{2},\frac{1}{2}}(\theta,\phi)$:

$$\begin{aligned}
F^j_{-\frac{1}{2},-\frac{3}{2}} &= \frac{1}{2M}\langle N\pi, \lambda_2 = \frac{1}{2}|e^\alpha_L J^V_\alpha|N, \lambda_1 = \frac{1}{2}\rangle\\
&= \frac{1}{2M}\langle N\pi, \lambda_2 = \frac{1}{2}|R, \lambda = -\frac{3}{2}\rangle\langle R, \lambda = -\frac{3}{2}|e^\alpha_L J^V_\alpha|N, \lambda_1 = \frac{1}{2}\rangle\\
&= \frac{M_R}{M}\sigma^D C^j_{N\pi}\sqrt{\chi_E}\kappa C^I_{N\pi} f_{\mathrm{BW}}(-f^V_{+3}).
\end{aligned} \tag{1.22}$$

1.1.3 Derivation by using symmetry properties

Using the symmetry property (1.20) for the multipole expansion coefficients, one obtains:

$$F^{j}_{-\frac{1}{2},-\frac{3}{2}} = -F^{j}_{\frac{1}{2},\frac{3}{2}}. \tag{1.23}$$

The values $F^{j}_{\frac{1}{2},\frac{3}{2}}$ are multipole expansion coefficients for the amplitude $\widetilde{F}^{+1}_{-\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$. Then

$$\begin{aligned} F^{j}_{-\frac{1}{2},-\frac{3}{2}} &= -\frac{1}{2M}\langle N\pi, \lambda_2 = -\frac{1}{2}|R, \lambda = \frac{3}{2}\rangle\langle R, \lambda = \frac{3}{2}|e^{\alpha}_{R} J^{V}_{\alpha}|N, \lambda_1 = -\frac{1}{2}\rangle \\ &= \frac{M_R}{M}\sigma^{D} C^{j}_{N\pi}\sqrt{\chi_E}\kappa C^{I}_{N\pi} f_{\mathsf{BW}} f^{V}_{-3}. \end{aligned} \tag{1.24}$$

1.1.4 Paradox and probable explanation

Equations (1.22) and (1.24) are consistent only for the resonances with $j = l + \frac{1}{2}$ (for which $f^{V}_{-3} = -f^{V}_{+3}$) as it can be seen from Eq. (1.21). But there is evident contradiction for the resonances with $j = l - \frac{1}{2}$ (for which $f^{V}_{-3} = +f^{V}_{+3}$).

For example, we present in Table 1.1 the calculation of all amplitudes by the direct method, which, in our opinion, does not lead to any contradictions. The signs, which differ from Minoo's ones, are marked by red color. The latest versions of the amplitudes calculated by Minoo are presented in Table III of erratum [4] and reproduced in Table 1.2. One can see that almost all amplitude signs differ from those calculated by the direct method.

This contradiction can be resolved if one takes into account that the symmetry condition stated by Eqs. ((15a,15b)– [6]) are valid only for the resonances with $j = l + \frac{1}{2}$. Indeed, the symmetry property (1.12) in general case for reaction $a + b \to c + d$ is given by Eq. ((44)– [7]):

$$F_{-\lambda_c,-\lambda_d;-\lambda_a,-\lambda_b}(\theta,\phi) = \eta_g F_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}(\theta,\pi-\phi), \quad \eta_g = \frac{\eta_c\eta_d}{\eta_a\eta_b}(-1)^{s_c+s_d-s_a-s_b}, \tag{1.25}$$

where $\eta_a, \eta_b, \eta_c, \eta_d$ – parity factors and s_a, s_b, s_c, s_d – total angular momentum of particles a, b, c and d. For our case $a = N, b = W^{\pm}, c = N', d = \pi$. When we consider the final state $N'\pi$ with $j = l + \frac{1}{2}$ then $s_N + s_W = \frac{1}{2} + s_W$ (the W-boson hits nucleon at rest) and $s_{N'} + s_\pi = j = l + \frac{1}{2}$; and when the final state has $j = l - \frac{1}{2}$ then $s_{N'} + s_\pi = j = l - \frac{1}{2}$. The factors $\eta_N, \eta_W, \eta_{N'}, \eta_\pi$ are the same for both cases. Therefore

$$\begin{aligned} &\text{for } j = l + \frac{1}{2}: \quad (-1)^{s_{N'}+s_\pi-s_N-s_W} = (-1)^{l+\frac{1}{2}-\frac{1}{2}-s_W} = (-1)^{l-s_W}, \\ &\text{for } j = l - \frac{1}{2}: \quad (-1)^{s_{N'}+s_\pi-s_N-s_W} = (-1)^{l-\frac{1}{2}-\frac{1}{2}-s_W} = (-1)^{l-1-s_W}. \end{aligned}$$

Thus the factors η_g differ in sign for the two cases $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$, i.e. if the following symmetry relation $F^{j}_{-\mu,-\lambda}(\theta,\phi) = F^{j}_{\mu,\lambda}(\theta,\pi-\phi)$ is hold for $j = l+\frac{1}{2}$ then the analogous relation for $j = l - \frac{1}{2}$ is $F^{j}_{-\mu,-\lambda}(\theta,\phi) = -F^{j}_{\mu,\lambda}(\theta,\pi-\phi)$. We account for that $F^{j}_{\mu,\lambda}(\theta,\phi) \equiv F^{j}_{\lambda_q-\lambda_2,\lambda_k-\lambda_1}(\theta,\phi) \equiv F_{\lambda_q,\lambda_2;\lambda_k,\lambda_1}(\theta,\phi)$; the helicity of pion $\lambda_q = 0$, the helicity of initial and final nucleons $\lambda_1 = \lambda_2 = \frac{1}{2}$, the helicity of W-boson $\lambda_k \in \{0, 1, -1\}$ as stated above.

It is difficult to understand which set of the amplitude signs is used in Minoo's code. The problem is that the signs belonging to one terms is sometimes assigned to another term. Considering numerous versions that increased as a result of several corrections, the problem with signs becomes even more complicated than it was before corrections. Several examples of the problem with signs:

- the sign of the amplitude itself, which depends on whether j is equal to $l+\frac{1}{2}$ or $l-\frac{1}{2}$;
- the sign of the functions $d^j_{\lambda,\mu}$, because Minoo coded only $d^j_{\lambda,\mu}$ with $\lambda > 0$ and then she finds other values using the symmetry properties of the d-functions (1.14) and (1.15);
- the sign of σ^D, which sits in the term $\mathcal{D}^j(R)$ (see Eq.(25) of Ref. [3]) and is different from the original Rein's signs [2] (see Table 5.3 in Ref. [1]).

However, one thing is obvious: the amplitude signs in the code definitely differ from those in the erratum [3]. Let's note that it is impossible to control (adjust) the signs by comparison of calculations with experimental data since currently available data are very slowly sensible to these signs. But interference between the high-mass resonances is very responsible to them. This will be important in the future experiments.

Currently, we use the signs as is in Minoo's code. However it poses a problem as it, again, is very difficult (if not impossible) to check whether the signs are correct (see also sections 2.6 and 2.11).

λ_2	λ_1	$\widetilde{F}^L_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{G}^L_{\lambda_2,\lambda_1}(\theta,\phi)$	$\widetilde{F}^R_{\lambda_2,\lambda_1}(\theta,\phi)) - \widetilde{G}^R_{\lambda_2,\lambda_1}(\theta,\phi))$
$\frac{1}{2}$	$\frac{1}{2}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{+3}(R)d^j_{\frac{3}{2}\frac{1}{2}}(\theta)e^{-2i\phi}$	$+\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{-1}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{+3}(R)d^j_{\frac{3}{2}-\frac{1}{2}}(\theta)e^{-i\phi}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{-1}(R)d^j_{\frac{1}{2}\frac{1}{2}}(\theta)e^{i\phi}$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{+1}(R)d^j_{\frac{1}{2}\frac{1}{2}}(\theta)e^{-i\phi}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{-3}(R)d^j_{\frac{3}{2}-\frac{1}{2}}(\theta)e^{i\phi}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{+1}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$+\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f_{-3}(R)d^j_{\frac{3}{2}\frac{1}{2}}(\theta)e^{2i\phi}$
		$\widetilde{F}^-_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{G}^-_{\lambda_2,\lambda_1}(\theta,\phi)$	$\widetilde{F}^+_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{F}^+_{\lambda_2,\lambda_1}(\theta,\phi)$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(-)}_{0-}(R)d^j_{-\frac{1}{2}-\frac{1}{2}}(\theta)e^{-i\phi}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(+)}_{0-}(R)d^j_{-\frac{1}{2}-\frac{1}{2}}(\theta)e^{-i\phi}$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(-)}_{0-}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(+)}_{0-}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(-)}_{0+}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$+\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(+)}_{0+}(R)d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(-)}_{0+}(R)d^j_{\frac{1}{2}\frac{1}{2}}(\theta)e^{i\phi}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{(+)}_{0+}(R)d^j_{\frac{1}{2}\frac{1}{2}}(\theta)e^{i\phi}$

Table 1.1: Vector helicity amplitudes of resonant interactions calculated by direct method.

[2]The decay sign of resonance P$_{11}(1710)$ given in Ref. [5] is equal to "+". It was changed in Ref. [6] and is now equal to "–".

λ_2	λ_1	$\widetilde{F}^{L}_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{G}^{L}_{\lambda_2,\lambda_1}(\theta,\phi)$	$\widetilde{F}^{R}_{\lambda_2,\lambda_1}(\theta,\phi)) - \widetilde{G}^{R}_{\lambda_2,\lambda_1}(\theta,\phi))$
$\frac{1}{2}$	$\frac{1}{2}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{+3}(R) d^j_{\frac{3}{2}\frac{1}{2}}(\theta) e^{-2i\phi}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{-1}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$-\frac{1}{2}$	$\frac{1}{2}$	$\mp\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{+3}(R) d^j_{\frac{3}{2}-\frac{1}{2}}(\theta) e^{-i\phi}$	$\pm\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{-1}(R) d^j_{\frac{1}{2}\frac{1}{2}}(\theta) e^{i\phi}$
$\frac{1}{2}$	$-\frac{1}{2}$	$+\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{+1}(R) d^j_{\frac{1}{2}\frac{1}{2}}(\theta) e^{-i\phi}$	$-\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{-3}(R) d^j_{\frac{3}{2}-\frac{1}{2}}(\theta) e^{i\phi}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\pm\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{+1}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$\pm\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f_{-3}(R) d^j_{\frac{3}{2}\frac{1}{2}}(\theta) e^{2i\phi}$
		$\widetilde{F}^{-}_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{G}^{-}_{\lambda_2,\lambda_1}(\theta,\phi)$	$\widetilde{F}^{+}_{\lambda_2,\lambda_1}(\theta,\phi) - \widetilde{F}^{+}_{\lambda_2,\lambda_1}(\theta,\phi)$
$\frac{1}{2}$	$\frac{1}{2}$	$\mp\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(-)}_{0-}(R) d^j_{-\frac{1}{2}-\frac{1}{2}}(\theta) e^{-i\phi}$	$\mp\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(+)}_{0-}(R) d^j_{-\frac{1}{2}-\frac{1}{2}}(\theta) e^{-i\phi}$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(-)}_{0-}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(+)}_{0-}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$\frac{1}{2}$	$-\frac{1}{2}$	$\pm\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(-)}_{0+}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$	$\pm\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(+)}_{0+}(R) d^j_{\frac{1}{2}-\frac{1}{2}}(\theta)$
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(-)}_{0+}(R) d^j_{\frac{1}{2}\frac{1}{2}}(\theta) e^{i\phi}$	$-\frac{\lvert\mathbf{k}\rvert}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R) f^{(+)}_{0+}(R) d^j_{\frac{1}{2}\frac{1}{2}}(\theta) e^{i\phi}$

Table 1.2: Vector helicity amplitudes of resonant interactions calculated by using symmetry properties and presented in Table III of Ref. [4].

