

NEUTRINO PHYSICS

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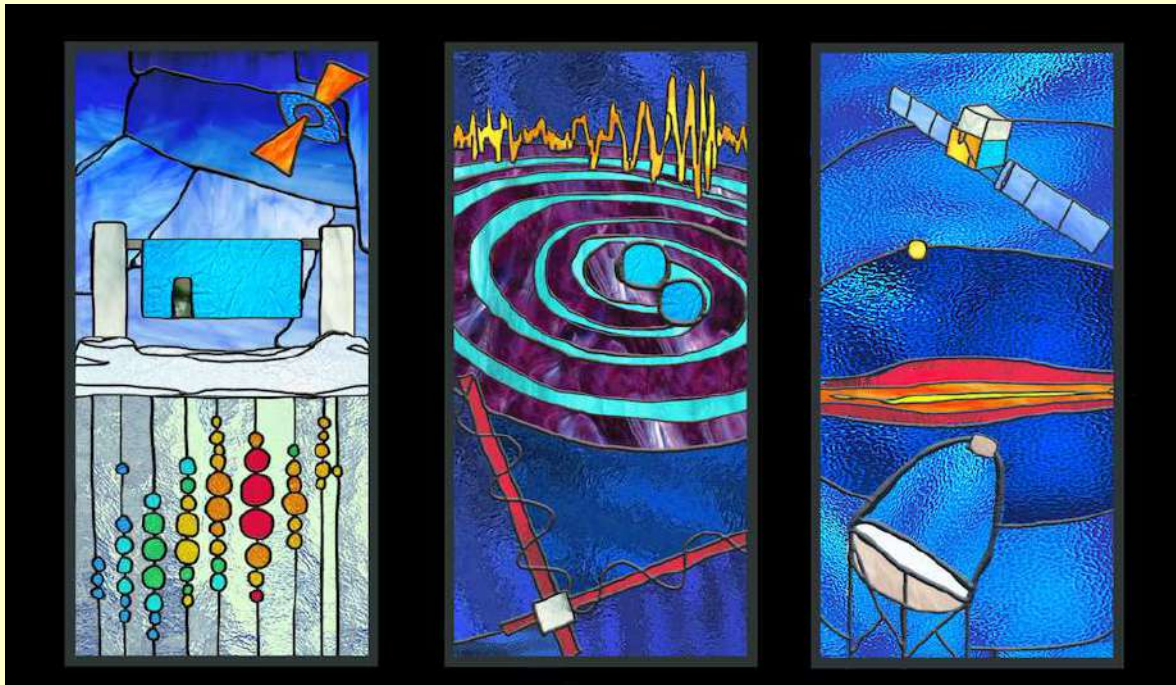
Moscow
International School
of Physics 2022



Skoltech
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Dubna, July 24 – 2 August 2, 2022

Preface



A piece of history: from V. Pauli (1930) to F. Reines & C. Cowan (1956).

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Abschrift/15.12.56 PM

Offener Brief an die Gruppe der Radioaktiven bei der Gauvereins-Tagung zu Tübingen.

Abschrift
Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Des. 1930
Cloriatstrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich mildvollst anhören bitte, Ihnen das näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweigten Ausweg verfallen um den "Wechselgats" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grössenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0,01 Protonenmasse.- Das kontinuierliche beta-Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, dazwischen, dass die Summe der Energien von Neutron und Elektron konstant ist.

Nun handelt es sich weiter darum, welche Kräfte auf die Neutronen wirken. Das wahrscheinlichste Modell für das Neutron scheint mir aus wellenmechanischen Gründen (näheres weiss der Ueberbringer dieser Zeilen) dieses zu sein, dass das ruhende Neutron ein magnetischer Dipol von einem gewissen Moment μ ist. Die Experimente verlangen wohl, dass die ionisierende Wirkung eines solchen Neutrons nicht grösser sein kann, als die eines gamma-Strahls und darf dann wohl nicht grösser sein als $e \cdot (10^{-13} \text{ cm})$.

Ich traue mich vorläufig aber nicht, etwas über diese Idee zu publizieren und wende mich erst vertrauensvoll an Euch, liebe Radioaktive, mit der Frage, wie es um den experimentellen Nachweis eines solchen Neutrons stände, wenn dieses ein ebensolches oder etwa 10mal grösseres Durchdringungsvermögen besitzen würde, wie ein gamma-Strahl.

Ich gebe zu, dass mein Ausweg vielleicht von vornherein wenig wahrscheinlich erscheinen wird, weil man die Neutronen, wenn sie existieren, wohl schon längst gesehen hätte. Aber nur wer wagt, gewinnt und der Ernst der Situation beim kontinuierlichen beta-Spektrum wird durch einen Ausspruch meines verehrten Vorgängers im Amt, Herrn Debye, beleuchtet, der mir kürzlich in Brüssel gesagt hat: "O, daran soll man am besten gar nicht denken, sowie an die neuen Steuern." Darum soll man jeden Weg zur Rettung ernstlich diskutieren.- Also, liebe Radioaktive, prüfet, und richtet.- Leider kann ich nicht persönlich in Tübingen erscheinen, da ich infolge eines in der Nacht vom 6. zum 7. Des. in Zürich stattfindenden Balles hier unabhklich bin.- Mit vielen Grüssen an Euch, sowie an Herrn Bask, Euer untertänigster Diener

gen. W. Pauli

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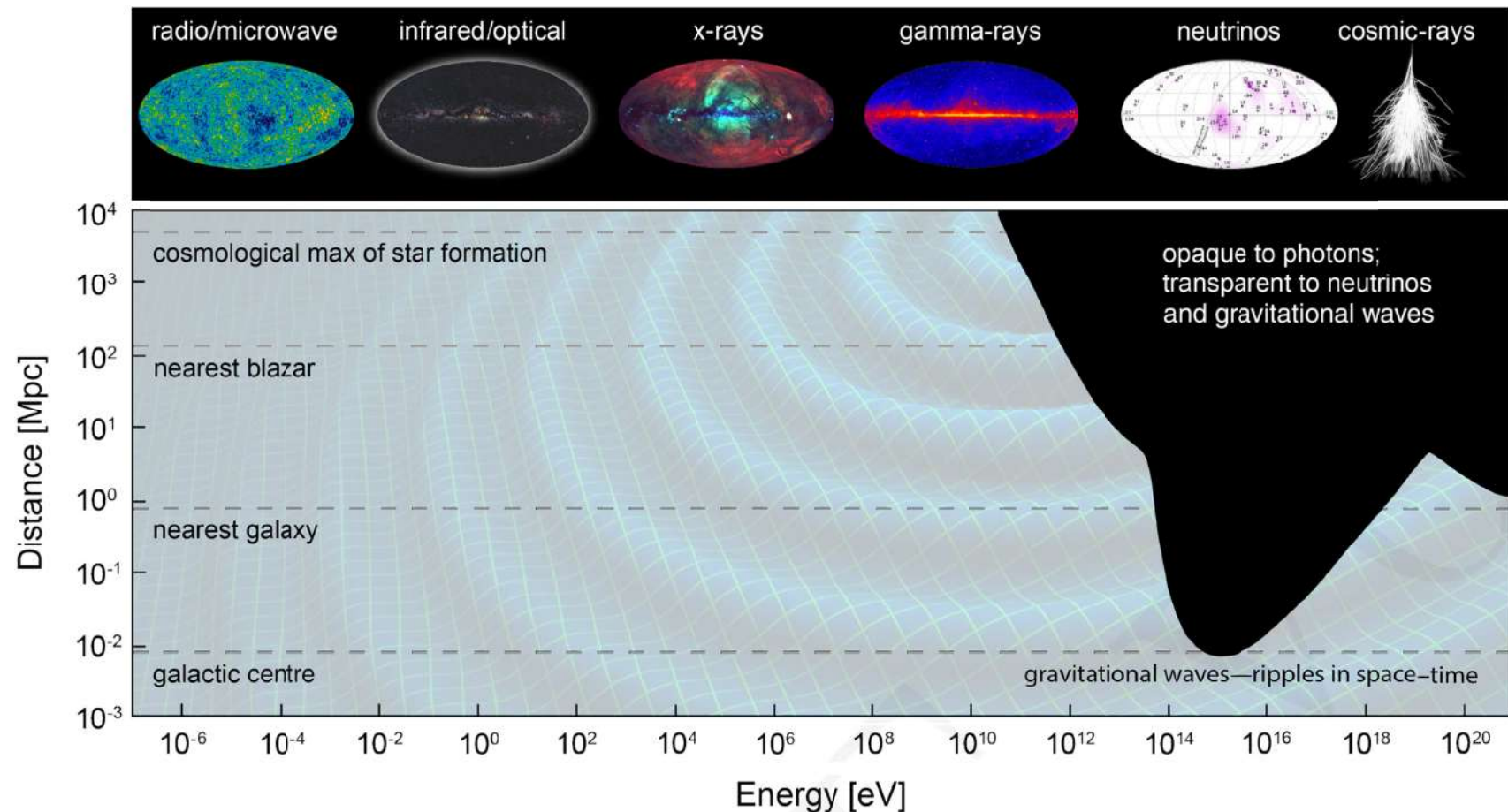
WE ARE HAPPY TO INFORM YOU THAT WE HAVE DEFINITELY DETECTED
NEUTRINOS FROM FISSION FRAGMENTS BY OBSERVING INVERSE BETA DECAY
OF PROTONS OBSERVED CROSS SECTION AGREES WELL WITH EXPECTED SIX
TIMES TEN TO MINUS FORTY FOUR SQUARE CENTIMETERS
FREDERICK REINES AND CLYDE COWAN
BOX 1663 LOS ALAMOS NEW MEXICO

Frederick REINES and Clyde COWAN
Box 1663, LOS ALAMOS, New Mexico

Thanks for message. Everything comes to
him who knows how to wait.

Pauli

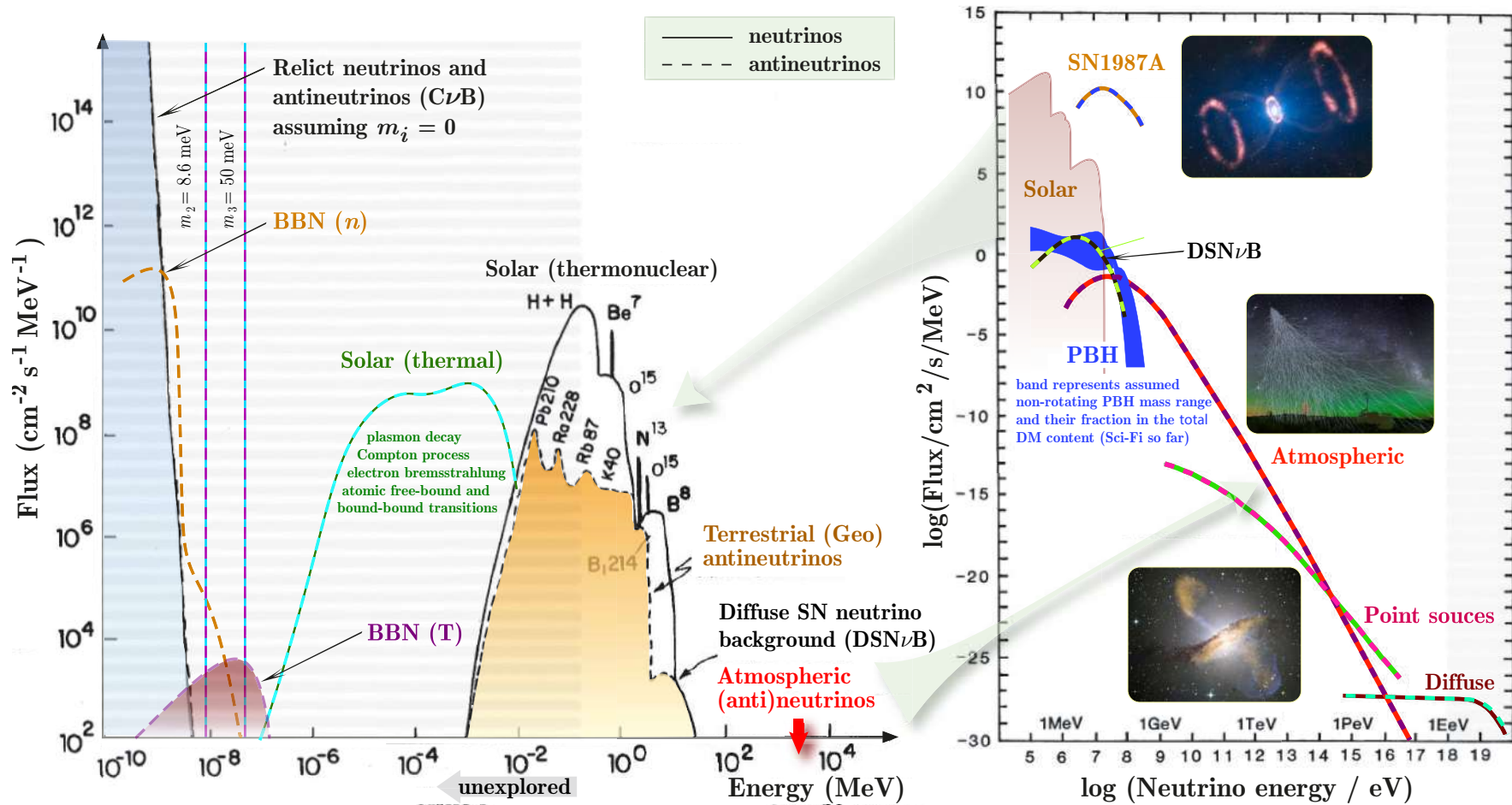
Horizons of multi-messenger high-energy astronomy & astrophysics



△ Figure shows the distances at which the Universe becomes opaque to electromagnetic radiation. While lower-energy photons can travel to us from the farthest corners of the Universe, the highest energy photons and cosmic rays are attenuated after short distances, obscuring our view of the most energetic cosmic events. In contrast, the Universe is transparent to gravitational waves and neutrinos, making them suitable probes of the high-energy sky.

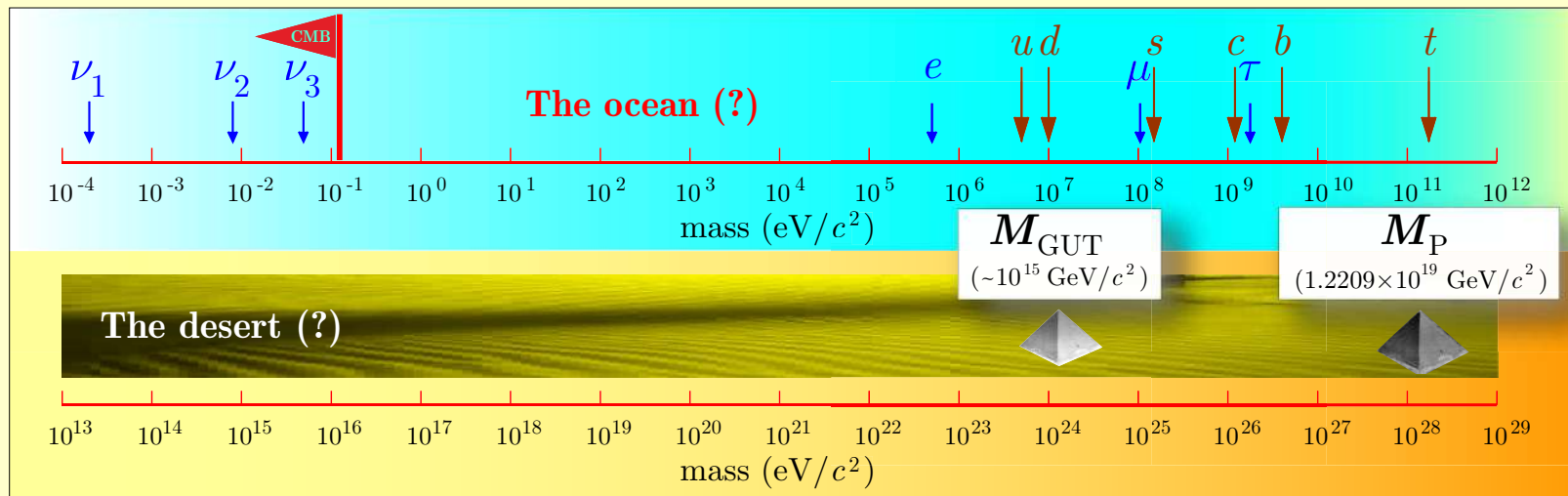
[From I. Bartos & M. Kowalski, “Multimessenger Astronomy” (Physics World Discovery, IoP Publishing, Bristol, 2017).]

Preview of local $\nu/\bar{\nu}$ flows in crude curves



[Constructed from the data of L. M. Krauss *et al.*, “Antineutrino astronomy and geophysics”, *Nature* **310** (1984) 191–198 and E. Vitagliano *et al.*, “Grand unified neutrino spectrum at Earth: Sources and spectral components,” *Rev. Mod. Phys.* **92** (2020) 45006, arXiv:1910.11878 [astro-ph.HE] (*left panel*) and A. M. Bakich, “Aspects of neutrino astronomy”, *Space Sci. Rev.* **49** (1989) 259–310 and R. Calabrese *et al.*, “Primordial black hole dark matter evaporating on the neutrino floor,” *Phys. Lett. B* **829** (2022) 137050, arXiv:2106.02492 [hep-ph] (*right panel*).]

Standard Model with Neutrino Masses



1 Interaction Lagrangian and weak currents.

In the Standard Model (SM), the charged and neutral current neutrino interactions with leptons are described by the following parts of the full Lagrangian:

$$\mathcal{L}_I^{\text{CC}}(x) = -\frac{g}{2\sqrt{2}}j_\alpha^{\text{CC}}(x)W^\alpha(x) + \text{H.c.} \quad \text{and} \quad \mathcal{L}_I^{\text{NC}}(x) = -\frac{g}{2\cos\theta_W}j_\alpha^{\text{NC}}(x)Z^\alpha(x).$$

Here g is the $SU(2)$ (electro-weak) gauge coupling constant

$$g^2 = 4\sqrt{2}m_W^2G_F, \quad g\sin\theta_W = |e|,$$

and θ_W is the weak mixing (Weinberg) angle, ($\sin^2\theta_W(M_Z) = 0.23120$).

The leptonic charged current and neutrino neutral current are given by the expressions:

$$j_\alpha^{\text{CC}}(x) = 2 \sum_{\ell=e,\mu,\tau,\dots} \bar{\nu}_{\ell,L}(x)\gamma_\alpha\ell_L(x) \quad \text{and} \quad j_\alpha^{\text{NC}}(x) = \sum_{\ell=e,\mu,\tau,\dots} \bar{\nu}_{\ell,L}(x)\gamma_\alpha\nu_{\ell,L}(x).$$

Phenomenologically, the charged and neutral currents may include (yet unknown) heavy neutrinos and corresponding heavy charged leptons. The left- and right-handed fermion fields are defined as usually:

$$\begin{cases} \nu_{\ell,L}(x) = P_L\nu_\ell(x), & \ell_L(x) = P_L\ell(x), & P_L \equiv \frac{1}{2}(1 - \gamma_5), \\ \nu_{\ell,R}(x) = P_R\nu_\ell(x), & \ell_R(x) = P_R\ell(x), & P_R \equiv \frac{1}{2}(1 + \gamma_5). \end{cases}$$

Physical meaning of chiral projections for a massive Dirac fermion.

$$(\hat{p} - m)\psi = 0 \implies \begin{pmatrix} p_0 - m & -\mathbf{p}\boldsymbol{\sigma} \\ \mathbf{p}\boldsymbol{\sigma} & -p_0 - m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \implies \begin{cases} (\mathbf{p}\boldsymbol{\sigma})\chi = (p_0 - m)\phi, \\ (\mathbf{p}\boldsymbol{\sigma})\phi = (p_0 + m)\chi. \end{cases}$$

⇓

$$\begin{aligned} \psi_L = P_L\psi &= \frac{1}{2} \begin{pmatrix} \phi - \chi \\ \chi - \phi \end{pmatrix} = \begin{pmatrix} \phi_- \\ -\phi_- \end{pmatrix} \\ \psi_R = P_R\psi &= \frac{1}{2} \begin{pmatrix} \phi + \chi \\ \phi + \chi \end{pmatrix} = \begin{pmatrix} \phi_+ \\ \phi_+ \end{pmatrix} \end{aligned} \quad \left| \quad \text{where } \phi_{\pm} = \frac{1}{2} \left(1 \pm \frac{\mathbf{p}\boldsymbol{\sigma}}{p_0 + m} \right) \phi. \right.$$

Let $p_0 \gg m$ and thus $1 - |\mathbf{v}| \ll 1$, where $\mathbf{v} = \mathbf{p}/p_0$. Then, directing \mathbf{v} along the z axis we obtain

$$\phi_- \simeq \frac{1 - \sigma_3}{2} \phi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{\rightarrow} \\ \phi_{\leftarrow} \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_{\leftarrow} \end{pmatrix}, \quad \phi_+ \simeq \frac{1 + \sigma_3}{2} \phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{\rightarrow} \\ \phi_{\leftarrow} \end{pmatrix} = \begin{pmatrix} \phi_{\rightarrow} \\ 0 \end{pmatrix}.$$

Reminder: Pauli & Dirac matrices

$$\sigma_0 \equiv \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\gamma^0 = \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \gamma^5 = \gamma_5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$



Note that the kinetic term of the Lagrangian includes both L and R handed neutrinos and moreover, it can include other sterile neutrinos:

$$\mathcal{L}_0 = \frac{i}{2} [\bar{\nu}(x)\gamma^\alpha\partial_\alpha\nu(x) - \partial_\alpha\bar{\nu}(x)\gamma^\alpha\nu(x)] \equiv \frac{i}{2}\bar{\nu}(x)\overleftrightarrow{\partial}\nu(x) = \frac{i}{2} [\bar{\nu}_L(x)\overleftrightarrow{\partial}\nu_L(x) + \bar{\nu}_R(x)\overleftrightarrow{\partial}\nu_R(x)],$$

$$\nu(x) = \nu_L(x) + \nu_R(x) = \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \\ \nu_\tau(x) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad \nu_{L/R}(x) = \begin{pmatrix} \nu_{e,L/R}(x) \\ \nu_{\mu,L/R}(x) \\ \nu_{\tau,L/R}(x) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \frac{1 \mp \gamma_5}{2} \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \\ \nu_\tau(x) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Neutrino chirality: $\gamma_5\nu_L = -\nu_L$ and $\gamma_5\nu_R = +\nu_R$.

The Lagrangian of the theory with massless neutrinos is invariant with respect to the global gauge transformations

$$\nu_\ell(x) \rightarrow e^{i\Lambda_\ell}\nu_\ell(x), \quad \ell(x) \rightarrow e^{i\Lambda_\ell}\ell(x) \quad \text{with} \quad \Lambda_\ell = \text{const.}$$

By Noether's theorem this leads to conservation of the individual lepton flavor numbers (more rarely called lepton flavor charges) L_ℓ . It is agreed that

$$L_\ell(\ell^-, \nu_\ell) = +1, \quad L_\ell(\ell^+, \bar{\nu}_\ell) = -1, \quad \ell^\pm = e^\pm, \mu^\pm, \tau^\pm, \text{ etc.}$$

Lepton flavor conservation is not the case for massive neutrinos.

There are **two** fundamentally different kinds of neutrino mass terms: **Dirac** and **Majorana**.

2 Dirac neutrinos

The conventional Dirac mass term for a single spinor field $\psi(x)$ is well known:

$$-m\bar{\psi}(x)\psi(x) = -m [\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R] = -m\bar{\psi}_R(x)\psi_L(x) + \text{H.c.}$$

(the identities $\bar{\psi}_L\psi_L = \bar{\psi}_R\psi_R = 0$ and $(\bar{\psi}_R\psi_L)^\dagger = \bar{\psi}_L\psi_R$ are used here).

The most general extension to the N -generation Dirac neutrino case reads:

$$\mathcal{L}_D(x) = -\bar{\nu}_R(x)\mathbf{M}_D\nu_L(x) + \text{H.c.},$$

where \mathbf{M}_D is a nonsingular [to exclude massless case] complex $N \times N$ matrix.

In general, $N \geq 3$ since the column ν_L may include both *active* and *sterile* neutrino fields which do not enter into the standard charged and neutral currents.

Any nonsingular complex matrix can be diagonalized by means of an appropriate **bi-unitary** transformation

$$\mathbf{M}_D = \tilde{\mathbf{V}}\mathbf{m}\mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k\delta_{kl}|| = \text{diag}(m_1, m_2, \dots, m_N),$$

where \mathbf{V} and $\tilde{\mathbf{V}}$ are unitary matrices and $m_k \geq 0$.

$$\implies \mathcal{L}_D(x) = -\bar{\nu}'_R(x)\mathbf{m}\nu'_L(x) + \text{H.c.} = -\bar{\nu}'(x)\mathbf{m}\nu'(x) = -\sum_{k=1}^N m_k \bar{\nu}_k(x)\nu_k(x),$$

where the new fields ν_k are defined by

$$\nu'_L(x) = \mathbf{V}^\dagger\nu_L(x), \quad \nu'_R(x) = \tilde{\mathbf{V}}^\dagger\nu_R(x), \quad \nu'(x) = (\nu_1, \nu_2, \dots, \nu_N)^T.$$

The fields $\nu'_R(x)$ do not enter into $\mathcal{L}_I \implies$ the matrix $\tilde{\mathbf{V}}$ remains out of play...



Since $\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V} = \mathbf{1}$ and $\tilde{\mathbf{V}}^\dagger\tilde{\mathbf{V}} = \tilde{\mathbf{V}}\tilde{\mathbf{V}}^\dagger = \mathbf{1}$, the neutrino kinetic term in the Lagrangian is transformed to

$$\mathcal{L}_0 = \frac{i}{2} \left[\bar{\nu}'_L(x) \overleftrightarrow{\partial} \nu'_L(x) + \bar{\nu}'_R(x) \overleftrightarrow{\partial} \nu'_R(x) \right] = \frac{i}{2} \bar{\nu}'(x) \overleftrightarrow{\partial} \nu'(x) = \frac{i}{2} \sum_k \bar{\nu}_k(x) \overleftrightarrow{\partial} \nu_k(x).$$



$\nu_k(x)$ is the field of a Dirac neutrino with the mass m_k and the flavor LH neutrino fields $\nu_{\ell,L}(x)$ involved into the SM weak lepton currents are linear combinations of the LH components of the fields of the neutrinos with definite masses:

$$\nu_L = \mathbf{V}\nu'_L \quad \text{or} \quad \nu_{\ell,L} = \sum_k V_{\ell k} \nu_{k,L}.$$

The matrix \mathbf{V} is referred to as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix while the matrix $\tilde{\mathbf{V}}$ is not honored with a personal name.

Quark-lepton complementarity (QLC): Of course the PMNS matrix it is not the same as the CKM (Cabibbo-Kobayashi-Maskawa) quark mixing matrix. However the PMNS and CKM matrices may be, in a sense, *complementary* to each other.

The QLC means that *in the same (PDG) parametrizations* the sums of (small) quark and (large) lepton mixing angles are almost (i.e., within errors) equal to $\pi/4$ for $(ij) = (12)$ and (23) :

$$\theta_{12}^{\text{CKM}} + \theta_{12}^{\text{PMNS}} = (46.49 \pm 0.77)^\circ, \quad \theta_{23}^{\text{CKM}} + \theta_{23}^{\text{PMNS}} = (44.48 \pm 1.10)^\circ, \quad \text{sum} = (90.97 \pm 1.34)^\circ.$$

The origin of the data (but not QLC) will be explained below.

2.1 Parametrization of mixing matrix for Dirac neutrinos.

It is well known that a complex $n \times n$ unitary matrix depends on n^2 real parameters.

The classical result by Francis Murnaghan [F. D. Murnaghan, “The unitary and rotation groups (Lectures on Applied Mathematics, Volume 3),” Spartan Books, Washington, D.C. (1962)] states that any $n \times n$ matrix from the unitary group $U(n)$ can be presented as product of the diagonal phase matrix

$$\Gamma = \text{diag} (e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_n}),$$

containing n phases α_k , and $n(n-1)/2$ matrices \mathbf{U} whose main building blocks have the form

$$\begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ -\sin \theta e^{+i\phi} & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{+i\phi} \end{pmatrix} \underbrace{\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}}_{\text{Euler rotation}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

Therefore any $n \times n$ unitary matrix can be parametrized in terms of

$$n(n-1)/2 \text{ “angles” (taking values within } [0, \pi/2])$$

and

$$n(n+1)/2 \text{ “phases” (taking values within } [0, 2\pi)).$$

The usual parametrization of both the CKM and PMNS matrices is of this type.

IMPORTANT: Murnaghan’s factorization method does not specify the sequence of the building blocks Γ and \mathbf{U} .

One can reduce the number of the phases further by taking into account that the Lagrangian with the Dirac mass term is invariant with respect to the transformation

$$\ell \mapsto e^{ia_\ell} \ell, \quad \nu_k \mapsto e^{ib_k} \nu_k, \quad V_{\ell k} \mapsto e^{i(b_k - a_\ell)} V_{\ell k},$$

and to the global gauge transformation

$$\ell \mapsto e^{i\Lambda} \ell, \quad \nu_k \mapsto e^{i\Lambda} \nu_k, \quad \text{with } \Lambda = \text{const.} \quad (1)$$

Therefore $2N - 1$ phases are unphysical and the number of physical (Dirac) phases is

$$n_D = \frac{N(N+1)}{2} - (2N-1) = \frac{N^2 - 3N + 2}{2} = \frac{(N-1)(N-2)}{2} \quad (N \geq 2);$$

$$n_D(2) = 0, \quad n_D(3) = 1, \quad n_D(4) = 3, \dots$$

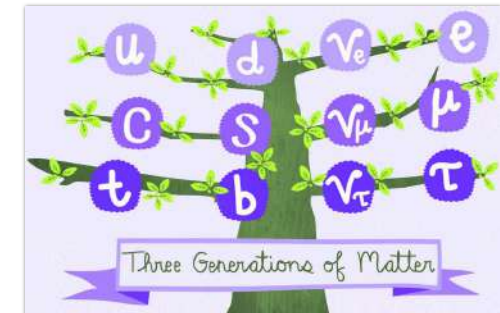
- The global symmetry (1) leads to conservation of the lepton charge

$$L = \sum_{\ell=e,\mu,\tau,\dots} L_\ell$$

common to all charged leptons and all neutrinos ν_k . However

The individual lepton flavor numbers L_ℓ are no longer conserved.

- The nonzero physical phases lead to the CP (and T) violation in the neutrino sector.^a This could have important implications for particle physics and cosmology (leptogenesis, baryogenesis,...).



^aThe proof can be found, e.g., in Sec. 4.6 of C. Giunti and C. W. Kim, “Fundamentals of neutrino physics and astrophysics” (Oxford University Press Inc., New York, 2007) or in Sec. 6.3 of S. M. Bilenky, “Introduction to the physics of massive and mixed neutrinos” (2nd ed.), Lect. Notes Phys. **947** (2018) 1–276. Note the differences in notation and in representation for the matrix C .

2.1.1 Three-neutrino case.

In the most interesting (today!) case of three lepton generations one defines the orthogonal rotation matrices in the ij -planes which depend upon the mixing angles θ_{ij} :

$$\mathbf{O}_{12} = \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Solar matrix}}, \quad \mathbf{O}_{13} = \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}}_{\text{Reactor matrix}}, \quad \mathbf{O}_{23} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{Atmospheric matrix}},$$

(where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$) and the diagonal matrix with the Dirac phase factor:

$$\Gamma_{\text{D}} = \text{diag}(1, 1, e^{i\delta}).$$

The parameter δ is commonly referred to as the Dirac CP -violation/violating phase.

Finally, by applying Murnaghan's factorization, the PMNS matrix for the Dirac neutrinos can be parametrized as

$$\mathbf{V}_{(\text{D})} = \mathbf{O}_{23} \Gamma_{\text{D}} \mathbf{O}_{13} \Gamma_{\text{D}}^{\dagger} \mathbf{O}_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

- ★ This is the *Chau–Keung presentation* advocated by the PDG for both CKM and PMNS matrices.
- ★ Remember that the positioning of the factors in $\mathbf{V}_{(\text{D})}$ is not fixed by the Murnaghan (or any other) algorithm and is just a subject-matter of agreement.
- ★ Today we believe we know a lot about the entries of this matrix.

2.1.2 Lepton numbers are not conserved, so what of it?.

Since the Dirac mass term violates conservation of the individual lepton numbers, L_e, L_μ, L_τ , it allows many lepton family number violating processes, like

$$\begin{aligned} \mu^\pm &\rightarrow e^\pm + \gamma, \quad \mu^\pm \rightarrow e^\pm + e^+ + e^-, \\ K^+ &\rightarrow \pi^+ + \mu^\pm + e^\mp, \quad K^- \rightarrow \pi^- + \mu^\pm + e^\mp, \\ \mu^- + (A, Z) &\rightarrow e^- + (A, Z), \quad \tau^- + (A, Z) \rightarrow \mu^- + (A, Z), \dots \end{aligned}$$

However the $(\beta\beta)_{0\nu}$ decay or the kaon semileptonic decays like

$$K^+ \rightarrow \pi^- + \mu^+ + e^+, \quad K^- \rightarrow \pi^+ + \mu^- + e^-,$$

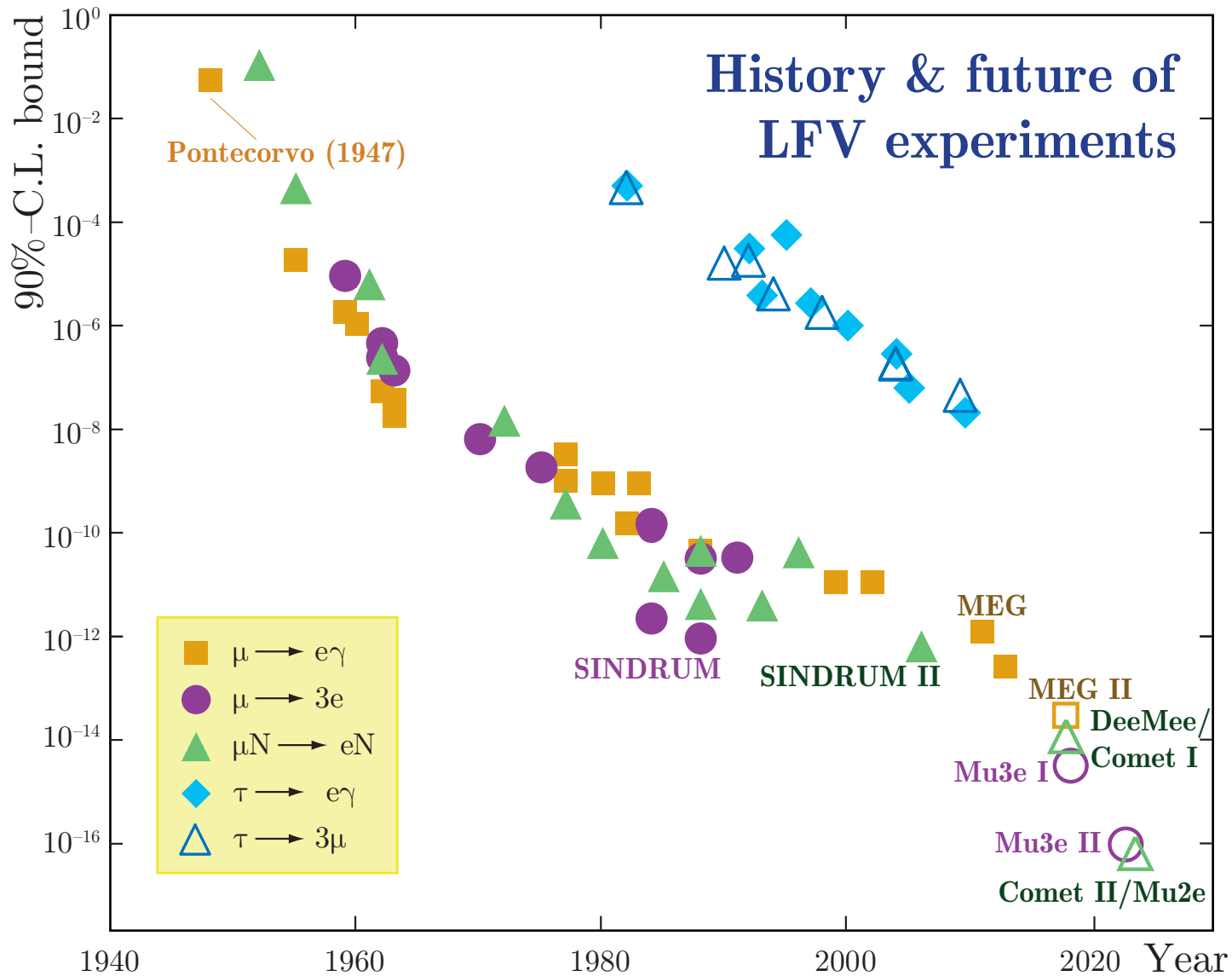
etc. are still *forbidden* as a consequence of the total lepton charge conservation.

Current limits on the simplest lepton family number violating μ and τ decays (2020). ^a

Decay Modes	Fraction	C.L.	Decay Modes	Fraction	C.L.
$\mu^- \rightarrow e^- \nu_e \bar{\nu}_\mu$	< 1.2%	90%	$\tau^- \rightarrow e^- \gamma$	< 3.3×10^{-8}	90%
$\mu^- \rightarrow e^- \gamma$	< 4.2×10^{-13}	90%	$\tau^- \rightarrow \mu^- \gamma$	< 4.4×10^{-8}	90%
$\mu^- \rightarrow e^- e^+ e^-$	< 1.0×10^{-12}	90%	$\tau^- \rightarrow e^- \pi^0$	< 8.0×10^{-8}	90%
$\mu^- \rightarrow e^- 2\gamma$	< 7.2×10^{-11}	90%	$\tau^- \rightarrow \mu^- \pi^0$	< 1.1×10^{-7}	90%

These limits are not quite as impressive as might appear at first glance.

^aP. A. Zyla *et al.* (Particle Data Group), “Review of Particle Physics”, PTEP **2020** (2020) 083C01.



[From N. Berger, "Charged lepton flavour violation experiments," talk at the Zürich Phenomenology Workshop, January 2015. For details, see W. J. Marciano, T. Mori, and J. M. Roney, "Charged lepton flavor violation experiments," *Ann. Rev. Nucl. Part. Sci.* **58** (2008) 315–341. Is not yet updated!]