1.2 Problem with transition from neutrino to antineutrino case

1.2.1 Derivation of the transition rules

Since this problem is one of the key points, we consider it in some detail. Let's start with a reminder of several basic formulas derived in Ref. [8]. The components of the leptonic current with the lepton helicity λ measured in lab. frame, can be written in the resonance rest frame (RRF)[3] as

$$
\begin{aligned}
j_0^\star &= N_\lambda m_\ell \frac{E_\nu}{W}(1-\lambda\cos\theta)\left(M-E_\ell-\lambda P_\ell\right), \\
j_x^\star &= N_\lambda m_\ell \frac{E_\nu}{|\mathbf{q}|}\sin\theta\left(P_\ell-\lambda E_\nu\right), \\
j_y^\star &= i\lambda N_\lambda m_\ell E_\nu \sin\theta, \\
j_z^\star &= N_\lambda m_\ell \frac{E_\nu}{|\mathbf{q}|\,W}(1-\lambda\cos\theta)\left[\left(E_\nu+\lambda P_\ell\right)\left(M-E_\ell\right)+P_\ell\left(\lambda E_\nu+2E_\nu\cos\theta-P_\ell\right)\right].
\end{aligned}
\tag{1.26}
$$

Here E_ν, E_ℓ, θ, and m_ℓ are, respectively, the incident neutrino energy, lepton energy, scattering angle, and mass of the lepton, $P_\ell=\sqrt{E_\ell^2-m_\ell^2}$, $\mathbf{q}=\mathbf{k}_\nu-\mathbf{k}_\ell$, the vectors $\mathbf{k}_\nu$ and $\mathbf{k}_\ell$ are the 3-momenta of the neutrino and lepton, respectively. All variables are written in lab. frame. The normalization constant N_λ is expressed in terms of the kinematic variables,

$$
N_\lambda = \frac{1}{m_\ell}\sqrt{\frac{E_\ell \mp \lambda P_\ell}{E_\nu\left(1\mp\lambda\cos\theta\right)}},
\tag{1.27}
$$

where the upper (lower) sign is for ν_ℓ ($\overline{\nu}_\ell$).

On the other hand, in the spirit of the RS model, the leptonic current can be treated as the intermediate W boson polarization 4-vector and may be decomposed into three polarization 4-vectors e_L, e_R, and $e_S(\lambda)\equiv e_{(\lambda)}$ corresponding to left-handed, right-handed and scalar polarizations:

$$
j_\lambda^\alpha = K^{-1}\left[c_L^\lambda e_L^\alpha + c_R^\lambda e_R^\alpha + c_S^\lambda e_{(\lambda)}^\alpha\right],
\tag{1.28}
$$

$$
\begin{aligned}
e_L^\alpha &= \frac{1}{\sqrt{2}}(0,1,-i,0), \\
e_R^\alpha &= \frac{1}{\sqrt{2}}(0,-1,-i,0), \\
e_{(\lambda)}^\alpha &= \frac{1}{\sqrt{Q^2}}\left(\mathcal{Q}_{(\lambda)}^\star,0,0,\nu_{(\lambda)}^\star\right).
\end{aligned}
\tag{1.29}
$$

Note that the 4-vectors e_L and e_R are exactly the same as in the RS model [5], while $e_{(\lambda)}$ has been modified to include the lepton mass effect; its components are given by

$$
\mathcal{Q}_{(\lambda)}^\star = \frac{K\sqrt{Q^2}}{c_S^\lambda}j_0^\star, \quad \nu_{(\lambda)}^\star = \frac{K\sqrt{Q^2}}{c_S^\lambda}j_z^\star, \quad K=\frac{|\mathbf{q}|}{E_\nu\sqrt{2Q^2}}.
\tag{1.30}
$$

[3]Here and below we mark this frame with asterisk ($\star$).

The polarization vectors form an orthonormal set:

$$e_i^\alpha e_{j\alpha} = \delta_{ij}, \quad i,j = L,R,S.$$

Taking this into account, the coefficients c_i^λ are explicitly defined through the components j_α^λ in RRF as

$$c_L^\lambda = \frac{K}{\sqrt{2}}\left(j_x^\star + i j_y^\star\right), \quad c_R^\lambda = -\frac{K}{\sqrt{2}}\left(j_x^\star - i j_y^\star\right), \quad c_S^\lambda = K\sqrt{|(j_0^\star)^2 - (j_z^\star)^2|}. \tag{1.31}$$

The sign in the last relation is chosen to be positive, by definition. The currents for neutrino and antineutrino can be related with Eqs. (2) and (3) in Ref. [5]

$$\overline{j}_\alpha^{\,\lambda} = -\lambda\left(j_\alpha^{-\lambda}\right)^*. \tag{1.32}$$

This identity is an intrinsic property of the leptonic current and does not depend on the nature of the process in which it participates. By sing Eqs. (1.26) and (1.32) we can write

$$\begin{aligned}
\left[j_0^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda\left[j_0^{\star-\lambda}\right]_\nu, \\
\left[j_x^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda\left[j_x^{\star-\lambda}\right]_\nu, \\
\left[j_y^{\star\lambda}\right]_{\overline{\nu}} &= +\lambda\left[j_y^{\star-\lambda}\right]_\nu, \\
\left[j_z^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda\left[j_z^{\star-\lambda}\right]_\nu.
\end{aligned} \tag{1.33}$$

The identities (1.31) and (1.33) allow us to derive relations between the coefficients c_L, c_R and c_S for the neutrino and antineutrino cases:

$$\begin{aligned}
\left[c_L^\lambda\right]_{\overline{\nu}} &= \frac{K}{\sqrt{2}}\left(\left[j_x^{\star\lambda}\right]_{\overline{\nu}} + i\left[j_y^{\star\lambda}\right]_{\overline{\nu}}\right) = \frac{K}{\sqrt{2}}\left(-\lambda\left[j_x^{\star-\lambda}\right]_{\overline{\nu}} + i\lambda\left[j_y^{\star-\lambda}\right]_{\overline{\nu}}\right) = \lambda\left[c_R^{-\lambda}\right]_\nu, \\
\left[c_R^\lambda\right]_{\overline{\nu}} &= -\frac{K}{\sqrt{2}}\left(\left[j_x^{\star\lambda}\right]_{\overline{\nu}} - i\left[j_y^{\star\lambda}\right]_{\overline{\nu}}\right) = -\frac{K}{\sqrt{2}}\left(-\lambda\left[j_x^{\star-\lambda}\right]_{\overline{\nu}} - i\lambda\left[j_y^{\star-\lambda}\right]_{\overline{\nu}}\right) = \lambda\left[c_L^{-\lambda}\right]_\nu, \\
\left[c_S^\lambda\right]_{\overline{\nu}} &= K\sqrt{\left|(\left[j_0^{\star\lambda}\right]_{\overline{\nu}})^2 - (\left[j_z^{\star\lambda}\right]_{\overline{\nu}})^2\right|} = K\sqrt{\left|(-\lambda\left[j_0^{\star-\lambda}\right]_\nu)^2 - (-\lambda\left[j_z^{\star-\lambda}\right]_\nu)^2\right|} \\
&= \pm\lambda K\sqrt{\left|(\left[j_0^{\star-\lambda}\right]_\nu)^2 - (\left[j_z^{\star-\lambda}\right]_\nu)^2\right|} = \pm\lambda\left[c_S^{-\lambda}\right]_\nu.
\end{aligned}$$

Considering that

$$\begin{aligned}
K^{-1}\left[c_S^\lambda\right]_{\overline{\nu}}\left[e_{(\lambda)}^\alpha\right]_{\overline{\nu}} &= \left(\left[j_0^{\star\lambda}\right]_{\overline{\nu}}, 0, 0, \left[j_z^{\star\lambda}\right]_{\overline{\nu}}\right) = -\lambda\left(\left[j_0^{\star-\lambda}\right]_\nu, 0, 0, \left[j_z^{\star-\lambda}\right]_\nu\right) \\
&= -\lambda K^{-1}\left[c_S^{-\lambda}\right]_\nu\left[e_{(-\lambda)}^\alpha\right]_\nu,
\end{aligned}$$

we obtain

$$\left[c_S^\lambda\right]_{\overline{\nu}}\left[e_{(\lambda)}^\alpha\right]_{\overline{\nu}} = -\lambda\left[c_S^{-\lambda}\right]_\nu\left[e_{(-\lambda)}^\alpha\right]_\nu. \tag{1.34}$$

To derive the exact relation for the coefficient c_S^λ, let's consider the limiting case of a massless final-state lepton. Since the massless lepton helicity can only be -1 (for incoming neutrino) or +1 (for incoming antineutrino), we have

$$\left.\begin{aligned}
\left[\mathcal{Q}_{(-1)}^\star\right]_\nu \to |\mathbf{q}^\star| \quad \text{and} \quad \left[\nu_{(-1)}^\star\right]_\nu \to \nu^\star \\
\left[\mathcal{Q}_{(+1)}^\star\right]_{\overline{\nu}} \to |\mathbf{q}^\star| \quad \text{and} \quad \left[\nu_{(+1)}^\star\right]_{\overline{\nu}} \to \nu^\star
\end{aligned}\right\} \quad \text{as} \quad m_\ell \to 0.$$

This implies

$$\left[\mathcal{Q}^{\star}_{(-1)}\right]_{\nu} \to \left[\mathcal{Q}^{\star}_{(+1)}\right]_{\overline{\nu}} \quad \text{and} \quad \left[\nu^{\star}_{(-1)}\right]_{\nu} \to \left[\nu^{\star}_{(+1)}\right]_{\overline{\nu}} \quad \text{as} \quad m_\ell \to 0. \tag{1.35}$$

If the equality $\left[c_S^{\lambda}\right]_{\overline{\nu}} = \lambda\left[c_S^{-\lambda}\right]_{\nu}$ is valid then, according to Eq. (1.34),

$$\left[e^{\alpha}_{(\lambda)}\right]_{\overline{\nu}} = -\left[e^{\alpha}_{(-\lambda)}\right]_{\nu}. \tag{1.36}$$

In the opposite case, from the equality $\left[c_S^{\lambda}\right]_{\overline{\nu}} = -\lambda\left[c_S^{-\lambda}\right]_{\nu}$ it follows that

$$\left[e^{\alpha}_{(\lambda)}\right]_{\overline{\nu}} = \left[e^{\alpha}_{(-\lambda)}\right]_{\nu}. \tag{1.37}$$

Since Eq. (1.36) contradicts the conditions (1.35), only the equality $\left[c_S^{\lambda}\right]_{\overline{\nu}} = -\lambda\left[c_S^{-\lambda}\right]_{\nu}$ and Eq. (1.37) is correct, from which it follows that

$$\left[\mathcal{Q}^{\star}_{(\lambda)}\right]_{\overline{\nu}} = \left[\mathcal{Q}^{\star}_{(-\lambda)}\right]_{\nu}, \quad \left[\nu^{\star}_{(\lambda)}\right]_{\overline{\nu}} = \left[\nu^{\star}_{(-\lambda)}\right]_{\nu}.$$

As a result, we get

$$\left[\mathcal{Q}^{\star}_{(\lambda)}\right]_{\overline{\nu}} = \left[\mathcal{Q}^{\star}_{(-\lambda)}\right]_{\nu}, \quad \left[\nu^{\star}_{(\lambda)}\right]_{\overline{\nu}} = \left[\nu^{\star}_{(-\lambda)}\right]_{\nu}, \tag{1.38}$$

$$\left[c_L^{\lambda}\right]_{\overline{\nu}} = \lambda\left[c_R^{-\lambda}\right]_{\nu}, \quad \left[c_R^{\lambda}\right]_{\overline{\nu}} = \lambda\left[c_L^{-\lambda}\right]_{\nu}, \quad \left[c_S^{\lambda}\right]_{\overline{\nu}} = -\lambda\left[c_S^{-\lambda}\right]_{\nu}. \tag{1.39}$$

We want to stress that Eqs. (1.38) and (1.39) were obtained by using only the properties of the leptonic current and do not depend on features of the hadronic current.

Minoo works in the similar framework. The differences between her notations and those used in Ref. [8] are:

- leptonic current denoted as $\varepsilon^{\alpha}_{\lambda}$ is equal to $2j^{\lambda}_{\alpha}$ (compare Eq. (1) in Ref. [8] and Eq. (5) in Ref. [3]) and is expressed in terms of kinematic variables defined in RRF, while in Ref. [8] it is expressed through the lab. frame kinematic variables;
- The coefficients c_L^{λ}, c_R^{λ}, and c_S^{λ} used in Ref. [8] are denoted in Minoo's paper as C_{L_λ}, C_{R_λ}, and C_λ, respectively;
- the polarization vectors e^{α}_L, e^{α}_R, and $e^{\alpha}_{(\lambda)}$ used in Ref. [8] are denoted as, respectively, e^{α}_L, e^{α}_R, and e^{α}_{λ} in Minoo's paper.

The corresponding values defined in Minoo's paper [3] are following:

$$\begin{aligned}
e^{\alpha}_L &= \frac{1}{\sqrt{2}}\left(0, \;\; 1, \;\; -i, \;\; 0\right), \\
e^{\alpha}_R &= \frac{1}{\sqrt{2}}\left(0, \;\; -1, \;\; -i, \;\; 0\right), \\
e^{\alpha}_{\lambda} &= \frac{1}{\sqrt{|(\varepsilon^0_\lambda)^2 - (\varepsilon^3_\lambda)^2|}}\left(\varepsilon^0_\lambda, \;\; 0, \;\; 0, \;\; \varepsilon^3_\lambda\right).
\end{aligned} \tag{1.40}$$

and

$$\begin{aligned}
C_{L_\lambda} &= \frac{1}{\sqrt{2}}\left(\varepsilon^1_\lambda + i\varepsilon^2_\lambda\right), \\
C_{R_\lambda} &= -\frac{1}{\sqrt{2}}\left(\varepsilon^1_\lambda - i\varepsilon^2_\lambda\right), \\
C_\lambda &= \sqrt{|(\varepsilon^0_\lambda)^2 - (\varepsilon^3_\lambda)^2|}.
\end{aligned} \tag{1.41}$$

It can be seen that the values e_L^α and e_R^α from Refs. [3] and [8] are identical. Taking into account Eqs. (1.28)–(1.31), (1.40), (1.41) and $\varepsilon_\lambda^\alpha = 2j_\alpha^\lambda$ we can relate the corresponding values:

$$C_{L_\lambda} e_L^\alpha + C_{R_\lambda} e_R^\alpha + C_\lambda e_\lambda^\alpha = 2K^{-1}\left[c_L^\lambda e_L^\alpha + c_R^\lambda e_R^\alpha + c_S^\lambda e_{(\lambda)}^\alpha\right]$$
$$\Downarrow$$
$$C_{L_\lambda} = 2K^{-1} c_L^\lambda, \quad C_{R_\lambda} = 2K^{-1} c_R^\lambda, \quad C_\lambda e_\lambda^\alpha = 2K^{-1} c_S^\lambda e_{(\lambda)}^\alpha,$$
$$C_\lambda = \sqrt{|(\varepsilon_\lambda^0)^2 - (\varepsilon_\lambda^3)^2|} = 2K^{-1}K\sqrt{|(j_0^\star)^2 - (j_z^\star)^2|} = 2K^{-1} c_S^\lambda$$
$$\Downarrow$$
$$e_\lambda^\alpha = e_{(\lambda)}^\alpha.$$

Thus we obtained

$$C_{L_\lambda} = 2K^{-1} c_L^\lambda, \quad C_{R_\lambda} = 2K^{-1} c_R^\lambda, \quad C_\lambda = 2K^{-1} c_S^\lambda, \quad e_\lambda^\alpha = e_{(\lambda)}^\alpha. \tag{1.42}$$

Therefore the relation similar to (1.39) must be hold:

$$\left[C_{L_\lambda}\right]_{\overline{\nu}} = \lambda\left[C_{R_{-\lambda}}\right]_\nu, \quad \left[C_{R_\lambda}\right]_{\overline{\nu}} = \lambda\left[C_{L_{-\lambda}}\right]_\nu, \quad \left[C_{S_\lambda}\right]_{\overline{\nu}} = -\lambda\left[C_{S_{-\lambda}}\right]_\nu. \tag{1.43}$$

For example, let us prove the first relation:

$$\left[C_{L_\lambda}\right]_{\overline{\nu}} = 2K^{-1}\left[c_L^\lambda\right]_{\overline{\nu}} = 2K^{-1}\lambda\left[c_R^{-\lambda}\right]_\nu = \lambda\left[C_{R_{-\lambda}}\right]_\nu.$$

However, in Minoo's paper it is proposed to use different relation (see words after Eq. (18) in Ref. [3] or words after Eq. (3.60) in Ref. [2]):

$$\left[C_{L_\lambda}\right]_{\overline{\nu}} = \left[C_{R_\lambda}\right]_\nu, \quad \left[C_{R_\lambda}\right]_{\overline{\nu}} = \left[C_{L_\lambda}\right]_\nu, \quad \left[C_{S_\lambda}\right]_{\overline{\nu}} = \left[C_{S_\lambda}\right]_\nu. \tag{1.44}$$

The last conditions can be rewritten in terms of Ref. [8] as follows

$$\left[c_L^\lambda\right]_{\overline{\nu}} = \left[c_R^\lambda\right]_\nu, \quad \left[c_R^\lambda\right]_{\overline{\nu}} = \left[c_L^\lambda\right]_\nu, \quad \left[c_S^\lambda\right]_{\overline{\nu}} = \left[c_S^\lambda\right]_\nu, \tag{1.45}$$

and one should apply this "recipe" to final formula for the differential cross section. This means that one can do all calculation exactly as as for the neutrino case and then replace the coefficients c_i^λ for neutrino by ones for antineutrino.

1.2.2 Application to the cross sections

Let's see what this leads to. First, we study whether there is a difference between the double-differential cross section $d\sigma/dW dQ^2$ and the polarization density matrix

$$\boldsymbol{\rho} = \frac{1}{2}\left(1 + \boldsymbol{\sigma}\boldsymbol{\mathcal{P}}\right)$$

obtained by using the relations (1.39) and (1.45) (or (1.43) and (1.44)) in the formalism developed in Ref. [8]. Again let us remind all needed points here.

The elements of the polarization matrix are given by following formulas (see Ref. [8]):

$$\rho_{\lambda\lambda'} = \frac{\Sigma_{\lambda\lambda'}}{\Sigma_{++} + \Sigma_{--}}, \quad \Sigma_{\lambda\lambda'} = \sum_{i=L,R,S} c_i^\lambda c_i^{\lambda'} \sigma_i^{\lambda\lambda'}, \tag{1.46}$$

and the differential cross section of unpolarized lepton production is given by

$$\frac{d^2\sigma}{dQ^2 dW^2} = \frac{G_F^2 \cos^2\theta_C Q^2}{2\pi^2 M \left|\mathbf{q}\right|^2} \left(\varSigma_{++} + \varSigma_{--}\right). \tag{1.47}$$

The partial cross sections are found to be the bilinear superpositions of the reduced amplitudes for producing a $N\pi$ final state with allowed isospin by a charged isovector current:

$$\sigma_{L,R}^{\lambda\lambda'} = \frac{\pi W}{2M} \left(A_{\pm 3}^{\lambda} A_{\pm 3}^{\lambda'} + A_{\pm 1}^{\lambda} A_{\pm 1}^{\lambda'}\right), \tag{1.48}$$

$$\sigma_{S}^{\lambda\lambda'} = \frac{\pi M \left|\mathbf{q}\right|^2}{2WQ^2} \left(A_{0+}^{\lambda} A_{0+}^{\lambda'} + A_{0-}^{\lambda} A_{0-}^{\lambda'}\right). \tag{1.49}$$

The amplitudes for neutrino induced reactions are

$$A_{\varkappa}^{\lambda}\left(p\pi^+\right) = \sqrt{3} \sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(N_3^*\right), \tag{1.50}$$

$$A_{\varkappa}^{\lambda}\left(p\pi^0\right) = \sqrt{\tfrac{2}{3}} \sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(N_3^*\right) - \sqrt{\tfrac{1}{3}} \sum_{(I=1/2)} a_{\varkappa}^{\lambda}\left(N_1^*\right), \tag{1.51}$$

$$A_{\varkappa}^{\lambda}\left(n\pi^+\right) = \sqrt{\tfrac{1}{3}} \sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(N_3^*\right) + \sqrt{\tfrac{2}{3}} \sum_{(I=1/2)} a_{\varkappa}^{\lambda}\left(N_1^*\right). \tag{1.52}$$

Here $\varkappa = \pm 3,\ \pm 1,\ 0\pm$ and only those resonances are allowed to interfere which have the same spin and orbital angular momentum.