2.1.3 Neutrinoless muon decay in SM.

The L_μ and L_e violating muon decay $\mu^- \rightarrow e^- \gamma$ is allowed if $V_{\mu k}^* V_{ek} \neq 0$ for $k = 1, 2$ or 3 . The corresponding Feynman diagrams include W loops and thus the decay width is strongly suppressed by the neutrino to W boson mass ratios:

$$R = \frac{\Gamma(\mu^- \rightarrow e^- \gamma)}{\Gamma(\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e)} = \frac{3\alpha}{32\pi} \left| \sum_k V_{\mu k}^* V_{ek} \frac{m_k^2}{m_W^2} \right|^2.$$

Since $m_k/m_W \approx 1.244 \times 10^{-12} (m_k/0.1 \text{ eV})$, the ratio can be estimated as

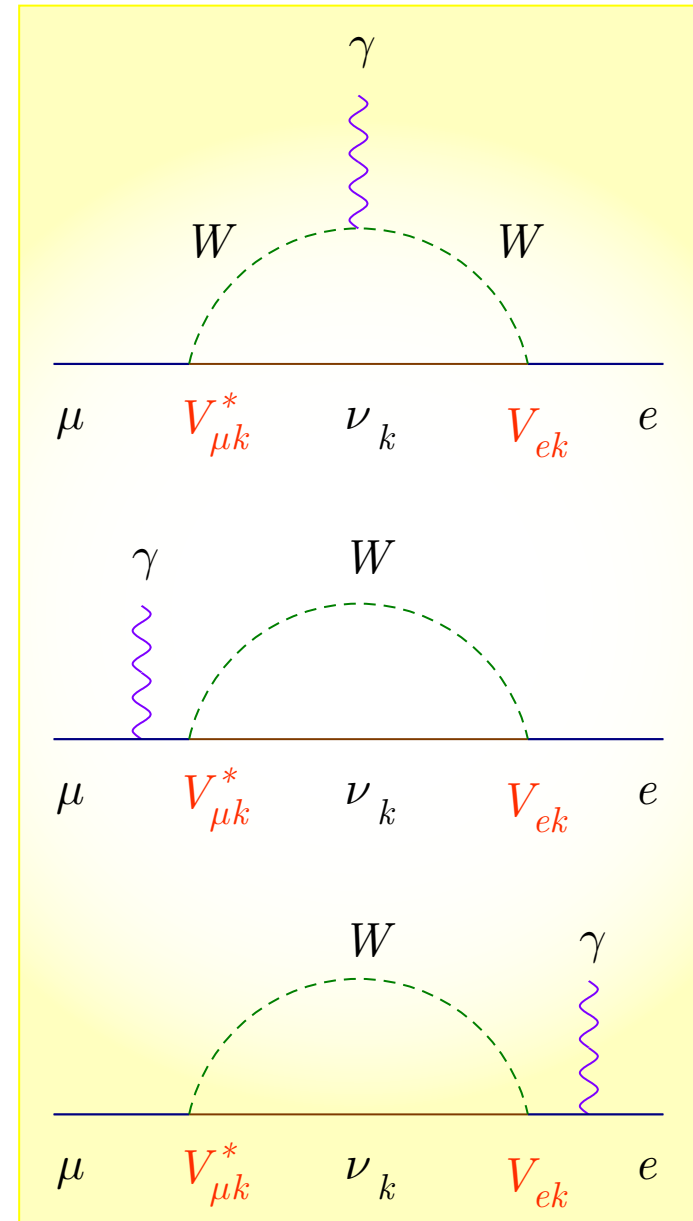
$$R \approx 5.22 \times 10^{-52} \left| \sum_k V_{\mu k}^* V_{ek} \left(\frac{m_k}{0.1 \text{ eV}} \right)^2 \right|^2 \lesssim 8 \times 10^{-54},$$

while the current experimental upper limit is (at least!) 40 orders of magnitude larger (see Table in p. 16):

$$R_{(\text{exp})} < 4.2 \times 10^{-13} \text{ at } 90\% \text{ C.L. } \quad (\text{NO GO!})$$

Some nonstandard models are much more optimistic.

We must deeply appreciate the oscillation phenomenon which makes the miserable ν mass effect measurable.



2.2 Nuclear beta decay.

The method of measurement of the (anti)neutrino mass through the investigation of the high-energy part of the β -spectrum was proposed by Perrin (1933) and Fermi (1934).

The first experiments on the measurement of the neutrino mass with this method have been done by Curran, Angus and Cockcroft (1948) and Hanna and Pontecorvo (1949).

The energy spectrum of electrons in the decay $(A, Z) \rightarrow (A, Z + 1) + e^- + \bar{\nu}_e$ is^a

$$\frac{d\Gamma}{dT} = \sum_k |V_{ek}|^2 \frac{d\Gamma_k}{dT}, \quad (2)$$

$$\frac{d\Gamma_k}{dT} = \frac{(G_F \cos \theta_C)^2}{2\pi^3} p p_k (T + m_e) (Q - T) |\mathcal{M}|^2 F(T, Z) \theta(Q - T - m_k). \quad (3)$$

Here G_F is the Fermi constant, θ_C is the Cabibbo angle, m_e , p and T are the mass, magnitude of the momentum and kinetic energy of the electron, respectively,

$$p_k = \sqrt{E_k^2 - m_k^2} = \sqrt{(Q - T)^2 - m_k^2} \quad \text{and} \quad Q = E_k + T = E_{A,Z} - E_{A,Z+1} - m_e$$

are, respectively, the magnitude of the neutrino momentum and energy released in the decay (the endpoint of the β spectrum in case $m_k = 0$), \mathcal{M} is the nuclear matrix element, and $F(T, Z)$ is the Fermi function, which describes the Coulomb interaction of the final-state nucleus and electron.

The step function in Eq. (3) ensures that a neutrino state ν_k is only produced if its total energy is larger than its mass: $E_k = Q - T \geq m_k$.

^aThe recoil of the final nucleus and radiative corrections (luckily small) are neglected.

As it is seen from Eq. (2), the largest distortion of the β -spectrum due to neutrino masses can be observed in the region

$$Q - T \sim m_k. \quad (4)$$

However, for $\max(m_k) \simeq 0.1 \text{ eV}$ only a very small part (about $10^{-(13-14)}$) of the decays give contribution to the region (4). This is the reason why in the analysis of the results of the measurement of the β -spectrum a relatively large part of the spectrum is used.^a

Taking this into account and applying unitarity of the mixing matrix, we can write

$$\begin{aligned} \sum_k |V_{ek}|^2 p_k &\approx \sum_k |V_{ek}|^2 (Q - T) \left[1 - \frac{m_k^2}{2(Q - T)^2} \right] && \Leftarrow 4E_k^2 \gg m_k^2 \\ &= (Q - T) \left[1 - \frac{1}{2(Q - T)^2} \sum_k |V_{ek}|^2 m_k^2 \right] && \Leftarrow \sum_k |V_{ek}|^2 = 1 \\ &\approx \sqrt{(Q - T)^2 - m_\beta^2}, \end{aligned}$$

where the effective neutrino mass m_β is defined by

$$m_\beta^2 = \sum_k |V_{ek}|^2 m_k^2$$

and it was assumed that

$$\max_k (m_k^2) \ll 4(Q - T)^2.$$

^aFor example, in the Mainz tritium experiment (see below) the last 70 eV of the spectrum is used.

Finally, the β -spectrum that is used for fitting the data can be presented as

$$\frac{d\Gamma}{dT} \propto p(T + m_e) |\mathcal{M}|^2 F(T) K^2(T),$$

where we have defined the *Kurie function* (sometimes called Fermi-Kurie function)

$$K(T) \propto \sqrt{\frac{d\Gamma/dT}{p(T + m_e) |\mathcal{M}|^2 F(T)}} \\ \approx (Q - T) \left[1 - \frac{m_\beta^2}{(Q - T)^2} \right]^{1/4}$$

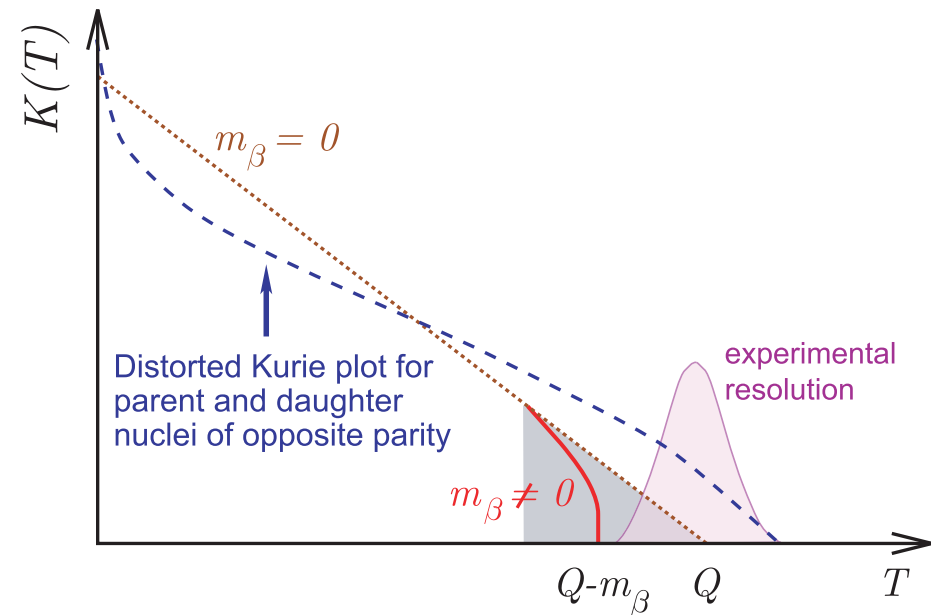
developed by Franz Newell Devereux Kurie.

Unfortunately, the real-life situation is much more complicated.

In an actual experiment, the measurable quantity is a sum of β spectra, leading each with probability $P_n = P_n(E_0 - V_n - E)$ to a final state n of excitation energy V_n :

$$\frac{d\Gamma(T, Q)}{dT} \mapsto \sum_n P_n (E_0 - V_n - E) \frac{d\Gamma(T, E_0 - V_n)}{dT}.$$

Here $E_0 = Q - \mathcal{E}$ the ground-state energy and \mathcal{E} is the recoil energy of the daughter nucleus.



Kurie plot for allowed processes is a sensitive test of m_β , while the first order forbidden processes should have a distorted Kurie plot.

2.2.1 Tritium beta decay.

An important issue is the decay of molecular tritium $T_2 \rightarrow ({}^3\text{HeT})^+ + e^- + \bar{\nu}_e$. Considering the most precise direct determination of the mass difference

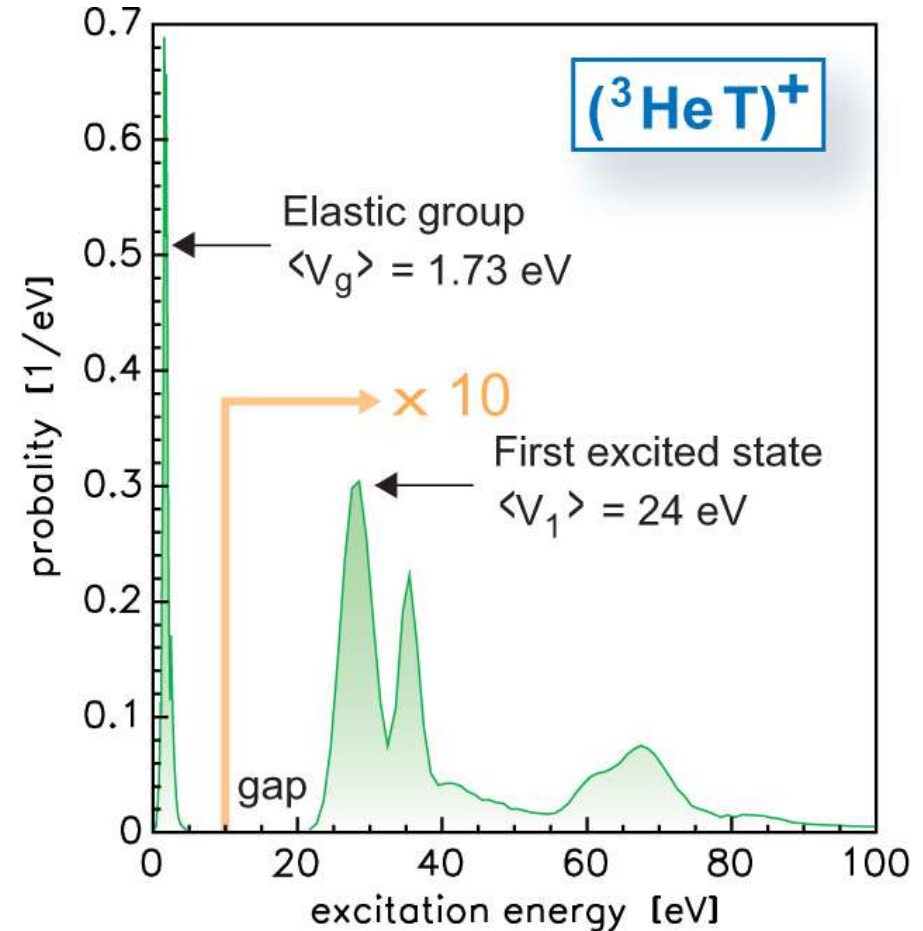
$$m(T) - m({}^3\text{He}) = (18590.1 \pm 1.7) \text{ eV}/c^2$$

and taking into account the recoil and apparative effects (these are taken for the Mainz experiment) one derives an endpoint energy of the molecular ion $({}^3\text{HeT})^+$ ground state:

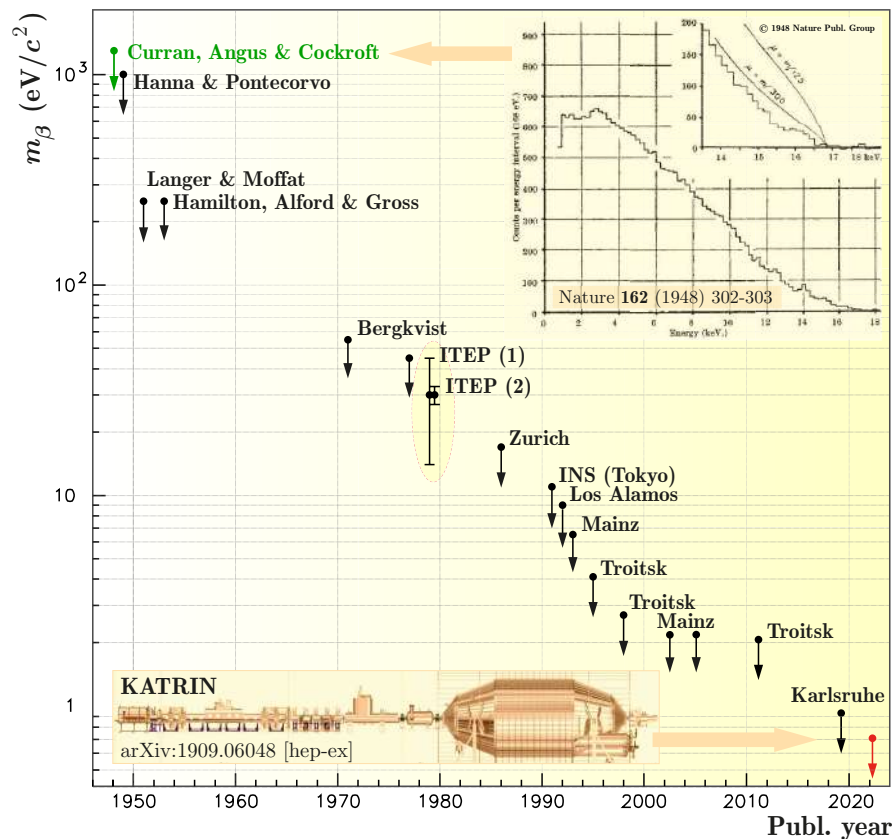
$$E_0 = (18574.3 \pm 1.7) \text{ eV}.$$

The excitation spectrum is shown in the figure. The first group concerns rotational and vibrational excitation of the molecule in its electronic ground state; it comprises a fraction of $P_g = 57.4\%$ of the total rate.

For more details, see C. Kraus *et al.*, "Final results from phase II of the Mainz neutrino mass search in tritium β decay," *Eur. Phys. J. C* **40** (2005) 447–468, hep-ex/0412056.



Excitation spectrum of the daughter molecular ion $({}^3\text{HeT})^+$ in β decay of molecular tritium.



Progress of the neutrino mass measurements in tritium β decay, including the final Mainz phase II, Troitsk, and KATRIN upper limits (see below).

[The compilation is taken from V. M. Lobashev, “Direct search for mass of neutrino,” in Proceedings of the 18th International Conference on Physics in Collision (“PIC 98”), Frascati, June 17–19, 1998, pp. 179–194 and supplemented with the recent data.]

◁ The history of the search for the neutrino mass in the tritium β decay counts more than 60 years. In 1980, the steady improvement of the upper limit was suddenly speeded up by a report of the ITEP group (Moscow) on the observation of the nonzero neutrino mass effect in the β -spectrum in the valine molecule ($C_5H_9T_2NO_2$). The reported result was^a

$$14 \leq m_\beta \leq 46 \text{ eV}/c^2 \quad (99\% \text{ C.L.})$$

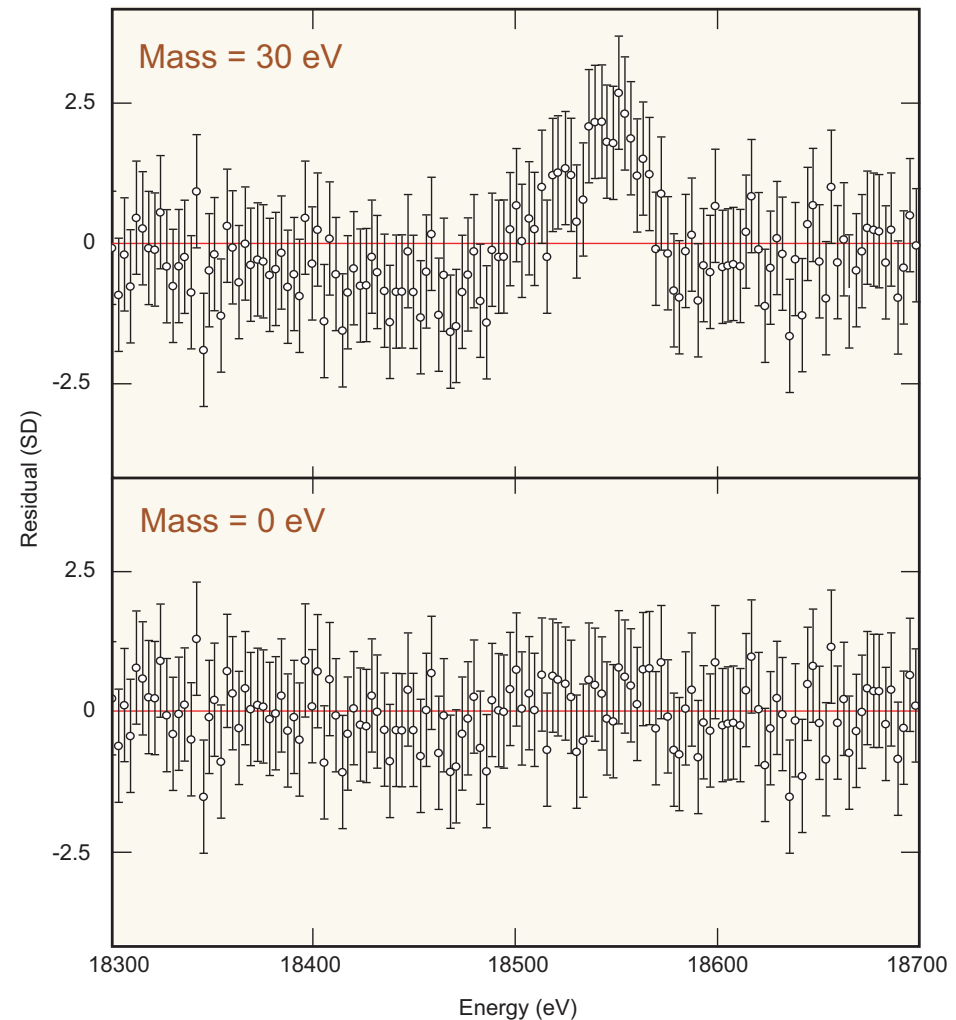
This research stimulated more than 20 experimental proposals with an intention to check this claim. Alas!... in several years the experimental groups from Zürich, Tokyo, Los Alamos, and then Livermore refuted the ITEP result.

^aV. A. Lyubimov, E. G. Novikov, V. Z. Nozik, E. F. Tretyakov, and V. S. Kosik, “An estimate of the ν_e mass from the β -spectrum of tritium in the valine molecule,” Phys. Lett. B **94** (1980) 266–268 (~ 500 citations in InSPIRE! by the end of 2021).

The top figure shows the data points from the tail of the β -spectrum measured in the Los Alamos tritium experiment compared with the expected values (the straight line) for $m_\beta = 30$ eV. The data wander from the line, ruling out the possibility of a 30-eV neutrino.

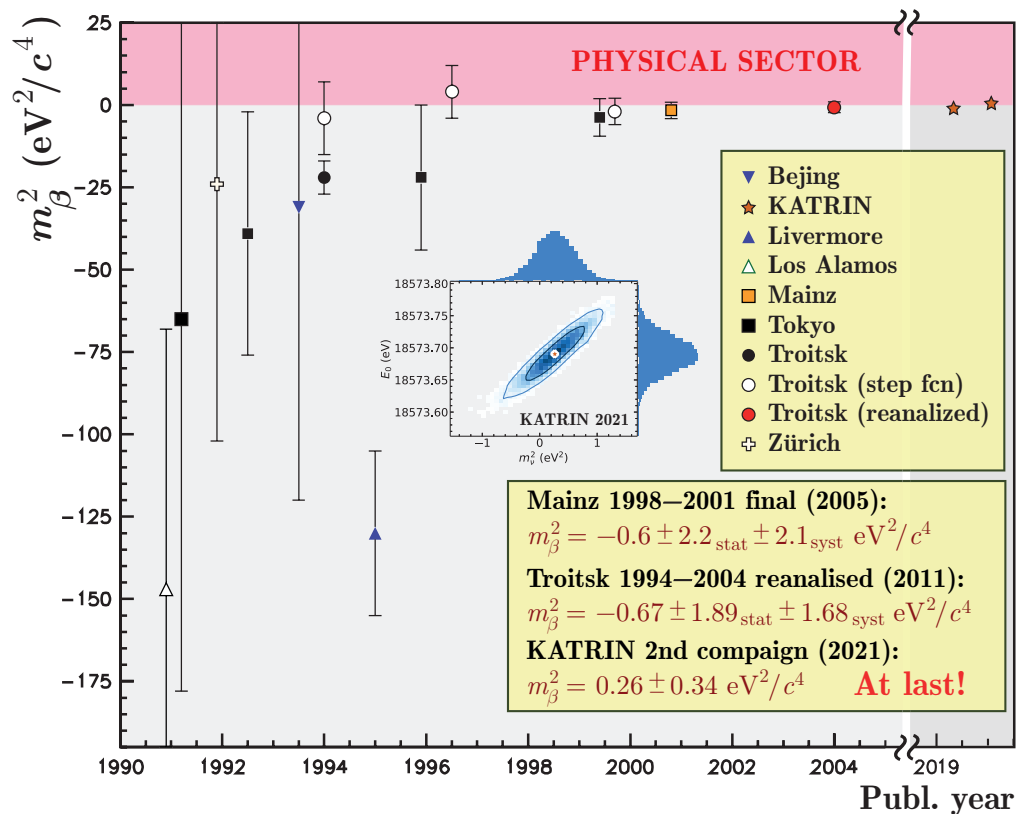
The bottom figure shows the same data points compared with the expectation for $m_\beta = 0$. While the data clearly favor a neutrino mass of zero, the best fit is actually for a slightly negative m_β . (Note that in the bottom plot, the data points lie, on average, slightly above the line, so this is not a perfect fit.)

Both plots display “residuals,” which indicate how many standard deviations each data point is from a particular hypothesis.



Did the neutrino weigh 30 electron volts?

[Borrowed from T. J. Bowles and R. G. H. Robertson, “Tritium beta decay and the search for neutrino mass,” *Los Alamos Sci.* **25** (1997) 6–11.]



◁ The figure shows the results on the m_β^2 measurements in the tritium β decay experiments reported after 1990.

The already finished experiments at Los Alamos, Zürich, Tokyo, Beijing and Livermore used magnetic spectrometers, while the experiments at Troitsk (ν mass), Mainz, and Karlsruhe (KATRIN) are using high-resolution electrostatic filters with magnetic adiabatic collimation.

The progress in the observable m_β of the latest Mainz, Troitsk, and KATRIN results as compared to the most sensitive earlier experiments approaches **two orders of magnitude**.

[The figure in this slide includes the data from C. Kraus *et al.*, *Eur. Phys. J. C* **40** (2005) 447–468, hep-ex/0412056; V. N. Aseev *et al.*, *Phys. Rev. D* **84** (2011) 112003, arXiv:1108.5034 [hep-ex]; M. Aker *et al.*, *Phys. Rev. Lett.* **123** (2019) 221802, arXiv:1909.06048 [hep-ex] M. Aker *et al.*, arXiv:2105.08533 [hep-ex].]

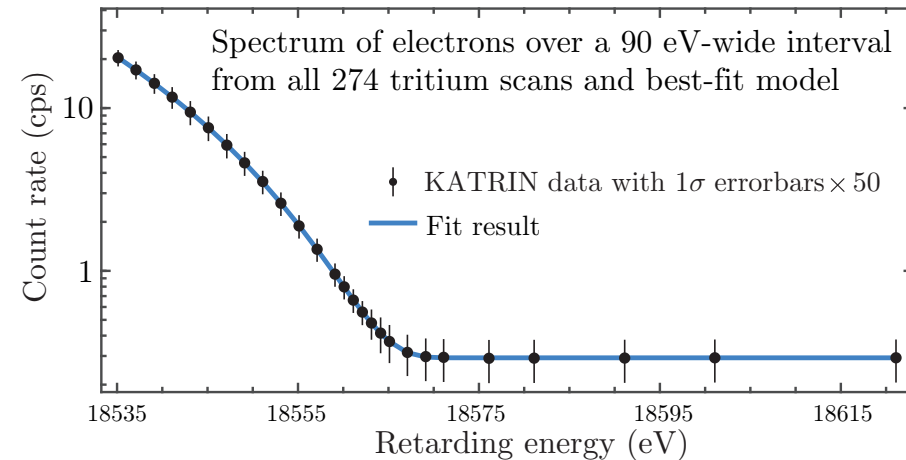
The negative m_β^2 most probably was “instrumental”. After KATRIN (2021), only a very small space remains for fans of heterodox models with *tachyonic* neutrino states (more generally – superpositions of *bradyon-luxon-tachyon* states), *pseudotachyonic* ($m_\nu^2 < 0$, $v = E/p$), or perhaps *superbradyonic* ($m_\nu > 0$, $v > 1$) neutrinos.

2.2.2 Summary of the KATRIN result from the first science run (KNM1).

The best fit value of the effective neutrino mass square was found to be^a

$$m_{\beta}^2 = (-1.0_{-1.1}^{+0.9}) \text{ eV}^2.$$

This result corresponds to a 1σ statistical fluctuation to negative values of m_{β}^2 possessing a p -value of **0.16**. The total uncertainty budget of m_{β}^2 is largely dominated by σ_{stat} (**0.97 eV²**) as compared to σ_{syst} (**0.32 eV²**). These uncertainties are smaller by a factor of **2** and **6**, respectively, compared to the final results of Troitsk and Mainz.



The methods of Lokhov and Tkachov (LT) and of Feldman and Cousins (FC) are then used to calculate the upper limit on the absolute mass scale of neutrino:

$$m_{\beta} < 1.1 \text{ eV at } 90\% \text{ C.L. (LT), } m_{\beta} < 0.8 \text{ (0.9) eV at } 90 \text{ (95)\% C.L. (FC).}$$

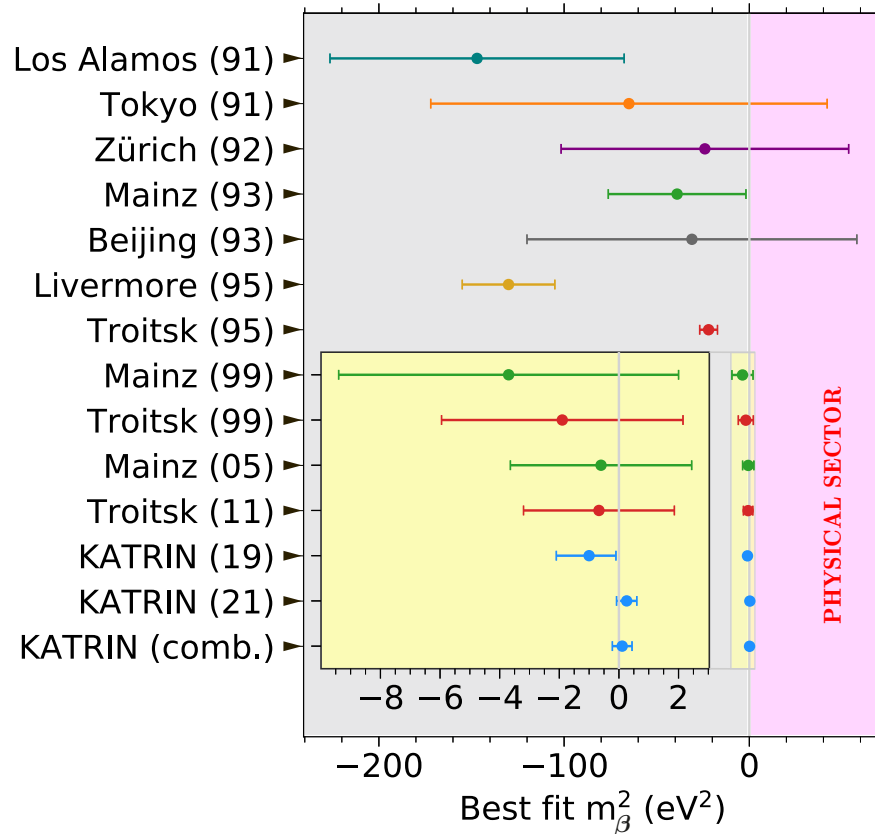
The LT value (the central result of the experiment) coincides with the KATRIN sensitivity. It is based on a purely kinematic method and improves upon previous works by almost a factor of two after a measuring period of only **four weeks** while operating at reduced column density.

After **1000 days** of data taking at nominal column density and further reductions of systematics the Karlsruhe Tritium Neutrino experiment KATRIN will reach a sensitivity of **0.2 eV** (90% C.L.) on m_{β} .

^aM. Aker *et al.*, “An improved upper limit on the neutrino mass from a direct kinematic method by KATRIN,” Phys. Rev. Lett. **123** (2019) 221802, arXiv:1909.06048 [hep-ex].

2.2.3 Summary of the KATRIN result from the second science run (KNM2).

In the 2nd physics run, the source activity was increased by a factor of 3.8 and the background was reduced by 25% with respect to the 1st campaign.^a A sensitivity on m_β of 0.7 eV at 90% C.L. was reached. This is the first sub-eV sensitivity from a direct neutrino-mass experiment.



The best fit to the spectral data yields $m_\beta = 0.26 \pm 0.34$ eV, resulting in an upper limit of $m_\beta < 0.9$ eV (90% C.L.), using the Lokhov-Tkachov method. The Feldman-Cousins technique yields the same limit. The resulting Bayesian limit at 90% C.L. is $m_\beta < 0.85$ eV.

A simultaneous fit of both KNM1 and KNM2 data sets yields $m_\beta = 0.1 \pm 0.3$ eV, resulting in an improved limit of $m_\beta < 0.8$ eV (90% C.L.).

As both data sets are statistics-dominated, correlated systematic uncertainties between both campaigns are negligible.

◁ The figure displays the evolution of best-fit m_β results from historical ν -mass measurements (c.f. p. 25).

$m_\beta < 0.9$ eV at 90 % C.L. (KNM2), $m_\beta < 0.8$ eV at 90 % C.L. (KNM1+KNM2).

^aM. Aker *et al.*, “First direct neutrino-mass measurement with sub-eV sensitivity”, Nature Phys. **18** (2022) 160–166, arXiv:2105.08533 [hep-ex]; see also arXiv:2203.08059 [nucl-ex], submitted to Nature Physics.

3 Majorana neutrinos.

The charge conjugated bispinor field ψ^c is defined by the transformation

$$\psi \mapsto \psi^c = C\bar{\psi}^T, \quad \bar{\psi} \mapsto \bar{\psi}^c = -\psi^T C,$$

where C is the charge-conjugation matrix which satisfies the conditions

$$C\gamma_\alpha^T C^\dagger = -\gamma_\alpha, \quad C\gamma_5^T C^\dagger = \gamma_5, \quad C^\dagger = C^{-1} = C, \quad C^T = -C,$$

and thus coincides (up to a phase factor) with the inversion of the axes x_0 and x_2 : $C = \gamma_0\gamma_2$.

Clearly the **charged** fermion field ψ is different from the charge-conjugated field ψ^c but a **neutral** fermion field ν can coincide with the charge-conjugated one ν^c . In other words: for a **neutral** fermion (neutrino, neutralino) field $\nu(x)$ the following condition is not forbidden: ^a

$$\nu^c(x) = \nu(x) \quad (\text{Majorana condition}) \quad \iff \quad \text{Majorana neutrino and antineutrino coincide!}$$

A few more details: In the chiral representation

$$\nu = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad \nu^c = C\bar{\nu}^T = \begin{pmatrix} -\sigma_2\chi^* \\ +\sigma_2\phi^* \end{pmatrix}. \quad \implies \quad \begin{cases} \phi = -\sigma_2\chi^*, \\ \chi = +\sigma_2\phi^* \end{cases} \quad \implies \quad \phi + \chi = \sigma_2(\phi - \chi)^*.$$

The Majorana neutrino is two-component, i.e., it is defined by only one chiral projection. Then (c.f. p. 9)

$$\nu_L = P_L\nu = \begin{pmatrix} \phi - \chi \\ \chi - \phi \end{pmatrix} \quad \text{and} \quad \nu_R = P_R\nu = \begin{pmatrix} \phi + \chi \\ \phi + \chi \end{pmatrix} = \nu_L^c. \quad \implies \quad \nu = \nu_L + \nu_R = \nu_L + \nu_L^c.$$

^aThe simplest generalization of the Majorana condition, $\nu^c(x) = e^{i\varphi}\nu(x)$ ($\varphi = \text{const}$), is not very interesting.



The Majorana mass term in the general N -neutrino case is [Gribov & Pontecorvo (1969)]:

$$\mathcal{L}_M(x) = -\frac{1}{2}\bar{\nu}_L^c(x)\mathbf{M}_M\nu_L(x) + \text{H.c.},$$

Here \mathbf{M}_M is a $N \times N$ complex *nondiagonal* matrix and, in general, $N \geq 3$.

It can be proved that the \mathbf{M}_M should be symmetric, $\mathbf{M}_M^T = \mathbf{M}_M$. Assuming for simplicity that its spectrum is non-degenerated, the mass matrix can be diagonalized by means of the following transformation [Bilenky & Petcov (1987)]

$$\mathbf{M}_M = \mathbf{V}^* \mathbf{m} \mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k \delta_{kl}|| = \text{diag}(m_1, m_2, \dots, m_N),$$

where \mathbf{V} is a unitary matrix and $m_k \geq 0$. Therefore

$$\begin{aligned} \mathcal{L}_M(x) &= -\frac{1}{2} [(\bar{\nu}'_L)^c \mathbf{m} \nu'_L + \bar{\nu}'_L \mathbf{m} (\nu'_L)^c] = -\frac{1}{2} \bar{\nu}' \mathbf{m} \nu' = -\frac{1}{2} \sum_{k=1}^N m_k \bar{\nu}_k \nu_k, \\ \nu'_L &= \mathbf{V}^\dagger \nu_L, \quad (\nu'_L)^c = C (\bar{\nu}'_L)^T, \quad \nu' = \nu'_L + (\nu'_L)^c. \end{aligned}$$

The last equality means that the fields $\nu_k(x)$ are Majorana neutrino fields. Considering that the kinetic term in the neutrino Lagrangian is transformed to^a

$$\mathcal{L}_0 = \frac{i}{2} \bar{\nu}'(x) \overleftrightarrow{\partial} \nu'(x) = \frac{i}{2} \sum_k \bar{\nu}_k(x) \overleftrightarrow{\partial} \nu_k(x),$$

one can conclude that $\nu_k(x)$ is the field with the definite mass m_k .

^aThis also explains the origin of the factor 1/2 in the Majorana mass term.

The flavor LH neutrino fields $\nu_{\ell,L}(x)$ present in the standard weak lepton currents are linear combinations of the LH components of the fields of neutrinos with definite masses:

$$\nu_L = \mathbf{V}\nu'_L \quad \text{or} \quad \nu_{\ell,L} = \sum_k V_{\ell k} \nu_{k,L}.$$

Of course **neutrino mixing matrix** \mathbf{V} is not the same as in the case of Dirac neutrinos.

There is no global gauge transformations under which the Majorana mass term (in its most general form) could be invariant. This implies that there are no conserved lepton charges that could allow us to distinguish Majorana ν s and $\bar{\nu}$ s. In other words,

Majorana neutrinos are truly neutral fermions.

3.1 Parametrization of mixing matrix for Majorana neutrinos.

Since the Majorana neutrinos are not rephasable, there may be a lot of extra phase factors in the mixing matrix. The Lagrangian with the Majorana mass term is invariant with respect to the transformation

$$l \mapsto e^{ia_\ell} l, \quad V_{\ell k} \mapsto e^{-ia_\ell} V_{\ell k}$$

Therefore N phases are unphysical and the number of the physical phases now is

$$\frac{N(N+1)}{2} - N = \frac{N(N-1)}{2} = \underbrace{\frac{(N-1)(N-2)}{2}}_{\text{Dirac phases}} + \underbrace{(N-1)}_{\text{Majorana phases}} = n_D + n_M;$$

$$n_M(2) = 1, \quad n_M(3) = 2, \quad n_M(4) = 3, \dots$$

| In fact all phases are Majorana and the above notation is provisional and unorthodox.

In the case of three lepton generations one defines the diagonal matrix with the extra phase factors: $\Gamma_M = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$, where $\alpha_{1,2}$ are commonly referred to as the Majorana CP -violation phases. Then the PMNS matrix can be parametrized as

$$\begin{aligned} \mathbf{V}_{(M)} &= \mathbf{O}_{23} \Gamma_D \mathbf{O}_{13} \Gamma_D^\dagger \mathbf{O}_{12} \Gamma_M = \mathbf{V}_{(D)} \Gamma_M \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

Neither L_ℓ nor $L = \sum_\ell L_\ell$ is now conserved allowing a lot of new processes, for example, $\tau^- \rightarrow e^+(\mu^+)\pi^-\pi^-$, $\tau^- \rightarrow e^+(\mu^+)\pi^-K^-$, $\pi^- \rightarrow \mu^+\bar{\nu}_e$, $K^+ \rightarrow \pi^-\mu^+e^+$, $K^+ \rightarrow \pi^0e^+\bar{\nu}_e$, $D^+ \rightarrow K^-\mu^+\mu^+$, $B^+ \rightarrow K^-e^+\mu^+$, $\Xi^- \rightarrow p\mu^-\mu^-$, $\Lambda_c^+ \rightarrow \Sigma^-\mu^+\mu^+$, etc.

Needless to say that **no one was discovered yet** [see RPP] but (may be!?) the $(\beta\beta)_{0\nu}$ decay. The following section will discuss this issue with some detail.

3.2 Neutrinoless double beta decay.

The theory with Majorana neutrinos allows the decay

$$(A, Z) \rightarrow (A, Z + 2) + 2e^- \quad [0\nu\beta\beta \equiv (\beta\beta)_{0\nu}]$$

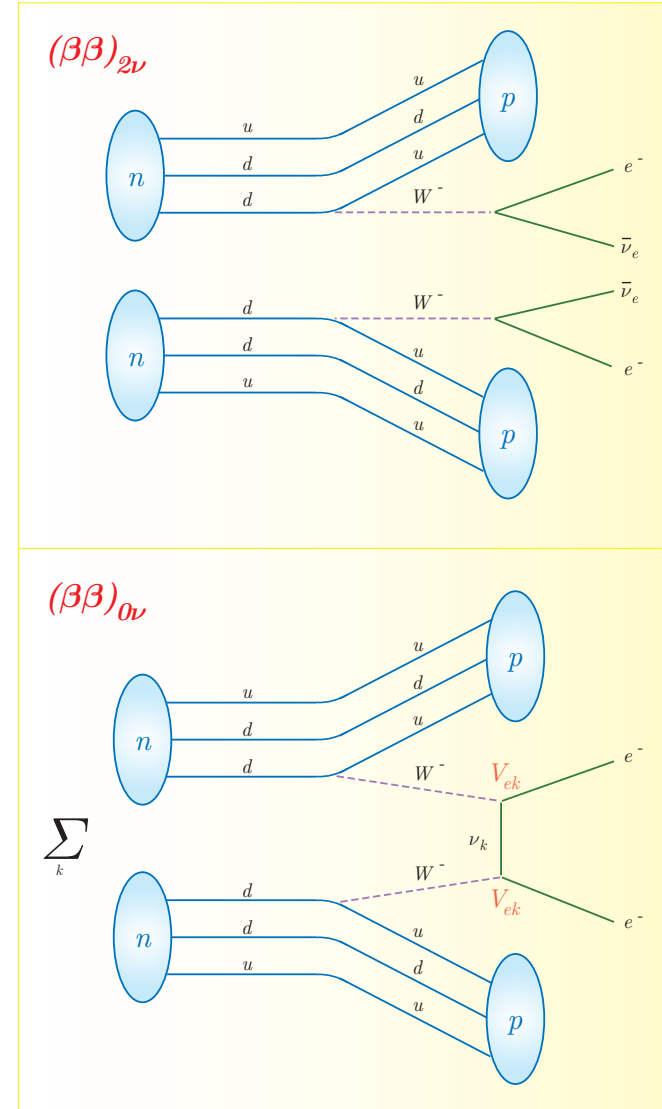
with $\Delta L = 2$. The decay rate for this process is expressed as follows:

$$[T_{1/2}^{0\nu}]^{-1} = G_Z^{0\nu} |m_{\beta\beta}|^2 |\mathcal{M}_F^{0\nu} - (g_A/g_V)^2 \mathcal{M}_{GT}^{0\nu}|^2,$$

where $G_Z^{0\nu}$ is the two-body phase-space factor including coupling constant, $\mathcal{M}_{F/GT}^{0\nu}$ are the Fermi/Gamow-Teller nuclear matrix elements. The constants g_V and g_A are the vector and axial-vector relative weak coupling constants, respectively. The complex parameter $m_{\beta\beta}$ is the *effective Majorana electron neutrino mass* given by

$$\begin{aligned} m_{\beta\beta} &= \sum_k V_{ek}^2 m_k = \sum_k |V_{ek}|^2 e^{i\phi_k} m_k \\ &= |V_{e1}|^2 m_1 + |V_{e2}|^2 m_2 e^{i\phi_2} + |V_{e3}|^2 m_3 e^{i\phi_3}. \end{aligned}$$

Here $\phi_1 = 0$, $\phi_2 = \alpha_2 - \alpha_1$ (pure Majorana phase) and $\phi_3 = -(\alpha_2 + 2\delta)$ (mixture of Dirac and Majorana CP -violation phases).