Any amplitude $a_{\varkappa}^{\lambda}\left(N_\imath^*\right)$ referring to one single resonance $N_\imath^*$ in a definite state of isospin, charge and helicity consists of two factors which describe the production and subsequent decay of the resonance:

$$a_{\varkappa}^{\lambda}\left(N_\imath^*\right) = f_{\varkappa}^{\lambda}\left(\nu N \to N_\imath^*\right)\, \eta(N_\imath^* \to N\pi) \equiv f_{\varkappa}^{\lambda(\imath)}\, \eta^{(\imath)}.$$

The decay amplitudes, $\eta^{(\imath)}$, can be split into three factors,

$$\eta^{(\imath)} = \operatorname{sign}(N_\imath^*)\, \sqrt{\chi_\imath}\, \eta_{\mathsf{BW}}^{(\imath)}(W),$$

irrespective of isospin, charge or helicity of the resonance. Here, the first factor is the decay sign for resonance $N_\imath^*$ (see Table III of Ref. [5]), $\chi_\imath$ is the elasticity of the resonance taking care of the branching ratio into the πN final state and $\eta_{\mathsf{BW}}^{(\imath)}(W)$ is the properly normalized Breit-Wigner term with the running width specified by the πN partial wave from which the resonance arises (Eq. (2.31) in Ref. [5]).

The resonance production amplitudes, $f_{\varkappa}^{\lambda(\imath)}$, can be calculated within the FKR quark model in exactly the same way as in Ref. [5]. It can be shown that they have the same structure as that given in Table II of Ref. [5] with the only important difference: the three coefficient functions S, B and C involved into the definitions of the amplitudes have to be modified. Indeed, since the structure of the polarization 4-vector $e_{(\lambda)}^{\alpha}$ has been changed with respect to that of the original RS model (by including the lepton mass and spin), we have to recalculate its inner products with the vector and axial hadronic currents. To do this, we used the explicit form for the FKR currents given by Ravndal [9]. As a result, the coefficients S, B and C (and thus the resonance production amplitudes) become parametricaly dependent of the lepton mass and helicity:

$$
\begin{aligned}
S_{(\lambda)} = S^V_{(\lambda)} = &\ \left(\nu^\star_{(\lambda)}\nu^\star - \mathcal{Q}^\star_{(\lambda)}\left|\mathbf{q}^\star\right|\right)\left(1+\frac{Q^2}{M^2}-\frac{3W}{M}\right)\frac{G^V\left(Q^2\right)}{6\left|\mathbf{q}\right|^2},\\
B_{(\lambda)} = B^A_{(\lambda)} = &\ \sqrt{\frac{\Omega}{2}}\left(\mathcal{Q}^\star_{(\lambda)}+\nu^\star_{(\lambda)}\frac{\left|\mathbf{q}^\star\right|}{aM}\right)\frac{ZG^A\left(Q^2\right)}{3W\left|\mathbf{q}^\star\right|},\\
C_{(\lambda)} = C^A_{(\lambda)} = &\ \left[\left(\mathcal{Q}^\star_{(\lambda)}\left|\mathbf{q}^\star\right|-\nu^\star_{(\lambda)}\nu^\star\right)\left(\frac{1}{3}+\frac{\nu^\star}{aM}\right)+\nu^\star_{(\lambda)}\left(\frac{2}{3}W-\frac{Q^2}{aM}+\frac{n\Omega}{3aM}\right)\right]\frac{ZG^A\left(Q^2\right)}{2W\left|\mathbf{q}^\star\right|}.
\end{aligned}
$$

Here

$$
\nu^\star = E^\star_\nu - E^\star_\ell = \frac{M\nu - Q^2}{W}, \quad a = 1+\frac{W^2+Q^2+M^2}{2MW},
$$

$G^{V,A}\left(Q^2\right)$ are the vector and axial transition form factors and the remaining notation is explained in Ref. [5]. Other 5 coefficients listed in Eq. (3.11) of Ref. [5] are left unchanged.

The resonance production amplitudes, $f^\lambda_\varkappa$ with $\varkappa = \pm 3, \pm 1$ depend on neither final lepton helicity nor initial lepton (neutrino or antineutrino)[4]:

$$
\left[f^\lambda_\varkappa\right]_\nu = \left[f^{-\lambda}_\varkappa\right]_\nu = \left[f^\lambda_\varkappa\right]_{\overline{\nu}} = \left[f^{-\lambda}_\varkappa\right]_{\overline{\nu}}, \quad \varkappa = \pm 3, \pm 1. \tag{1.53}
$$

However, it is not so for $f^\lambda_{0\pm}$. Indeed, to find the amplitude $f^\lambda_{0\pm}$ one need use the values of $S_{(\lambda)}$, $B_{(\lambda)}$ and $C_{(\lambda)}$. Due to (1.38) we have

$$
\left[S_{(\lambda)}\right]_{\overline{\nu}} = \left[S_{(-\lambda)}\right]_\nu, \quad \left[B_{(\lambda)}\right]_{\overline{\nu}} = \left[B_{(-\lambda)}\right]_\nu, \quad \left[C_{(\lambda)}\right]_{\overline{\nu}} = \left[C_{(-\lambda)}\right]_\nu. \tag{1.54}
$$

Consequently

$$
\left[f^\lambda_{0\pm}\right]_\nu = \left[f^{-\lambda}_{0\pm}\right]_{\overline{\nu}}. \tag{1.55}
$$

Using Eqs. (1.53) and (1.55) we obtain

$$
\left[a^\lambda_\varkappa\right]_\nu = \left[a^{-\lambda}_\varkappa\right]_\nu = \left[a^\lambda_\varkappa\right]_{\overline{\nu}} = \left[a^{-\lambda}_\varkappa\right]_{\overline{\nu}}, \quad \varkappa = \pm 3, \pm 1 \tag{1.56}
$$

$$
\left[a^\lambda_\varkappa\right]_\nu = \left[a^{-\lambda}_\varkappa\right]_{\overline{\nu}}, \quad \varkappa = 0\pm, \tag{1.57}
$$

and accounting for (1.50)

$$
\begin{aligned}
&\left[A^\lambda_\varkappa\right]_\nu = \left[A^{-\lambda}_\varkappa\right]_\nu = \left[A^\lambda_\varkappa\right]_{\overline{\nu}} = \left[A^{-\lambda}_\varkappa\right]_{\overline{\nu}}, \quad \varkappa = \pm 3, \pm 1\\
&\left[A^\lambda_\varkappa\right]_\nu = \left[A^{-\lambda}_\varkappa\right]_{\overline{\nu}}, \quad \varkappa = 0\pm\,.
\end{aligned}
$$

For the partial cross sections (1.48) the following identities are hold:

$$
\begin{aligned}
&\left[\sigma^{++}_{L,R}\right]_\nu = \left[\sigma^{+-}_{L,R}\right]_\nu = \left[\sigma^{-+}_{L,R}\right]_\nu = \left[\sigma^{--}_{L,R}\right]_\nu =\\
&\left[\sigma^{++}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{+-}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{-+}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{--}_{L,R}\right]_{\overline{\nu}},\\
&\left[\sigma^{++}_S\right]_\nu = \left[\sigma^{--}_S\right]_{\overline{\nu}}, \left[\sigma^{--}_S\right]_\nu = \left[\sigma^{++}_S\right]_{\overline{\nu}},\\
&\left[\sigma^{+-}_S\right]_\nu = \left[\sigma^{-+}_S\right]_\nu = \left[\sigma^{+-}_S\right]_{\overline{\nu}} = \left[\sigma^{-+}_S\right]_{\overline{\nu}}.
\end{aligned} \tag{1.58}
$$

[4]The explicit form of the amplitudes can be found in Ref. [5].

Next (see Eq. (1.46))

$$[\Sigma_{++}+\Sigma_{--}]_\nu = \left[c_L^+\right]_\nu^2\left[\sigma_L^{++}\right]_\nu + \left[c_R^+\right]_\nu^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^+\right]_\nu^2\left[\sigma_S^{++}\right]_\nu + \left[c_L^-\right]_\nu^2\left[\sigma_L^{--}\right]_\nu + \left[c_R^-\right]_\nu^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^-\right]_\nu^2\left[\sigma_S^{--}\right]_\nu \tag{1.59}$$

Using the transition conditions (1.39) and (1.58)

$$\begin{aligned}[\Sigma_{++}+\Sigma_{--}]_{\overline{\nu}} = {} & \left[c_L^+\right]_{\overline{\nu}}^2\left[\sigma_L^{++}\right]_{\overline{\nu}} + \left[c_R^+\right]_{\overline{\nu}}^2\left[\sigma_R^{++}\right]_{\overline{\nu}} + \left[c_S^+\right]_{\overline{\nu}}^2\left[\sigma_S^{++}\right]_{\overline{\nu}} + \\ & \left[c_L^-\right]_{\overline{\nu}}^2\left[\sigma_L^{--}\right]_{\overline{\nu}} + \left[c_R^-\right]_{\overline{\nu}}^2\left[\sigma_R^{--}\right]_{\overline{\nu}} + \left[c_S^-\right]_{\overline{\nu}}^2\left[\sigma_S^{--}\right]_{\overline{\nu}} = \\ & \left[c_R^-\right]_\nu^2\left[\sigma_L^{++}\right]_\nu + \left[c_L^-\right]_\nu^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^-\right]_\nu^2\left[\sigma_S^{--}\right]_\nu + \\ & \left[c_R^+\right]_\nu^2\left[\sigma_L^{--}\right]_\nu + \left[c_L^+\right]_\nu^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^+\right]_\nu^2\left[\sigma_S^{++}\right]_\nu .\end{aligned} \tag{1.60}$$

Using Minoo's "recipe" (1.45) one finds

$$\begin{aligned}[\Sigma_{++}+\Sigma_{--}]_{\overline{\nu}} = {} & \left[c_L^+\right]_{\overline{\nu}}^2\left[\sigma_L^{++}\right]_\nu + \left[c_R^+\right]_{\overline{\nu}}^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^+\right]_{\overline{\nu}}^2\left[\sigma_S^{++}\right]_\nu + \\ & \left[c_L^-\right]_{\overline{\nu}}^2\left[\sigma_L^{--}\right]_\nu + \left[c_R^-\right]_{\overline{\nu}}^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^-\right]_{\overline{\nu}}^2\left[\sigma_S^{--}\right]_\nu = \\ & \left[c_R^+\right]_\nu^2\left[\sigma_L^{++}\right]_\nu + \left[c_L^+\right]_\nu^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^+\right]_\nu^2\left[\sigma_S^{++}\right]_\nu + \\ & \left[c_R^-\right]_\nu^2\left[\sigma_L^{--}\right]_\nu + \left[c_L^-\right]_\nu^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^-\right]_\nu^2\left[\sigma_S^{--}\right]_\nu ,\end{aligned} \tag{1.61}$$

what at first glance is different from Eq. (1.60), but again using the relations (1.58) we obtain

$$\begin{aligned}[\Sigma_{++}+\Sigma_{--}]_{\overline{\nu}} = {} & \left[c_R^+\right]_\nu^2\left[\sigma_L^{++}\right]_\nu + \left[c_L^+\right]_\nu^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^+\right]_\nu^2\left[\sigma_S^{++}\right]_\nu + \\ & \left[c_R^-\right]_\nu^2\left[\sigma_L^{--}\right]_\nu + \left[c_L^-\right]_\nu^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^-\right]_\nu^2\left[\sigma_S^{--}\right]_\nu = \\ & \left[c_R^+\right]_\nu^2\left[\sigma_L^{--}\right]_\nu + \left[c_L^+\right]_\nu^2\left[\sigma_R^{--}\right]_\nu + \left[c_S^+\right]_\nu^2\left[\sigma_S^{--}\right]_\nu + \\ & \left[c_R^-\right]_\nu^2\left[\sigma_L^{++}\right]_\nu + \left[c_L^-\right]_\nu^2\left[\sigma_R^{++}\right]_\nu + \left[c_S^-\right]_\nu^2\left[\sigma_S^{++}\right]_\nu .\end{aligned} \tag{1.62}$$

It can be seen that Eqs. (1.60) and (1.62) are coincide. Thus differential cross section of unpolarized lepton production (1.47) does not depend on what transition conditions is used.

Now, by applying transition conditions (1.39) and (1.58) we get:

$$\begin{aligned}[\Sigma_{\lambda\lambda'}]_{\overline{\nu}} = {} & \left[c_L^\lambda\right]_{\overline{\nu}}\left[c_L^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_L^{\lambda\lambda'}\right]_{\overline{\nu}} + \left[c_R^\lambda\right]_{\overline{\nu}}\left[c_R^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_R^{\lambda\lambda'}\right]_{\overline{\nu}} + \left[c_S^\lambda\right]_{\overline{\nu}}\left[c_S^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_S^{\lambda\lambda'}\right]_{\overline{\nu}} = \\ & \lambda\lambda'\left(\left[c_R^{-\lambda}\right]_\nu\left[c_R^{-\lambda'}\right]_\nu\left[\sigma_L^{\lambda\lambda'}\right]_\nu + \left[c_L^{-\lambda}\right]_\nu\left[c_L^{-\lambda'}\right]_\nu\left[\sigma_R^{\lambda\lambda'}\right]_\nu + \left[c_S^{-\lambda}\right]_\nu\left[c_S^{-\lambda'}\right]_\nu\left[\sigma_S^{-\lambda-\lambda'}\right]_\nu\right).\end{aligned} \tag{1.63}$$

But, according to Eqs. (1.45), Minoo's "recipe" gives different (in fact wrong) result:

$$[\Sigma_{\lambda\lambda'}]_{\overline{\nu}} = \left[c_L^\lambda\right]_{\overline{\nu}}\left[c_L^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_L^{\lambda\lambda'}\right]_\nu + \left[c_R^\lambda\right]_{\overline{\nu}}\left[c_R^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_R^{\lambda\lambda'}\right]_\nu + \left[c_S^\lambda\right]_{\overline{\nu}}\left[c_S^{\lambda'}\right]_{\overline{\nu}}\left[\sigma_S^{\lambda\lambda'}\right]_\nu, \text{ i.e.}$$

$$[\Sigma_{\lambda\lambda'}]_{\overline{\nu},\text{Minoo's rule}} = +1\left(\left[c_R^\lambda\right]_\nu\left[c_R^{\lambda'}\right]_\nu\left[\sigma_L^{\lambda\lambda'}\right]_\nu + \left[c_L^\lambda\right]_\nu\left[c_L^{\lambda'}\right]_\nu\left[\sigma_R^{\lambda\lambda'}\right]_\nu + \left[c_S^\lambda\right]_\nu\left[c_S^{\lambda'}\right]_\nu\left[\sigma_S^{\lambda\lambda'}\right]_\nu\right).$$

Now let us consider the case of the $\nu \mapsto \overline{\nu}$ transition for the differential cross section of single pion production derived in Refs. [2,3]:

$$\frac{d\sigma(\nu N \to l N\pi)}{dQ^2 dW d\Omega_\pi} = \frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Bigg\{$$

$$
\begin{aligned}
&\Big|C_{L-}(\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))+C_{R-}(\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))+\\
&C_{-}(\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\Big|^2\\
+&\Big|C_{L+}(\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))+C_{R+}(\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))+\\
&C_{+}(\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)-\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\Big|^2\Big\}. \qquad (1.64)
\end{aligned}
$$

First, we suppose, that the amplitudes $\tilde{F}^{\lambda_k}_{\lambda_2\lambda_1}(\theta,\phi)$ and $\tilde{G}^{\lambda_k}_{\lambda_2\lambda_1}(\theta,\phi)$ correspond only to the resonance case. Using formulas from Table 1.1 and transition relations (1.53) and (1.55), we can find how the amplitudes should be transformed when we want to calculate the cross sections for antineutrino:

$$
\begin{aligned}
&\left[\tilde{F}^L_{\lambda_2\lambda_1}\right]_{\overline{\nu}}-\left[\tilde{G}^L_{\lambda_2\lambda_1}\right]_{\overline{\nu}}=\left[\tilde{F}^L_{\lambda_2\lambda_1}\right]_{\nu}-\left[\tilde{G}^L_{\lambda_2\lambda_1}\right]_{\nu}, \quad \left[\tilde{F}^R_{\lambda_2\lambda_1}\right]_{\overline{\nu}}-\left[\tilde{G}^R_{\lambda_2\lambda_1}\right]_{\overline{\nu}}=\left[\tilde{F}^R_{\lambda_2\lambda_1}\right]_{\nu}-\left[\tilde{G}^R_{\lambda_2\lambda_1}\right]_{\nu},\\
&\left[\tilde{F}^+_{\lambda_2\lambda_1}\right]_{\overline{\nu}}-\left[\tilde{G}^+_{\lambda_2\lambda_1}\right]_{\overline{\nu}}=\left[\tilde{F}^-_{\lambda_2\lambda_1}\right]_{\nu}-\left[\tilde{G}^-_{\lambda_2\lambda_1}\right]_{\nu}, \quad \left[\tilde{F}^-_{\lambda_2\lambda_1}\right]_{\overline{\nu}}-\left[\tilde{G}^-_{\lambda_2\lambda_1}\right]_{\overline{\nu}}=\left[\tilde{F}^+_{\lambda_2\lambda_1}\right]_{\nu}-\left[\tilde{G}^+_{\lambda_2\lambda_1}\right]_{\nu}. \qquad (1.65)
\end{aligned}
$$

For example, it is easy to check that

$$
\begin{aligned}
\left[\tilde{F}^L_{\frac12\frac12}\right]_{\overline{\nu}}-\left[\tilde{G}^L_{\frac12\frac12}\right]_{\overline{\nu}}&=\sum_j\frac{2j+1}{\sqrt2}\mathcal{D}^j(R)\left[f^\lambda_{+3}(R)\right]_{\overline{\nu}}d^j_{\frac32\frac12}(\theta)e^{-2i\phi}=\\
&\sum_j\frac{2j+1}{\sqrt2}\mathcal{D}^j(R)\left[f^\lambda_{+3}(R)\right]_{\nu}d^j_{\frac32\frac12}(\theta)e^{-2i\phi}=\left[\tilde{F}^L_{\frac12\frac12}\right]_{\nu}-\left[\tilde{G}^L_{\frac12\frac12}\right]_{\nu},\\
\left[\tilde{F}^-_{\frac12\frac12}\right]_{\overline{\nu}}-\left[\tilde{G}^-_{\frac12\frac12}\right]_{\overline{\nu}}&=+\frac{|\mathbf{k}|}{\sqrt{Q^2}}\sum_j\frac{2j+1}{\sqrt2}\mathcal{D}^j(R)\left[f^{(-)}_{0-}(R)\right]_{\overline{\nu}}d^j_{\frac12\frac12}(\theta)e^{-i\phi}=\\
&+\frac{|\mathbf{k}|}{\sqrt{Q^2}}\sum_j\frac{2j+1}{\sqrt2}\mathcal{D}^j(R)\left[f^{(+)}_{0-}(R)\right]_{\nu}d^j_{\frac12\frac12}(\theta)e^{-i\phi}=\left[\tilde{F}^+_{\frac12\frac12}\right]_{\nu}-\left[\tilde{G}^+_{\frac12\frac12}\right]_{\nu},
\end{aligned}
$$

etc. Applying the transition relations (1.43) to Eq. (1.64) one gets:

$$
\begin{aligned}
&\frac{d\sigma(\overline{\nu}N\to lN\pi)}{dQ^2dWd\Omega_\pi}=\frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Big\{\\
&\Big|\left[C_{L-}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)+\left[C_{R-}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)+\\
&\left[C_{-}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)\Big|^2+\\
&\Big|\left[C_{L+}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)+\left[C_{R+}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)+\\
&\left[C_{+}\right]_{\overline{\nu}}\Big(\left[\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}}-\left[\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\Big)\Big|^2\Big\}=\\
&\frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Big\{\\
&\Big|-\left[C_{R+}\right]_{\nu}\Big(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\nu}-\left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\nu}\Big)-\left[C_{L+}\right]_{\nu}\Big(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\nu}-\left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\nu}\Big)+\\
&\left[C_{+}\right]_{\nu}\Big(\left[\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\nu}-\left[\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\nu}\Big)\Big|^2+
\end{aligned}
$$

$$\Big| [C_{R-}]_\nu \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{L-}]_\nu \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) -$$
$$[C_-]_\nu \left(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 \Big\} =$$
$$\frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Big\{$$
$$\Big| [C_{R+}]_\nu \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{L+}]_\nu \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) -$$
$$[C_+]_\nu \left(\left[\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 +$$
$$\Big| [C_{R-}]_\nu \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{L-}]_\nu \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) -$$
$$[C_-]_\nu \left(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 \Big\}. \tag{1.66}$$

If we use the transition relations (1.44), we get

$$\frac{d\sigma(\overline{\nu}N \to lN\pi)}{dQ^2 dW d\Omega_\pi} = \frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Big\{$$
$$\Big| [C_{L-}]_{\overline{\nu}} \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{R-}]_{\overline{\nu}} \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) +$$
$$[C_-]_{\overline{\nu}} \left(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 +$$
$$\Big| [C_{L+}]_{\overline{\nu}} \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{R+}]_{\overline{\nu}} \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) +$$
$$[C_+]_{\overline{\nu}} \left(\left[\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 \Big\}$$
$$\frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{|\mathbf{q}|}{4}\frac{Q^2}{(k^L)^2}\sum_{\lambda_2,\lambda_1}\Big\{$$
$$\Big| [C_{R-}]_\nu \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{L-}]_\nu \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) {\color{red}+}$$
$$[C_-]_\nu \left(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 +$$
$$\Big| [C_{R+}]_\nu \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) + [C_{L+}]_\nu \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) {\color{red}+}$$
$$[C_+]_\nu \left(\left[\tilde{F}^+_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^+_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu \right) \Big|^2 \Big\}. \tag{1.67}$$

We see that Eqs. (1.66) and (1.67) differ in signs of the term $\left[\tilde{F}^\pm_{\lambda_2\lambda_1}(\theta,\phi)\right]_\nu - \left[\tilde{G}^\pm_{\lambda_2\lambda_1}(\theta,\phi))\right]_\nu$, which are marked in red color.