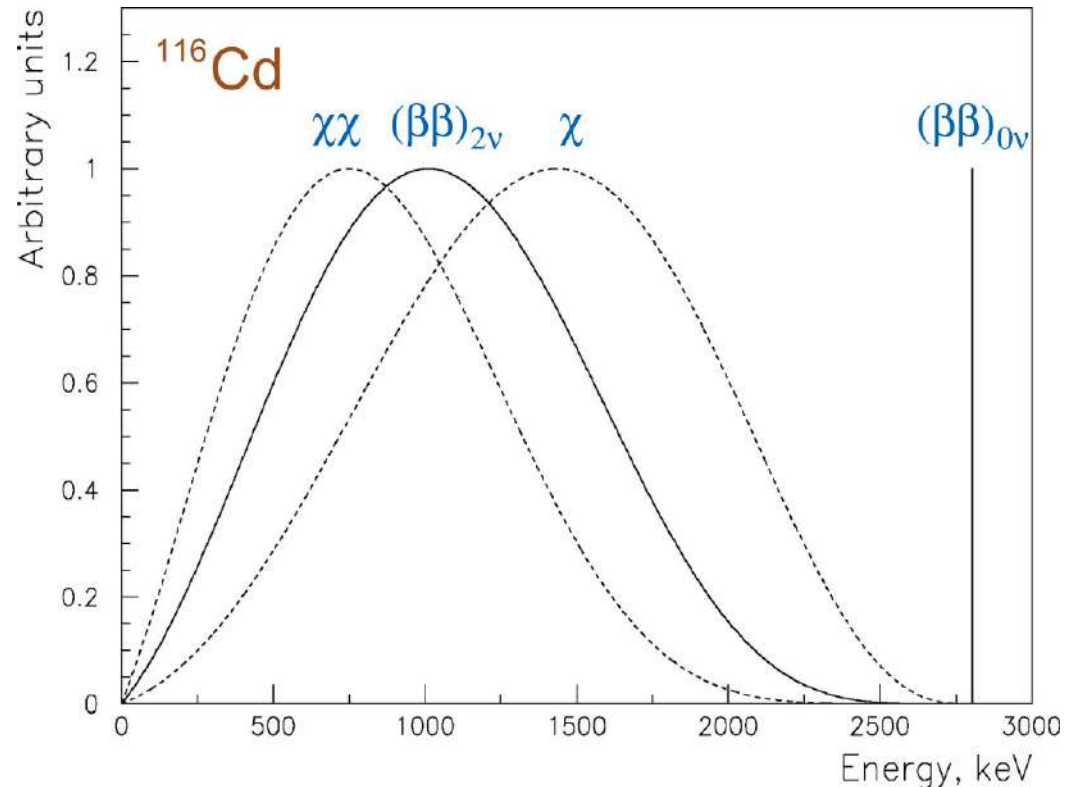


The electron sum energy spectrum of the $(\beta\beta)_{2\nu}$ mode as well as of the exotic modes with one or two majorons in final state,

$$(A, Z) \rightarrow (A, Z + 2) + 2e^- + \chi,$$

$$(A, Z) \rightarrow (A, Z + 2) + 2e^- + 2\chi,$$

is *continuous* because the available energy release ($Q_{\beta\beta}$) is shared between the electrons and other final state particles. In contrast, the two electrons from the $(\beta\beta)_{0\nu}$ decay carry the full available energy, and hence the electron sum energy spectrum has a sharp peak at the $Q_{\beta\beta}$ value. This feature allows one to distinguish the $(\beta\beta)_{0\nu}$ decay signal from the background.



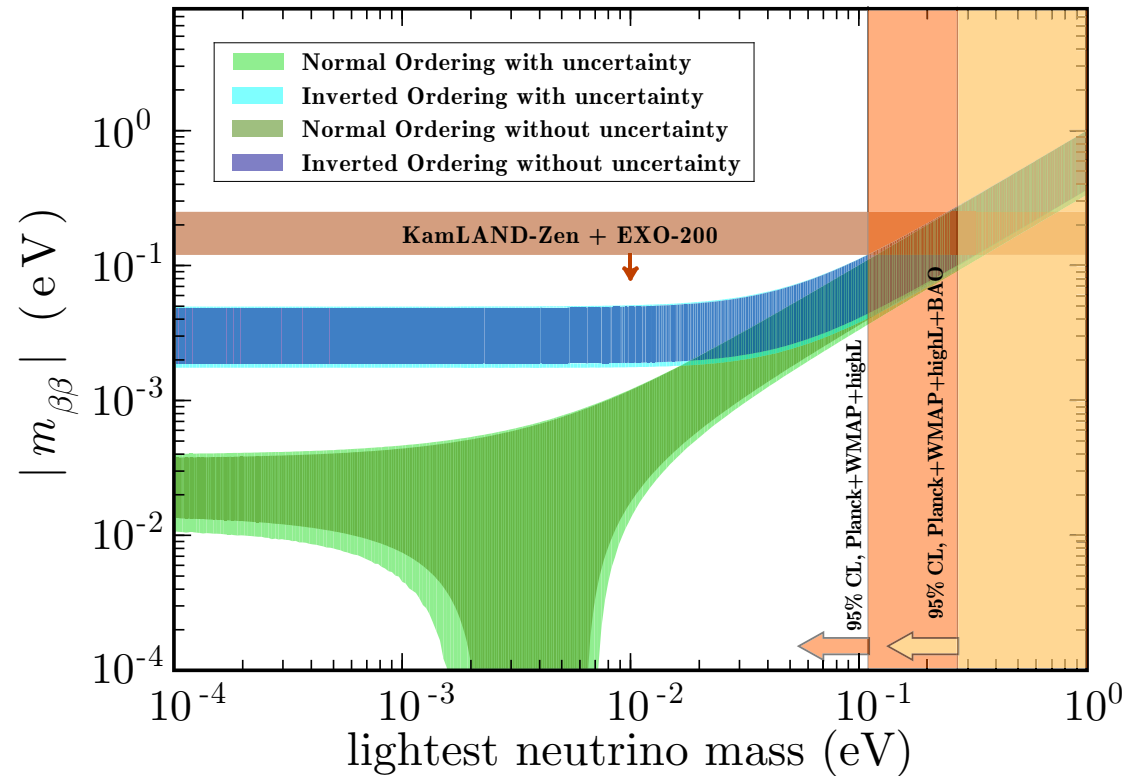
The electron sum energy spectra calculated for the different β decay modes of cadmium-116.

[From Y. Zdesenko, "Colloquium: The future of double beta decay research," Rev. Mod. Phys. 74 (2003) 663–684.]

Majoron is a Nambu-Goldstone boson, – a hypothetical neutral pseudoscalar zero-mass particle which couples to Majorana neutrinos and may be emitted in the neutrinoless β decay. It is a consequence of the spontaneous breaking of the global $B - L$ symmetry.

The currently allowed ranges of $m_{\beta\beta}$ observables of $0\nu\beta\beta$ decay is shown as a function of the lightest neutrino mass m_0 . In the case of normal (inverted) mass ordering the ranges are shown by green (blue) color. The light (dark) colored regions are computed by taking into account (without taking account) the current 1σ uncertainties of the relevant mixing parameters.

Also shown are the limits on $m_{\beta\beta}$ coming from KamLAND-Zen and EXO-200 (by the light brown band and arrow) and the bounds on m_0 obtained by *Planck*.



Note that the “KamLAND-Zen+EXO 200” bound spans a broad band (rather than a line) because of the nuclear matrix element uncertainty.

It is remarkable that the effect of the 1σ uncertainties of the mixing parameters is quite small. In contrast, variation over the Majorana phases gives much larger impact on allowed region of $m_{\beta\beta}$, not only producing sizeable width but also creating a down-going branch at $10^{-3} \text{ eV} \lesssim m_0 \lesssim 10^{-2} \text{ eV}$ for the case of the normal mass ordering due to the strong cancellation of the three mass terms.

[From H. Minakata, H. Nunokawa, and A. A. Quiroga, “Constraining Majorana CP phase in the precision era of cosmology and the double beta decay experiment,” PTEP 2015 (2015) 033B03, arXiv:1402.6014 [hep-ph].]

4 See-saw mechanism.

4.1 Dirac-Majorana mass term for one generation.

It is possible to consider mixed models in which both Majorana and Dirac mass terms are present. For simplicity sake we'll start with a **toy model** for one lepton generation.

Let us consider a theory containing two independent neutrino fields ν_L and ν_R :

$$\left\{ \begin{array}{l} \nu_L \text{ would generally represent any active neutrino (e.g., } \nu_L = \nu_{eL}), \\ \nu_R \text{ can represents a right handed field unrelated to any of these or} \\ \text{it can be charge conjugate of any of the active neutrinos (e.g., } \nu_R = (\nu_{\mu L})^c). \end{array} \right.$$

We can write the following generic mass term between ν_L and ν_R :

$$\mathcal{L}_m = - \underbrace{m_D \bar{\nu}_L \nu_R}_{\text{Dirac mass term}} - \underbrace{(1/2) [m_L \bar{\nu}_L \nu_L^c + m_R \bar{\nu}_R^c \nu_R]}_{\text{Majorana mass term}} + \text{H.c.} \quad (5)$$

- ★ As we know, the Dirac mass term respects L while the Majorana mass term violates it.
- ★ The parameter m_D in Eq. (5) is in general complex; to simplify matters, we'll assume it to be real but **not necessarily positive**.
- ★ The parameters m_L , and m_R in Eq. (5) can be chosen **real** and (by an appropriate rephasing the fields ν_L and ν_R) **non-negative**, but the latter is not assumed.
- ★ Obviously, neither ν_L nor ν_R is a mass eigenstate.

In order to obtain the *mass basis* we can apply the useful identity

$$\bar{\nu}_L \nu_R = (\bar{\nu}_R)^c (\nu_L)^c \quad (6)$$

The identity (6) is a particular case of the more general relation

$$\bar{\psi}_1 \Gamma \psi_2 = \bar{\psi}_2^c C \Gamma^T C^{-1} \psi_1^c,$$

in which $\psi_{1,2}$ are Dirac spinors and Γ represents an arbitrary combination of the Dirac γ matrices.

Relation (6) allows us to rewrite Eq. (5) as follows

$$\mathcal{L}_m = -\frac{1}{2} (\bar{\nu}_L, (\bar{\nu}_R)^c) \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + \text{H.c.} \equiv -\frac{1}{2} \bar{\nu}_L \mathbf{M} (\nu_L)^c + \text{H.c.}$$

If (again for simplicity) CP conservation is assumed the matrix \mathbf{M} can be diagonalized by the **orthogonal** transformation that is rotation

$$\mathbf{V} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{with} \quad \theta = \frac{1}{2} \arctan \left(\frac{2m_D}{m_R - m_L} \right).$$

and we have

$$\mathbf{V}^T \mathbf{M} \mathbf{V} = \text{diag}(m_1, m_2),$$

where $m_{1,2}$ are eigenvalues of \mathbf{M} given by

$$m_{1,2} = \frac{1}{2} \left(m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4m_D^2} \right).$$



The eigenvalues are real if (as we assume) $m_{D,L,R}$ are real, but **not necessarily positive**. Let us define $\zeta_k = \text{sign } m_k$ and rewrite the mass term in the new basis:

$$\mathcal{L}_m = -\frac{1}{2} [\zeta_1 |m_1| \bar{\nu}_{1L} (\nu_{1L})^c + \zeta_2 |m_2| (\bar{\nu}_{2R})^c \nu_{2R}] + \text{H.c.}, \quad (7)$$

The new fields ν_{1L} and ν_{2R} represent chiral components of two different neutrino states with “masses” m_1 and m_2 , respectively:

$$\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \mathbf{V} \begin{pmatrix} \nu_{1L} \\ \nu_{2R}^c \end{pmatrix} \implies \begin{cases} \nu_{1L} = \cos \theta \nu_L - \sin \theta \nu_R^c, \\ \nu_{2R} = \sin \theta \nu_L^c + \cos \theta \nu_R. \end{cases}$$

Now we define two 4-component fields

$$\nu_1 = \nu_{1L} + \zeta_1 (\nu_{1L})^c \quad \text{and} \quad \nu_2 = \nu_{2R} + \zeta_2 (\nu_{2R})^c.$$

Certainly, these fields are self-conjugate with respect to the C transformation:

$$\nu_k^c = \zeta_k \nu_k \quad (k = 1, 2)$$

and therefore they describe Majorana neutrinos. In terms of these fields Eq. (7) reads

$$\mathcal{L}_m = -\frac{1}{2} (|m_1| \bar{\nu}_1 \nu_1 + |m_2| \bar{\nu}_2 \nu_2). \quad (8)$$

We can conclude therefore that $\nu_k(x)$ is the Majorana neutrino field with the definite (physical) mass $|m_k|$.

There are several special cases of the Dirac-Majorana mass matrix \mathbf{M} which are of considerable phenomenological importance, in particular,

$$(A): \quad \mathbf{M} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \quad \Longrightarrow \quad |m_{1,2}| = m, \quad \theta = \frac{\pi}{4} \quad (\text{maximal mixing}).$$

Two Majorana fields are equivalent to one Dirac field.

A generalization $|m_{L,R}| \ll |m_D|$, leads to the so-called Pseudo-Dirac neutrinos.

$$(B): \quad \mathbf{M} = \begin{pmatrix} m_L & m \\ m & m_L \end{pmatrix} \quad \Longrightarrow \quad m_{1,2} = m_L \pm m_D, \quad \theta = \frac{\pi}{4} \quad (\text{maximal mixing});$$

$$(C): \quad \mathbf{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \quad \text{or, more generally,} \quad |m_L| \ll |m_R|, \quad m_D > 0.$$

The see-saw

The case (C) with $m \ll M$ is the simplest example of the see-saw mechanism. It leads to two masses, one very large, $m_1 \approx M$, other very small, $m_2 \approx -m^2/M \ll m$, suppressed compared to the entries in \mathbf{M} . In particular, one can assume

$$m \sim m_\ell \text{ or } m_q \quad (0.5 \text{ MeV to } 200 \text{ GeV}) \quad \text{and} \quad M \sim M_{\text{GUT}} \sim 10^{15-16} \text{ GeV}.$$

Then $|m_2|$ can ranges from $\sim 10^{-14} \text{ eV}$ to $\sim 0.04 \text{ eV}$. The mixing between the heavy and light neutrinos is extremely small: $\theta \approx m/M \sim 10^{-20} - 10^{-13} \lll 1$.



If one eigenvalue *goes up*, the other *goes down*, and vice versa. This is the reason of the term [see-saw](#)...
a bit intricate for so simple idea...

$$|m_2| \sim m^2/M \ll m \ll M$$

$$m_1 \sim M \sim M_{\text{GUT}}$$

4.2 More neutral fermions.

A generalization of the above scheme to N generations is almost straightforward but technically rather cumbersome. Let's consider it schematically for the $N = 3$ case.

- ▷ If neutral fermions are added to the set of the SM fields, then the flavour neutrinos can acquire mass by mixing with them.
- ▷ The additional fermions can be^a
 - Gauge chiral singlets per family \mathcal{N} (e.g., right-handed neutrinos) [Type I seesaw], or
 - $SU(2) \times U(1)$ doublets (e.g., Higgsino in SUSY), or
 - $Y = 0$, $SU(2)_L$ triplets Σ (e.g., Wino in SUSY) [Type III seesaw].
- ▷ Addition of three right-handed neutrinos \mathcal{N}_{iR} leads to the see-saw mechanism with the following mass terms:

$$\mathcal{L}_m = - \sum_{ij} \left[\bar{\nu}_{iL} M_{ij}^D \mathcal{N}_{jR} - \frac{1}{2} (\mathcal{N}_{iR})^c M_{ij}^R \mathcal{N}_{jR} + \text{H.c.} \right].$$

- ▷ The above equation leads to the following 6×6 see-saw mass matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{m}_D^T \\ \mathbf{m}_D & \mathbf{M}_R \end{pmatrix}.$$

Both \mathbf{m}_D and \mathbf{M}_R are 3×3 matrices in the generation space.

^aType II seesaw operates with additional $SU(2)_L$ scalar triplets Δ .

Similar to the one-generation case we assume that the eigenvalues of \mathbf{M}_R are large in comparison with the eigenvalues of \mathbf{m}_D . Then \mathbf{M} can be approximately block-diagonalized by an unitary transformation:

$$\mathbf{U}^\dagger \mathbf{M} \mathbf{U} = \text{diag}(\mathbf{M}_1, \mathbf{M}_2) + \mathcal{O}(\mathbf{m}_D \mathbf{M}_R^{-1}),$$

where

$$\mathbf{U} = \begin{pmatrix} 1 + \frac{1}{2} \mathbf{m}_D^\dagger (\mathbf{M}_R \mathbf{M}_R^\dagger)^{-1} \mathbf{m}_D & \mathbf{m}_D^\dagger (\mathbf{M}_R^\dagger)^{-1} \\ -\mathbf{M}_R^{-1} \mathbf{m}_D & 1 + \frac{1}{2} \mathbf{M}_R^{-1} \mathbf{m}_D \mathbf{m}_D^\dagger (\mathbf{M}_R^\dagger)^{-1} \end{pmatrix}.$$

$$\mathbf{M}_1 \simeq \mathbf{M}_R \quad \text{and} \quad \mathbf{M}_2 \simeq -\mathbf{m}_D^T \mathbf{M}_R^{-1} \mathbf{m}_D$$

The mass eigenfields are surely Majorana neutrinos.

- **Quadratic see-saw:** If eigenvalues of \mathbf{M}_R are of the order of a large scale parameter $M \sim M_{\text{GUT}}$ ^a [e.g., $\mathbf{M}_R = \mathbf{M}_1$] than the standard neutrino masses are suppressed:

$$m_i \sim \frac{m_{Di}^2}{M} \lll m_{Di},$$

Here $m_{Di} \sim Y_i \langle H \rangle$ are the eigenvalues of \mathbf{m}_D . As long as these eigenvalues (or Yukawa couplings Y_i) are hierarchical, the Majorana neutrino masses display quadratic hierarchy:

$$m_1 : m_2 : m_3 \propto m_{D1}^2 : m_{D2}^2 : m_{D3}^2.$$

^aLarge M is natural in, e.g., $SO(10)$ inspired GUT models which therefore provide a nice framework to understand small neutrino masses.

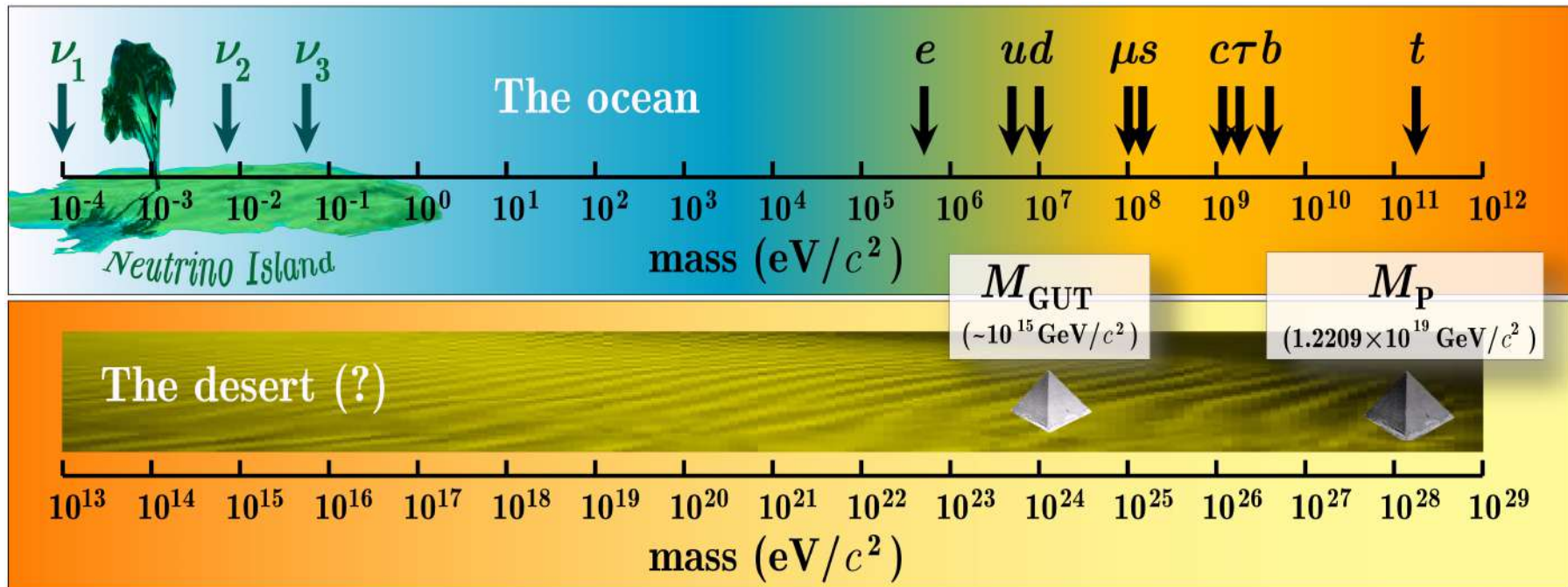
- **Linear see-saw:** In a more special case, $\mathbf{M}_R = (M/M_D)\mathbf{M}_D$, where M_D is the generic scale of the charged fermion masses than

$$m_i \sim \frac{M_D m_{Di}}{M} \lll m_{Di}$$

but the hierarchy is linear:

$$m_1 : m_2 : m_3 \propto m_{D1} : m_{D2} : m_{D3}.$$

The two mentioned possibilities are, in principle, experimentally distinguishable.



Beyond this section

- ◆ Double see-saw*
- ◆ Inverse see-saw*
- ◆ Radiative see-saw*
- ◆ SUSY & SUGRA see-saw
- ◆ TeV-scale gauged $B - L$ symmetry*
- ◆ TeV see-saw & large extra dimensions
- ◆ See-saw & Dark Matter
- ◆ See-saw & Leptogenesis
- ◆ See-saw & Baryogenesis
- ◆ Dirac see-saw
- ◆ Top (top-bottom) see-saw
- ◆ Cascade see-saw
- ◆ ...



* See Backup.

Conclusions (not really confirmed)


- The “mainstream” ν mass models, defined as see-saw models, are capable of describing the atmospheric–reactor–accelerator ν oscillation data, the LMA MSW solar neutrino solution, and cosmological limits. The SM and MSSM may naturally be extended to incorporate the see-saw mechanism.
- [A fly in the ointment] Wealth of the models (\gg number of the authors of the models) greatly complicates the choice of the best one.



What do we know and don't know about neutrinos?



4.3 Here's what we know today (we're getting ahead of ourselves).

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 7.0$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
With SK atmospheric data	$\sin^2 \theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.269 \rightarrow 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \rightarrow 0.343$
	$\theta_{12}/^\circ$	$33.45^{+0.77}_{-0.75}$	$31.27 \rightarrow 35.87$	$33.45^{+0.78}_{-0.75}$	$31.27 \rightarrow 35.87$
	$\sin^2 \theta_{23}$	$0.450^{+0.019}_{-0.016}$	$0.408 \rightarrow 0.603$	$0.570^{+0.016}_{-0.022}$	$0.410 \rightarrow 0.613$
	$\theta_{23}/^\circ$	$42.1^{+1.1}_{-0.9}$	$39.7 \rightarrow 50.9$	$49.0^{+0.9}_{-1.3}$	$39.8 \rightarrow 51.6$
	$\sin^2 \theta_{13}$	$0.02246^{+0.00062}_{-0.00062}$	$0.02060 \rightarrow 0.02435$	$0.02241^{+0.00074}_{-0.00062}$	$0.02055 \rightarrow 0.02457$
	$\theta_{13}/^\circ$	$8.62^{+0.12}_{-0.12}$	$8.25 \rightarrow 8.98$	$8.61^{+0.14}_{-0.12}$	$8.24 \rightarrow 9.02$
	$\delta_{CP}/^\circ$	230^{+36}_{-25}	$144 \rightarrow 350$	278^{+22}_{-30}	$194 \rightarrow 345$
	$\frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \rightarrow 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \rightarrow 8.04$
$\frac{\Delta m_{3\ell}^2}{10^{-3} \text{ eV}^2}$	$+2.510^{+0.027}_{-0.027}$	$+2.430 \rightarrow +2.593$	$-2.490^{+0.026}_{-0.028}$	$-2.574 \rightarrow -2.410$	

Three-flavor oscillation parameters from a recent fit to global data (“NuFIT 5.1”) performed by the NuFIT team. Note that $\Delta m_{3\ell}^2 \equiv \Delta m_{31}^2 > 0$ for NO and $\Delta m_{3\ell}^2 \equiv \Delta m_{32}^2 < 0$ for IO.

[See I. Esteban *et al.* (The NuFIT team), “The fate of hints: updated global analysis of three-flavor neutrino oscillations,” JHEP09(2020)178, arXiv:2007.14792 [hep-ph]. Present update (October 2021) is from <http://www.nu-fit.org/>.]

List of data used in the NuFIT 5.1 analysis (October 2021)



Solar experiments:

Homestake chlorine total rate (1 dp), Gallex & GNO total rates (2 dp), SAGE total rate (1 dp), SK-I full energy and zenith spectrum (44 dp), SK-II full energy and day/night spectrum (33 dp), SK-III full energy and day/night spectrum (42 dp), SK-IV 2970-day day-night asymmetry and energy spectrum (24 dp), SNO combined analysis (7 dp), Borexino Phase-I 741-day low-energy data (33 dp), Borexino Phase-I 246-day high-energy data (6 dp), Borexino Phase-II 408-day low-energy data (42 dp).

Atmospheric experiments:

IceCube/DeepCore 3-year data (64 dp), SK-I-IV 364.8 kiloton years + χ^2 map.

Reactor experiments:

KamLAND separate DS1, DS2, DS3 spectra with Daya-Bay reactor $\bar{\nu}_e$ fluxes (69 dp), Double-Chooz FD/ND spectral ratio, with 1276-day (FD), 587-day (ND) exposures (26 dp), Daya-Bay 1958-day EH2/EH1 and EH3/EH1 spectral ratios (52 dp), RENO 2908-day FD/ND spectral ratio (45 dp).

Accelerator experiments:

MINOS 10.71 PoT₂₀ ν_μ -disappearance data (39 dp), MINOS 3.36 PoT₂₀ $\bar{\nu}_\mu$ -disappearance data (14 dp), MINOS 10.60 PoT₂₀ ν_e -appearance data (5 dp), MINOS 3.30 PoT₂₀ $\bar{\nu}_e$ -appearance (5 dp), T2K 19.7 PoT₂₀ ν_μ -disappearance data (35 dp), T2K 19.7 PoT₂₀ ν_e -appearance data (23 dp for the CCQE and 16 dp for CC1 π samples), T2K 16.3 PoT₂₀ $\bar{\nu}_\mu$ -disappearance data (35 dp), T2K 16.3 PoT₂₀ $\bar{\nu}_e$ -appearance data (23 dp), NOvA 13.6 PoT₂₀ ν_μ -disappearance data (76 dp), NOvA 13.6 PoT₂₀ ν_e -appearance data (13 dp), NOvA 12.5 PoT₂₀ $\bar{\nu}_\mu$ -disappearance data (76 dp), NOvA 12.5 PoT₂₀ $\bar{\nu}_e$ -appearance data (13 dp).

Here dp = data point(s), PoT₂₀ = 10²⁰ PoT (Protons on Target), and EH = Experiment Hall.

4.3.1 Neutrino oscillation parameter plot.

The regions of neutrino squared-mass splitting

$$\Delta m^2 = |\Delta m_{ij}^2| = |m_j^2 - m_i^2|$$

and $\tan^2 \theta$ (where θ is one of the mixing angles θ_{ij} corresponding to a particular experiment) favored or excluded by various experiments. Contributed to RPP-2018^a by Hitoshi Murayama (University of California, Berkeley).

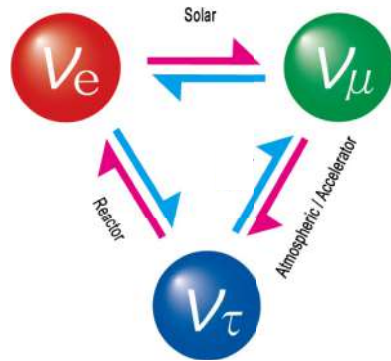
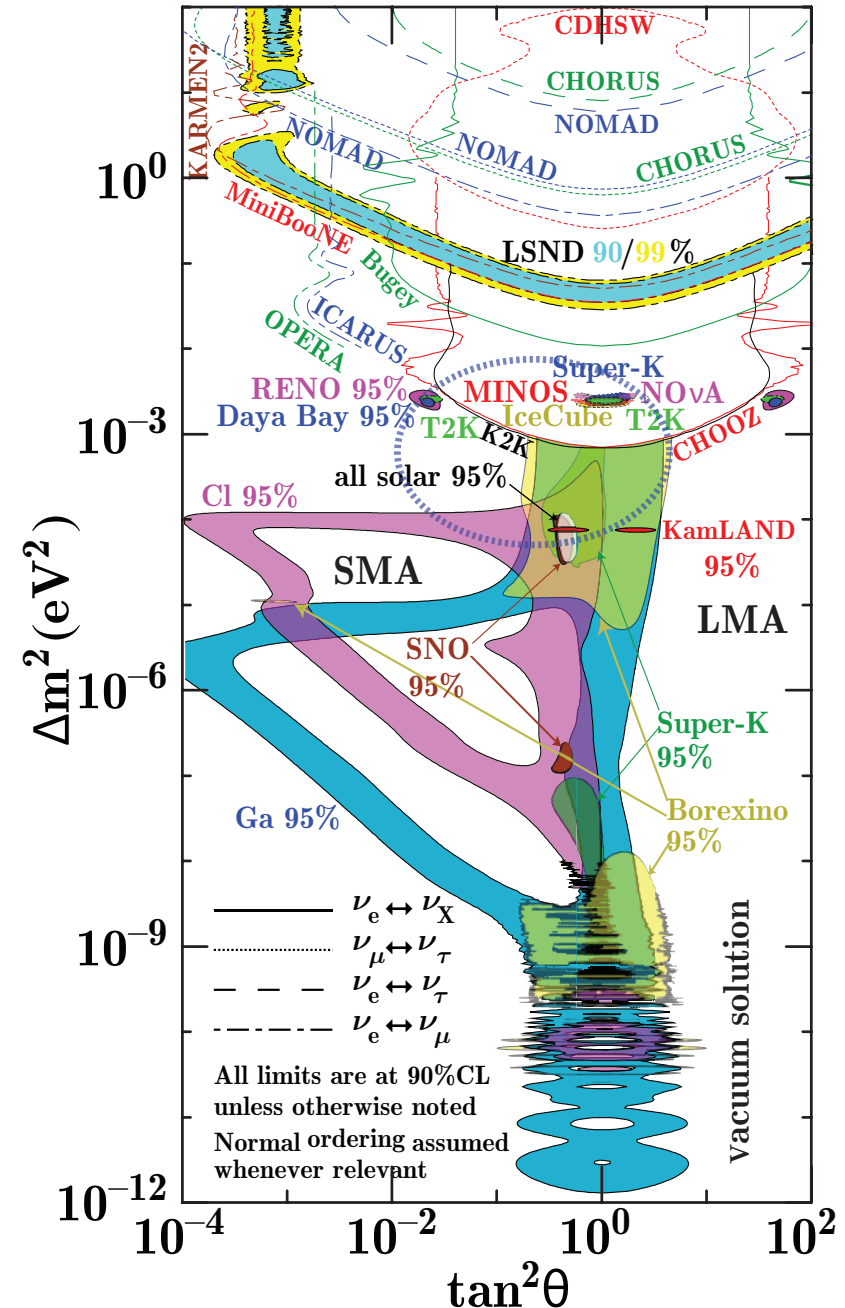


Figure includes the most rigorous results from before 2018, but data from many earlier experiments (e.g., BUST, NUSEX, Fréjus, IMB, Kamiokande, MACRO, SOUDAN 2) are ignored.

^aM. Tanabashi *et al.* (Particle Data Group), “Review of Particle Physics”, Phys. Rev. D **98** (2018) 030001.



In the absence of CP violation, the mixing angles may be represented as Euler angles relating the flavor eigenstates to the mass eigenstates. \blacktriangleright

According to the NuFIT analysis (p.45), the best-fit mixing angles and δ for the normal mass ordering (a bit preferred) are:

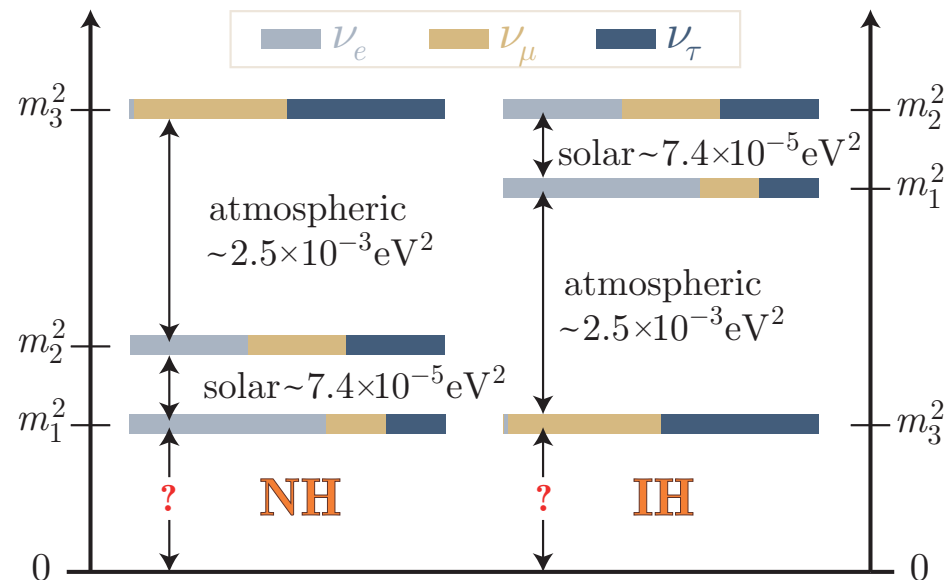
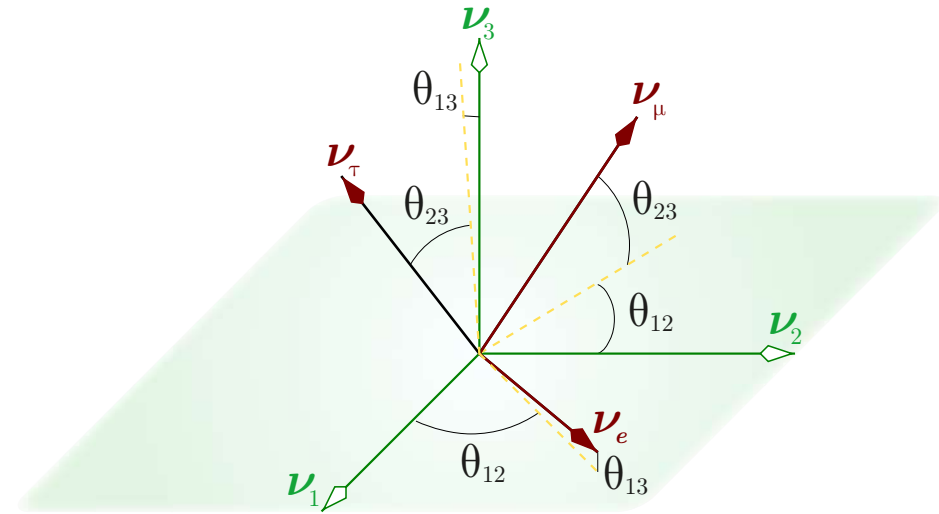
	PNMS	CKM
$\theta_{12}/^\circ$	$33.45^{+0.77}_{-0.75}$	13.04 ± 0.05
$\theta_{23}/^\circ$	$42.1^{+1.1}_{-0.9}$	2.38 ± 0.06
$\theta_{13}/^\circ$	$8.62^{+0.12}_{-0.12}$	0.201 ± 0.011
δ°	230^{+36}_{-25}	68.8 ± 4.5

The CKM angles and CP phase are also shown for comparison.

It should be stressed that the neutrino mass spectrum is still undetermined. \blacktriangleright

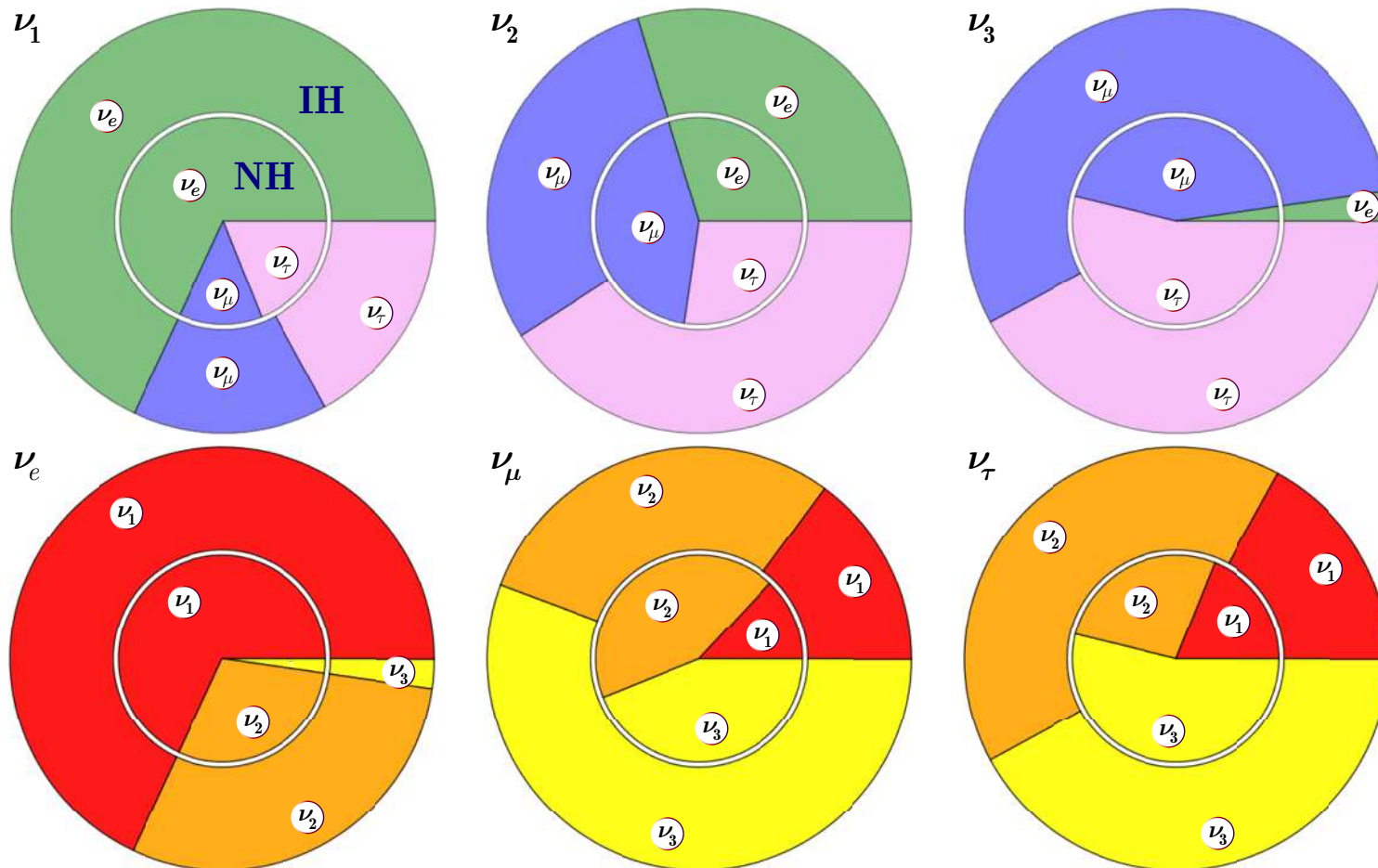
[Figures (slightly modified and updated) are taken from S. F. King, "Neutrino mass and mixing in the seesaw playground," arXiv:1511.03831 [hep-ph].]

Flavor content of mass states and mass content of flavor states is the same for Dirac ν and $\bar{\nu}$ (CP phase δ only changes the sign for $\bar{\nu}$) and for Majorana left/right ν_s ($|V_{\alpha i}^D| = |V_{\alpha i}^M|$).



4.3.2 Flavor content of mass states and mass content of flavor states.

$$\left(|V_{\alpha i}|^2\right)_{\text{NH}} = \begin{pmatrix} 0.681 & 0.297 & 0.0225 \\ 0.130 & 0.430 & 0.439 \\ 0.189 & 0.273 & 0.538 \end{pmatrix}, \quad \left(|V_{\alpha i}|^2\right)_{\text{IH}} = \begin{pmatrix} 0.681 & 0.297 & 0.0224 \\ 0.149 & 0.294 & 0.557 \\ 0.170 & 0.409 & 0.421 \end{pmatrix}.$$



4.3.3 Current status of the neutrino masses from oscillation experiments.

So, NuFIT 5.1 provides the following constraints for the mass squared splittings:

$$m_2^2 - m_1^2 = 7.42_{-0.20}^{+0.21} \times 10^{-5} \text{ eV}^2 \quad (\text{"solar" for NH and IH})$$

$$m_3^2 - m_1^2 = 2.51_{-0.027}^{+0.027} \times 10^{-3} \text{ eV}^2 \quad (\text{"atmospheric" for NH})$$

$$m_2^2 - m_3^2 = 2.49_{-0.028}^{+0.026} \times 10^{-3} \text{ eV}^2 \quad (\text{"atmospheric" for IH})$$

These result imply that **at least two of the neutrino eigenfields have nonzero masses** and thus there are (at least) two very different possible scenarios related to the mass ordering:

$$m_1 \ll m_2 < m_3 \quad (\text{for NH}) \quad \text{or} \quad m_3 \ll m_1 < m_2 \quad (\text{for IH}).$$

The data on Δm_{ij}^2 give the following estimates (henceforth $\sum m_\nu \equiv \sum_{i=1}^3 m_i$):

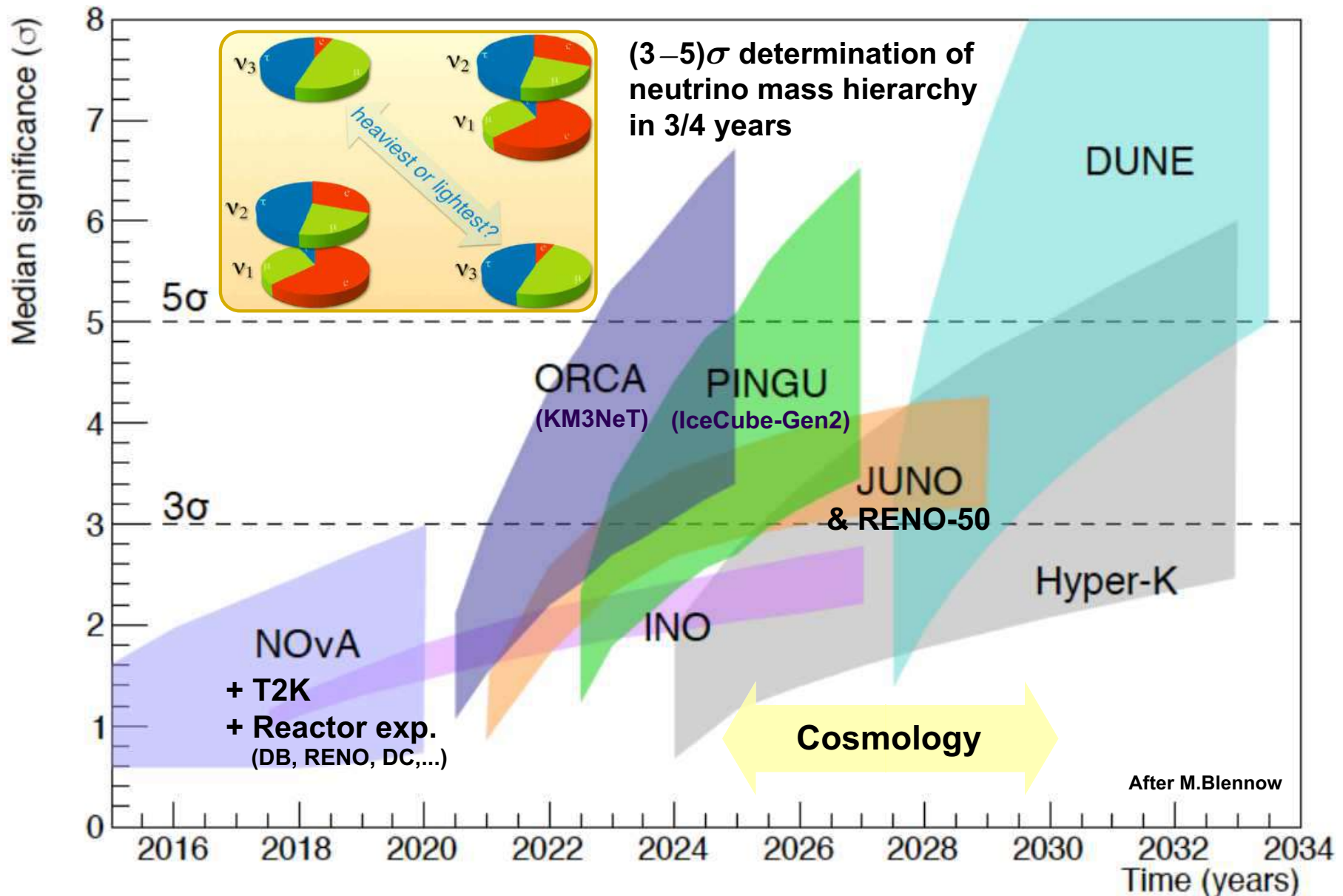
$$\begin{cases} m_2 = (8.61 \pm 0.122) \times 10^{-3} \text{ eV}, \\ m_3 = (5.01 \pm 0.027) \times 10^{-2} \text{ eV}, \end{cases} \implies \sum m_\nu \geq m_2 + m_3 = 0.0587 \pm 0.0003 \text{ eV} \quad (\text{for NH}) \quad (9)$$

$$\begin{cases} m_2 = (4.99 \pm 0.028) \times 10^{-2} \text{ eV}, \\ m_1 = (4.92 \pm 0.029) \times 10^{-2} \text{ eV}, \end{cases} \implies \sum m_\nu \geq m_1 + m_2 = 0.0983 \pm 0.0006 \text{ eV} \quad (\text{for IH}) \quad (10)$$

Therefore, the lower bounds on $\sum m_\nu$ at 1σ C.L. are:

$$\sum m_\nu^{\text{NH}} > 0.0584 \text{ eV} \quad \text{and} \quad \sum m_\nu^{\text{IH}} > 0.0977 \text{ eV}.$$

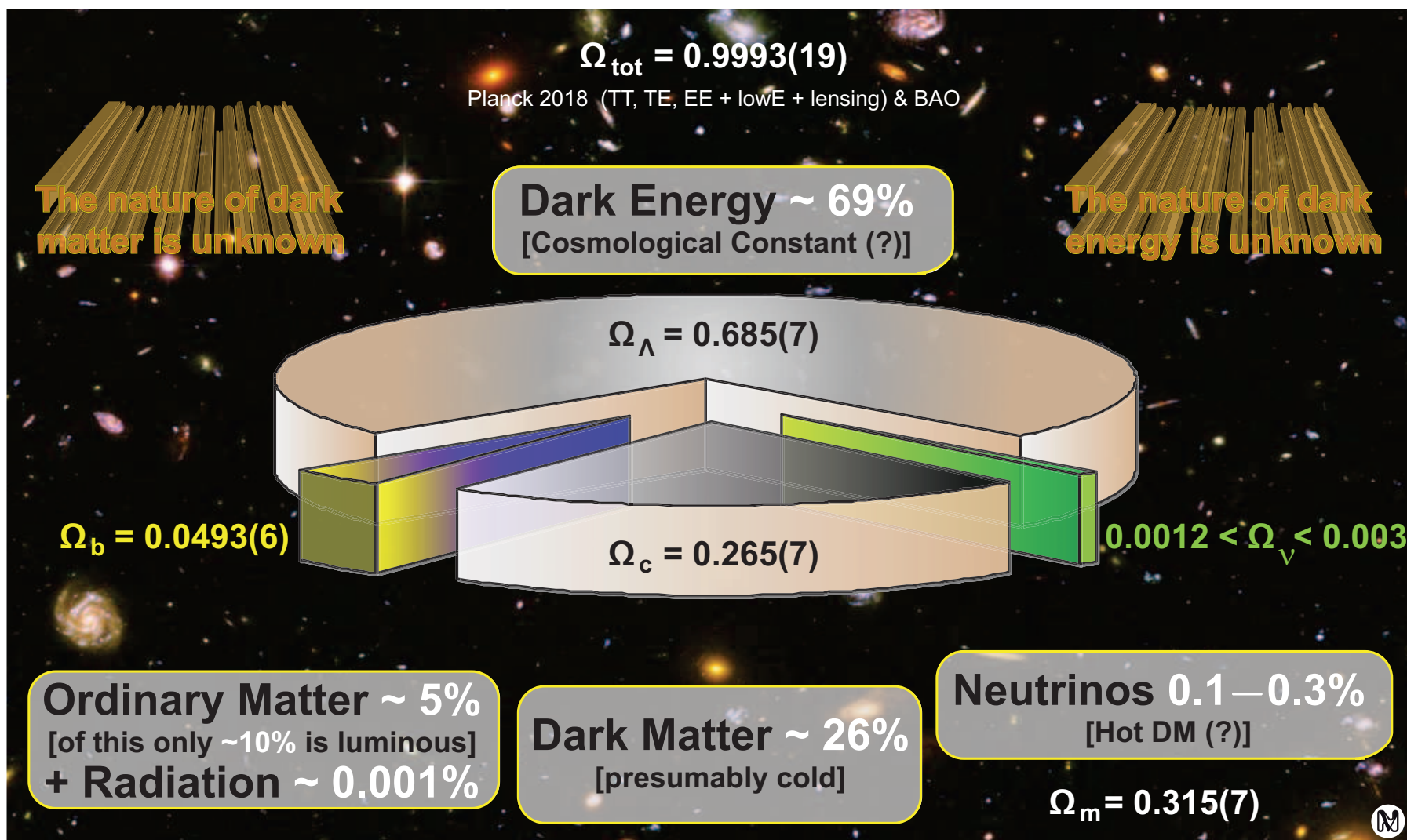
Note: Current accelerator and reactor data favor the NH scenario, but the question is not yet closed.



A summary of sensitivities to the neutrino mass hierarchy for various experimental approaches, with timescales, as claimed by the proponents in each case. Widths indicate main expected uncertainty.

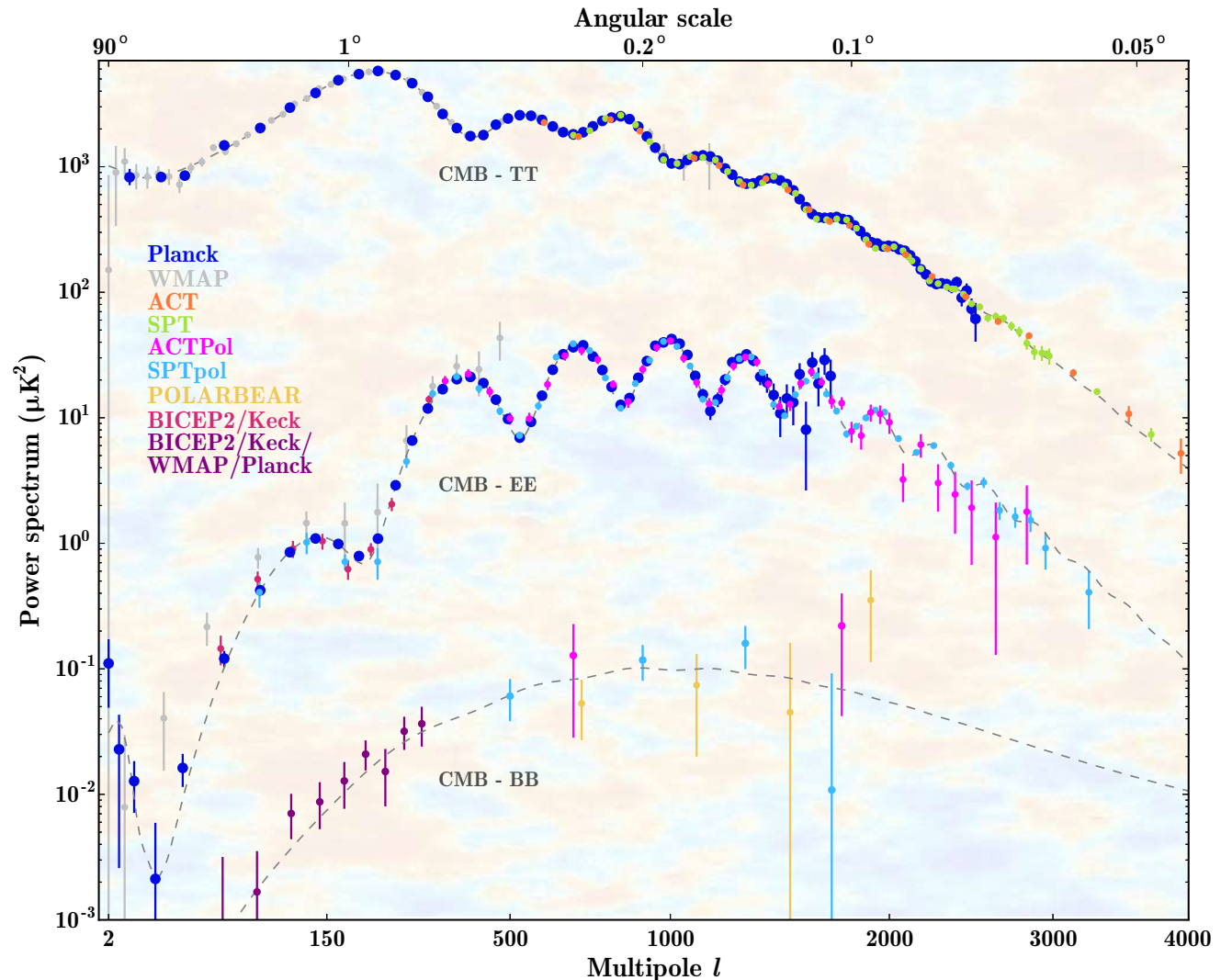
$C\nu B$.

Relict neutrinos (or Cosmic Neutrino Background, or CNB, or $C\nu B$) produce the largest neutrino flux on Earth, but compose only a very small fraction of invisible (non-luminous) matter in the Universe.



CMB as a probe of $C\nu B$.

It is not yet realistic to directly detect the νs created within the first second after the Big Bang, and which have too little energy now. However, for the first time, *Planck*, ESA's mission has unambiguously detected the effect $C\nu B$ has on relic radiation maps. The quality of these maps is now such that the imprints left by dark matter and relic νs are clearly visible.^a



^aSee N. Aghanim *et al.* (*Planck* Collaboration), “Planck 2018 results. I. Overview and the cosmological legacy of Planck”, *Astron. Astrophys.* **641** (2020) A1, arXiv:1807.06205 [astro-ph.CO]; “Planck 2018 results. VI. Cosmological parameters”, *Astron. Astrophys.* **641** (2020) A6, arXiv:1807.06209 [astro-ph.CO].

The relic photon spectrum almost exactly follows the blackbody spectrum with temperature

$$T_0 = 2.7255 \pm 0.0006 \text{ K.}$$

After many decades of experimental and theoretical efforts, the CMB is known to be almost isotropic but having small temperature fluctuations (called CMB anisotropy) with amplitude

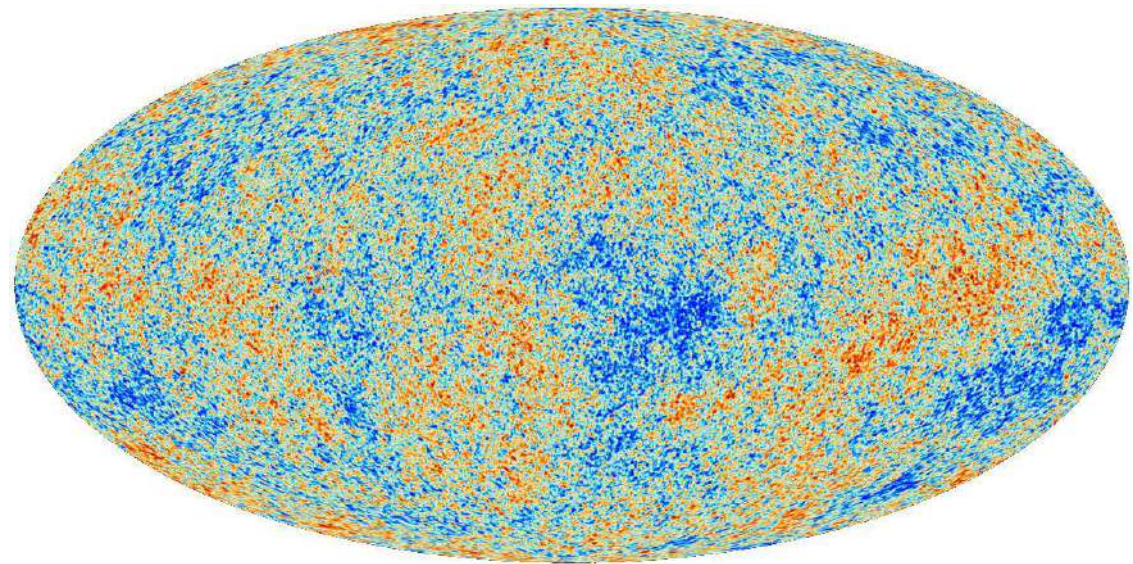
$$\delta T \sim (10^{-5} - 10^{-3}).$$

These fluctuations can be decomposed in a sum of spherical harmonics $Y_{lm}(\theta, \phi)$

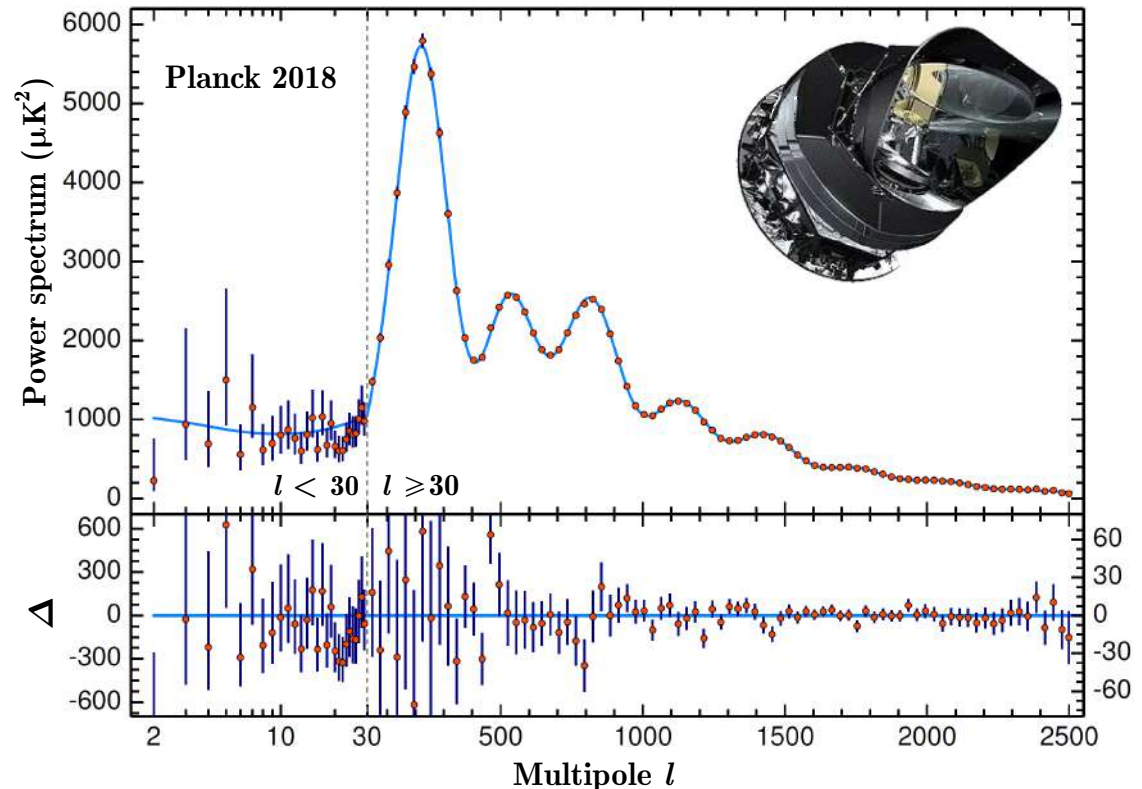
$$\delta T(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\theta, \phi).$$

The averaged squared coefficients a_{lm} give the variance

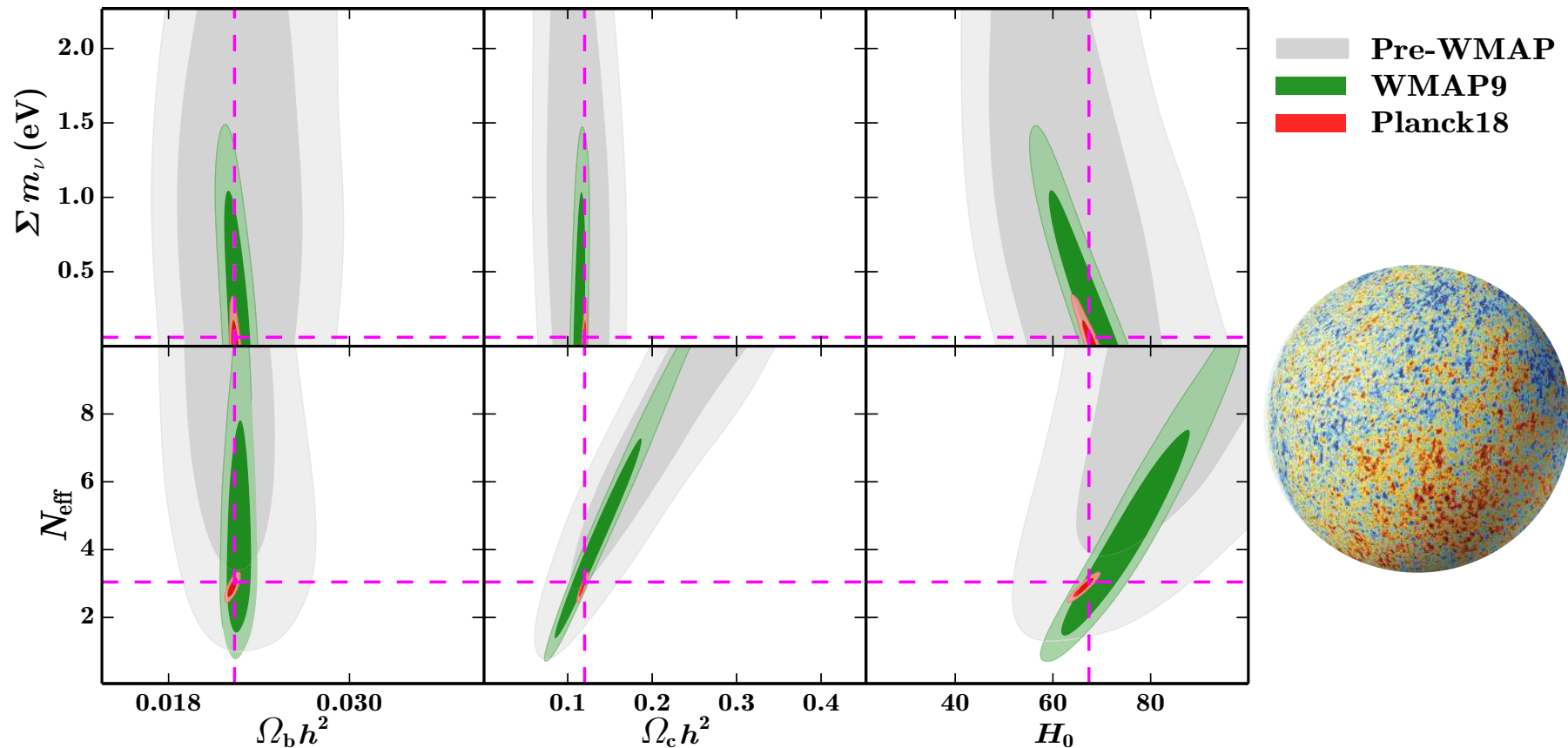
$$C_l = \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \sum_{m=-l}^l |a_{lm}|^2.$$



CMB maps can be compressed into the power spectrum



Planck 2018: neutrino summary.



Successive reductions in the allowed parameter space for various one-parameter extensions to Λ CDM, from pre-WMAP (MAXIMA, DASI, BOOMERANG, VSA, CBI) to *Planck*. The contours display the 68% and 95% C.L. for the extra parameter vs. five other base- Λ CDM parameters. The dashed lines indicate the Λ CDM best-fit parameters or fixed default values of the extended parameters.

[Adopted from Aghanim *et al.* (*Planck* Collaboration), “Planck 2018 results. I. Overview and the cosmological legacy of Planck”, *Astron. Astrophys.* **641** (2020) A1, arXiv:1807.06205 [astro-ph.CO];]

Finally *Planck* 2018 (+BAO) sets:

$$\sum m_\nu < 0.12 \text{ eV}, \quad N_{\text{eff}} = 2.99 \pm 0.17, \quad \Delta N_{\text{eff}} < 0.3.$$

Here N_{eff} is the effective number of neutrino species; roughly speaking, $N_{\text{eff}} \simeq 3$ means that additional light neutrinos are not supported (although not excluded) by *Planck*.

But(!) this constraint implies degenerate mass hierarchy (DH), $m_i = \sum m_\nu / 3$, and many other model assumptions. Results for other ν mass spectra have been obtained recently ($m_0 \equiv m_{\text{min}}$):^a

	Base			Base+SNe		
	DH	NH	IH	DH	NH	IH
$\Lambda\text{CDM} + \sum m_\nu$						
ω_c	0.1194 ± 0.0009	0.1192 ± 0.0009	0.1191 ± 0.0009	0.1193 ± 0.0009	0.1191 ± 0.0009	0.1189 ± 0.0009
ω_b	0.02242 ± 0.00013	$0.02242^{+0.00013}_{-0.00014}$	0.02243 ± 0.00013	0.02243 ± 0.00013	0.02244 ± 0.00013	0.02244 ± 0.00013
Θ_s	1.04100 ± 0.00029	1.04100 ± 0.00029	1.04100 ± 0.00029	1.04102 ± 0.00029	1.04103 ± 0.00029	1.04103 ± 0.00029
τ	$0.0554^{+0.0068}_{-0.0076}$	$0.0569^{+0.0066}_{-0.0076}$	$0.0585^{+0.0069}_{-0.0076}$	0.0556 ± 0.0071	$0.0573^{+0.0069}_{-0.0076}$	$0.0588^{+0.0068}_{-0.0077}$
n_s	0.9666 ± 0.0036	0.9668 ± 0.0037	0.9671 ± 0.0037	0.9669 ± 0.0036	0.9673 ± 0.0036	0.9675 ± 0.0037
$\ln[10^{10} A_s]$	$3.048^{+0.014}_{-0.015}$	$3.051^{+0.014}_{-0.015}$	3.053 ± 0.015	3.046 ± 0.014	3.049 ± 0.014	$3.052^{+0.014}_{-0.015}$
m_0 (eV)	< 0.040	< 0.040	< 0.042	< 0.038	< 0.038	< 0.039
$\sum m_\nu$ (eV)	< 0.12	< 0.15	< 0.17	< 0.11	< 0.14	< 0.16
H_0 (km/s/Mpc)	$67.81^{+0.54}_{-0.46}$	$67.50^{+0.49}_{-0.44}$	67.22 ± 0.45	$67.89^{+0.52}_{-0.45}$	67.59 ± 0.44	67.33 ± 0.43
σ_8	$0.814^{+0.010}_{-0.007}$	$0.806^{+0.009}_{-0.006}$	$0.799^{+0.008}_{-0.006}$	$0.815^{+0.010}_{-0.007}$	$0.806^{+0.008}_{-0.006}$	$0.799^{+0.008}_{-0.006}$
S_8	0.827 ± 0.011	0.823 ± 0.011	0.820 ± 0.011	0.826 ± 0.011	0.822 ± 0.011	0.818 ± 0.011
$\Delta\chi^2 = \chi^2 - \chi^2_{\text{IH}}$	-2.89	-0.95	0	-2.73	-1.27	0

Let's recall the latest oscillation lower limits: $\sum m_\nu^{\text{NH}} \gtrsim 0.058 \text{ eV}$ and $\sum m_\nu^{\text{IH}} \gtrsim 0.098 \text{ eV}$.

^aSh. R. Choudhury & S. Hannestad, "Updated results on neutrino mass and mass hierarchy from cosmology with *Planck* 2018 likelihoods," JCAP07(2020)037, arXiv:1907.12598 [astro-ph.CO].

Afterward: Open problems in neutrino physics.

- Are neutrinos Dirac or Majorana fermions?
- What is the absolute mass scale of (known) neutrinos?
Why neutrino masses are so small? [Does any version of see-saw work?]
What is the neutrino mass spectrum? [$\text{sign}(\Delta m_{32}^2) \iff \text{NH or IH.}$]
Can the lightest neutrinos be massless fermions? [Not quasiparticles in Weyl semimetals!]
- Why neutrino mixing is so different from quark mixing?
~~What physics is responsible for the octant degeneracy? [$\text{sign}(\theta_{23} = 45^\circ)$.]~~
- What are the source and scale of CP/T violation in the neutrino sector?
How many CP violating phases are there?
- Is CPT conserved in the neutrino sector?
- How many neutrino flavors are there?
- Whether the number of neutrinos with definite masses is equal to or greater than the number of flavor neutrinos? In other words, do sterile neutrinos exist? ^a If so,
 - What is their mass spectrum?
 - Do they mix with active neutrinos?
 - Do light (heavy) sterile neutrinos constitute hot (cold) dark matter?
- Are (all) neutrinos stable particles?

^aHints from LSND+MiniBooNE, Neutrino-4, SAGE+GALLEX+BEST are in tension with many other data.



Neutrino oscillations in vacuum



5 Quantum-mechanical treatment.

5.1 Angels & hippopotami.

According to the current theoretical understanding, the neutrino fields/states of definite flavor are superpositions of the fields/states with definite, generally different masses [and vice versa]:

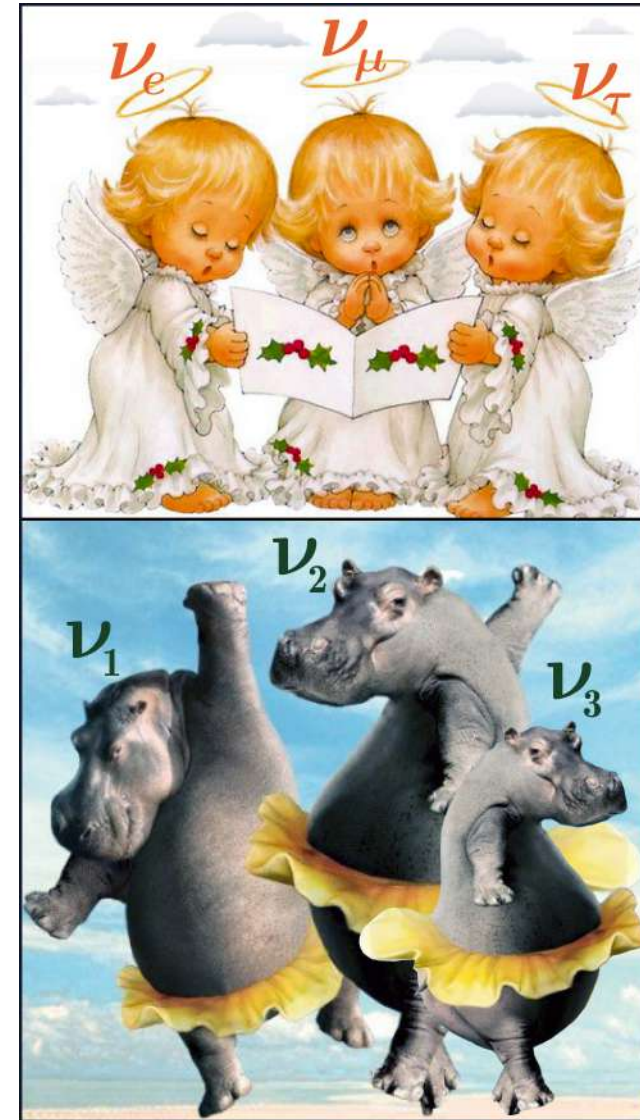
$$\nu_\alpha = \sum_i V_{\alpha i} \nu_i \quad \text{for neutrino fields,}$$

$$|\nu_\alpha\rangle = \sum_i V_{\alpha i}^* |\nu_i\rangle \quad \text{for neutrino states;}$$

$$\alpha = e, \mu, \tau, \quad i = 1, 2, 3, \dots$$

Here $V_{\alpha i}$ are the elements of the Pontecorvo-Maki-Nakagawa-Sakata neutrino vacuum mixing matrix \mathbf{V} .

This concept leads to the possibility of transitions between different flavor neutrinos, $\nu_\alpha \longleftrightarrow \nu_\beta$, phenomenon known as neutrino flavor oscillations.



Let us introduce two types of neutrino eigenstates:

- The **flavor** neutrino eigenstates which can be written as a vector

$$|\nu\rangle_f = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle, \dots)^T \equiv (|\nu_\alpha\rangle)^T$$

are defined as the states which correspond to the charge leptons $\alpha = e, \mu, \tau$. The correspondence is established through the charged current interactions of active neutrinos and charged leptons.

Together with the standard ν_s , $|\nu\rangle_f$ may include also neutrino states allied with additional heavy charged leptons, as well as the states not associated with charge leptons, like sterile neutrinos, ν_s .

In general, the flavor states have no definite masses. Therefore, they can have either definite momentum, or definite energy but not both.

- The neutrino **mass** eigenstates

$$|\nu\rangle_m = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \dots)^T \equiv (|\nu_k\rangle)^T$$

are, by definition, the states with the definite masses m_k , $k = 1, 2, 3, \dots$

Since $|\nu_\alpha\rangle$ and $|\nu_k\rangle$ are not identical, they are related to each other through a unitary transformation

$$|\nu_\alpha\rangle = \sum_k \hat{V}_{\alpha k} |\nu_k\rangle \quad \text{or} \quad |\nu\rangle_f = \hat{\mathbf{V}} |\nu\rangle_m,$$

where $\hat{\mathbf{V}} = \|\hat{V}_{\alpha k}\|$ is a unitary (in general, $N \times N$) matrix.

To find out the correspondence between $\hat{\mathbf{V}}$ and the PMNS mixing matrix \mathbf{V} we can normalize the “ f ” and “ m ” states by the following conditions

$$\langle 0|\nu_{\alpha L}(x)|\nu_{\alpha'}\rangle = \delta_{\alpha\alpha'} \quad \text{and} \quad \langle 0|\nu_{kL}(x)|\nu_{k'}\rangle = \delta_{kk'}.$$

From these conditions we obtain

$$\sum_k V_{\alpha k} \hat{V}_{\alpha' k} = \delta_{\alpha\alpha'} \quad \text{and} \quad \sum_{\alpha} V_{\alpha k} \hat{V}_{\alpha k'} = \delta_{kk'}.$$

Therefore

$$\hat{\mathbf{V}} \equiv \mathbf{V}^\dagger$$

and

$$\boxed{|\nu\rangle_f = \mathbf{V}^\dagger |\nu\rangle_m \iff |\nu\rangle_m = \mathbf{V} |\nu\rangle_f.} \quad (11)$$

The time evolution of a single mass eigenstate $|\nu_k\rangle$ with momentum p_ν is trivial,

$$i \frac{d}{dt} |\nu_k(t)\rangle = E_k |\nu_k(t)\rangle \implies |\nu_k(t)\rangle = e^{-iE_k(t-t_0)} |\nu_k(t_0)\rangle,$$

where $E_k = \sqrt{p_\nu^2 + m_k^2}$ is the total energy in the state $|\nu_k\rangle$. Now, assuming that all N states $|\nu_k\rangle$ have the same momentum, one can write

$$i \frac{d}{dt} |\nu(t)\rangle_m = \mathbf{H}_0 |\nu(t)\rangle_m, \quad \text{where} \quad \mathbf{H}_0 = \text{diag}(E_1, E_2, E_3, \dots). \quad (12)$$

From Eqs. (11) and (12) we have

$$i \frac{d}{dt} |\nu(t)\rangle_f = \mathbf{V}^\dagger \mathbf{H}_0 \mathbf{V} |\nu(t)\rangle_f. \quad (13)$$

Solution to this equation is obvious:

$$\begin{aligned} |\nu(t)\rangle_f &= \mathbf{V}^\dagger e^{-i\mathbf{H}_0(t-t_0)} \mathbf{V} |\nu(t_0)\rangle_f \\ &= \mathbf{V}^\dagger \text{diag} \left(e^{-iE_1(t-t_0)}, e^{-iE_2(t-t_0)}, \dots \right) \mathbf{V} |\nu(t_0)\rangle_f. \end{aligned} \quad (14)$$

Now we can derive the survival and transition probabilities

$$\begin{aligned} P_{\alpha\beta}(t-t_0) &= P[\nu_\alpha(t_0) \rightarrow \nu_\beta(t)] = |\langle \nu_\beta(t) | \nu_\alpha(t_0) \rangle|^2 \\ &= \left| \sum_k V_{\alpha k} V_{\beta k}^* \exp[iE_k(t-t_0)] \right|^2 \\ &= \sum_{jk} V_{\alpha j} V_{\beta k} (V_{\alpha k} V_{\beta j})^* \exp[i(E_j - E_k)(t-t_0)]. \end{aligned}$$

In the ultrarelativistic limit $p_\nu^2 \gg m_k^2$, which is undoubtedly valid for all interesting circumstances (except relic neutrinos),

$$E_k = \sqrt{p_\nu^2 + m_k^2} \approx p_\nu + \frac{m_k^2}{2p_\nu} \approx E_\nu + \frac{m_k^2}{2E_\nu}.$$

Therefore in *very* good approximation

$$P_{\alpha\beta}(t - t_0) = \sum_{jk} V_{\alpha j} V_{\beta k} (V_{\alpha k} V_{\beta j})^* \exp \left[\frac{i\Delta m_{jk}^2 (t - t_0)}{2E_\nu} \right].$$

As a rule, there is no way to measure t_0 and t in the same experiment.^a But it is usually possible to measure the distance L between the source and detector. So we have to connect $t - t_0$ with L . It is easy to do in the standard ultrarelativistic approximation,

$$v_k = \frac{p_\nu}{E_k} \simeq 1 - \frac{m_k^2}{2E_\nu^2} = 1 - 0.5 \times 10^{-14} \left(\frac{m_k}{0.1 \text{ eV}} \right)^2 \left(\frac{1 \text{ MeV}}{E_\nu} \right)^2 \simeq 1,$$

from which it almost evidently follows that $t - t_0 \approx L$. Finally we arrive at the following formula

$$P_{\alpha\beta}(L) = \sum_{jk} V_{\alpha j} V_{\beta k} (V_{\alpha k} V_{\beta j})^* \exp \left(\frac{2i\pi L}{L_{jk}} \right), \quad L_{jk} = \frac{4\pi E_\nu}{\Delta m_{jk}^2}, \quad (15)$$

where L_{jk} (or more exactly $|L_{jk}| = |L_{kj}|$) are the so-called *neutrino oscillation lengths*.