1.2.3 Last update of the MK model

Minoo recently suggested the new transitions rules:

$$\begin{aligned}
\left[j_0^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda \left[j_0^{\star-\lambda}\right]_\nu ,\\
\left[j_x^{\star\lambda}\right]_{\overline{\nu}} &= +\lambda \left[j_x^{\star-\lambda}\right]_\nu ,\\
\left[j_y^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda \left[j_y^{\star-\lambda}\right]_\nu ,\\
\left[j_z^{\star\lambda}\right]_{\overline{\nu}} &= -\lambda \left[j_z^{\star-\lambda}\right]_\nu ,
\end{aligned} \tag{1.68}$$

from which, as she stated in her note, it follows:

$$[C_{L_\lambda}]_{\overline{\nu}} = -\lambda \left[C_{R_{-\lambda}}\right]_\nu , \quad [C_{R_\lambda}]_{\overline{\nu}} = -\lambda \left[C_{L_{-\lambda}}\right]_\nu , \quad [C_{S_\lambda}]_{\overline{\nu}} = \left[C_{S_{-\lambda}}\right]_\nu . \tag{1.69}$$

It follows from (1.68) and (1.69)

$$\begin{aligned}
&\left[\mathcal{Q}^\star_{(\lambda)}\right]_{\overline{\nu}} = -\lambda \left[\mathcal{Q}^\star_{(-\lambda)}\right]_\nu , \quad \left[\nu^\star_{(\lambda)}\right]_{\overline{\nu}} = -\lambda \left[\nu^\star_{(-\lambda)}\right]_\nu , \\
&\left[S_{(\lambda)}\right]_{\overline{\nu}} = -\lambda \left[S_{(-\lambda)}\right]_\nu , \quad \left[B_{(\lambda)}\right]_{\overline{\nu}} = -\lambda \left[B_{(-\lambda)}\right]_\nu , \quad \left[C_{(\lambda)}\right]_{\overline{\nu}} = -\lambda \left[C_{(-\lambda)}\right]_\nu , \\
&\left[f^{\lambda}_{0\pm}\right]_\nu = -\lambda \left[f^{-\lambda}_{0\pm}\right]_{\overline{\nu}} , \\
&\left[\tilde{F}^L_{\lambda_2\lambda_1}\right]_{\overline{\nu}} - \left[\tilde{G}^L_{\lambda_2\lambda_1}\right]_{\overline{\nu}} = \left[\tilde{F}^L_{\lambda_2\lambda_1}\right]_\nu - \left[\tilde{G}^L_{\lambda_2\lambda_1}\right]_\nu , \quad \left[\tilde{F}^R_{\lambda_2\lambda_1}\right]_{\overline{\nu}} - \left[\tilde{G}^R_{\lambda_2\lambda_1}\right]_{\overline{\nu}} = \left[\tilde{F}^R_{\lambda_2\lambda_1}\right]_\nu - \left[\tilde{G}^R_{\lambda_2\lambda_1}\right]_\nu , \\
&\left[\tilde{F}^+_{\lambda_2\lambda_1}\right]_{\overline{\nu}} - \left[\tilde{G}^+_{\lambda_2\lambda_1}\right]_{\overline{\nu}} = -\left(\left[\tilde{F}^-_{\lambda_2\lambda_1}\right]_\nu - \left[\tilde{G}^-_{\lambda_2\lambda_1}\right]_\nu\right) , \\
&\left[\tilde{F}^-_{\lambda_2\lambda_1}\right]_{\overline{\nu}} - \left[\tilde{G}^-_{\lambda_2\lambda_1}\right]_{\overline{\nu}} = \left[\tilde{F}^+_{\lambda_2\lambda_1}\right]_\nu - \left[\tilde{G}^+_{\lambda_2\lambda_1}\right]_\nu , \\
&\left[\sigma^{++}_{L,R}\right]_\nu = \left[\sigma^{+-}_{L,R}\right]_\nu = \left[\sigma^{-+}_{L,R}\right]_\nu = \left[\sigma^{--}_{L,R}\right]_\nu = \\
&\left[\sigma^{++}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{+-}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{-+}_{L,R}\right]_{\overline{\nu}} = \left[\sigma^{--}_{L,R}\right]_{\overline{\nu}} , \\
&\left[\sigma^{++}_S\right]_\nu = \left[\sigma^{--}_S\right]_{\overline{\nu}} , \left[\sigma^{--}_S\right]_\nu = \left[\sigma^{++}_S\right]_{\overline{\nu}} , \\
&\left[\sigma^{+-}_S\right]_\nu = \left[\sigma^{-+}_S\right]_\nu = -\left[\sigma^{+-}_S\right]_{\overline{\nu}} = -\left[\sigma^{-+}_S\right]_{\overline{\nu}}
\end{aligned}$$

Again, this rules are not coincide with (1.33) and (1.39) and as before one can show that the sum is coincide with (1.60):

$$\begin{aligned}
[\varSigma_{++} + \varSigma_{--}]_{\overline{\nu}} = & \left[c^+_L\right]^2_{\overline{\nu}} \left[\sigma^{++}_L\right]_{\overline{\nu}} + \left[c^+_R\right]^2_{\overline{\nu}} \left[\sigma^{++}_R\right]_{\overline{\nu}} + \left[c^+_S\right]^2_{\overline{\nu}} \left[\sigma^{++}_S\right]_{\overline{\nu}} + \\
& \left[c^-_L\right]^2_{\overline{\nu}} \left[\sigma^{--}_L\right]_{\overline{\nu}} + \left[c^-_R\right]^2_{\overline{\nu}} \left[\sigma^{--}_R\right]_{\overline{\nu}} + \left[c^-_S\right]^2_{\overline{\nu}} \left[\sigma^{--}_S\right]_{\overline{\nu}} = \\
& \left[c^-_R\right]^2_\nu \left[\sigma^{++}_L\right]_\nu + \left[c^-_L\right]^2_\nu \left[\sigma^{++}_R\right]_\nu + \left[c^-_S\right]^2_\nu \left[\sigma^{--}_S\right]_\nu + \\
& \left[c^+_R\right]^2_\nu \left[\sigma^{--}_L\right]_\nu + \left[c^+_L\right]^2_\nu \left[\sigma^{--}_R\right]_\nu + \left[c^+_S\right]^2_\nu \left[\sigma^{++}_S\right]_\nu .
\end{aligned} \tag{1.70}$$

However the value of

$$\begin{aligned}
[\varSigma_{\lambda\lambda'}]_{\overline{\nu}} = & \left[c^\lambda_L\right]_{\overline{\nu}} \left[c^{\lambda'}_L\right]_{\overline{\nu}} \left[\sigma^{\lambda\lambda'}_L\right]_{\overline{\nu}} + \left[c^\lambda_R\right]_{\overline{\nu}} \left[c^{\lambda'}_R\right]_{\overline{\nu}} \left[\sigma^{\lambda\lambda'}_R\right]_{\overline{\nu}} + \left[c^\lambda_S\right]_{\overline{\nu}} \left[c^{\lambda'}_S\right]_{\overline{\nu}} \left[\sigma^{\lambda\lambda'}_S\right]_{\overline{\nu}} = \\
& \lambda\lambda' \left(\left[c^{-\lambda}_R\right]_\nu \left[c^{-\lambda'}_R\right]_\nu \left[\sigma^{\lambda\lambda'}_L\right]_\nu + \left[c^{-\lambda}_L\right]_\nu \left[c^{-\lambda'}_L\right]_\nu \left[\sigma^{\lambda\lambda'}_R\right]_\nu - \left[c^{-\lambda}_S\right]_\nu \left[c^{-\lambda'}_S\right]_\nu \left[\sigma^{-\lambda-\lambda'}_S\right]_\nu\right)
\end{aligned} \tag{1.71}$$

is not coincide with (1.63).

$$\begin{aligned}
&\frac{d\sigma(\overline{\nu}N \to lN\pi)}{dQ^2 dW d\Omega_\pi} = \frac{G_F^2}{2} \frac{1}{(2\pi)^4} \frac{|\mathbf{q}|}{4} \frac{Q^2}{(k^L)^2} \sum_{\lambda_2,\lambda_1} \Bigg\{ \\
&\Big| \left[C_{L_-}\right]_{\overline{\nu}} \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}} - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\right) + \left[C_{R_-}\right]_{\overline{\nu}} \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}} - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\right) + \\
&\left[C_-\right]_{\overline{\nu}} \left(\left[\tilde{F}^-_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}} - \left[\tilde{G}^-_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\right) \Big|^2 + \\
&\Big| \left[C_{L_+}\right]_{\overline{\nu}} \left(\left[\tilde{F}^L_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}} - \left[\tilde{G}^L_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\right) + \left[C_{R_+}\right]_{\overline{\nu}} \left(\left[\tilde{F}^R_{\lambda_2\lambda_1}(\theta,\phi)\right]_{\overline{\nu}} - \left[\tilde{G}^R_{\lambda_2\lambda_1}(\theta,\phi))\right]_{\overline{\nu}}\right) +
\end{aligned}$$

$$[C_+]_{\overline{\nu}} \left(\left[\tilde{F}^{+}_{\lambda_2\lambda_1}(\theta,\phi) \right]_{\overline{\nu}} - \left[\tilde{G}^{+}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_{\overline{\nu}} \right) \Big|^2 \Big\} =$$

$$\frac{G_F^2}{2} \frac{1}{(2\pi)^4} \frac{|\mathbf{q}|}{4} \frac{Q^2}{(k^L)^2} \sum_{\lambda_2,\lambda_1} \Big\{$$

$$\Big| [C_{R_+}]_\nu \left(\left[\tilde{F}^{L}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{L}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) + [C_{L_+}]_\nu \left(\left[\tilde{F}^{R}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{R}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) +$$

$$[C_+]_\nu \left(\left[\tilde{F}^{+}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{+}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) \Big|^2 +$$

$$\Big| - [C_{R_-}]_\nu \left(\left[\tilde{F}^{L}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{L}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) - [C_{L_-}]_\nu \left(\left[\tilde{F}^{R}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{R}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) -$$

$$[C_-]_\nu \left(\left[\tilde{F}^{-}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{-}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) \Big|^2 \Big\}.$$

Finally we obtain:

$$\left[\frac{d\sigma(\overline{\nu}N \to lN\pi)}{dQ^2 dW d\Omega_\pi} \right]_{\text{Minoo's transition rule}} = \frac{G_F^2}{2} \frac{1}{(2\pi)^4} \frac{|\mathbf{q}|}{4} \frac{Q^2}{(k^L)^2} \sum_{\lambda_2,\lambda_1} \Big\{$$

$$\Big| [C_{R_+}]_\nu \left(\left[\tilde{F}^{L}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{L}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) + [C_{L_+}]_\nu \left(\left[\tilde{F}^{R}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{R}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) +$$

$$[C_+]_\nu \left(\left[\tilde{F}^{+}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{+}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) \Big|^2 +$$

$$\Big| [C_{R_-}]_\nu \left(\left[\tilde{F}^{L}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{L}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) + [C_{L_-}]_\nu \left(\left[\tilde{F}^{R}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{R}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) +$$

$$[C_-]_\nu \left(\left[\tilde{F}^{-}_{\lambda_2\lambda_1}(\theta,\phi) \right]_\nu - \left[\tilde{G}^{-}_{\lambda_2\lambda_1}(\theta,\phi)) \right]_\nu \right) \Big|^2 \Big\}. \tag{1.72}$$

This result *is not coincide* with (1.66); the difference is in the two signs marked in red.