It is straightforward to prove that the QM formula satisfies the probability conservation law:

$$\sum_{\alpha} P_{\alpha\beta}(L) = \sum_{\beta} P_{\alpha\beta}(L) = 1.$$

The range of applicability of the standard quantum-mechanical approach is limited but enough for the interpretation of essentially all modern experiments with accelerator, reactor, atmospheric, solar, and astrophysical neutrino beams.

^aImportant exceptions will be discussed in the special section.

5.2 Energy conservation.

Although the energy of the state with definite flavor, $|\nu_\alpha(L)\rangle = |\nu_\alpha(t)\rangle$, is not defined, its mean energy, $\langle E_\alpha(t) \rangle = \langle \nu_\alpha(t) | \hat{H} | \nu_\alpha(t) \rangle$, is a well-defined and conserved quantity. Indeed,

$$\langle E_\alpha(t) \rangle = \sum_{ij} V_{\alpha i} V_{\alpha j}^* \langle \nu_i(p) | \hat{H} | \nu_j(p) \rangle = \sum_{ij} V_{\alpha i} V_{\alpha j}^* \langle \nu_i(p) | E_i | \nu_j(p) \rangle \equiv \langle E_\alpha \rangle = \text{inv.}$$

$$\langle E_\alpha \rangle = \sum_i |V_{\alpha i}|^2 E_i \simeq p + \sum_i |V_{\alpha i}|^2 \frac{m_i^2}{2p}, \quad \Rightarrow \quad \sum_\alpha \langle E_\alpha \rangle = \sum_i E_i \simeq 3 \left(p + \sum_i \frac{m_i^2}{2p} \right).$$

Moreover, the mean energy of an arbitrary entangled state characterized by a certain density matrix $\rho(t)$ is also conserved. Indeed, let the initial state have the form

$$\rho(0) = \sum_\alpha w_\alpha |\nu_\alpha(0)\rangle \langle \nu_\alpha(0)|,$$

The mean energy of the mixed state at arbitrary time t is then written as

$$\begin{aligned} \langle E(t) \rangle &= \text{Tr} (\hat{H} \rho(t)) = \text{Tr} \left(\hat{H} e^{-i\hat{H}t} \rho(0) e^{i\hat{H}t} \right) \\ &= \sum_\alpha w_\alpha \sum_{ij} V_{\alpha i}^* V_{\alpha j} e^{-i(E_i - E_j)t} E_i \text{Tr} |\nu_i(p)\rangle \langle \nu_j(p)| \\ &= \sum_\alpha w_\alpha \sum_i |V_{\alpha i}|^2 E_i = \text{inv.}, \quad \Rightarrow \quad \langle E(t) \rangle = \sum_\alpha w_\alpha \langle E_\alpha \rangle. \end{aligned}$$

Naturally, $\langle E(t) \rangle = \langle E_\alpha \rangle$ for the pure initial state $|\nu_\alpha(0)\rangle$ (when $\rho(0) = |\nu_\alpha(0)\rangle \langle \nu_\alpha(0)|$).

5.3 Simplest example: two-flavor oscillations.

Let's now consider the simplest (toy) 2-flavor model, e.g., with $i = 2, 3$ and $\alpha = \mu, \tau$ (the most favorable due to the SK and other underground experiments). The 2×2 vacuum mixing matrix can be parametrized (due to the unitarity) with a single parameter, θ ($= \theta_{23}$), the vacuum mixing angle,

$$\mathbf{V} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

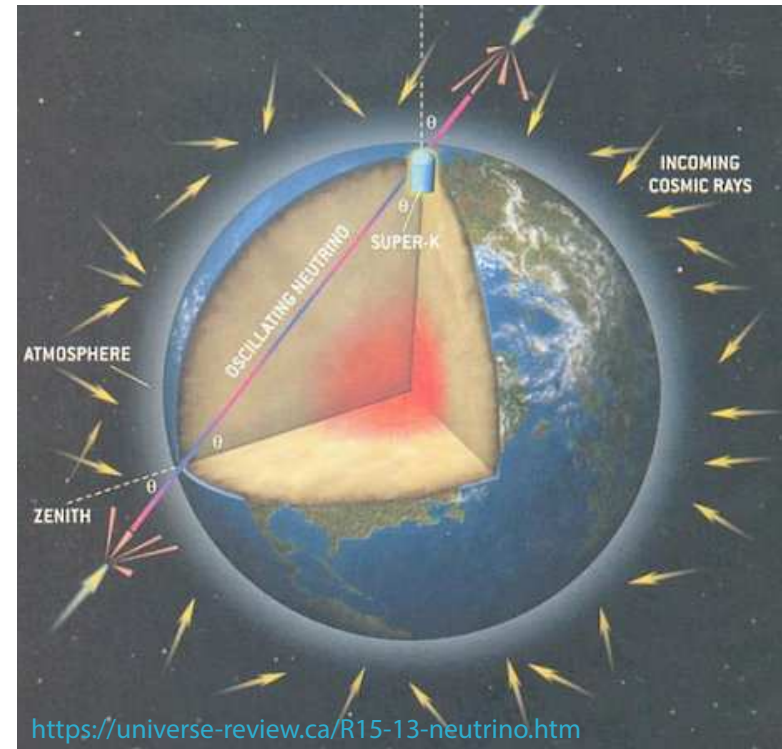
In this model, Eq. (15) then becomes very simple and transparent:

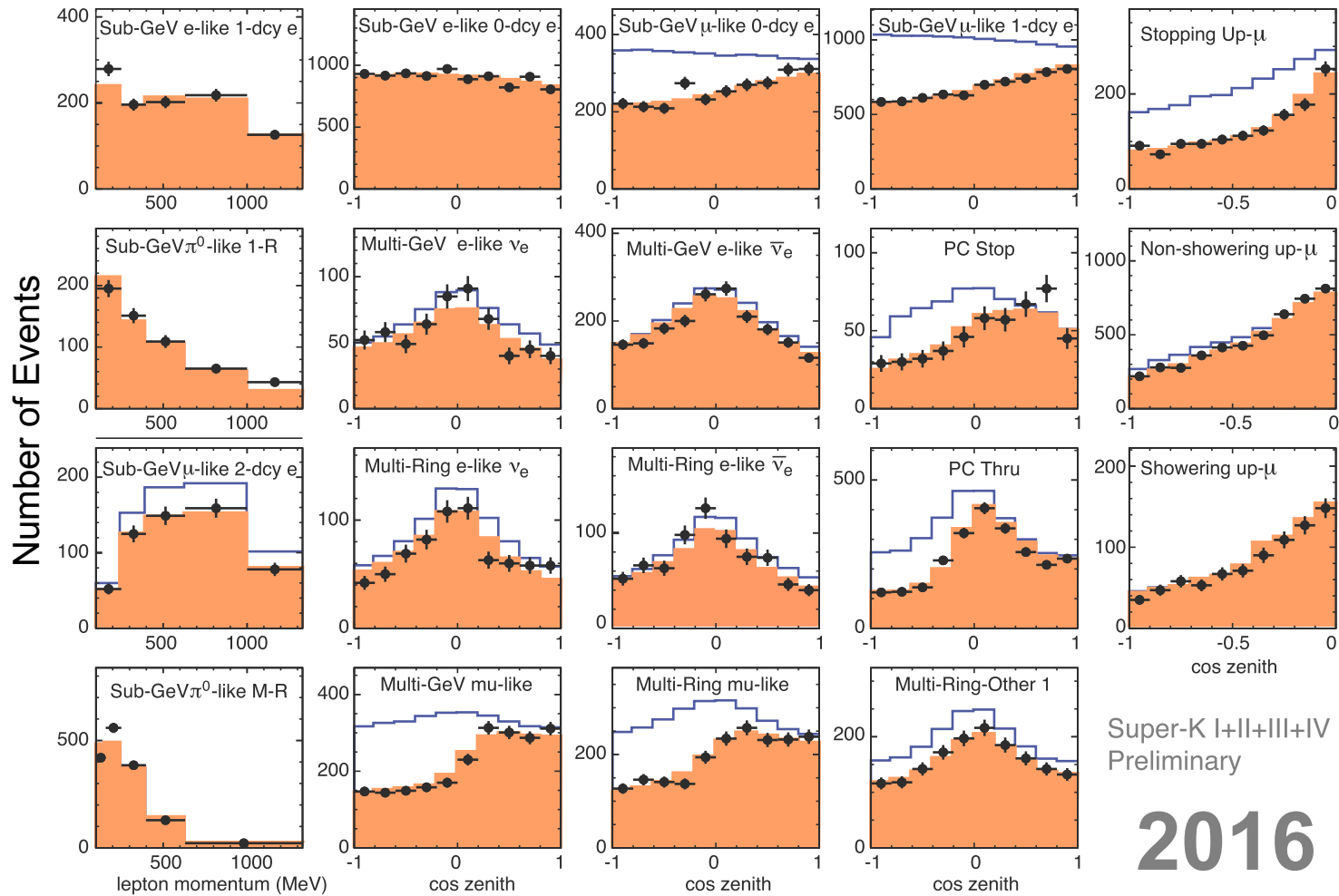
$$P_{\mu\tau}(L) = P_{\tau\mu}(L) = \frac{1}{2} \sin^2 2\theta \left[1 - \cos \left(\frac{2\pi L}{L_\nu} \right) \right],$$

$$L_\nu \equiv L_{23} = \frac{4\pi E_\nu}{\Delta m_{23}^2} \approx 2R_\oplus \left(\frac{E_\nu}{10 \text{ GeV}} \right) \left(\frac{0.002 \text{ eV}^2}{\Delta m_{23}^2} \right).$$

Here R_\oplus is the mean radius of Earth and 10 GeV is a typical energy in the (very wide) atmospheric neutrino spectrum.

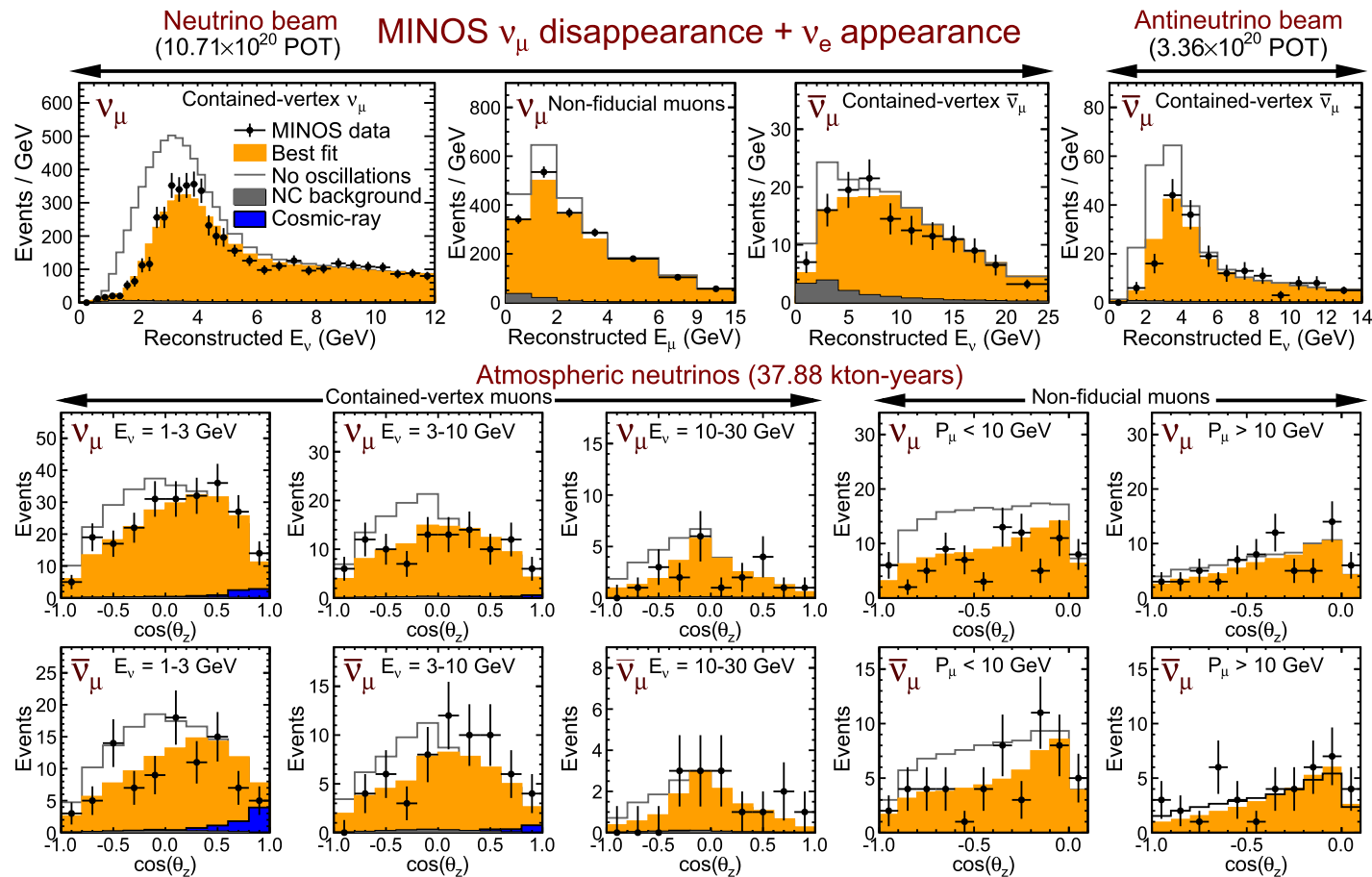
Since Earth provides variable "baseline" [from about 15 km to about 12700 km], it is surprisingly suitable for studying the atmospheric (as well as accelerator and reactor) neutrino oscillations in rather wide range of the oscillation parameters.





Zenith angle and momentum distributions for atmospheric neutrino subsamples used for an analyses by Super-Kamiokande to study subleading effects, preferences for mass hierarchy and δ_{CP} , as well as searches for astrophysical sources such as dark matter annihilation.

[From T. Kajita *et al.* (for the Super-Kamiokande Collaboration), "Establishing atmospheric neutrino oscillations with Super-Kamiokande," *Nucl. Phys. B* **908** (2016) 14–29.]



The event spectra at MINOS from 10.71×10^{20} POT FHC (ν_μ -dominated) mode, 3.36×10^{20} POT RHC ($\bar{\nu}_\mu$ -dominated) mode and 37.88 kt-yrs of atmospheric data. The data are shown compared to the prediction in absence of oscillations (grey lines) and to the best-fit prediction (red). The beam histograms (top) also include the NC background component (filled grey) and the atmospheric histograms (bottom) include the cosmic-ray background contribution filled blue).

[From L. H. Whitehead (For the MINOS Collaboration), "Neutrino oscillations with MINOS and MINOS+," Nucl. Phys. B 908 (2016) 130–150.]

5.4 Summary of the standard QM theory.

The standard assumptions are intuitively transparent and (almost) commonly accepted.

- [1] The neutrino flavor states $|\nu_\alpha\rangle$ associated with the charged leptons $\alpha = e, \mu, \tau$ (that is having definite lepton numbers) are not identical to the neutrino mass eigenstates $|\nu_i\rangle$ with the definite masses m_i ($i = 1, 2, 3$).

Both sets of states are orthonormal: $\langle \nu_\beta | \nu_\alpha \rangle = \delta_{\alpha\beta}$, $\langle \nu_j | \nu_i \rangle = \delta_{ij}$.



They are related to each other through a unitary transformation $\mathbf{V} = \|V_{\alpha i}\|$, $\mathbf{V}\mathbf{V}^\dagger = \mathbf{1}$,

$$|\nu_\alpha\rangle = \sum_i V_{\alpha i}^* |\nu_i\rangle, \quad |\nu_i\rangle = \sum_\alpha V_{\alpha i} |\nu_\alpha\rangle.$$

- [2] Massive neutrino states originated from any reaction or decay have the same definite momenta \mathbf{p}_ν ["equal momentum (EM) assumption"].^a

To simplify matter, we do not consider exotic processes with multiple neutrino production.



The flavor states $|\nu_\alpha\rangle$ have the same momentum \mathbf{p}_ν but have no definite mass and energy.

^aSometimes – the same definite energies ["equal energy (EE) assumption"].

- [3] Neutrino masses are so small that in essentially all experimental circumstances (or, more precisely, in a wide class of reference frames) the neutrinos are ultrarelativistic. Hence

$$E_k = \sqrt{\mathbf{p}_\nu^2 + m_k^2} \simeq |\mathbf{p}_\nu| + \frac{m_k^2}{2|\mathbf{p}_\nu|}.$$

- [4] Moreover, in the evolution equation, one can safely replace the time parameter t by the distance L between the neutrino source and detector. [Let's remind that $\hbar = c = 1$.]

The enumerated assumptions are sufficient to derive the nice and commonly accepted expression for the neutrino flavor transition probability [L_{jk} are the neutrino oscillation lengths]:

$$\begin{aligned}
 P(\nu_\alpha \rightarrow \nu_\beta; L) &\equiv P_{\alpha\beta}(L) = \sum_{jk} V_{\alpha j} V_{\beta k} (V_{\alpha k} V_{\beta j})^* \exp\left(\frac{2i\pi L}{L_{jk}}\right) \\
 &= \sum_j |V_{\alpha j}|^2 |V_{\beta j}|^2 + 2 \sum_{j>k} \left[\operatorname{Re}(V_{\alpha j}^* V_{\beta j} V_{\alpha k} V_{\beta k}^*) \cos\left(\frac{2\pi L}{L_{jk}}\right) \right. \\
 &\quad \left. + \operatorname{Im}(V_{\alpha j}^* V_{\beta j} V_{\alpha k} V_{\beta k}^*) \sin\left(\frac{2\pi L}{L_{jk}}\right) \right], \\
 L_{jk} &= \frac{4\pi E_\nu}{\Delta m_{jk}^2}, \quad E_\nu = |\mathbf{p}_\nu|, \quad \Delta m_{jk}^2 = m_j^2 - m_k^2.
 \end{aligned}$$

Just this result is the basis for the “oscillation interpretation” of the current experiments with the natural and artificial neutrino and antineutrino beams.

5.5 Some challenges against the QM approach.



Equal-momentum assumption

Massive neutrinos ν_i have, by assumption, **equal momenta**: $\mathbf{p}_i = \mathbf{p}_\nu$ ($i = 1, 2, 3$).

This key assumption seems to be **unphysical** being **reference-frame (RF) dependent**; if it is **true** in a certain RF then it is **false** in another RF moving with the velocity \mathbf{v} :

$$E'_i = \Gamma_{\mathbf{v}} [E_i - (\mathbf{v}\mathbf{p}_\nu)], \quad \mathbf{p}'_i = \mathbf{p}_\nu + \Gamma_{\mathbf{v}} \left[\frac{\Gamma_{\mathbf{v}}(\mathbf{v}\mathbf{p}_\nu)}{\Gamma_{\mathbf{v}} + 1} - E_i \right] \mathbf{v},$$

↓ [assuming, as necessary for oscillations, that $m_i \neq m_j$] ↓

$$\mathbf{p}'_i - \mathbf{p}'_j = (E'_j - E'_i) \mathbf{v} = \Gamma_{\mathbf{v}} (E_j - E_i) \mathbf{v} \neq 0.$$

Treating the Lorentz transformation as **active**, we conclude that the EM assumption cannot be applied to the **non-monoenergetic** ν beams (the case in real-life experiments).

* A similar objection exists against the alternative **equal-energy assumption**; in that case

$$E'_i - E'_j = \Gamma_{\mathbf{v}} (\mathbf{p}_j - \mathbf{p}_i) \mathbf{v} \neq 0, \quad |\mathbf{p}'_i - \mathbf{p}'_j| = \sqrt{|\mathbf{p}_i - \mathbf{p}_j|^2 + \Gamma_{\mathbf{v}}^2 [(\mathbf{p}_i - \mathbf{p}_j) \mathbf{v}]^2} \neq 0.$$

* Can the EM (or EE) assumption be at least a good approximation? Alas, **no, it cannot**.

Let ν_μ s arise from $\pi_{\mu 2}$ decays. If the pion beam has a wide momentum spectrum – from subrelativistic to ultrarelativistic (as it is, e.g., for cosmic-ray particles), the EM (or EE) condition cannot be valid even approximately within the whole spectral range of the pion neutrinos.



Light-ray approximation

The propagation time T is, by assumption, equal to the distance L traveled by the neutrino between production and detection points. But, if the massive neutrino components have **the same momentum \mathbf{p}_ν** , their velocities are in fact different:

$$\mathbf{v}_i = \frac{\mathbf{p}_\nu}{\sqrt{\mathbf{p}_\nu^2 + m_i^2}} \implies |\mathbf{v}_i - \mathbf{v}_j| \approx \frac{\Delta m_{ji}^2}{2E_\nu^2}.$$

One may naively expect that during the time T the neutrino ν_i travels the distance $L_i = |\mathbf{v}_i|T$; therefore, there must be a spread in distances of each neutrino pair

$$\delta L_{ij} = L_i - L_j \approx \frac{\Delta m_{ji}^2}{2E_\nu^2} L, \quad \text{where } L = cT = T.$$

Δm_{ji}^2	E_ν	L	L_{ij}	$ \delta L_{ij} $
Δm_{23}^2	1 GeV	$2R_\oplus$	$0.1R_\oplus$	$\sim 10^{-12}$ cm
Δm_{23}^2	1 TeV	$R_G \sim 100$ kps	$100R_\oplus$	$\sim 10^{-4}$ cm
Δm_{21}^2	1 MeV	1 AU	$0.25R_\oplus$	$\sim 10^{-3}$ cm

The values of δL_{ij} listed in the Table seem to be **fantastically** small. But

Are they sufficiently small to preserve the coherence in any circumstance?

In other words:

What is the natural scale of the distances and times?



Can light neutrinos oscillate into heavy ones or vice versa?
[Can active neutrinos oscillate into sterile ones or vice versa?]

The naive QM answer is **Yes**. **Why not?** If, at least, both ν_α (light) and ν_s (heavy) are ultrarelativistic [$|\mathbf{p}_\nu| \gg \max(m_1, m_2, m_3, \dots, M)$,] one obtains the **same** formula for the oscillation probability $P_{\alpha s}(L)$, since the QM formalism has no any limitation to the neutrino mass hierarchy.

Possibility of such transitions is a basis for many speculations in astrophysics and cosmology.

But! Assume again that the neutrino source is $\pi_{\mu 2}$ decay and $M > m_\pi$. Then the transition $\nu_\alpha \rightarrow \nu_s$ in the pion rest frame is forbidden by the energy conservation.



There must be some limitations & flaws in the QM formula. What are they?



Do relic neutrinos oscillate?

Most likely the lightest relic neutrinos are always relativistic or even ultrarelativistic, while heavier ones become subrelativistic and then non-relativistic as the universe expands.

The naive QM approach does not know how to handle such a set of neutrinos.



Does the motion of the neutrino source affect the transition probabilities?

To answer these and similar questions

one has to unload the UR approximation & develop a covariant formalism.

In the QFT approach: the **effective** (most probable) energies and momenta of **virtual** ν_i s are *found to be* functions of the **masses**, **most probable momenta** and **momentum spreads** of all particles (wave packets) involved into the neutrino production and detection processes.

In particular, in the two limiting cases – ultrarelativistic (**UR**) and nonrelativistic (**NR**):

Ultrarelativistic case

$$(|q_{s,d}^0| \sim |\mathbf{q}_{s,d}| \gg m_i)$$

$$\left\{ \begin{array}{l} E_i = E_\nu \left[1 - nr_i - mr_i^2 + \dots \right], \\ |\mathbf{p}_i| = E_\nu \left[1 - (n+1)r_i - \left(m + n + \frac{1}{2} \right) r_i^2 + \dots \right], \\ v_i = 1 - r_i - \left(2n + \frac{1}{2} \right) r_i^2 + \dots < 1, \end{array} \right.$$

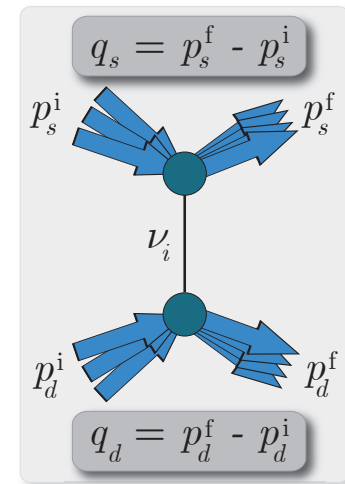
Nonrelativistic case

$$(|q_{s,d}^0| \sim m_i \gg |\mathbf{q}_{s,d}|)$$

$$\left\{ \begin{array}{l} E_i = m_i + \frac{m_i v_i^2}{2} \left(1 + \frac{3}{4} \delta_i + \dots \right), \\ |\mathbf{p}_i| = m_i v_i \left(1 + \frac{1}{2} \delta_i + \dots \right), \\ v_i \approx \frac{e_i^l}{1 + e_i^0} \ll 1, \end{array} \right.$$

$$E_\nu \approx q_s^0 \approx -q_d^0, \quad r_i = \frac{m_i^2}{2E_\nu^2} \ll 1 \text{ (UR),}$$

$$e_i^\mu = \frac{1}{m_i \mathcal{R}} \left[\tilde{\mathcal{R}}_s^{\mu 0} (m_i - q_s^0) + \tilde{\mathcal{R}}_d^{\mu 0} (m_i + q_d^0) - \tilde{\mathcal{R}}_s^{\mu k} q_s^k + \tilde{\mathcal{R}}_d^{\mu k} q_d^k \right], \quad |e_i^\mu| \ll 1 \text{ (NR).}$$





Definite momentum assumption

In the naive QM approach, the assumed **definite momenta** of neutrinos (both ν_α and ν_i) imply that the spatial coordinates of neutrino production (\mathbf{X}_s) and detection (\mathbf{X}_d) are **fully uncertain** (Heisenberg's principle).



The distance $L = |\mathbf{X}_d - \mathbf{X}_s|$ is uncertain too, that makes the standard QM formula for the flavor transition probabilities to be strictly speaking **senseless**.

In the correct theory, the neutrino momentum uncertainty $\delta|\mathbf{p}_\nu|$ must be **at least** of the order of $\min(1/D_s, 1/D_d)$, where D_s and D_d are the characteristic dimensions of the source and detector "machines" along the neutrino beam.



The neutrino states must be some **wave packets** (WP) [though having very small spreads] dependent, in general, on the quantum states of the particles [or, more exactly, also WPs] which participate in the production and detection processes.

In the QFT approach: the **effective** WPs of virtual UR ν_i s are *found to be*

$$\psi_i^{(*)} = \exp \left\{ \pm i(p_i X_{s,d}) - \frac{\tilde{\mathcal{D}}_i^2}{E_\nu^2} \left[(p_i X)^2 - m_i^2 X^2 \right] \right\}, \quad X = X_d - X_s,$$

where $p_i = (E_i, \mathbf{p}_i)$ and $X_{s,d}$ are the 4-vectors which characterize the space-time location of the ν production and detection processes, while $\tilde{\mathcal{D}}_i$ are certain (in general, complex-valued) functions of the masses, mean momenta and momentum spreads of all particles involved into these processes. [$\tilde{\mathcal{D}}_i/E_\nu$ and thereby ψ_i are Lorentz invariants.]

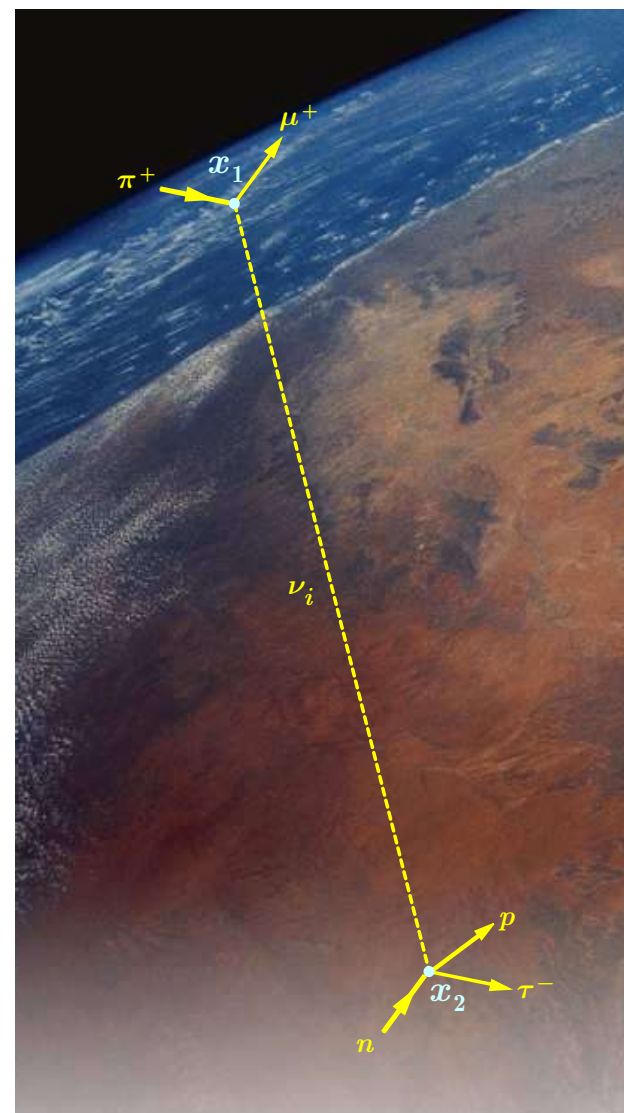
5.6 The aims and concepts of the field-theoretical approach.

The main purposes:

To define the domain of applicability of the standard quantum-mechanical (QM) theory of **vacuum neutrino oscillations** and obtain the QFT corrections to it.

The basic concepts:

- The “ ν -oscillation” phenomenon in QFT is nothing else than a result of **interference of the macroscopic Feynman diagrams** perturbatively describing the lepton number violating processes with the **massive neutrino fields** as **internal lines** (propagators).
- The **external lines** of the macrodiagrams are **wave packets** rather than **plane waves** (therefore the standard S matrix approach should be revised).
- The external **wave packet states** are the **covariant superpositions** of the standard one-particle Fock states, satisfying a **correspondence principle**.

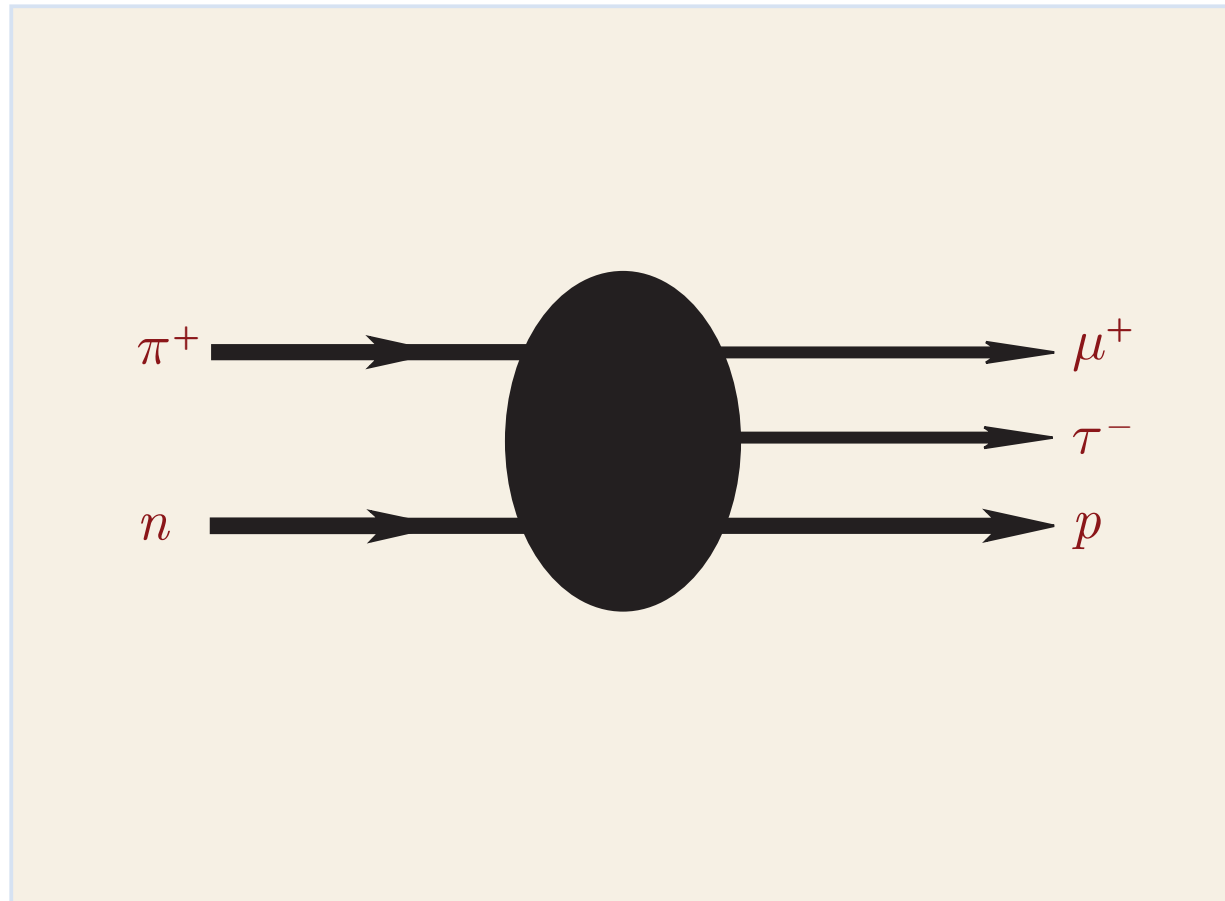


References: D. V. Naumov & VN, J. Phys. G **37** (2010) 105014, arXiv:1008.0306 [hep-ph]; Russ. Phys. J. **53** (2010) 549–574; arXiv:1110.0989 [hep-ph]; ЭЧАЯ **51** (2020) 1–209 [Phys. Part. Nucl. **51** (2020) 1–106].

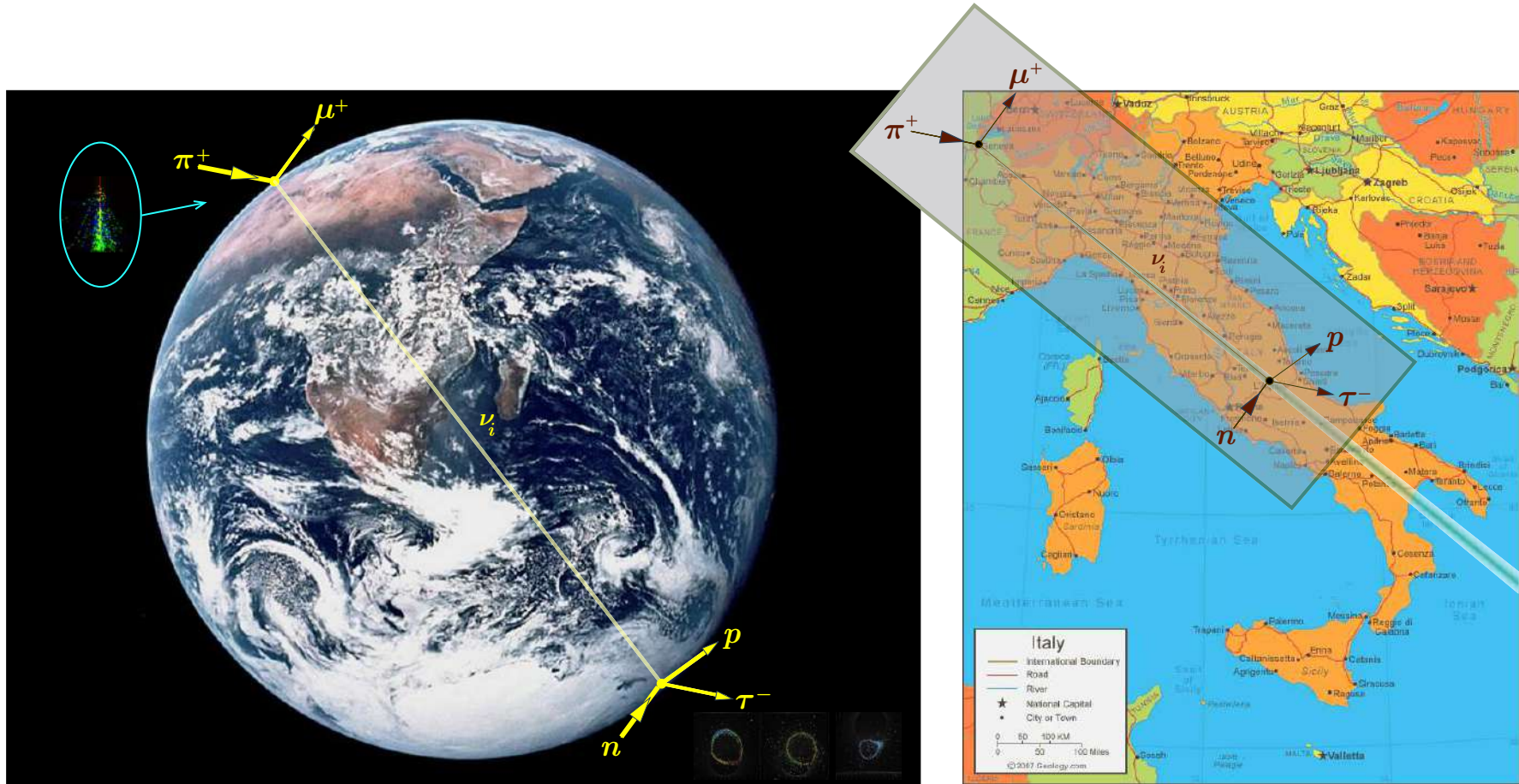
5.7 A sketch of the approach.

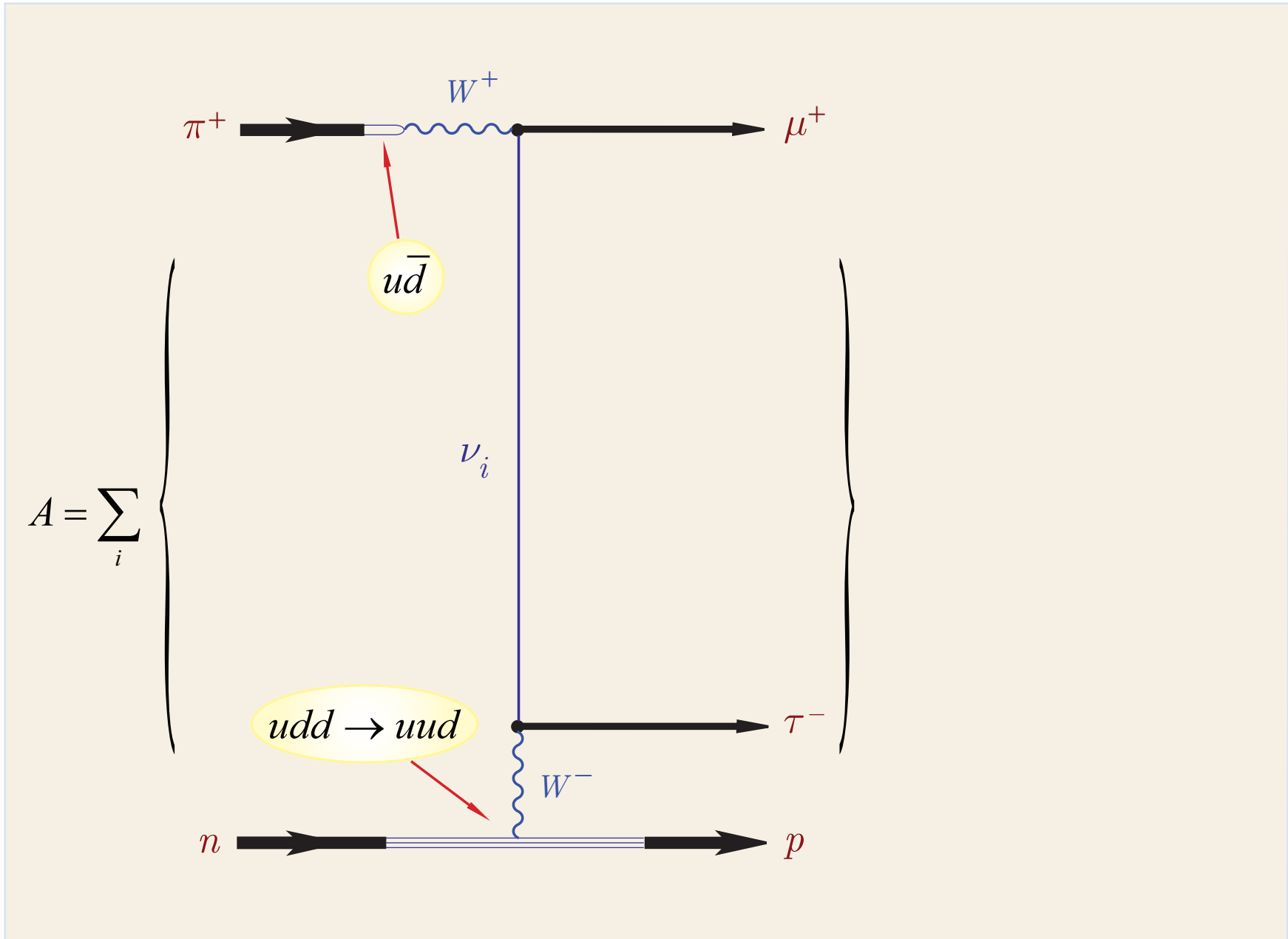
Let us first consider the basics of the QFT approach using the simplest example.

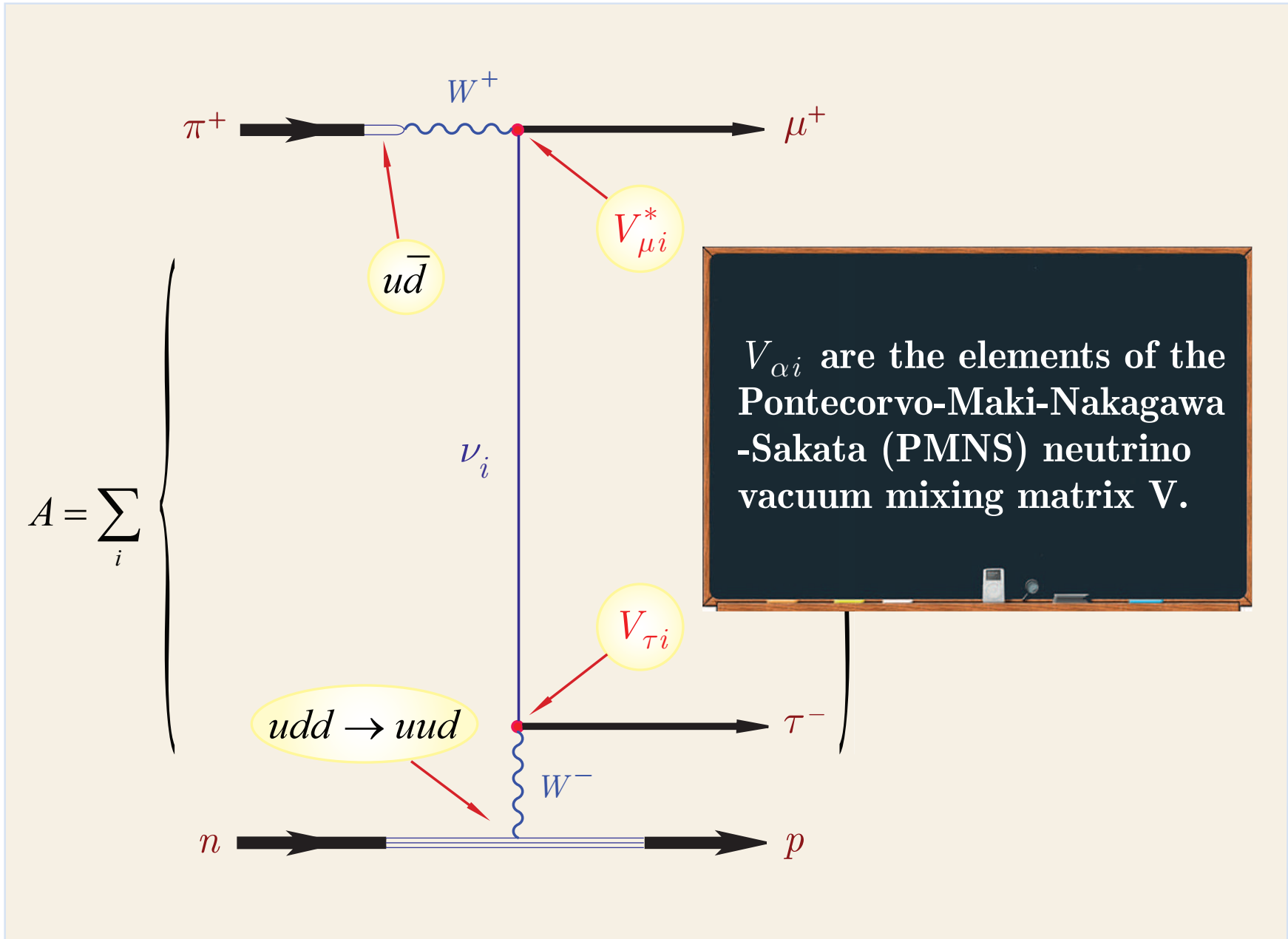
5.7.1 QFT approach by the example of the reaction $\pi^+ n \rightarrow \mu^+ \tau^- p$.

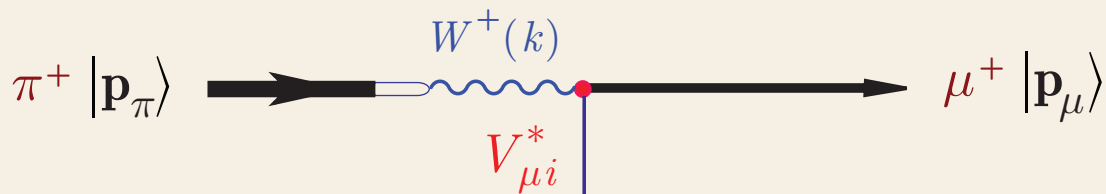


The rare reactions $\pi^+ \oplus n \rightarrow \mu^+ \oplus \tau^- p + \dots$ were (indirectly) detected by several underground experiments (Kamiokande, IMB, Super-Kamiokande) with atmospheric neutrinos. In 2010, OPERA experiment (INFN, LNGS) with the CNGS neutrino beam announced the direct observation of the first τ^- candidate event; six candidates were recorded in several years of the detector operation.









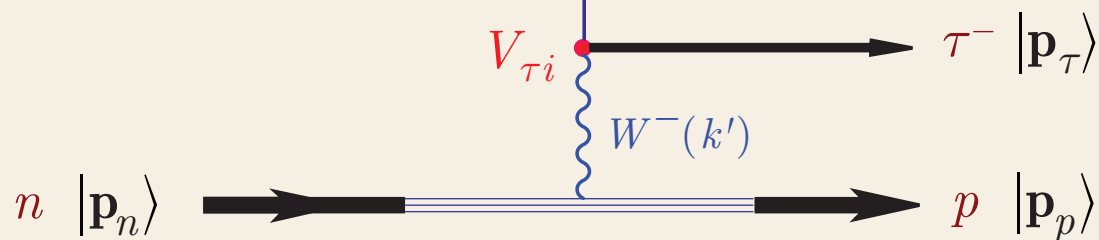
In the standard S matrix perturbation theory the **in** & **out** states are one-particle **Fock** states:

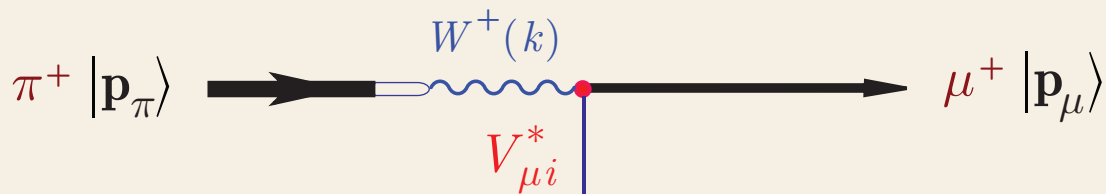
$$|\mathbf{p}_\alpha\rangle = \sqrt{2E_{\mathbf{p}_\alpha}} a_\alpha^+(\mathbf{p}_\alpha) |0\rangle$$

$$E_{\mathbf{p}_\alpha} = \sqrt{\mathbf{p}_\alpha^2 + m_\alpha^2}, \quad \alpha = \pi, \mu, n, \dots$$

$$\langle \mathbf{q} | \mathbf{k} \rangle = (2\pi)^3 2E_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q})$$

$$\nu_i(q)$$



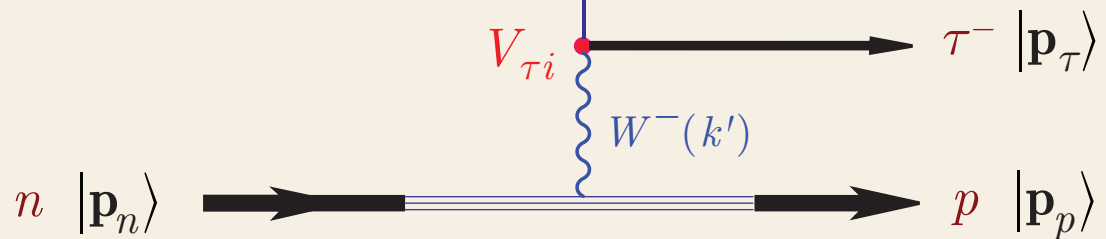


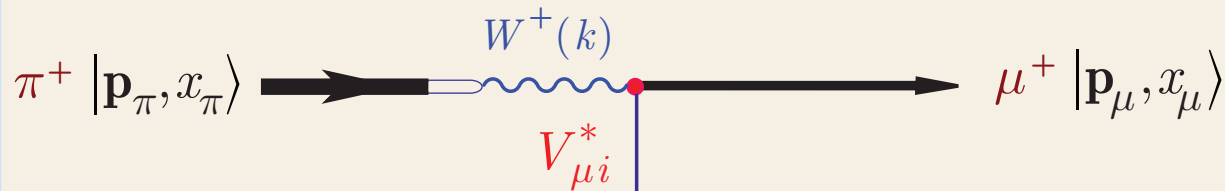
Feynman graphs with Fock legs cannot reproduce the ν -oscillation phenomenon.

In the standard S matrix perturbation theory the in & out states are one-particle Fock states:

$$|\mathbf{p}_\nu\rangle = \sqrt{2E_{\mathbf{p}_\nu}} a_\nu^+(\mathbf{p}_\nu) |0\rangle$$

$$E_{\mathbf{p}_\nu} = \sqrt{\mathbf{p}_\nu^2 + m_\nu^2}, \quad \nu = \pi, \mu, n, \dots$$

$$\langle \mathbf{q} | \mathbf{k} \rangle = (2\pi)^3 2E_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q})$$




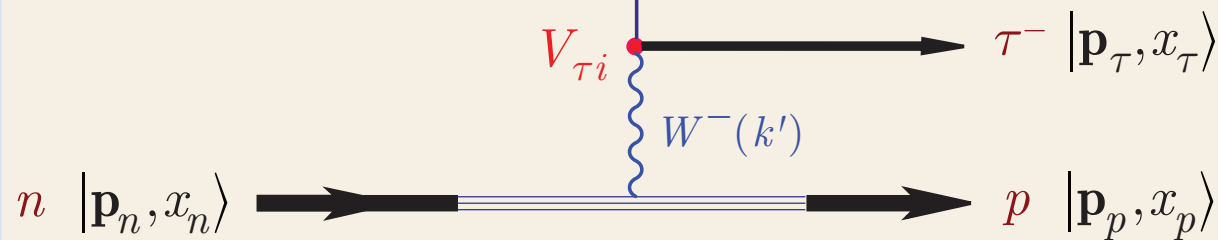
$$\nu_i(q)$$

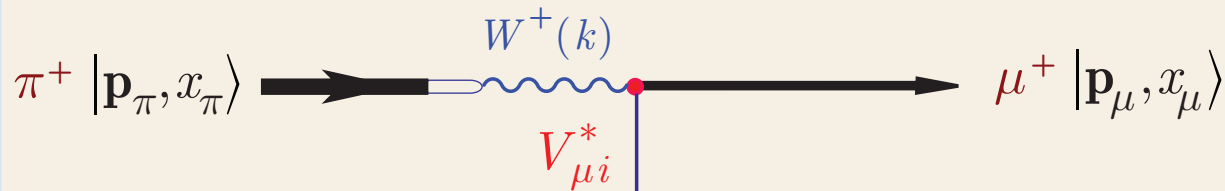
In our approach the **in** and **out** states are covariant wave packets:

$$| \mathbf{p}_\nu, x_\nu \rangle = \sqrt{2E_{\mathbf{p}_\nu}} A_\nu^+(\mathbf{p}_\nu, x_\nu) | 0 \rangle$$

$$A_\nu^+(\mathbf{p}, x) = \int \frac{d\mathbf{k} \phi(\mathbf{k}, \mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3 \sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} a_\nu^+(\mathbf{k})$$

$$A_\nu^+(\mathbf{p}, x) \xrightarrow{\text{PWL}} a_\nu^+(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x | \mathbf{p}, x \rangle = 2mV_*$$



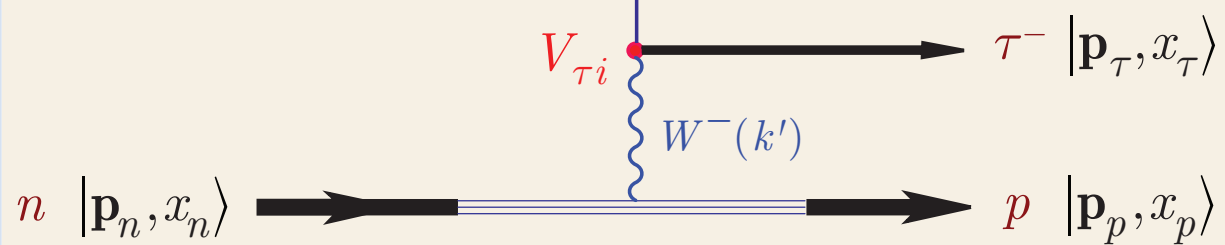


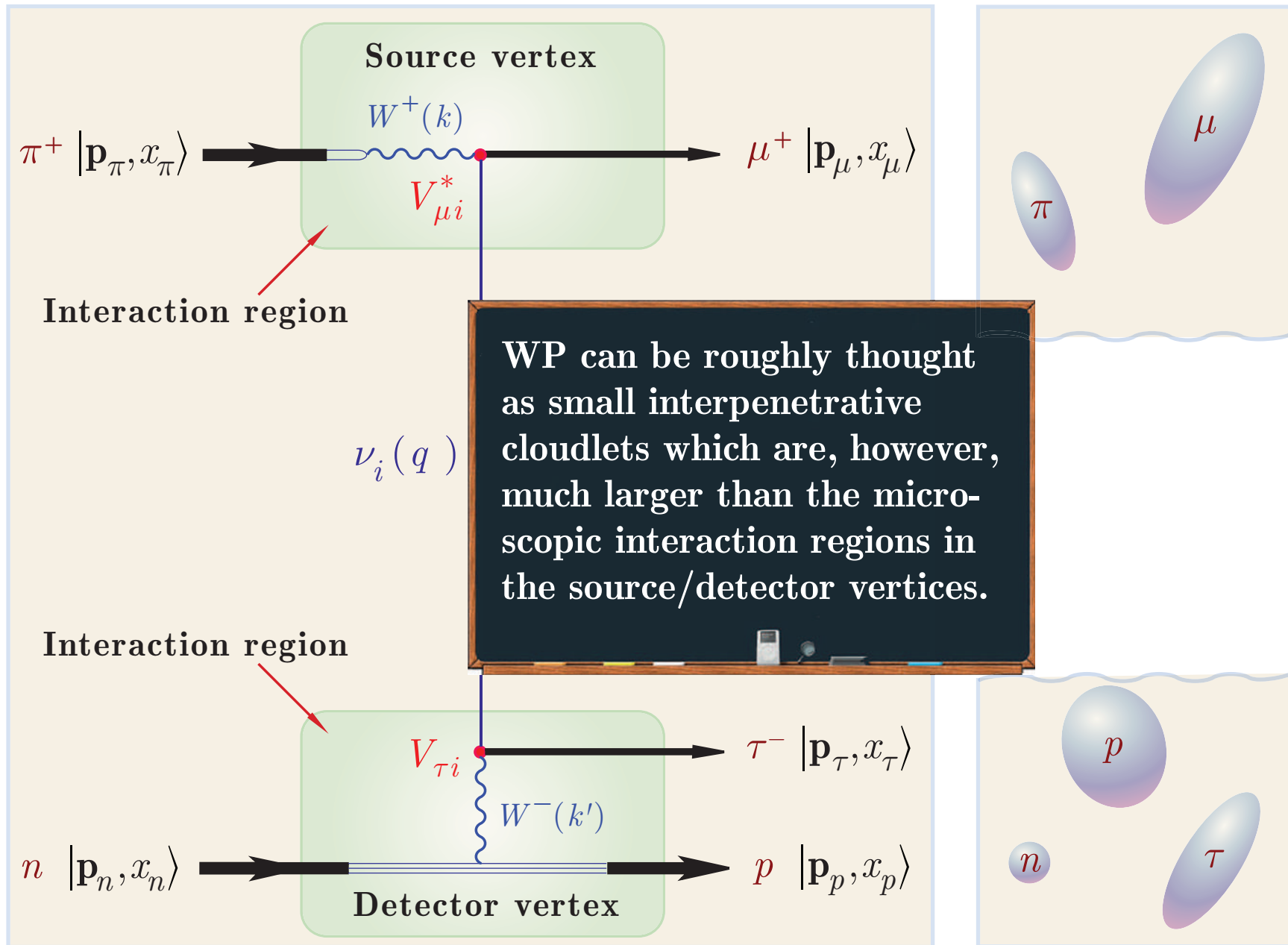
For simplicity we omit the spin and other discrete variables in the WP states

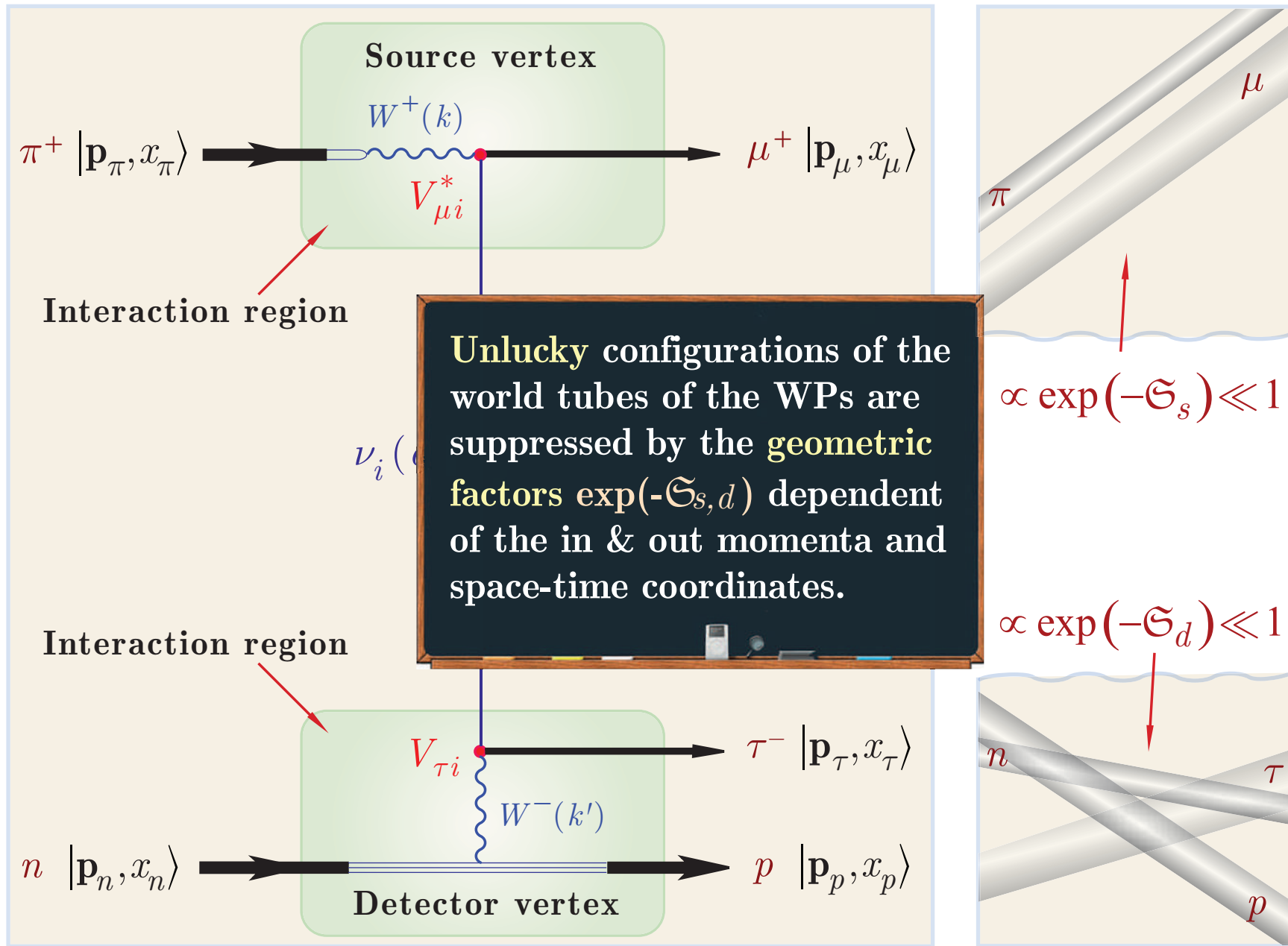
In our approach the **in** and **out** states are covariant wave packets:

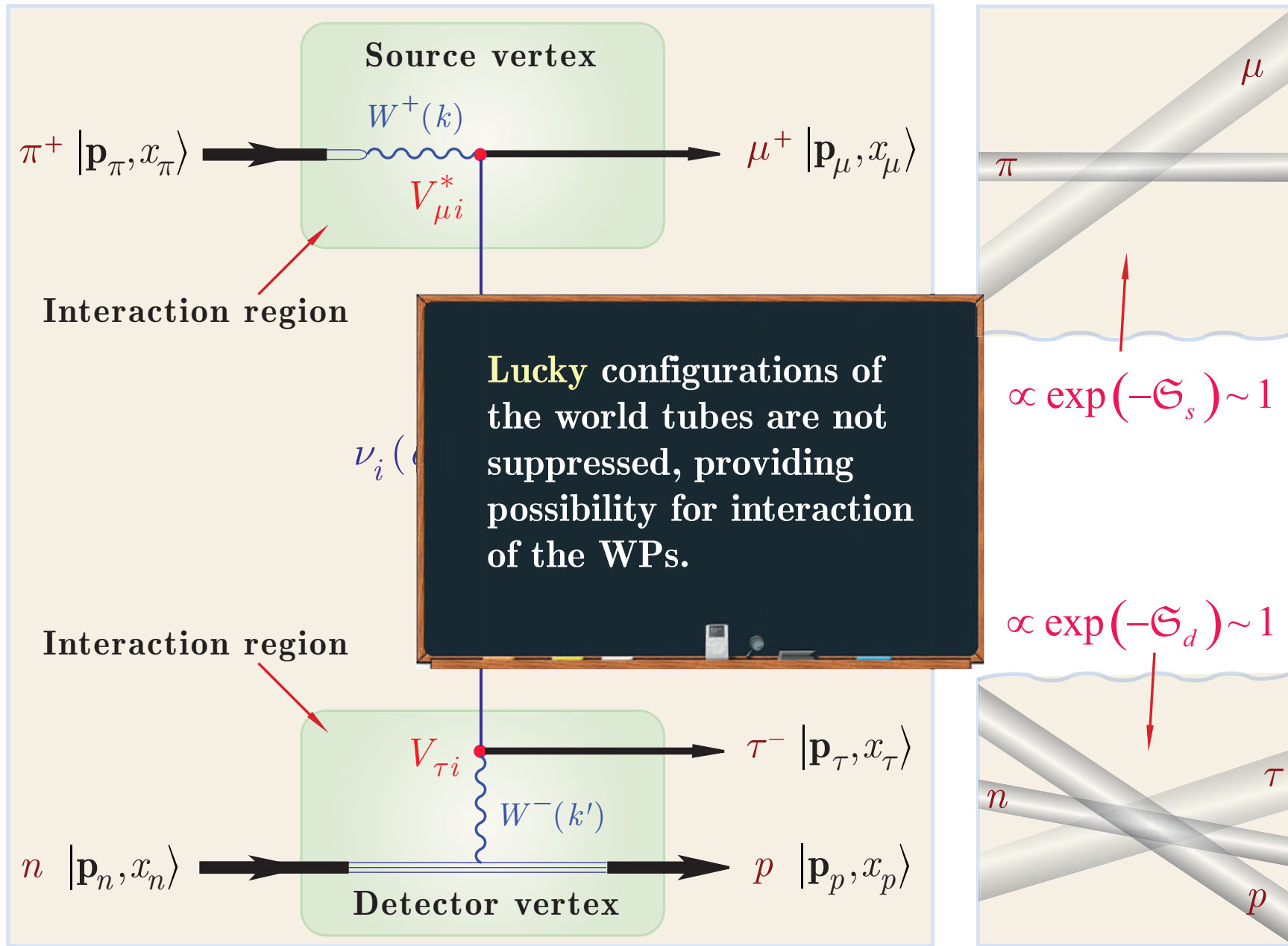
$$| \mathbf{p}_\nu, x_\nu \rangle = \sqrt{2E_{\mathbf{p}_\nu}} A_\nu^+(\mathbf{p}_\nu, x_\nu) | 0 \rangle$$

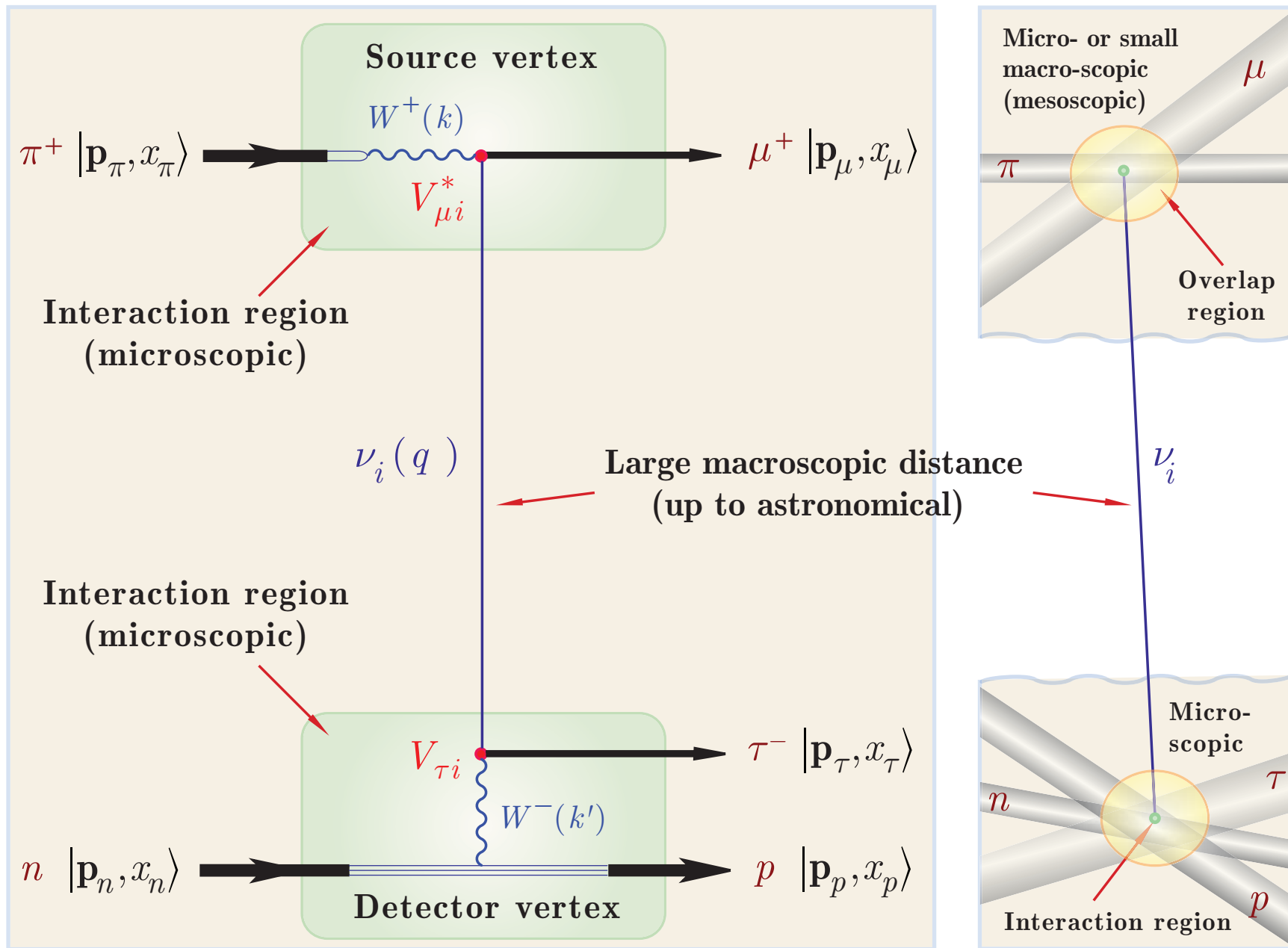
$$A_\nu^+(\mathbf{p}, x) = \int \frac{d\mathbf{k} \phi(\mathbf{k}, \mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3 \sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} a_\nu^+(\mathbf{k})$$

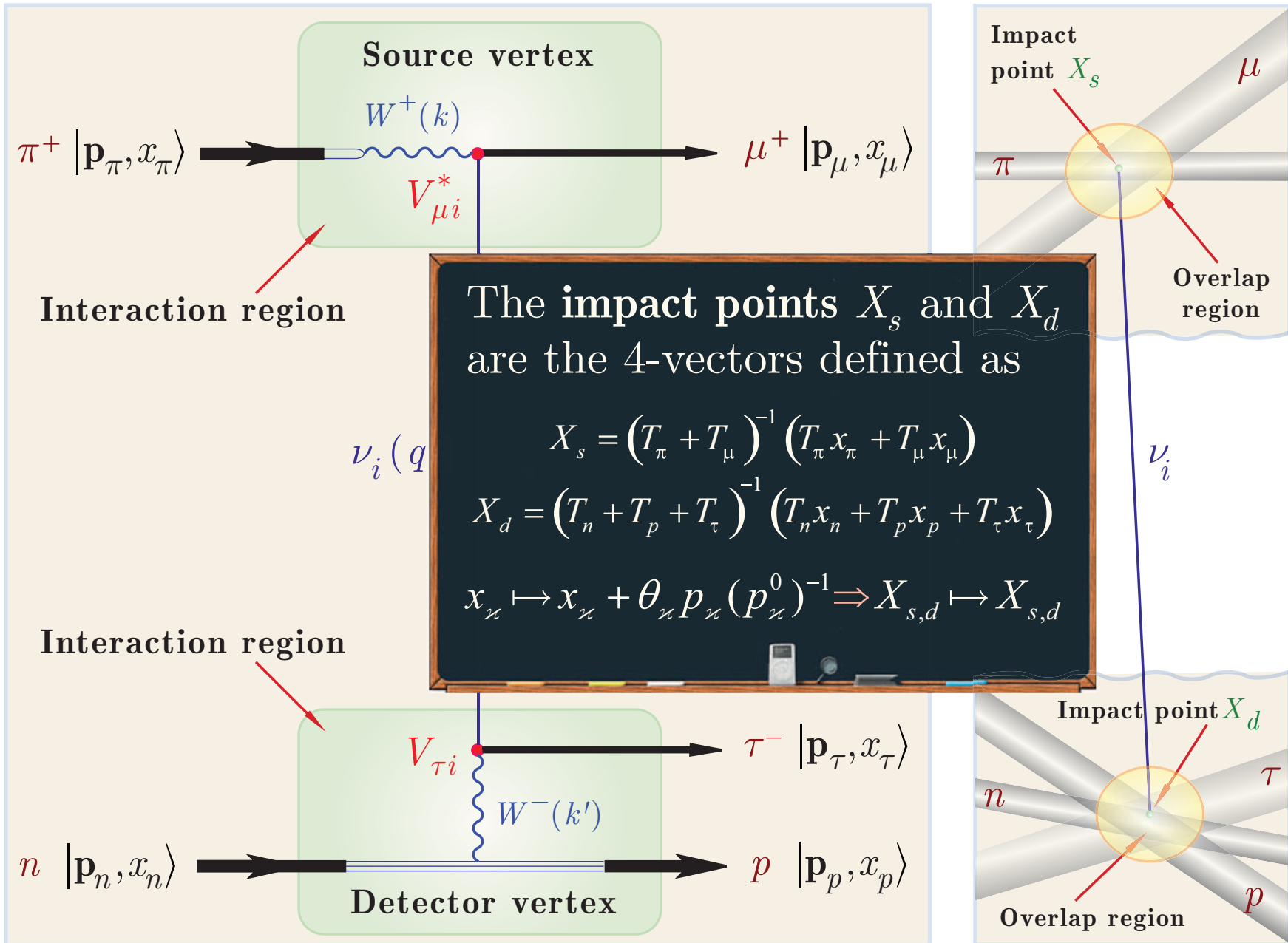
$$A_\nu^+(\mathbf{p}, x) \xrightarrow{\text{PWL}} a_\nu^+(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x | \mathbf{p}, x \rangle = 2mV_*$$












5.7.2 Space-time scales.

In the covariant WP approach there are several space-time scales:

- $T_I^{s,d}$ and $R_I^{s,d}$ – **microscopic** interaction time and radius defined by the Lagrangian.
- $T_O^{s,d}$ and $R_O^{s,d}$ – **microscopic** or **small macroscopic** dimensions of the overlap space-time regions of the interacting **in** and **out** packets in the source and detector vertices, defined by the effective dimensions of the packets.

The suppression of the “unlucky” configurations of world tubes of the external packets is governed by the geometric factor in the amplitude:

$$\exp[-(\mathfrak{G}_s + \mathfrak{G}_d)],$$

where $\mathfrak{G}_{s,d}$ are the positive Lorentz and translation invariant functions of $\{\mathbf{p}_\kappa\}$ and $\{x_\kappa\}$. In the simplest one-parameter model of WP (relativistic Gaussian packet)

$$\mathfrak{G}_{s,d} = \sum \sigma_\kappa^2 |\mathbf{b}_\kappa^*|^2, \quad \kappa \in S, D,$$

where σ_κ are the momentum speeds of the packet κ and \mathbf{b}_κ^* is the classical impact vector in the rest frame of the packet κ relative to the corresponding impact point.

- $T = X_d^0 - X_s^0$ and $L = |\mathbf{X}_d - \mathbf{X}_s|$ – **large macroscopic** neutrino time of flight and way between the impact points X_s and X_d .

For light neutrinos, the impact points lie very close to the light cone $T^2 = L^2$.

- In usual circumstance (terrestrial experiments) $T_I^{s,d} \ll T_O^{s,d} \ll T$ and $R_I^{s,d} \ll R_O^{s,d} \ll L$.

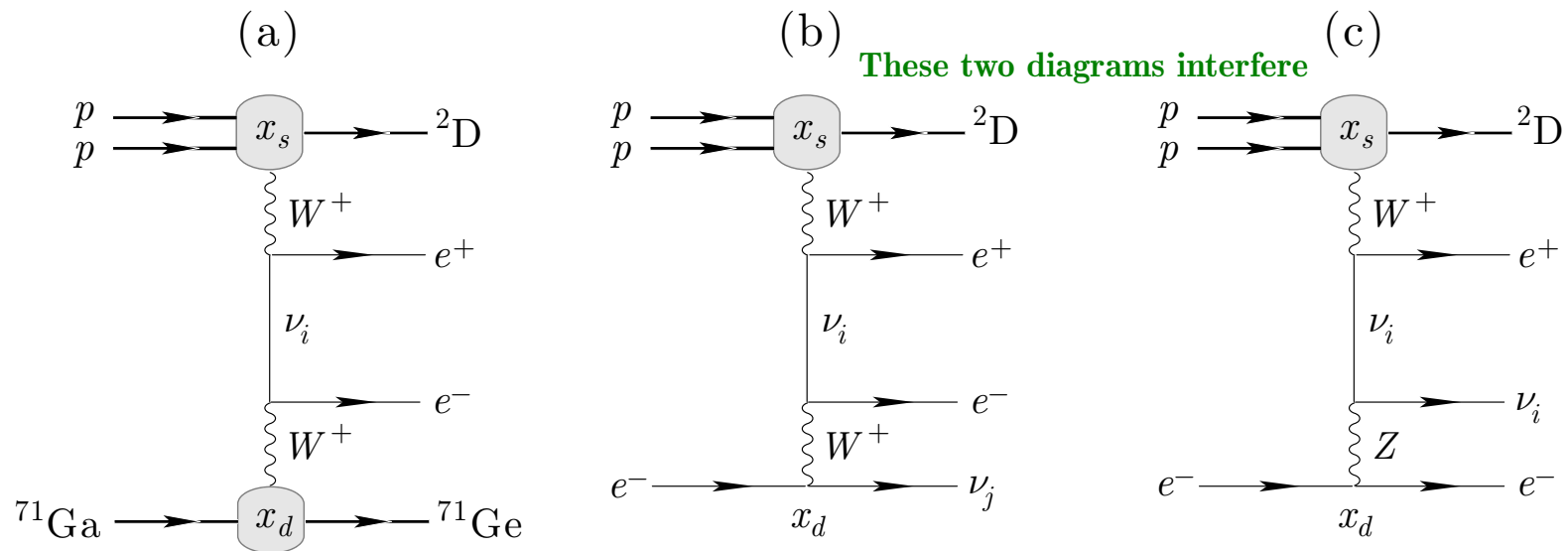
5.7.3 Examples of macroscopic diagrams.

- The pp fusion.

The first reaction of the pp I branch



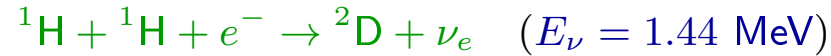
lights the Sun and can be detected in Ga-Ge detectors like SAGE and GALLEX.