1.2.4 Conclusions

Ultimately, Minoo's "recipe" for the $\nu \mapsto \overline{\nu}$ transition gives correct result only for the double differential cross sections, like $d\sigma/dWdQ^2$. For the cross section $d\sigma/dQ^2dWd\Omega_\pi$ and for the polarization density matrix her "recipe" does not work. **The same situation with the last updated rules.**

Let's emphasize that calculation of the polarization matrix is by no means an academic issue because in the long-awaited high-precision experiments with atmospheric and astrophysical neutrinos (like Hyper-Kamiokande, DUNE, PINGU, ORCA) and in possible future dedicated accelerator experiments, one will need to perform detailed calculations of production and decay of polarized τ-leptons in the detector, Earth, atmosphere, and astrophysical sources. In fact, this was one of the main goals in generalizing the RS model (KLN, BS, etc.).[5] We think, this option will certainly need to be added to GENIE. So we plan to return to this problem in the future.

Considering that the transition rules (1.39) are consequence of the leptonic current properties only, they are valid for any neutrino-induced CC process. The coincidence of the cross section $d\sigma/dWdQ^2$ obtained by using this rules with Minoo's one is due to underlying symmetry of the Rein-Sehgal model [5]. This may not be so in other models or after some modifications of the KLN-BS or MK models. In particular, since in the MK-model, the resonance amplitudes are coherently added to the non-resonance ones, it is not in general obvious that in this case even the cross section $d\sigma/dWdQ^2$ calculated by using Minoo's prescription will remain correct.

[5] More details can be found in Refs. [8, 10].

We would like to remind that the values $\widetilde{F}^{\lambda_k}_{\lambda_2,\lambda_1}(\theta,\phi)$ and $\widetilde{G}^{\lambda_k(p)}_{\lambda_2,\lambda_1}(\theta,\phi)$, which we consider as correct, are different from Minoo's ones (see Tables 1.1 and 1.2), but all derived formulas, in particular Eq. (1.72), are generic. This difference could either enhance or partially offset the discrepancy between Eqs. (1.66) and (1.72).

Let us also note that agreement of the MK model predictions with the experimental data cannot confirm or disconfirm the $\nu \mapsto \overline{\nu}$ transition rules just because the currently available data are too fragmentary and uncertain and are either slowly sensible or (as the single and double-differential cross sections) fully insensible to the differences in the two versions of the rules under consideration. Moreover, after refitting the MK model parameters, the agreement with the more detailed data may seem to be quite satisfactory even with wrong formulas. But the predictive power of the model will be under big question. So we suggest to use the $\nu \mapsto \overline{\nu}$ transition rules derived in Ref. [8] and reproduced above... or to refute our derivations.

1.3 Erroneous Cj_signs

We reported about this problem in correspondence on 02 Oct 2020 and 15 Oct 2020, but the problem still remains. In the lines starting from number 498 of file *imode1.cc*

```
HA_realM11[i]  =         -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f3*Cjsgn_plus[i]   ;
HA_realM_11[i] = -Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f3*Cjsgn_minus[i];
HA_realM1_1[i] =          JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f1*Cjsgn_plus[i]  ;
HA_realM_1_1[i]=  Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f1*Cjsgn_minus[i]  ;

HA_realP11[i]  =         -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f_1*Cjsgn_plus[i]  ;
HA_realP_11[i] = Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f_1*Cjsgn_minus[i];
HA_realP1_1[i] =         -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f_3*Cjsgn_plus[i];
HA_realP_1_1[i]= Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f_3*Cjsgn_minus[i]  ;

HA0_realM11[i]  = -Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0_minus
                                                                        *Cjsgn_plus[i]  ;
HA0_realM_11[i] =         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0_minus
                                                                        *Cjsgn_minus[i];
HA0_realM1_1[i] =  Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0_plus
                                                                        *Cjsgn_plus[i]  ;
HA0_realM_1_1[i]=         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0_plus
                                                                        *Cjsgn_minus[i]  ;

HA0_realP11[i]  = -Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0R_minus
                                                                        *Cjsgn_plus[i];
HA0_realP_11[i] =         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0R_minus
                                                                        *Cjsgn_minus[i];
HA0_realP1_1[i] =  Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0R_plus
                                                                        *Cjsgn_plus[i]  ;
HA0_realP_1_1[i]=         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_real[i]*f0R_plus
                                                                        *Cjsgn_minus[i];

HA_ImM11[i]  =         -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f3*Cjsgn_plus[i]  ;
HA_ImM_11[i] = -Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f3*Cjsgn_minus[i];
HA_ImM1_1[i] =          JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f1*Cjsgn_plus[i]  ;
```

```
HA_ImM_1_1[i]=  Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f1*Cjsgn_minus[i] ;

HA_ImP11[i]  =        -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f_1*Cjsgn_plus[i] ;
HA_ImP_11[i] = Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f_1*Cjsgn_minus[i];
HA_ImP1_1[i] =        -JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f_3*Cjsgn_plus[i] ;
HA_ImP_1_1[i]= Jsgn[i]*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f_3*Cjsgn_minus[i];

HA0_ImM11[i]  = -Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0_minus
                                                                          *Cjsgn_plus[i] ;
HA0_ImM_11[i] =         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0_minus
                                                                          *Cjsgn_minus[i];
HA0_ImM1_1[i] =  Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0_plus
                                                                          *Cjsgn_plus[i] ;
HA0_ImM_1_1[i]=         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0_plus
                                                                          *Cjsgn_minus[i] ;

HA0_ImP11[i]  = -Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0R_minus
                                                                          *Cjsgn_plus[i] ;
HA0_ImP_11[i] =         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0R_minus
                                                                          *Cjsgn_minus[i];
HA0_ImP1_1[i] =  Jsgn[i]*(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0R_plus
                                                                          *Cjsgn_plus[i] ;
HA0_ImP_1_1[i]=         -(abs_mom_k/sqrt(Q))*JPsq2[i]*Dsgn[i]*kapa[i]*f_BW_Im[i]*f0R_plus
                                                                       *Cjsgn_minus[i];

// CG signes \lambda2 = +
const int Cjsgn_plus[17]}  ={1,-1,-1, 1, 1, 1, 1,-1, -1,-1,-1, 1, -1, -1, 1, 1,1};
const int Jsgn[17] =       {1,-1,-1, 1, 1, 1, 1,-1, -1,-1,-1, 1, -1, -1, 1, 1,1};
 // CG signes \lambda2 = -
const int Cjsgn_minus[17] ={1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1};
```

We marked signs ± (which correspond to the signs in Tables 1.1 and 1.2) by red. Also, we marked by green Jsgn and Cjsgn_plus which coincide and by magenta Cjsgn_minus, which are always equal to 1. The Cjsgn_plus and Cjsgn_minus correspond to the value $C^j_{N\pi}$ from formula (25) of [4]. The fact of presence Jsgn is an obvious error and it does not have any relation to signs ±.

Chapter 2

Fixed problems, bugs and mistakes

2.1 Incorrect phase factors for resonance and background amplitudes

The expression for the differential cross section is obtained assuming incorrect phase factors for resonance and background amplitudes in the C++ code provided by Minoo. The proof of this fact can be founded in accompanying *Wolfram Mathematica* notebook **XSec_MK_Diff.nb** with all explanatory comments. In brief, the phase factors of the amplitudes $\widetilde{F}^{e_L}_{\lambda_2,\lambda 1}(\theta, \phi)$ and $\widetilde{F}^{e_R}_{\lambda_2,\lambda 1}(\theta, \phi)$ are mixed up. This lead to incorrect expression for the cross section.

There is a similar problem in the latest version of Minoo's code. We checked differential cross section only for one channel (*MK_mode1.cc*) and found that it is wrong (see accompanying file **XSec_MK_Diff_v.2.nb**), but we didn't check other channels yet.

2.2 Incorrect helicity amplitudes

The helicity amplitudes:

- for resonance $S_{11}(1650)$ instead of

  ```
  f1       = sqrt(1./24.)*L*(R1_minus + 4*sin2Wein*R1_V);
  ```

  should be

  ```
  f1       = sqrt(1./24.)*L*(R1_plus + 4*sin2Wein*R1_V);
  ```

- for resonance $P_{13}(1720)$ instead of

  ```
  f_1      = -sqrt(27./40.)*L*T2_minus +
                sqrt(5./12.)*L*L*(R2_minus+2.*sin2Wein*(4./5.)*R2_V);
  f1       = +sqrt(27./40.)*L*T2_plus -
                sqrt(5./12.)*L*L*(R2_plus +2.*sin2Wein*(4./5.)*R2_V);
  f3       = -sqrt(9. /40.)*L*T2_plus;
  f0_plus  = -sqrt(3./20.)*L*L*S2_KLM_minus +
  ```

```
                 sqrt(5./12.)*L*(L*C2_minus-5.*B2_minus);
f0_minus =       sqrt(3./20.)*L*L*S2_KLM_minus +
                 sqrt(5./12.)*L*(L*C2_minus-5.*B2_minus);
```

should be

```
f_1      =  sqrt(27./40.)*L*T2_minus +
            sqrt(5./12.)*L*L*(R2_minus+2.*sin2Wein*(4./5.)*R2_V);
f1       = -sqrt(27./40.)*L*T2_plus -
            sqrt(5./12.)*L*L*(R2_plus +2.*sin2Wein*(4./5.)*R2_V);
f3       =  sqrt(9. /40.)*L*T2_plus;
f0_plus  =  sqrt(3./20.)*L*L*S2_KLM_minus +
            sqrt(5./12.)*L*(L*C2_minus-5.*B2_minus);
f0_minus = -sqrt(3./20.)*L*L*S2_KLM_minus +
            sqrt(5./12.)*L*(L*C2_minus-5.*B2_minus);
```

- for resonance $F_{15}(1680)$ instead of

```
f3        = -sqrt(9./10.)*L*T2_plus;
```

should be

```
f3        = sqrt(9./10.)*L*T2_plus;
```

defined in the file *MK_imode6.cc, MK_imode7.cc, MK_imode16.cc, MK_imode17.cc* are wrong (see Ref. [6]). In the latest version this is corrected.

2.3 Erroneous dynamical form factor B

In the MK-model the wrong form factor B was used (see Minoo's erratum to Ref. [10]). It was fixed in the latest version of code.

2.4 Mistaken cross-section formula for NC-processes in the code

The values Fem_zero_minus is not used in the expression for the cross section. As a result some parts of the expression for the cross section are incorrect, for example, instead of lines 1003-1006 from *mk_imode4_new.cc*:

```
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus11   - G_zero_minus11   + C3*sum3Re_OM11  + C1*sum1Re_OM11) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus_11  - G_zero_minus_11  + C3*sum3Re_OM_11 + C1*sum1Re_OM_11) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus1_1  - G_zero_minus1_1  + C3*sum3Re_OM1_1 + C1*sum1Re_OM1_1) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus_1_1 - G_zero_minus_1_1 + C3*sum3Re_OM_1_1+ C1*sum1Re_OM_1_1) , 2)
```

should be the following lines:

```
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus11   - G_zero_minus11  - 4.*sin2Wein*Fem_zero_minus11
+ C3*sum3Re_OM11  + C1*sum1Re_OM11 ) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus_11  - G_zero_minus_11 - 4.*sin2Wein*Fem_zero_minus_11
+ C3*sum3Re_OM_11 + C1*sum1Re_OM_11) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus1_1  - G_zero_minus1_1 - 4.*sin2Wein*Fem_zero_minus1_1
+ C3*sum3Re_OM1_1 + C1*sum1Re_OM1_1) , 2)
+ pow( ((1. - 2.*sin2Wein)*F_zero_minus_1_1 - G_zero_minus_1_1- 4.*sin2Wein*Fem_zero_minus_1_1
+ C3*sum3Re_OM_1_1+ C1*sum1Re_OM_1_1), 2
```

The error has been fixed in the most recent version of the code, where the file *mk_imode4_new.cc* is renamed to *mk_imode4.cc*.

2.5 Incorrect C_j-signs

The C_j-signs in the following fragments of code, which are denoted as Jsgn, are wrong, because for the amplitudes $\widetilde{F}^{e_R}_{-\frac{1}{2},\lambda 1}(\theta,\phi))$ (denoted as HV_(real|Im)P_1(1|_1)) the corresponding signs are always positive (see Table 7 from Ref. [6]).

```
const int Jsgn[nRes] = {1,-1,-1, 1, 1, 1, 1,-1,-1,-1,-1, 1,-1,-1, 1, 1, 1};
HV_realP11[i]   =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_1;
HV_realP_11[i]  = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_1*Jsgn[i];
HV_realP1_1[i]  = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_3;
HV_realP_1_1[i] =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_3*Jsgn[i];

HV_ImP11[i]     =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_1;
HV_ImP_11[i]    = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_1*Jsgn[i];
HV_ImP1_1[i]    = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_3;
HV_ImP_1_1[i]   =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_3*Jsgn[i];

HA_realP11[i]   =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_1;
HA_realP_11[i]  = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_1*Jsgn[i];
HA_realP1_1[i]  = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_3;
HA_realP_1_1[i] =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_real[i]*fV_3*Jsgn[i];

HA_ImP11[i]     =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_1;
HA_ImP_11[i]    = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_1*Jsgn[i];
HA_ImP1_1[i]    = -sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_3;
HA_ImP_1_1[i]   =  sqrt(2.)*JP[i]*Dsgn[i]*kapa[i]*Vf_BW_Im[i]*fV_3*Jsgn[i];
```

It was fixed in the file *mk_imode1_erratum.cc*, which is renamed with *mk_imode1.cc* in the recent version of code, where the author get rid of HV-values and everything is correct there.

2.6 Problem with multipole expansion of some amplitudes in the code

The multipole expansion of the following amplitudes $\widetilde{F}^{+}_{\text{res}\,-\frac{1}{2},\frac{1}{2}}$ and $\widetilde{F}^{-}_{\text{res}\,-\frac{1}{2},\frac{1}{2}}$ is definitely incorrect as they were defined in the first version of Erratum (this is not the same definition as in paper [3]):

$$\begin{aligned}\widetilde{F}^{-}_{\text{res}\,-\frac{1}{2},\frac{1}{2}} &= \frac{|\mathbf{k}|}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{V(-)}_{0-}(R)d^j_{-\frac{1}{2}\frac{1}{2}}(\theta),\\ \widetilde{F}^{+}_{\text{res}\,-\frac{1}{2},\frac{1}{2}} &= \frac{|\mathbf{k}|}{\sqrt{Q^2}}\sum_j \frac{2j+1}{\sqrt{2}}\mathcal{D}^j(R)f^{V(-)}_{0-}(R)d^j_{-\frac{1}{2}\frac{1}{2}}(\theta).\end{aligned} \tag{2.1}$$

But in the MK-code they are defined different way. For example, in line 823 of the file *mk_imode1_new.cc*:

```
double sum3Re_OM_11 =   HA0_realM_11[0] *d31_1 + HA0_realM_11[4]*d11_1 +
                        HA0_realM_11[9] *d31_1 + HA0_realM_11[12]*d51_1+
                        HA0_realM_11[13]*d11_1 + HA0_realM_11[14]*d31_1+
                        HA0_realM_11[15]*d71_1 + HA0_realM_11[16]*d31_1;
```

where `dj1_1` denotes $d^j_{\frac{1}{2},-\frac{1}{2}}$. But, according to Eq. (1.15), $d^j_{\frac{1}{2},-\frac{1}{2}} = -d^j_{-\frac{1}{2},\frac{1}{2}}$. So the above line should be rewritten as follows:

```
double sum3Re_OM_11 = -(HA0_realM_11[0] *d31_1 + HA0_realM_11[4]*d11_1 +
                        HA0_realM_11[9] *d31_1 + HA0_realM_11[12]*d51_1+
                        HA0_realM_11[13]*d11_1 + HA0_realM_11[14]*d31_1+
                        HA0_realM_11[15]*d71_1 + HA0_realM_11[16]*d31_1);
```

Lines 828, 843, 848, 864, 869, 884, and 889 should be corrected in the same way.

The mistake has been fixed in the second version of Erratum [4] but appears again in the latest versions of erratum for paper [3] and code. However, it is unclear how one should fix it (see section 1.1.4).

2.7 Expressions for EM-amplitudes contains mistake

There is an error in the lines 207–210 of the file *mk_imode4_new.cc*, instead of C_S_plus_square it should be C_S_minus_square:

```
double Fem_zero_minus11   =  (1./sqrt(C_S_plus_square))*sqrt((1+t)/2.)*(k_0*eps_z_L - abs_mom_k*eps_zero_L)*(sFem_5 + sFem_6);
double Fem_zero_minus_11  = -(1./sqrt(C_S_plus_square))*sqrt((1-t)/2.)*(k_0*eps_z_L - abs_mom_k*eps_zero_L)*(sFem_5 - sFem_6);
double Fem_zero_minus1_1  = -(1./sqrt(C_S_plus_square))*sqrt((1-t)/2.)*(k_0*eps_z_L - abs_mom_k*eps_zero_L)*(sFem_5 - sFem_6);
double Fem_zero_minus_1_1 = -(1./sqrt(C_S_plus_square))*sqrt((1+t)/2.)*(k_0*eps_z_L - abs_mom_k*eps_zero_L)*(sFem_5 + sFem_6);
```

2.8 Error in the dynamical amplitudes for resonance $D_{13}(1520)$

The error is in lines 448, 449 of the file *mk_imode6.cc*:

```
fV_1 = -sqrt(3./2.)*(-T1_V[2]+2.*sin2Wein*T1_V[2])+sqrt(4./3.)*L*(-R1_V[2]+3.*sin2Wein*R1_V[2]);
fV1  = -sqrt(3./2.)*(-T1_V[2]+2.*sin2Wein*T1_V[2])+sqrt(4./3.)*L*(-R1_V[2]+3.*sin2Wein*R1_V[2]);
```

The corrected lines should be [5]:

```
fV_1 = -sqrt(3./2.)*(-T1_V[2]+2.*sin2Wein*T1_V[2])+sqrt(4./3.)*L*(-R1_V[2]+sin2Wein*R1_V[2]);
fV1  = -sqrt(3./2.)*(-T1_V[2]+2.*sin2Wein*T1_V[2])+sqrt(4./3.)*L*(-R1_V[2]+sin2Wein*R1_V[2]);
```

2.9 Mistype in a formula for cross section

A mistype is in line 1201 of the file *mk_imode2.cc*, namely, the amplitude sum1Re0PMM should be instead of sum1Re0MMM. For more details see the accompanying *Wolfram Mathematica* notebook **XSec_MK_Diff.nb**.

2.10 Problem with phase factors for resonance amplitudes

The phase factors $\exp[n\phi]$ of the resonance amplitudes in Table 3 in Ref. [3] are incorrect. They should be the same as for the background contribution presented in Table 6 of the same paper. This mistake has been fixed in Ref. [4].

2.11 The values of some resonance parameters are incorrect

The values of resonance masses defined in the penultimate version of the file *mkcons.h* are wrong:

```
const double MR[17] = {1.232, 1.440, 1.515, 1.53, 1.57,
1.61, 1.65, 1.675, 1.685, 1.72,
1.71, 1.71, 1.72, 1.88, 1.9, 1.92, 1.93};
```

because they should be

```
const double MR[17] =  {1.232, 1.440, 1.515, 1.530, 1.610,
1.650, 1.675, 1.685, 1.720, 1.710,
1.710, 1.720, 1.880, 1.900, 1.920, 1.930, 1.570};
```

according to the order in which resonance amplitudes are calculated in the files *MK_imode1.cc–MK_imode17.cc*:

```
P_{33}(1232) **** IBLOCK=0   ****
P_{11}(1440) **** IBLOCK=1   ****
D_{13}(1520) **** IBLOCK=2   ****
S_{11}(1535) **** IBLOCK=3   ****
S_{31}(1620) **** IBLOCK=4   ****
S_{11}(1650) **** IBLOCK=5   ****
D_{15}(1675) **** IBLOCK=6   ****
F_{15(1680)  **** IBLOCK=7   ****
D_{13}(1700) **** IBLOCK=8   ****
D_{33}(1700) **** IBLOCK=9   ****
P_{11}(1710) **** IBLOCK=10  ****
P_{13}(1720) **** IBLOCK=11  ****
F_{35}(1905) **** IBLOCK=12  ****
P_{31}(1910) **** IBLOCK=13  ****
P_{33}(1920) **** IBLOCK=14  ****
F_{37}(1950) **** IBLOCK=15  ****
P_{33}(1600) **** IBLOCK=16  ****
```

The same is also true for the widths and branching ratios. However such inputs as a signs of the angular Clebsch–Gordan coefficients, angular momentum and total angular momentum projections are defined correctly. Since it is not quite clear which set of resonance amplitudes is used in the latest version of the code, it becomes problematic to determine whether the values of σ^D is correct or not since they differ in the previous and latest code version (see also section 1.1.4). This bug has been fixed in the latest version of code.

2.12 Value of $\hat{\kappa}$-factor is incorrect

Recall that the MK model is essentially based on Rein model [6]. It is obvious from comparison of Eqs. (4.15) in Ref. [1] and (40a) in Ref. [6] that the κ-factor should be replaced by $\hat{\kappa}$, which is defined after Eq. (39c) in Ref. [6].

To calculate the $\hat{\kappa}$-factor one needs know the ratio of isospin coefficients

$$\zeta = \frac{c_I}{a_I},$$

where a_I and c_I are defined in Eqs. (8) and (24) of Rein's paper [6]. Note, that the isospin coefficients a_I and c_I[1] are given in the explicit form only for several channel in Ref. [6].

As we stated in a previous version of this report: "It is therefore necessary to calculate them for other channels. The absence of this factor leads to very significant diversities with the original theory." But in fact it is, fortunately, not so, because it reduced in the final formula for cross section. So the explicit form of this factor is only of academic interest, and is not needed for practical calculations of the cross sections.

[1]We want to note here, that in Rein's paper [6], the coefficient c_I for the decay channel $\nu p \to \ell p \pi^+$ is equal to 1, while in Ref. [5] it equal to $\sqrt{3}$. The origin of the factor $\sqrt{3}$ in the Rein-Sehgal paper is clear (it arises due to the isospin symmetry), but it is not clear (for us!) why it is absent in Rein's paper. Maybe somebody can clarify this question for us?

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