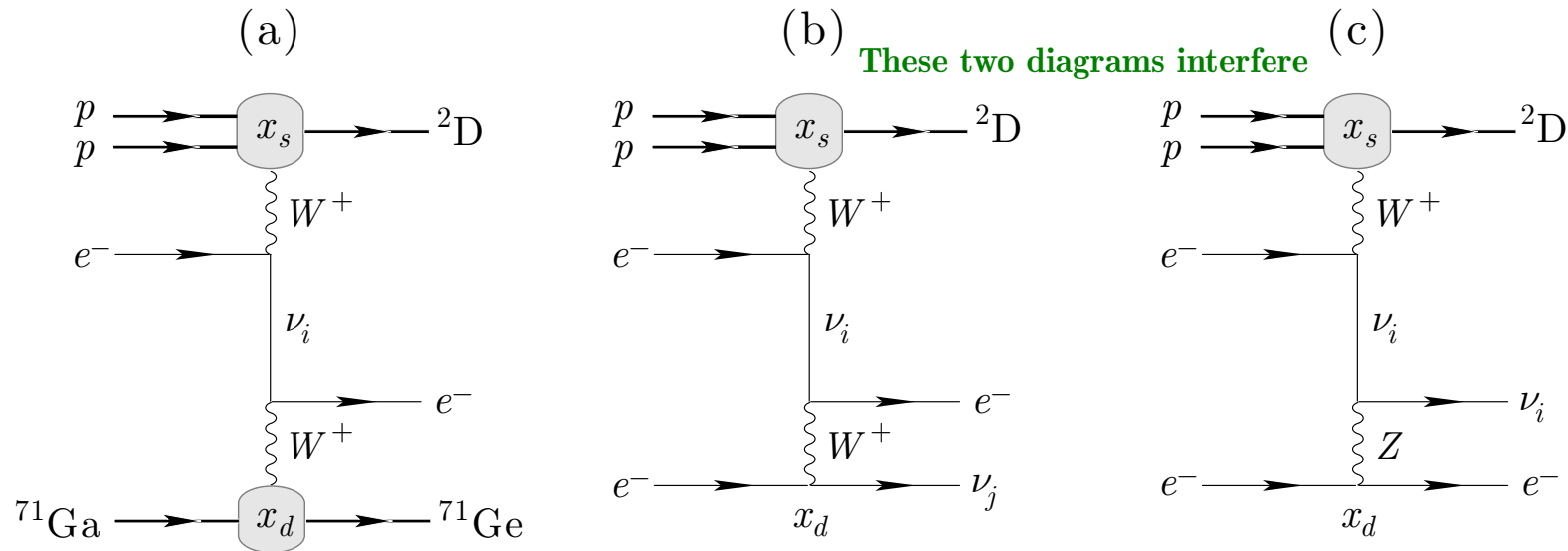
The Figure illustrates the detection of pp neutrinos with gallium (a) and electron (b,c) targets. Unfortunately, the final electron energies in the reactions (b,c) are too low to be detected by Cherenkov method.

- The *pep* fusion.

The reaction



accounts for about 0.25% of the deuterium created in the Sun in the *pp* chain. It has a characteristic time scale $\sim 10^{12}$ yr that is larger than the age of the Universe. So it is *insignificant* in the Sun as far as energy generation is concerned. Enough *pep* fusions happen to produce a detectable number of neutrinos in Ga-Ge detectors. Hence the reaction must be accounted for by those interested in the *solar neutrino problem*.



The Figure illustrates the detection of *pep* neutrinos with gallium (a) and electron (b,c) targets. Similar to the *pp* neutrino case, the diagram sets (c) and (d) interfere. While the final electron in the detector vertices of the diagrams (b,c) may have a momentum above the Cherenkov threshold, the current water-Cherenkov detectors SK and SNO+ are insensitive to the *pep* neutrinos.

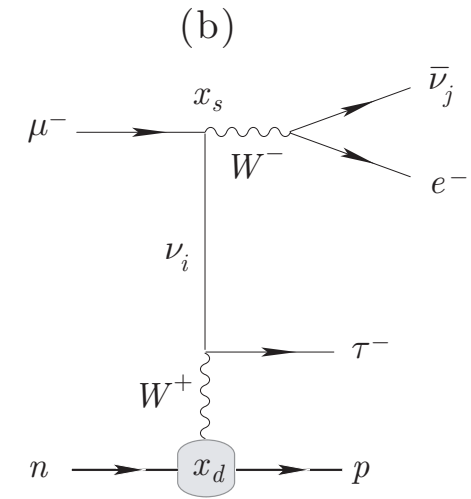
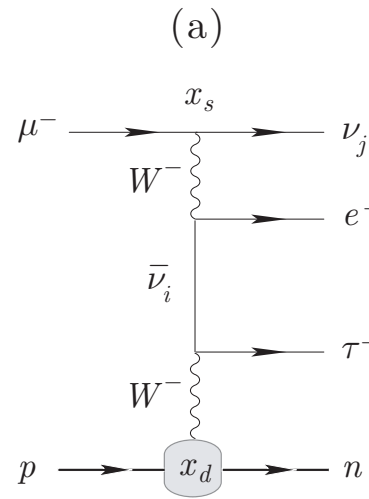
- The $\mu e3$ decay

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

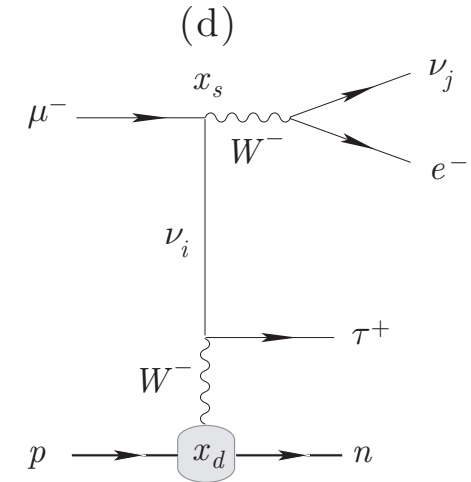
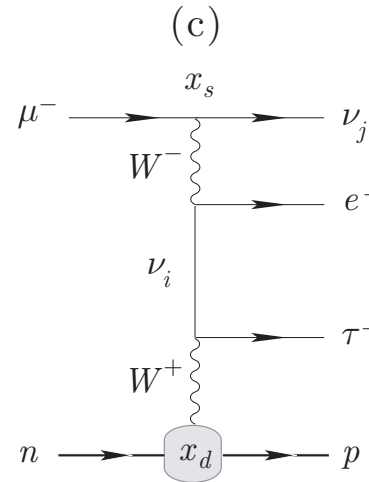
in the source can be detected through quasielastic scattering with production of e^\pm , μ^\pm , or τ^\pm ; of course, only μ^\pm production is permitted in SM. The diagrams (a) and (b) are for both Dirac and Majorana (anti)neutrinos, while diagrams (c) and (d) are only for Majorana neutrinos.

In the Majorana case, the diagrams (a), (d) and (b), (c) *interfere*. Potentially this provides a way for distinguishing between the Dirac and Majorana cases. Unfortunately, the diagrams (c) and (d) are suppressed by a factor $\propto m_i/E_\nu$.

Dirac or Majorana



Majorana

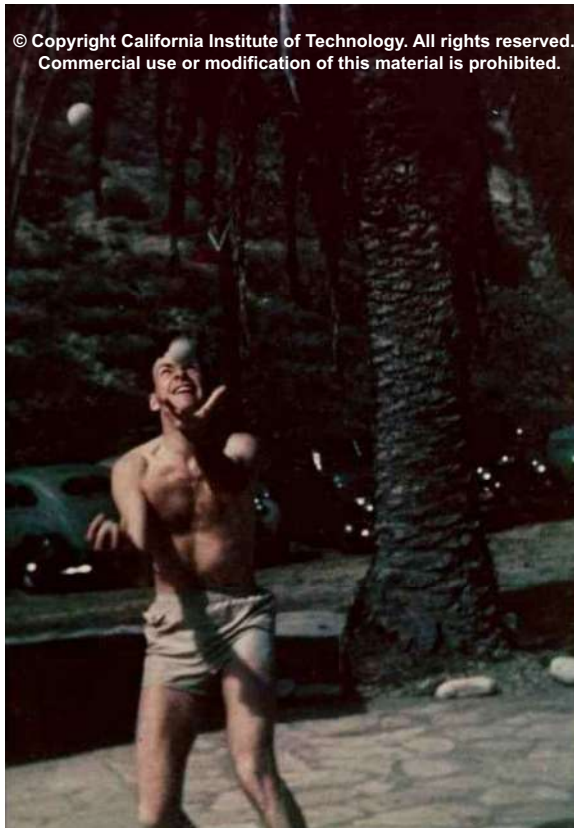
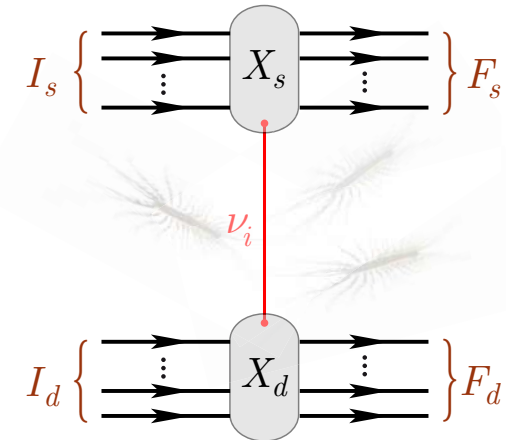


Similar diagrams can be drawn for $\tau e3$ and $\tau\mu3$ decays.

5.8 Shortest summary.

The QFT-based neutrino oscillation theory deals with generic Feynman's macrodiagrams (“myriapods”). ▷

The external legs correspond to asymptotically free **incoming** (“in”) and **outgoing** (“out”) wave packets (WP) in the coordinate representation. Here and below: I_s (F_s) is the set of in (out) WPs in X_s (“source”), I_d (F_d) is the set of in (out) WPs in X_d (“detector”).



The internal line denotes the causal Green's function of the **neutrino mass eigenfield** ν_i ($i = 1, 2, 3, \dots$). The blocks (vertices) X_s and X_d must be **macroscopically separated** in space-time. This explains the term “**macroscopic Feynman diagram**”.

For narrow WPs, the Feynman rules in the formalism are to be modified^a in a rather trivial way: for each external line, the *standard* (plain-wave) factor must be multiplied by

$$\begin{cases} e^{-ip_a(x_a-x)} \psi_a(\mathbf{p}_a, x_a - x) & \text{for } a \in I_s \oplus I_d, \\ e^{+ip_b(x_b-x)} \psi_b^*(\mathbf{p}_b, x_b - x) & \text{for } b \in F_s \oplus F_d, \end{cases} \quad (16)$$

where each function $\psi_\chi(\mathbf{p}_\chi, x)$ ($\chi = a, b$) is specified by the mass m_χ and momentum spread σ_χ . The lines inside X_s and X_d (including possible loops) and vertex factors remain unchanged.

^aFor non-commercial purposes.

5.8.1 Important class of macrodiagrams.

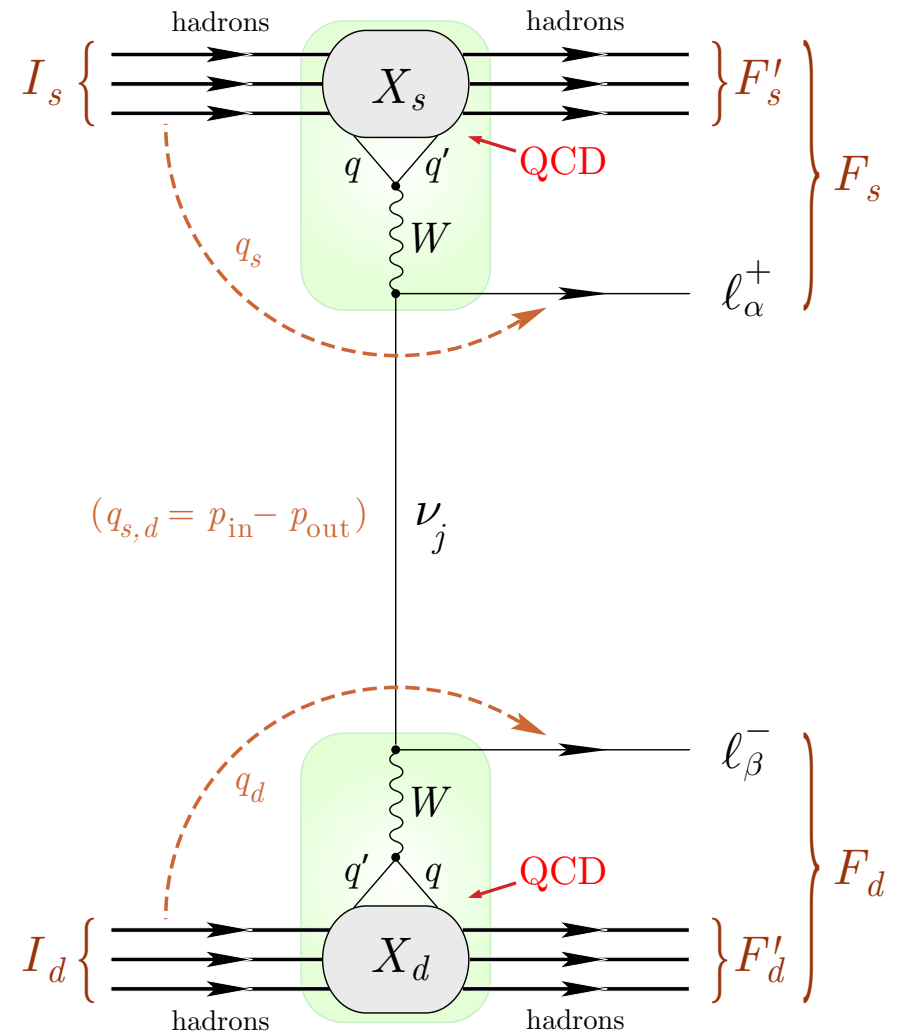
As a practically important example, we consider the charged-current induced production of charged leptons l_α^+ and l_β^- ($l_{\alpha,\beta} = e, \mu, \tau$) in the process

$$I_s \oplus I_d \rightarrow F'_s + l_\alpha^+ \oplus F'_d + l_\beta^-, \quad (17)$$

We assume for definiteness that all the external substates I_s , I_d , F'_s , and F'_d consist exclusively of (asymptotically free) hadronic WPs.

Consequently, if $\alpha \neq \beta$, the process (17) violates the lepton numbers L_α and L_β that is only possible via exchange of massive neutrinos (no matter whether they are Dirac or Majorana particles).

In the lowest nonvanishing order in electroweak interactions, the process (17) is described by the sum of the diagrams shown in the figure. \blacktriangleright



The impact points X_s and X_d in the figure are macroscopically separated and the asymptotic conditions are assumed to be fulfilled.

5.8.2 Main result.

A rather general (while not the most general) expression for the number of neutrino-induced events corresponding to the diagram shown in previous page, is of the form

$$\frac{N_{\beta\alpha}}{\tau_d} = \sum_{\text{spins}} \int d\mathbf{x} \int d\mathbf{y} \int d\mathfrak{P}_s \int d\mathfrak{P}_d \int d|\mathbf{q}| \frac{\mathcal{P}_{\alpha\beta} (|\mathbf{q}|, |\mathbf{y} - \mathbf{x}|)}{4(2\pi)^3 |\mathbf{y} - \mathbf{x}|^2},$$

$$\mathcal{P}_{\alpha\beta} (|\mathbf{q}|, |\mathbf{y} - \mathbf{x}|) = \sum_{ij} V_{\beta j} V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* \exp(i\varphi_{ij} - \mathcal{A}_{ij}^2 - \mathcal{C}_{ij}^2 - \Theta_{ij}) S_{ij},$$

$$S_{ij} = \frac{\exp(-\mathcal{B}_{ij}^2)}{4\mathfrak{D}\tau_d} \sum_{l,l'=1}^2 (-1)^{l+l'+1} \text{lerf} [2\mathfrak{D} (x_l^0 - y_{l'}^0 + |\mathbf{y} - \mathbf{x}|) + i\mathcal{B}_{ij}],$$

$$\mathfrak{D} = 1/\sqrt{2\tilde{\mathfrak{K}}^{\mu\nu} l_\mu l_\nu},$$

$$d\mathfrak{P}_s = (2\pi)^4 \delta_s(q - q_s) |M_s|^2 \prod_{a \in I_s} \frac{d\mathbf{p}_a f_a(\mathbf{p}_a, s_a, x)}{(2\pi)^3 2E_a} \prod_{b \in F_s} \frac{d\mathbf{p}_b}{(2\pi)^3 2E_b},$$

$$d\mathfrak{P}_d = (2\pi)^4 \delta_d(q + q_d) |M_d|^2 \prod_{a \in I_d} \frac{d\mathbf{p}_a f_a(\mathbf{p}_a, s_a, y)}{(2\pi)^3 2E_a} \prod_{b \in F_d} \frac{[d\mathbf{p}_b]}{(2\pi)^3 2E_b}.$$

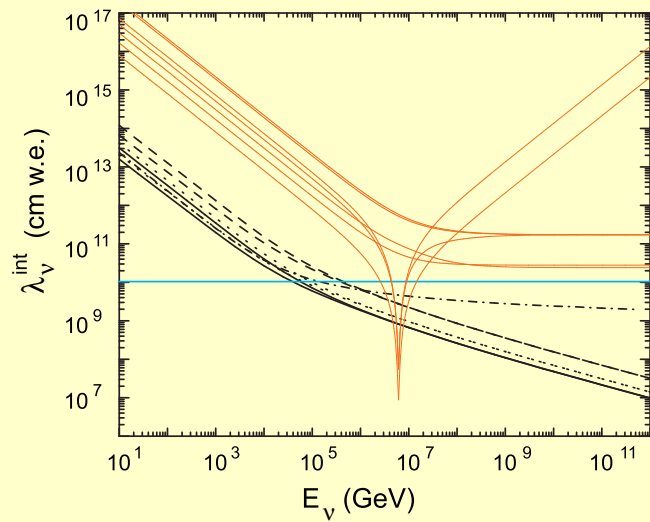
The ingredients are listed on p. 96. These formulas do not take into account the inverse-square law violation corrections, for which we unfortunately do not have enough time to discuss.^a

^aSee VN & D. S. Shkirmanov, Eur. Phys. J. C **73** (2013) 2627; Universe **7** (2021) 246 and refs. therein.

Таблица 1: Ingredients of the equations shown in p. 95, in the leading order for the off-mass-shell (short distances) and on-mass-shell (long distances) regimes. Here $L = |\mathbf{y} - \mathbf{x}|$, $\Delta m_{ij}^2 = m_i^2 - m_j^2$, $Q^4 = (\mathcal{R}^{00}\mathcal{R}^{\mu\nu} - \mathcal{R}^{0\mu}\mathcal{R}^{0\nu}) l_\mu l_\nu$, $Y^\mu = \tilde{\mathcal{R}}_s^{\mu\nu} q_{s\nu} - \tilde{\mathcal{R}}_d^{\mu\nu} q_{d\nu}$, $\tilde{\mathcal{R}}_{s,d}$ are the so-called inverse overlap tensors of in and out WPs in the source and detector vertices, $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_s + \tilde{\mathcal{R}}_d$, \mathcal{R} is the tensor inverse to $\tilde{\mathcal{R}}$ (that is $\mathcal{R}^{\mu\lambda}\tilde{\mathcal{R}}_{\lambda\nu} = \delta_\nu^\mu$), and $\Sigma = \det(\mathcal{R})^{1/8}$ is the scale of the energy-momentum dispersion of the effective neutrino WP. Last column shows the order of magnitude (OoM) of the quantity. Evidently, $E_\nu \simeq q_0 \simeq |\mathbf{q}|$ in the UR approximation.

Quantity	Off-shell regime	On-shell regime	OoM
φ_{ij}	$\frac{\Delta m_{ij}^2 L}{2 \mathbf{q} }$	$\frac{\Delta m_{ij}^2 L}{2E_\nu}$	$\frac{ \Delta m_{ij}^2 L}{E_\nu}$
\mathcal{A}_{ij}^2	$\left(\frac{\Delta m_{ij}^2 L}{2 \mathbf{q} ^2}\right)^2 \frac{Q^4}{2\mathcal{R}^{\mu\nu} l_\mu l_\nu}$	$\left(\frac{\Delta m_{ij}^2 L}{2E_\nu^2}\right)^2 \frac{1}{2\tilde{\mathcal{R}}^{\mu\nu} l_\mu l_\nu}$	$\left(\frac{\Delta m_{ij}^2}{E_\nu^2} \Sigma L\right)^2$
\mathcal{B}_{ij}	$\frac{\Delta m_{ij}^2}{4 \mathbf{q} } \sqrt{\frac{\tilde{\mathcal{R}}^{\mu\nu} l_\mu l_\nu}{2}} \frac{\mathcal{R}^{0\mu} l_\mu}{\mathcal{R}^{\mu\nu} l_\mu l_\nu}$	$\frac{\Delta m_{ij}^2}{4E_\nu} \sqrt{\frac{\tilde{\mathcal{R}}^{\mu\nu} l_\mu l_\nu}{2}} \frac{Y_k l_k}{Y^\mu l_\mu}$	$\frac{ \Delta m_{ij}^2 }{\Sigma E_\nu}$
\mathcal{C}_{ij}^2	$\left(\frac{\Delta m_{ij}^2}{2 \mathbf{q} }\right)^2 \frac{1}{8\mathcal{R}^{\mu\nu} l_\mu l_\nu}$	0	$\left(\frac{\Delta m_{ij}^2}{\Sigma E_\nu}\right)^2$
Θ_{ij}	$\frac{m_i^2 + m_j^2}{4 \mathbf{q} } \left[\tilde{\mathcal{R}}_s^{0\mu} (q - q_s)_\mu + \tilde{\mathcal{R}}_d^{0\mu} (q + q_d)_\mu \right]$	$\frac{m_i^2 + m_j^2}{4q_0} \left[\tilde{\mathcal{R}}_s^{\mu k} l^k (q_0 l - q_s)_\mu + \tilde{\mathcal{R}}_d^{\mu k} l^k (q_0 l + q_d)_\mu \right]$	$\frac{m_i^2 + m_j^2}{\Sigma E_\nu}$

Neutrino oscillations in matter



6 Neutrino refraction.

It has been noted by Wolfenstein^a that neutrino oscillations in a medium are affected by interactions even if the thickness of the medium is negligible in comparison with the neutrino mean free path.

Let us forget for the moment about the inelastic collisions and consider the simplest case of a ultrarelativistic neutrino which moves in an external (effective) potential W formed by the matter background. If the neutrino momentum in vacuum was \mathbf{p} then its energy was $\simeq p = |\mathbf{p}|$. When the neutrino enters into the medium, its energy becomes $E = p + W$. Let us now introduce the index of refraction $n = p/E$ which is a positive value in the absence of inelastic collisions. Therefore

$$W = (1 - n)E \simeq (1 - n)p. \quad (18)$$

In the last step, we took into account that neutrino interaction with matter is very weak, $|W| \ll E$, and thus $E \simeq p$ is a good approximation.

The natural generalization of Eq. (13) for the time evolution of neutrino flavor states in matter then follows from this simple consideration and the quantum-mechanical correspondence principle.

^aL. Wolfenstein, Phys. Rev. D **17** (1978) 2369.

This is the famous **Wolfenstein equation**:

$$i \frac{d}{dt} |\nu(t)\rangle_f = [\mathbf{VH}_0\mathbf{V}^\dagger + \mathbf{W}(t)] |\nu(t)\rangle_f, \quad (19)$$

where

$$\mathbf{W}(t) = \text{diag} (1 - n_{\nu_e}, 1 - n_{\nu_\mu}, 1 - n_{\nu_\tau}, \dots) p \quad (20)$$

is the interaction Hamiltonian.

It will be useful for the following to introduce the *time-evolution operator* for the flavor states defined by

$$|\nu(t)\rangle_f = \mathbf{S}(t) |\nu(0)\rangle_f.$$

Taking into account that $|\nu(t)\rangle_f$ must satisfy Eq. (19) for any initial condition

$|\nu(t=0)\rangle_f = |\nu(0)\rangle_f$, the Wolfenstein equation can be immediately rewritten in terms of the evolution operator:

$$i\dot{\mathbf{S}}(t) = [\mathbf{VH}_0\mathbf{V}^\dagger + \mathbf{W}(t)] \mathbf{S}(t), \quad \mathbf{S}(0) = \mathbf{1}. \quad (21)$$

This equation (or its equivalent (19)) cannot be solved analytically in the general case of a medium with a varying (along the neutrino pass) density. But for a medium with a slowly (adiabatically) varying density distribution the approximate solution can be obtained by a diagonalization of the effective Hamiltonian. Below we will consider this method for a rather general 2-flavor case but now let us illustrate (without derivation) the simplest situation with a matter of constant density.

6.1 Matter of constant density.

In the 2-flavor case, the transition probability is given by the formula very similar to that for vacuum:

$$P_{\alpha\alpha'}(L) = \frac{1}{2} \sin^2 2\theta_m \left[1 - \cos \left(\frac{2\pi L}{L_m} \right) \right],$$

$$L_m = L_v \left[1 - 2\kappa (L_v/L_0) \cos 2\theta + (L_v/L_0)^2 \right]^{-1/2}.$$

The L_m is called the **oscillation length in matter** and is defined through the following quantities:

$$L_v \equiv L_{23} = \frac{4\pi E}{\Delta m^2}, \quad L_0 = \frac{\sqrt{2}\pi A}{G_F N_A Z \rho} \approx 2R_\oplus \left(\frac{A}{2Z} \right) \left(\frac{2.5 \text{ g/cm}^3}{\rho} \right),$$

$$\kappa = \text{sign} (m_3^2 - m_2^2), \quad \Delta m^2 = |m_3^2 - m_2^2|.$$

The parameter θ_m is called the **mixing angle in matter** and is given by

$$\sin 2\theta_m = \sin 2\theta \left(\frac{L_m}{L_v} \right),$$

$$\cos 2\theta_m = \left(\cos 2\theta - \kappa \frac{L_v}{L_0} \right) \left(\frac{L_m}{L_v} \right).$$

The solution for antineutrinos is the same but with the replacement

$$\kappa \longmapsto -\kappa.$$

The closeness of the value of L_0 to the Earth's diameter is even more surprising than that for L_v . The matter effects are therefore important for atmospheric neutrinos.

7 Propagation of high-energy mixed neutrinos through matter.

“The matter doesn’t matter”

Lincoln Wolfenstein, lecture given at 28th SLAC Summer Institute on Particle Physics “Neutrinos from the Lab, the Sun, and the Cosmos”, Stanford, CA, Aug. 14-25, 2000.

When neutrinos propagate through vacuum there is a phase change $\exp(-im_i^2 t/2p_\nu)$. For two mixed flavors there is a resulting oscillation with length

$$L_{\text{vac}} = \frac{4\pi E_\nu}{\Delta m^2} \approx D_\oplus \left(\frac{E_\nu}{10 \text{ GeV}} \right) \left(\frac{0.002 \text{ eV}^2}{\Delta m^2} \right).$$

In matter there is an additional phase change due to refraction associated with forward scattering $\exp[ip_\nu(\text{Re } n - 1)t]$.

The characteristic length (for a normal medium) is

$$L_{\text{ref}} = \frac{\sqrt{2}A}{G_F N_A Z \rho} \approx D_\oplus \left(\frac{A}{2Z} \right) \left(\frac{2.5 \text{ g/cm}^2}{\rho} \right).$$

It is generally believed that the imaginary part of the index of refraction n which describes the neutrino absorption due to inelastic interactions *does not affect the oscillation probabilities* or at the least inelastic interactions can be somewhat decoupled from oscillations.

The conventional arguments are

- $\text{Re } n - 1 \propto G_F$ while $\text{Im } n \propto G_F^2$;
- Only Δn may affect the oscillations and $\Delta \text{Im } n$ is all the more negligible.

It will be shown that these arguments **do not work** for sufficiently high neutrino energies and/or for thick media \implies in general absorption cannot be decoupled from refraction and mixing.^a By using another cant phrase of Wolfenstein, one can say that

“In some circumstances the matter could matter.”

7.1 Generalized MSW equation.

Let

$f_{\nu_\alpha A}(0)$ be the amplitude for the ν_α zero-angle scattering from particle A of the matter background ($A = e, p, n, \dots$),

$\rho(t)$ be the matter density (in g/cm^3),

$Y_A(t)$ be the number of particles A per amu in the point t of the medium, and

$N_0 = 6.02214199 \times 10^{23} \text{ cm}^{-3}$ be the reference particle number density (numerically equal to Avogadro's number).

Then the index of refraction of ν_α for small $|n - 1|$ (for normal media $|n - 1| \lll 1$) is given by

$$n_\alpha(t) = 1 + \frac{2\pi N_0 \rho(t)}{p_\nu^2} \sum_A Y_A(t) f_{\nu_\alpha A}(0),$$

where p_ν is the neutrino momentum.

^a $p_\nu \text{Im } n \propto \sigma^{\text{tot}}(p_\nu)$ grows fast with energy while $p_\nu (\text{Re } n - 1)$ is a constant or decreasing function of E_ν .

Since the amplitude $f_{\nu_\alpha A}(0)$ is in general a **complex number**, the index of refraction is also **complex**. Its real part is responsible for neutrino refraction while the imaginary part – for absorption. From the optical theorem of quantum mechanics we have

$$\text{Im} [f_{\nu_\alpha A}(0)] = \frac{p_\nu}{4\pi} \sigma_{\nu_\alpha A}^{\text{tot}}(p_\nu).$$

This implies that

$$p_\nu \text{Im} [n_\alpha(t)] = \frac{1}{2} N_0 \rho(t) \sum_A Y_A(t) \sigma_{\nu_\alpha A}^{\text{tot}}(p_\nu) = \frac{1}{2\Lambda_\alpha(p_\nu, t)},$$

where

$$\Lambda_\alpha(p_\nu, t) = \frac{1}{\Sigma_\alpha^{\text{tot}}(p_\nu, t)} = \frac{\lambda_a^{\text{tot}}(p_\nu, t)}{\rho(t)}.$$

is the mean free path [in **cm**] of ν_α in the point t of the medium. Since the neutrino momentum, p_ν , is an extrinsic variable in Eq. (22), we will sometimes omit this argument to simplify formulas.

The generalized MSW equation for the time-evolution operator

$$\mathbf{S}(t) = \begin{pmatrix} S_{\alpha\alpha}(t) & S_{\alpha\beta}(t) \\ S_{\beta\alpha}(t) & S_{\beta\beta}(t) \end{pmatrix}$$

of two mixed stable neutrino flavors ν_α and ν_β propagating through an absorbing medium can be written as

$$\boxed{i \frac{d}{dt} \mathbf{S}(t) = [\mathbf{V} \mathbf{H}_0 \mathbf{V}^T + \mathbf{W}(t)] \mathbf{S}(t), \quad (\mathbf{S}(0) = \mathbf{1}).} \quad (22)$$

Here

$\mathbf{V} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ is the vacuum mixing matrix ($0 \leq \theta \leq \pi/2$),

$\mathbf{H}_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$ is the vacuum Hamiltonian for ν mass eigenstates,

$E_i = \sqrt{p_\nu^2 + m_i^2} \simeq p_\nu + m_i^2/2p_\nu$ is the energy of the ν_i eigenstate,

$\mathbf{W}(t) = -p_\nu \begin{pmatrix} n_\alpha(t) - 1 & 0 \\ 0 & n_\beta(t) - 1 \end{pmatrix}$ is the interaction Hamiltonian.

7.2 Master equation.

It is useful to transform MSW equation into the one with a traceless Hamiltonian. For this purpose we define the matrix

$$\tilde{\mathbf{S}}(t) = \exp \left\{ \frac{i}{2} \int_0^t \text{Tr} [\mathbf{H}_0 + \mathbf{W}(t')] dt' \right\} \mathbf{S}(t).$$

The master equation (ME) for this matrix then is

$$\boxed{i \frac{d}{dt} \tilde{\mathbf{S}}(t) = \mathbf{H}(t) \tilde{\mathbf{S}}(t), \quad \tilde{\mathbf{S}}(0) = \mathbf{1}.} \quad (23)$$

The effective Hamiltonian is defined by

$$\mathbf{H}(t) = \begin{pmatrix} q(t) - \Delta_c & \Delta_s \\ \Delta_s & -q(t) + \Delta_c \end{pmatrix},$$

$$\Delta_c = \Delta \cos 2\theta, \quad \Delta_s = \Delta \sin 2\theta, \quad \Delta = \frac{m_2^2 - m_1^2}{4p_\nu},$$

$$q(t) = q_R(t) + iq_I(t) = \frac{1}{2}p_\nu [n_\beta(t) - n_\alpha(t)].$$

The Hamiltonian for antineutrinos is of the same form as $\mathbf{H}(t)$ but

$$\text{Re} [f_{\bar{\nu}_\alpha A}(0)] = -\text{Re} [f_{\nu_\alpha A}(0)] \quad \text{and} \quad \text{Im} [f_{\bar{\nu}_\alpha A}(0)] \neq \text{Im} [f_{\nu_\alpha A}(0)].$$

The neutrino oscillation probabilities are

$$P [\nu_\alpha(0) \rightarrow \nu_{\alpha'}(t)] \equiv P_{\alpha\alpha'}(t) = |S_{\alpha'\alpha}(t)|^2 = A(t) \left| \tilde{S}_{\alpha'\alpha}(t) \right|^2, \quad (24)$$

where

$$A(t) = \exp \left[- \int_0^t \frac{dt'}{\Lambda(t')} \right], \quad \frac{1}{\Lambda(t)} = \frac{1}{2} \left[\frac{1}{\Lambda_\alpha(t)} + \frac{1}{\Lambda_\beta(t)} \right].$$

Owing to the complex potential q , the Hamiltonian $\mathbf{H}(t)$ is **non-Hermitian** and the new evolution operator $\tilde{\mathbf{S}}(t)$ is **nonunitary**. As a result, there are no conventional relations between $P_{\alpha\alpha'}(t)$.

Since

$$q_I(t) = \frac{1}{4} \left[\frac{1}{\Lambda_\beta(t)} - \frac{1}{\Lambda_\alpha(t)} \right],$$

the matrix $\mathbf{H}(t)$ becomes Hermitian when $\Lambda_\alpha = \Lambda_\beta$. If this is the case at any t , the ME reduces to the standard MSW equation and inelastic scattering results in the common exponential attenuation of the probabilities. From here, we shall consider the more general and more interesting case, when $\Lambda_\alpha \neq \Lambda_\beta$.

7.3 Examples.

$$\nu_\alpha - \nu_s$$

This is the extreme example. Since $\Lambda_s = \infty$, we have $\Lambda = 2\Lambda_\alpha$ and $q_I = -1/4\Lambda_\alpha$. So $q_I \neq 0$ at any energy. Even without solving the evolution equation, one can expect the penetrability of active neutrinos to be essentially modified in this case because, roughly speaking, they spend a certain part of life in the sterile state. In other words, sterile neutrinos “tow” their active companions through the medium as a tugboat. On the other hand, the active neutrinos “retard” the sterile ones, like a bulky barge retards its tugboat. As a result, the sterile neutrinos undergo some absorption.

$$\nu_{e,\mu} - \nu_\tau$$

Essentially at all energies, $\sigma_{\nu_{e,\mu}N}^{\text{CC}} > \sigma_{\nu_\tau N}^{\text{CC}}$. This is because of large value of the τ lepton mass, m_τ , which leads to several consequences:

1. high neutrino energy threshold for τ production;
2. sharp shrinkage of the phase spaces for CC $\nu_\tau N$ reactions;
3. kinematic correction factors ($\propto m_\tau^2$) to the nucleon structure functions (the corresponding structures are negligible for e production and small for μ production).

The neutral current contributions are canceled out from q_I . Thus, in the context of the master equation, ν_τ can be treated as (almost) sterile within the energy range for which $\sigma_{\nu_{e,\mu}N}^{\text{CC}} \gg \sigma_{\nu_\tau N}^{\text{CC}}$ (see Figures in pp. 109–110).

$$\bar{\nu}_e - \bar{\nu}_\alpha$$

A similar situation, while in quite a different and narrow energy range, holds in the case of mixing of $\bar{\nu}_e$ with some other flavor. This is a particular case for a normal *C asymmetric* medium, because of the *W boson resonance* formed in the neighborhood of $E_\nu^{\text{res}} = m_W^2/2m_e \approx 6.33$ PeV through the reactions

$$\bar{\nu}_e e^- \rightarrow W^- \rightarrow \text{hadrons} \quad \text{and} \quad \bar{\nu}_e e^- \rightarrow W^- \rightarrow \bar{\nu}_\ell l^- \quad (\ell = e, \mu, \tau).$$

Let's remind that $\sigma_{\bar{\nu}_e e}^{\text{tot}} \approx 250 \sigma_{\bar{\nu}_e N}^{\text{tot}}$ just at the resonance peak.

7.4 Total cross sections.

According to Albright and Jarlskog^a

$$\frac{d\sigma_{\nu,\bar{\nu}}^{\text{CC}}}{dxdy} = \frac{G_F^2 m_N E_\nu}{\pi} (A_1 F_1 + A_2 F_2 \pm A_3 F_3 + A_4 F_4 + A_5 F_5),$$

where $F_i = F_i(x, Q^2)$ are the nucleon structure functions and A_i are the kinematic factors ($i = 1, \dots, 5$). These factors were calculated by many authors^b and the most accurate formulas were given by Paschos and Yu:

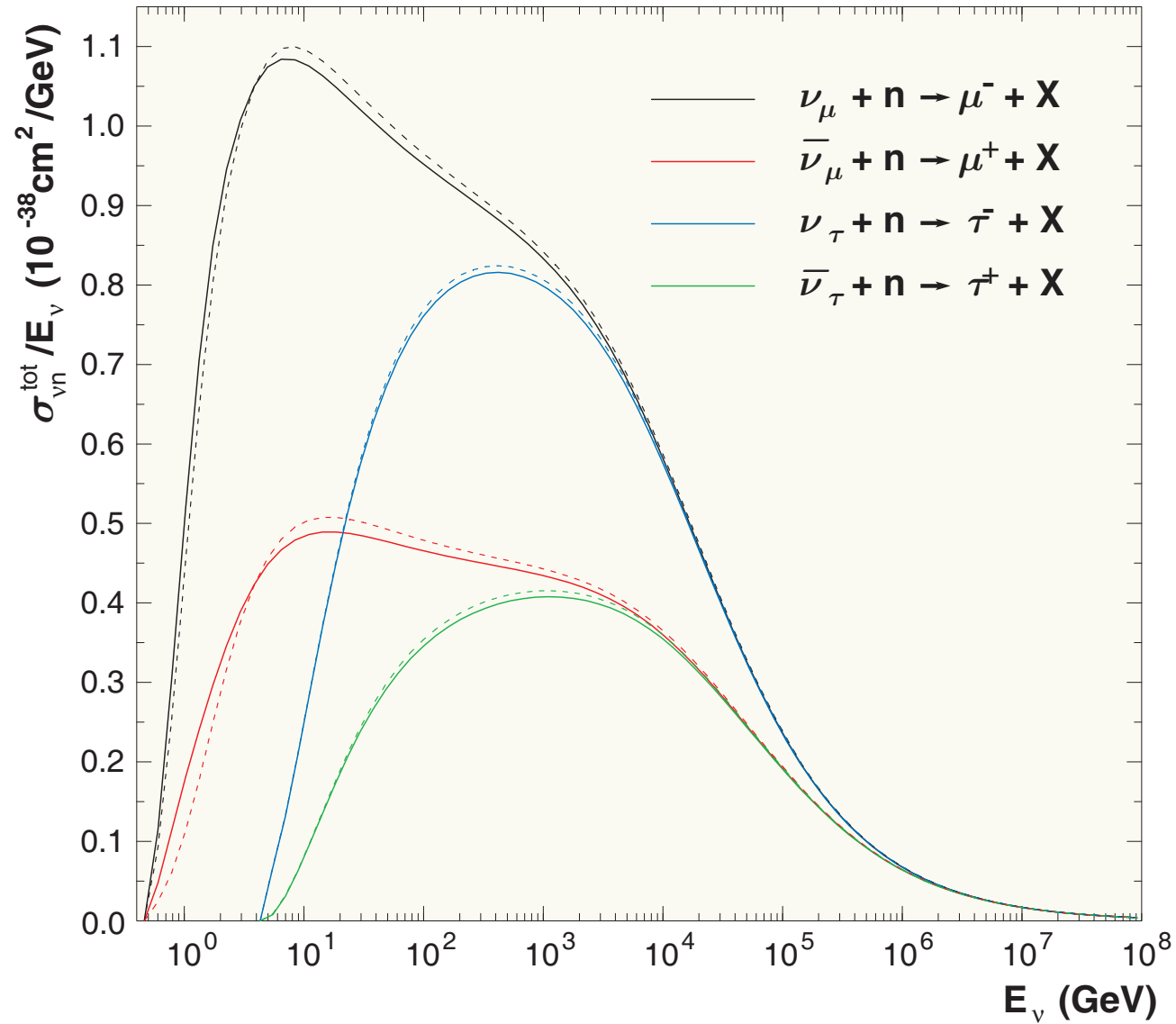
$$A_1 = xy^2 + \frac{m_l^2 y}{2m_N E_\nu}, \quad A_2 = 1 - y - \frac{m_N}{2E_\nu} xy - \frac{m_l^2}{4E_\nu^2}, \quad A_3 = xy \left(1 - \frac{y}{2}\right) - \frac{m_l^2 y}{4m_N E_\nu},$$

$$A_4 = \frac{m_l^2}{2m_N E_\nu} \left(xy + \frac{m_l^2}{2m_N E_\nu}\right), \quad A_5 = -\frac{m_l^2}{2m_N E_\nu}.$$

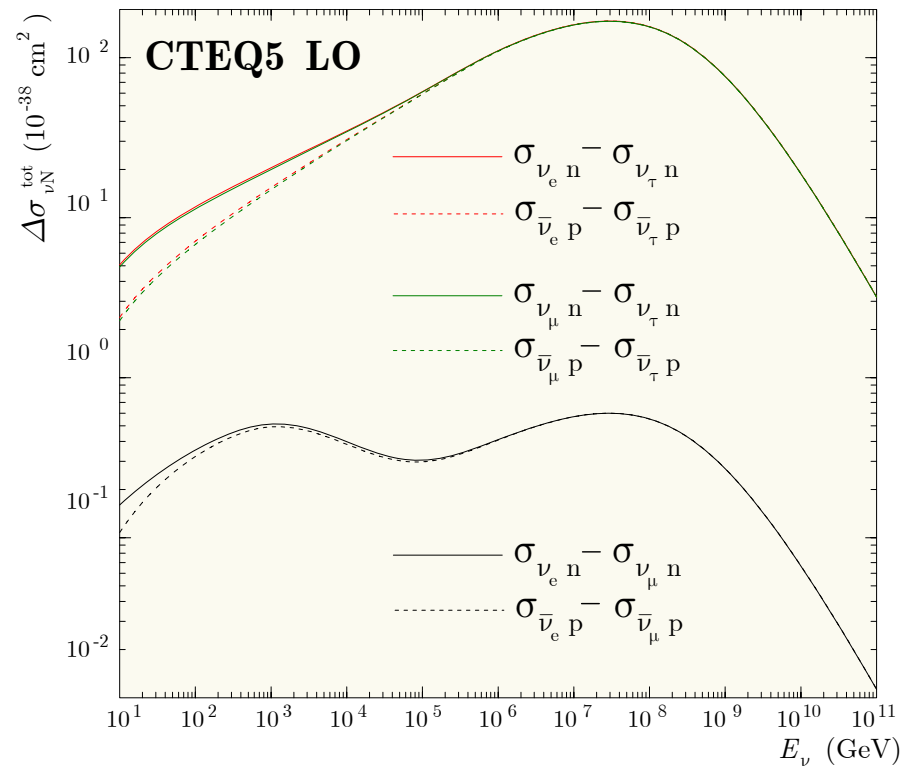
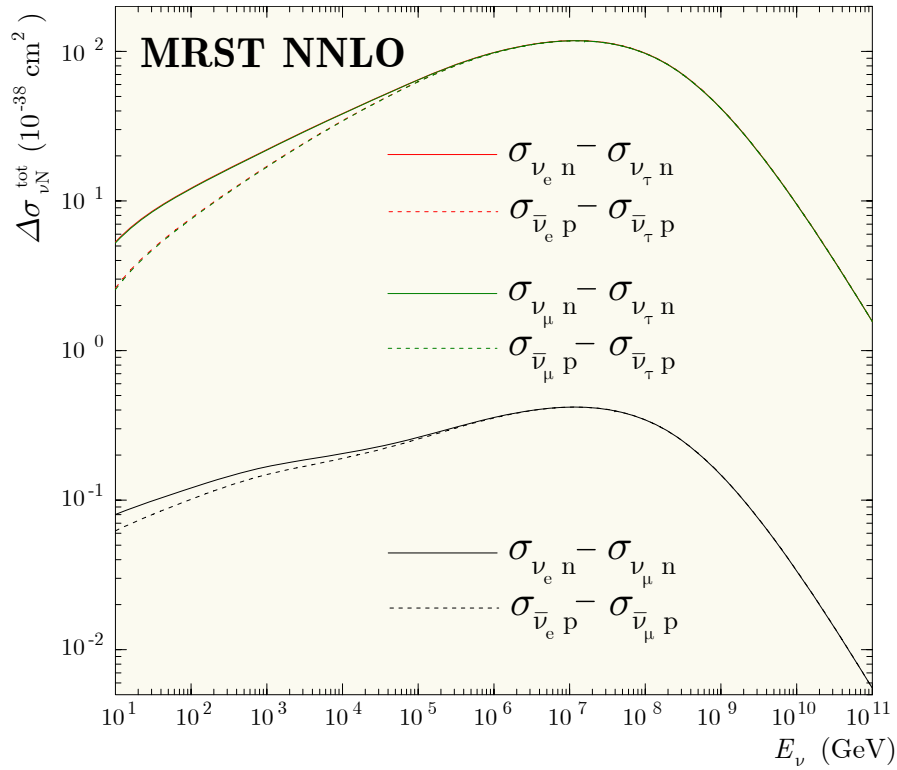
The contributions proportional to m_l^2 must vanish as $E_\nu \gg m_l$. However they remain surprisingly important even at very high energies.

^aC. H. Albright and C. Jarlskog, Nucl. Phys. B **84** (1975) 467–492; see also I. Ju, Phys. Rev. D **8** (1973) 3103–3109 and V. D. Barger *et al.*, Phys. Rev. D **16** (1977) 2141–2157.

^bSee previous footnote and also the more recent papers: S. Dutta, R. Gandhi, and B. Mukhopadhyaya, Eur. Phys. J. C **18** (2000) 405–416, hep-ph/9905475; N. I. Starkov, J. Phys. G **27** (2001) L81–L85; E. A. Paschos and J. Y. Yu, Phys. Rev. D **65** (2002) 033002, hep-ph/0107261.



Total inelastic νn cross sections evaluated with the MRST 2002 NNLO PDF model modified according to Bodek–Yang prescription (solid lines) and unmodified (dashed lines).



Differences between the total neutrino cross sections for proton and neutron targets evaluated with the MRST 2002 NNLO (*left panel*) and CTEQ 5-DIS LO (*right panel*) PDF models.

7.5 Indices of refraction.

For $E_\nu \ll \min(m_{W,Z}^2/2m_A)$ and for an electroneutral nonpolarized cold medium, the q_R is energy independent. In the leading orders of the standard electroweak theory it is

$$q_R = \begin{cases} \frac{1}{2}V_0 Y_p \rho & \text{for } \alpha = e \text{ and } \beta = \mu \text{ or } \tau, \\ -\frac{1}{2}a_\tau V_0 (Y_p + b_\tau Y_n) \rho & \text{for } \alpha = \mu \text{ and } \beta = \tau, \\ \frac{1}{2}V_0 (Y_p - \frac{1}{2}Y_n) \rho & \text{for } \alpha = e \text{ and } \beta = s, \\ \frac{1}{4}V_0 Y_n \rho & \text{for } \alpha = \mu \text{ or } \tau \text{ and } \beta = s, \end{cases}$$

where

$$\begin{aligned} V_0 &= \sqrt{2}G_F N_0 \simeq 7.63 \times 10^{-14} \text{ eV} \\ \left(L_0 = \frac{2\pi}{V_0} \simeq 1.62 \times 10^4 \text{ km} \sim D_\oplus \right), \\ a_\tau &= \frac{3\alpha r_\tau [\ln(1/r_\tau) - 1]}{4\pi \sin^2 \theta_W} \simeq 2.44 \times 10^{-5}, \\ b_\tau &= \frac{\ln(1/r_\tau) - 2/3}{\ln(1/r_\tau) - 1} \simeq 1.05, \end{aligned}$$

α is the fine-structure constant, θ_W is the weak-mixing angle and $r_\tau = (m_\tau/m_W)^2$.

Notes:

- For an isoscalar medium the $|q_R|$ is of the same order of magnitude for any pair of flavors but $\nu_\mu - \nu_\tau$.
- For an isoscalar medium $q_R^{(\nu_\mu - \nu_\tau)} / q_R^{(\nu_e - \nu_\mu)} \approx -5 \times 10^{-5}$.
- For certain regions of a neutron-rich medium the value of $q_R^{(\nu_e - \nu_s)}$ may become vanishingly small. In this case, the one-loop radiative corrections must be taken into account.
- For very high energies the q_R have to be corrected for the gauge boson propagators and strong-interaction effects.

One can expect $|q_R|$ to be either an energy-independent or decreasing function for any pair of mixed neutrino flavors. On the other hand, there are several cases of much current interest when $|q_I|$ either increases with energy without bound (mixing between active and sterile neutrino states) or has a broad or sharp maximum (as for $\nu_\mu - \nu_\tau$ or $\bar{\nu}_e - \bar{\nu}_\mu$ mixings, respectively).

Numerical estimations suggest that for every of these cases there is an energy range in which q_R and q_I are **comparable in magnitude**. Since $q_R \propto \rho$ and $q_I \propto$ and are dependent upon the composition of the medium (Y_A) there may exist some more specific situations, when

$$|q_R| \sim |q_I| \sim |\Delta|$$

or even

$$|q_R| \sim |\Delta_c| \quad \text{and} \quad |q_I| \sim |\Delta_s|.$$

If this is the case, the refraction, absorption and mixing become interestingly superimposed.

7.6 Eigenproblem and mixing matrix in matter.

7.6.1 Eigenvalues.

The matrix $\mathbf{H}(t)$ has two complex instantaneous eigenvalues, $\varepsilon(t)$ and $-\varepsilon(t)$, with $\varepsilon = \varepsilon_R + i\varepsilon_I$ satisfying the characteristic equation

$$\varepsilon^2 = (q - q_+)(q - q_-),$$

where

$$q_{\pm} = \Delta_c \pm i\Delta_s = \Delta e^{\pm 2i\theta}.$$

The solution is

$$\begin{aligned}\varepsilon_R^2 &= \frac{1}{2} (\varepsilon_0^2 - q_I^2) + \frac{1}{2} \sqrt{(\varepsilon_0^2 - q_I^2)^2 + 4q_I^2 (\varepsilon_0^2 - \Delta_s^2)}, \\ \varepsilon_I &= \frac{q_I (q_R - \Delta_c)}{\varepsilon_R} \quad (\text{provided } q_R \neq \Delta_c),\end{aligned}$$

with

$$\varepsilon_0 = \sqrt{\Delta^2 - 2\Delta_c q_R + q_R^2} \geq |\Delta_s|, \quad \text{sign}(\varepsilon_R) \stackrel{\text{def}}{=} \text{sign}(\Delta) \equiv \zeta.$$

(At that choice $\varepsilon = \Delta$ for vacuum and $\varepsilon = \zeta\varepsilon_0$ if $q_I = 0$.)

In the vicinity of the MSW resonance, $q_R = q_R(t_*) = \Delta_c$

$$\lim_{q_R \rightarrow \Delta_c \pm 0} \varepsilon_R = \Delta_s \sqrt{\max(1 - \Delta_I^2/\Delta_s^2, 0)},$$

$$\lim_{q_R \rightarrow \Delta_c \pm 0} \varepsilon_I = \pm \zeta \Delta_I \sqrt{\max(1 - \Delta_s^2/\Delta_I^2, 0)},$$

where $\Delta_I = q_I(t_*)$. Therefore the resonance value of $|\varepsilon_R|$ (which is inversely proportional to the neutrino oscillation length in matter) is always **smaller** than the conventional MSW value $|\Delta_s|$ and *vanishes* if $\Delta_I^2 < \Delta_s^2$ (ε_I remains finite in this case). In neutrino transition through the region of resonance density $\rho = \rho(t_*)$, ε_I undergoes discontinuous jump while ε_R remains continuous. The corresponding cuts in the q plane are placed outside the circle $|q| \leq |\Delta|$. If $\Delta_I^2 > \Delta_s^2$, the imaginary part of ε vanishes while the real part remains finite.

A distinctive feature of the characteristic equation is the existence of two mutually conjugate “super-resonance” points q_{\pm} in which ε vanishes giving rise to the **total degeneracy** of the levels of the system (impossible in the “standard MSW” solution). Certainly, the behavior of the system in the vicinity of these points must be dramatically different from the conventional pattern.

The “super-resonance” conditions are physically realizable for various meaningful mixing scenarios.

Some useful relations:

$$\varepsilon_R^2 = \frac{2q_I^2 (\varepsilon_0^2 - \Delta_s^2)}{\sqrt{(\varepsilon_0^2 - q_I^2)^2 + 4q_I^2 (\varepsilon_0^2 - \Delta_s^2)} - \varepsilon_0^2 + q_I^2},$$

$$\varepsilon_I = \frac{\sqrt{(\varepsilon_0^2 - q_I^2)^2 + 4q_I^2 (\varepsilon_0^2 - \Delta_s^2)} - \varepsilon_0^2 + q_I^2}{2q_I (q_R - \Delta_c)},$$

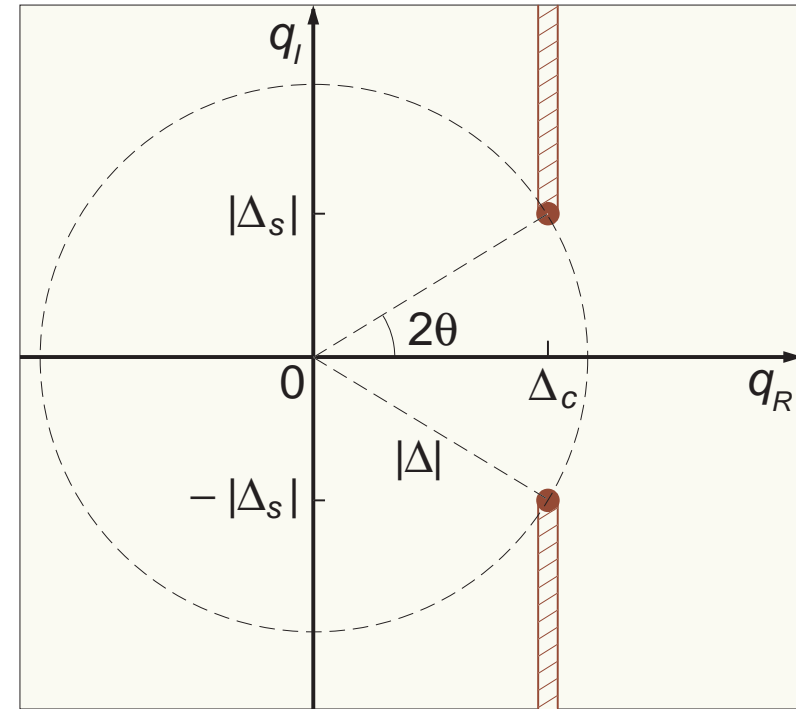
$$\frac{\partial \varepsilon_R}{\partial q_R} = \frac{\partial \varepsilon_I}{\partial q_I} = \frac{q_I \varepsilon_I + (q_R - \Delta_c) \varepsilon_R}{\varepsilon_R^2 + \varepsilon_I^2},$$

$$\frac{\partial \varepsilon_I}{\partial q_R} = -\frac{\partial \varepsilon_R}{\partial q_I} = \frac{q_I \varepsilon_R - (q_R - \Delta_c) \varepsilon_I}{\varepsilon_R^2 + \varepsilon_I^2},$$

$$\operatorname{Re} \left[\frac{q(t) - \Delta_c}{\varepsilon} \right] = \left(\frac{q_R - \Delta_c}{\varepsilon_R} \right) \left(\frac{\varepsilon_R^2 + q_I^2}{\varepsilon_R^2 + \varepsilon_I^2} \right),$$

$$\operatorname{Im} \left[\frac{q(t) - \Delta_c}{\varepsilon} \right] = \left(\frac{q_I}{\varepsilon_R} \right) \left(\frac{\varepsilon_R^2 - \varepsilon_0^2 + \Delta_s^2}{\varepsilon_R^2 + \varepsilon_I^2} \right),$$

$$(q_R - \Delta_c)^2 = \varepsilon_0^2 - \Delta_s^2.$$



Zeros and cuts of ε in the q plane for $\Delta_c > 0$. The cuts are placed outside the circle $|q| \leq |\Delta|$ parallel to axis $q_R = 0$. The MSW resonance point is $(\Delta_c, 0)$ and the two “super-resonance” points are $(\Delta_c, \pm\Delta_s)$.

7.6.2 Eigenstates.

In order to simplify the solution to the eigenstate problem we'll assume that the phase trajectory $q = q(t)$ does not cross the points q_{\pm} at any t . In non-Hermitian quantum dynamics one has to consider the two pairs of instantaneous eigenvectors $|\Psi_{\pm}\rangle$ and $|\bar{\Psi}_{\pm}\rangle$ which obey the relations

$$\mathbf{H}|\Psi_{\pm}\rangle = \pm \varepsilon |\Psi_{\pm}\rangle \quad \text{and} \quad \mathbf{H}^{\dagger}|\bar{\Psi}_{\pm}\rangle = \pm \varepsilon^* |\bar{\Psi}_{\pm}\rangle. \quad (25)$$

and (for $q \neq q_{\pm}$) form a complete biorthogonal and biorthonormal set,

$$\langle \bar{\Psi}_{\pm} | \Psi_{\pm} \rangle = 1, \quad \langle \bar{\Psi}_{\pm} | \Psi_{\mp} \rangle = 0, \quad |\Psi_{+}\rangle \langle \bar{\Psi}_{+}| + |\Psi_{-}\rangle \langle \bar{\Psi}_{-}| = \mathbf{1}.$$

Therefore, the eigenvectors are defined up to a gauge transformation

$$|\Psi_{\pm}\rangle \mapsto e^{if_{\pm}} |\Psi_{\pm}\rangle, \quad |\bar{\Psi}_{\pm}\rangle \mapsto e^{-if_{\pm}^*} |\bar{\Psi}_{\pm}\rangle,$$

with arbitrary complex functions $f_{\pm}(t)$ such that $\text{Im}(f_{\pm})$ vanish as $q = 0$.^a Thus it is sufficient to find any particular solution of Eqs. (25). Taking into account that $\mathbf{H}^{\dagger} = \mathbf{H}^*$, we may set $|\bar{\Psi}_{\pm}\rangle = |\Psi_{\pm}^*\rangle$ and hence the eigenvectors can be found from the identity

$$\mathbf{H} = \varepsilon |\Psi_{+}\rangle \langle \Psi_{+}^*| - \varepsilon |\Psi_{-}\rangle \langle \Psi_{-}^*|.$$

^aFor our aims, the class of the gauge functions may be restricted without loss of generality by the condition $f_{\pm}|_{q=0} = 0$.

Setting $|\Psi_{\pm}\rangle = (v_{\pm}, \pm v_{\mp})^T$ we arrive at the equations

$$v_{\pm}^2 = \frac{\varepsilon \pm (q - \Delta_c)}{2\varepsilon}, \quad v_+ v_- = \frac{\Delta_s}{2\varepsilon},$$

a particular solution of which can be written as

$$v_+ = \sqrt{\left| \frac{\varepsilon + q - \Delta_c}{2\varepsilon} \right|} e^{i(\varphi - \psi)/2},$$

$$v_- = \zeta \sqrt{\left| \frac{\varepsilon - q + \Delta_c}{2\varepsilon} \right|} e^{i(-\varphi - \psi)/2}.$$

where

$$\varphi = \arg(\varepsilon + q - \Delta_c) = -\arg(\varepsilon - q + \Delta_c) = \arctan\left(\frac{q_I}{\varepsilon_R}\right),$$

$$\psi = \arg(\varepsilon) = \arctan\left(\frac{\varepsilon_I}{\varepsilon_R}\right).$$

We have fixed the remaining gauge ambiguity by a comparison with the vacuum case.

7.6.3 Mixing angle in matter.

It may be sometimes useful to define the complex mixing angle in matter $\Theta = \Theta_R + i\Theta_I$ by the relations

$$\sin \Theta = v_+ \quad \text{and} \quad \cos \Theta = v_-$$

or, equivalently,

$$\sin 2\Theta = \frac{\Delta_s}{\varepsilon}, \quad \cos 2\Theta = \frac{\Delta_c - q}{\varepsilon},$$

The real and imaginary parts of Θ are found to be

$$\text{Re}(\Theta) \equiv \Theta_R = \frac{1}{2} \arctan \left[\frac{(q_I - \Delta_s) \varepsilon_R - (q_R - \Delta_c) \varepsilon_I}{(q_R - \Delta_c) \varepsilon_R + (q_I - \Delta_s) \varepsilon_I} \right],$$

$$\text{Im}(\Theta) \equiv \Theta_I = \frac{1}{4} \ln \left[\frac{\varepsilon_R^2 + \varepsilon_I^2}{(q_R - \Delta_c)^2 + (q_I - \Delta_s)^2} \right].$$

$$\cos \Theta = \cos \Theta_R \cosh \Theta_I - i \sin \Theta_R \sinh \Theta_I,$$

$$\sin \Theta = \sin \Theta_R \cosh \Theta_I + i \cos \Theta_R \sinh \Theta_I.$$

Having regard to the prescription for the sign of ε_R , one can verify that $\Theta = \theta$ if $q = 0$ (vacuum case) and $\Theta = 0$ if $\Delta_s = 0$ (no mixing or $m_1^2 = m_2^2$). It is also clear that Θ becomes the standard MSW mixing angle with $\text{Im}(\Theta) = 0$ when $q_I = 0$ ($\Lambda_\alpha = \Lambda_\beta$).

7.6.4 Mixing matrix in matter.

In order to build up the solution to ME for the nondegenerated case one has to diagonalize the Hamiltonian. Generally a non-Hermitian matrix cannot be diagonalized by a single unitary transformation. But in our simple case this can be done by a complex orthogonal matrix (extended mixing matrix in matter)

$$\mathbf{U}_f = \mathbf{U} \exp(i\mathbf{f}),$$

where $\mathbf{f} = \text{diag}(f_-, f_+)$ and

$$\begin{aligned} \mathbf{U} = (|\Psi_-\rangle, |\Psi_+\rangle) &= \begin{pmatrix} v_- & v_+ \\ -v_+ & v_- \end{pmatrix} \\ &= \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix}. \end{aligned}$$

Properties of \mathbf{U} :

$$\begin{aligned} \mathbf{U}^T \mathbf{H} \mathbf{U} &= \text{diag}(-\varepsilon, \varepsilon), \\ \mathbf{U}^T \mathbf{U} &= \mathbf{1}, \quad \mathbf{U}|_{q=0} = \mathbf{V}. \end{aligned}$$

From CE it follows that

$$\frac{\partial \varepsilon}{\partial q} = \frac{(q - \Delta_c)}{\varepsilon}$$

and thus

$$\frac{\partial v_{\pm}}{\partial q} = \pm \frac{\Delta_s^2 v_{\mp}}{2\varepsilon^2}.$$

We therefore have

$$\begin{aligned} i\mathbf{U}^T \dot{\mathbf{U}} &= -\Omega \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\Omega \boldsymbol{\sigma}_2, \\ \Omega &= \frac{\dot{q} \Delta_s}{2\varepsilon^2} = \frac{i}{4} \frac{d}{dt} \ln \left(\frac{q - q_+}{q - q_-} \right). \end{aligned}$$

Properties of \mathbf{U}_f :

$$\begin{aligned} \mathbf{U}_f^T \mathbf{H} \mathbf{U}_f &= \text{diag}(-\varepsilon, \varepsilon), \\ \mathbf{U}_f^T \mathbf{U}_f &= \mathbf{1}, \quad \mathbf{U}_f|_{q=0} = \mathbf{V}, \\ i\mathbf{U}_f^T \dot{\mathbf{U}}_f &= -\Omega e^{-i\mathbf{f}} \boldsymbol{\sigma}_2 e^{i\mathbf{f}} - \dot{\mathbf{f}}. \end{aligned}$$

7.7 Adiabatic solution.

Formal solution to ME in the most general form:

$$\tilde{\mathbf{S}}(t) = \mathbf{U}_f(t) \exp[-i\Phi(t)] \mathbf{X}_f(t) \mathbf{U}_f^T(0). \quad (26)$$

Here $\Phi(t) = \text{diag}(-\Phi(t), \Phi(t))$ and $\Phi(t) = \Phi_R(t) + i\Phi_I(t)$ is the complex dynamical phase, defined by

$$\Phi_R(t) = \int_0^t \varepsilon_R(t') dt', \quad \Phi_I(t) = \int_0^t \varepsilon_I(t') dt',$$

and $\mathbf{X}_f(t)$ must satisfy the equation

$$i\dot{\mathbf{X}}_f(t) = \left[\Omega(t)e^{-i\mathbf{f}(t)} \mathbf{F}(t)e^{i\mathbf{f}(t)} + \dot{\mathbf{f}}(t) \right] \mathbf{X}_f(t), \quad \mathbf{X}_f(0) = \mathbf{1},$$

where

$$\mathbf{F}(t) = e^{i\Phi(t)} \boldsymbol{\sigma}_2 e^{-i\Phi(t)} = \begin{pmatrix} 0 & -ie^{-2i\Phi(t)} \\ ie^{2i\Phi(t)} & 0 \end{pmatrix}.$$

It can be proved now that the right side of Eq. (26) is gauge-invariant i.e. it **does not depend** on the unphysical complex phases $f_{\pm}(t)$. This crucial fact is closely related to the absence of the **Abelian topological phases** in the system under consideration.

Finally, we can put $f_{\pm} = 0$ in Eq. (26) and the result is

$$\tilde{\mathbf{S}}(t) = \mathbf{U}(t) \exp[-i\Phi(t)] \mathbf{X}(t) \mathbf{U}^T(0), \quad (27a)$$

$$i\dot{\mathbf{X}}(t) = \Omega(t)\mathbf{F}(t)\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{1}. \quad (27b)$$

These equations, being equivalent to the ME, have nevertheless a restricted range of practical usage on account of poles and cuts as well as decaying and increasing exponents in the “Hamiltonian” $\Omega\mathbf{F}$.

7.7.1 Adiabatic theorem.

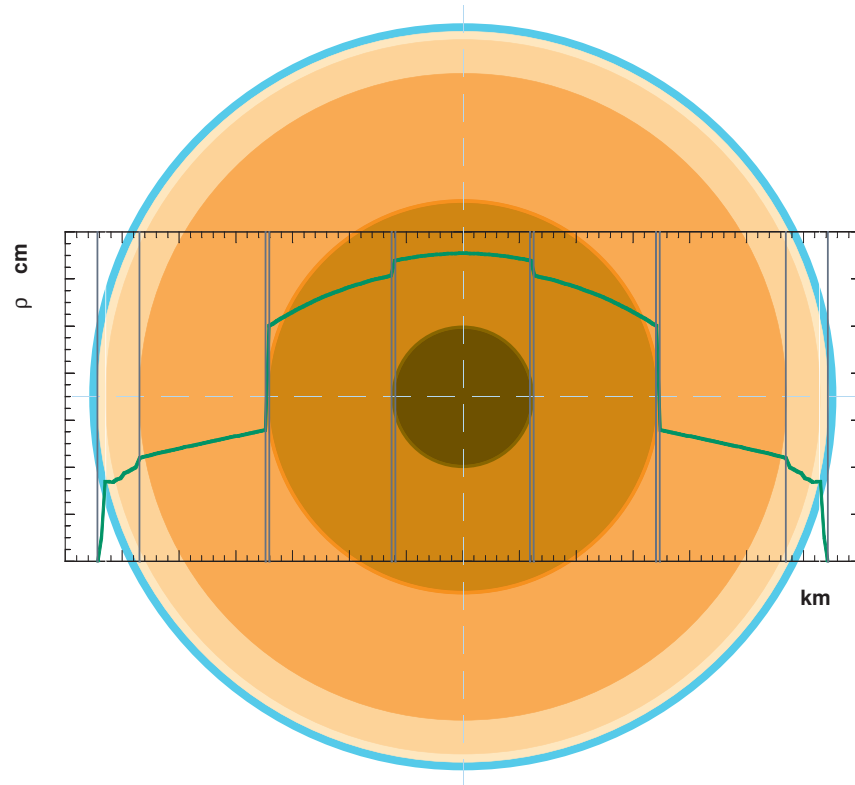
The adiabatic theorem of Hermitian quantum mechanics can almost straightforwardly be extended to ME under the requirements:

- (a) the potential q is a sufficiently smooth and slow function of t ;
- (b) the imaginary part of the dynamical phase is a bounded function i.e. $\lim_{t \rightarrow \infty} |\Phi_I(t)|$ is finite;
- (c) the phase trajectory $q = q(t)$ is placed far from the singularities for any t .

The first requirement breaks down for a condensed medium with a sharp boundary or layered structure (like the Earth). If however the requirement (a) is valid inside each layer (t_i, t_{i+1}) , the problem reduces to Eqs. (27) by applying the rule

$$\tilde{\mathbf{S}}(t) \equiv \tilde{\mathbf{S}}(t, 0) = \tilde{\mathbf{S}}(t, t_n) \dots \tilde{\mathbf{S}}(t_2, t_1) \tilde{\mathbf{S}}(t_1, 0),$$

where $\tilde{\mathbf{S}}(t_{i+1}, t_i)$ is the time-evolution operator for the i -th layer.



The requirement (b) alone is not too restrictive considering that for many astrophysical objects (like stars, galactic nuclei, jets and so on) the density ρ exponentially disappears to the periphery and, on the other hand, $\varepsilon_I \rightarrow 0$ as $\rho \rightarrow 0$. In this instance, the function $\Phi_I(t)$ must be t independent for sufficiently large t . But, in the case of a steep density profile, the requirements (a) and (b) may be inconsistent. The important case of violation of the requirement (c) is the subject of a special study which is beyond the scope of this study.

It is interesting to note in this connection that, in the Hermitian case, a general adiabatic theorem has been proved without the traditional gap condition^a.

^aJ. E. Avron and A. Elgart, Commun. Math. Phys. **203** (1999) 445–467.

7.7.2 The solution.

Presume that all necessary conditions do hold for $0 \leq t \leq T$. Then, in the adiabatic limit, we can put $\Omega = 0$ in Eq. (27b). Therefore $\mathbf{X} = \mathbf{1}$ and Eq. (27a) yields

$$\begin{aligned}\tilde{S}_{\alpha\alpha}(t) &= v_+(0)v_+(t)e^{-i\Phi(t)} + v_-(0)v_-(t)e^{i\Phi(t)}, \\ \tilde{S}_{\alpha\beta}(t) &= v_-(0)v_+(t)e^{-i\Phi(t)} - v_+(0)v_-(t)e^{i\Phi(t)}, \\ \tilde{S}_{\beta\alpha}(t) &= v_+(0)v_-(t)e^{-i\Phi(t)} - v_-(0)v_+(t)e^{i\Phi(t)}, \\ \tilde{S}_{\beta\beta}(t) &= v_-(0)v_-(t)e^{-i\Phi(t)} + v_+(0)v_+(t)e^{i\Phi(t)},\end{aligned}$$

Taking into account Eq. (24) we obtain the survival and transition probabilities:

$$\begin{aligned}P_{\alpha\alpha}(t) &= A(t) \left\{ \left[I_+^+(t)e^{\Phi_I(t)} + I_-^-(t)e^{-\Phi_I(t)} \right]^2 - I^2(t) \sin^2 [\Phi_R(t) - \varphi_+(t)] \right\}, \\ P_{\alpha\beta}(t) &= A(t) \left\{ \left[I_+^-(t)e^{\Phi_I(t)} - I_-^+(t)e^{-\Phi_I(t)} \right]^2 + I^2(t) \sin^2 [\Phi_R(t) - \varphi_-(t)] \right\}, \\ P_{\beta\alpha}(t) &= A(t) \left\{ \left[I_-^+(t)e^{\Phi_I(t)} - I_+^-(t)e^{-\Phi_I(t)} \right]^2 + I^2(t) \sin^2 [\Phi_R(t) + \varphi_-(t)] \right\}, \\ P_{\beta\beta}(t) &= A(t) \left\{ \left[I_-^-(t)e^{\Phi_I(t)} + I_+^+(t)e^{-\Phi_I(t)} \right]^2 - I^2(t) \sin^2 [\Phi_R(t) + \varphi_+(t)] \right\},\end{aligned}\tag{28}$$

where we have denoted for compactness ($\varsigma, \varsigma' = \pm$)

$$I_{\varsigma}^{\varsigma'}(t) = |v_{\varsigma}(0)v_{\varsigma'}(t)|, \quad \varphi_{\pm}(t) = \frac{\varphi(0) \pm \varphi(t)}{2}, \quad I^2(t) = 4I_+^+(t)I_-^-(t) = 4I_+^-(t)I_-^+(t) = \frac{\Delta_s^2}{|\varepsilon(0)\varepsilon(t)|}.$$

7.7.3 Limiting cases.

In the event that the conditions

$$\left| \frac{1}{\Lambda_\beta(t)} - \frac{1}{\Lambda_\alpha(t)} \right| \ll 4\varepsilon_0(t) \quad \text{and} \quad t \ll \min [\Lambda_\alpha(t), \Lambda_\beta(t)]$$

are satisfied for any $t \in [0, T]$, the formulas (28) reduce to the standard MSW adiabatic solution

$$\left. \begin{aligned} P_{\alpha\alpha}(t) &= P_{\beta\beta}(t) = \frac{1}{2} [1 + J(t)] - I_0^2(t) \sin^2 [\Phi_0(t)], \\ P_{\alpha\beta}(t) &= P_{\beta\alpha}(t) = \frac{1}{2} [1 - J(t)] + I_0^2(t) \sin^2 [\Phi_0(t)], \end{aligned} \right\} \quad (\text{MSW})$$

where

$$J(t) = \frac{\Delta^2 - \Delta_c [q_R(0) + q_R(t)] + q_R(0)q_R(t)}{\varepsilon_0(0)\varepsilon_0(t)},$$

$$I_0^2(t) = \frac{\Delta_s^2}{\varepsilon_0(0)\varepsilon_0(t)}, \quad \Phi_0(t) = \int_0^t \varepsilon_0(t') dt'.$$

Needless to say either of the above conditions or both may be violated for sufficiently high neutrino energies and/or for thick media, resulting in radical differences between the two solutions. These differences are of obvious interest to high-energy neutrino astrophysics.

It is perhaps even more instructive to examine the distinctions between the general adiabatic solution (28) and its “classical limit”

$$\left. \begin{aligned} P_{\alpha\alpha}(t) &= \exp \left[- \int_0^t \frac{dt'}{\Lambda_\alpha(t')} \right], & P_{\alpha\beta}(t) &= 0, \\ P_{\beta\beta}(t) &= \exp \left[- \int_0^t \frac{dt'}{\Lambda_\beta(t')} \right], & P_{\beta\alpha}(t) &= 0, \end{aligned} \right\} \quad (\Delta_s = 0)$$

which takes place either in the absence of mixing or for $m_1^2 = m_2^2$.

Note:

Considering that $\Omega \propto \Delta_s$, the classical limit is the exact solution to the master equation (for $\Delta_s = 0$). Therefore it can be derived directly from Eq. (23). To make certain that the adiabatic solution has correct classical limit, the following relations are useful:

$$\lim_{\Delta_s \rightarrow 0} \varepsilon(t) = \zeta \zeta_R [q(t) - \Delta_c]$$

and

$$\lim_{\Delta_s \rightarrow 0} |v_\pm(t)|^2 = \frac{1}{2} (\zeta \zeta_R \pm 1),$$

where

$$\zeta_R = \text{sign} [q_R(t) - \Delta_c].$$

7.8 Matter of constant density and composition.

In this simple case, the adiabatic approximation becomes **exact** and thus free from the above-mentioned conceptual difficulties. For definiteness sake we assume $\Lambda_\alpha < \Lambda_\beta$ (and thus $q_I < 0$) from here. The opposite case can be considered in a similar way. Let's denote

$$\frac{1}{\Lambda_\pm} = \frac{1}{2} \left(\frac{1}{\Lambda_\alpha} + \frac{1}{\Lambda_\beta} \right) \pm \frac{\xi}{2} \left(\frac{1}{\Lambda_\alpha} - \frac{1}{\Lambda_\beta} \right),$$

$$I_\pm^2 = \frac{1}{4} \left(1 + \frac{\varepsilon_0^2 + q_I^2 - \Delta_s^2}{\varepsilon_R^2 + \varepsilon_I^2} \right) \pm \frac{\xi}{2} \left(\frac{\varepsilon_R^2 + q_I^2}{\varepsilon_R^2 + \varepsilon_I^2} \right),$$

$$L = \frac{\pi}{|\varepsilon_R|} \quad \text{and} \quad \xi = \left| \frac{q_R - \Delta_c}{\varepsilon_R} \right|.$$

As is easy to see,

$$I_\pm^\pm = \begin{cases} I_\pm & \text{if } \text{sign}(q_R - \Delta_c) = +\zeta, \\ I_\mp & \text{if } \text{sign}(q_R - \Delta_c) = -\zeta, \end{cases}$$

$$I_+^- = I_-^+ = \sqrt{I_+ I_-} = \frac{I}{2} = \left| \frac{\Delta_s}{2\varepsilon} \right|$$

and $\text{sign}(\varphi) = -\zeta$.

By applying the above identities, the neutrino oscillation probabilities can be written as

$$P_{\alpha\alpha}(t) = \left(I_+ e^{-t/2\Lambda_+} + I_- e^{-t/2\Lambda_-} \right)^2 - I^2 e^{-t/\Lambda} \sin^2 \left(\frac{\pi t}{L} + |\varphi| \right),$$

$$P_{\beta\beta}(t) = \left(I_- e^{-t/2\Lambda_+} + I_+ e^{-t/2\Lambda_-} \right)^2 - I^2 e^{-t/\Lambda} \sin^2 \left(\frac{\pi t}{L} - |\varphi| \right),$$

$$P_{\alpha\beta}(t) = P_{\beta\alpha}(t) = \frac{1}{4} I^2 \left(e^{-t/2\Lambda_-} - e^{-t/2\Lambda_+} \right)^2 + I^2 e^{-t/\Lambda} \sin^2 \left(\frac{\pi t}{L} \right).$$

The difference between the survival probabilities for ν_α and ν_β is

$$P_{\alpha\alpha}(t) - P_{\beta\beta}(t) = -\zeta \operatorname{Re} \left(\frac{q - \Delta_c}{\varepsilon} \right) \left(e^{-t/2\Lambda_-} - e^{-t/2\Lambda_+} \right) + I^2 e^{-t/\Lambda} \sin \varphi \sin \left(\frac{2\pi t}{L} \right).$$

7.8.1 Case $|q| \gtrsim |\Delta_s|$.

Let's examine the case when Λ_+ and Λ_- are vastly different in magnitude. This will be true when $\Lambda_\beta \gg \Lambda_\alpha$ and the factor ξ is not too small. The second condition holds if q_R is away from the MSW resonance value Δ_c and the following dimensionless parameter

$$\varkappa = \frac{\Delta_s}{|q|} \approx 0.033 \times \sin 2\theta \left(\frac{\Delta m^2}{10^{-3} \text{ eV}^2} \right) \left(\frac{100 \text{ GeV}}{E_\nu} \right) \left(\frac{V_0}{|q|} \right)$$

is **sufficiently small**. In fact we assume $|\varkappa| \lesssim 1$ and impose no specific restriction for the ratio q_R/q_I . This spans several possibilities:

- ★ small Δm^2 ,
- ★ small mixing angle,
- ★ high energy,
- ★ high matter density.

The last two possibilities are of special interest because the inequality $|\varkappa| \lesssim 1$ may be fulfilled for a wide range of the mixing parameters Δm^2 and θ by changing E_ν and/or ρ . In other words, this condition is by no means artificial or too restrictive.

After elementary while a bit tedious calculations we obtain

$$\xi = 1 - \frac{1}{2}\varkappa^2 + \mathcal{O}(\varkappa^3), \quad I^2 = \varkappa^2 + \mathcal{O}(\varkappa^3),$$

$$I_+ = 1 + \mathcal{O}(\varkappa^2), \quad I_- = \frac{1}{4}\varkappa^2 + \mathcal{O}(\varkappa^3);$$

$$\Lambda \approx 2\Lambda_\alpha, \quad \Lambda_+ \approx \left(1 + \frac{\varkappa^2}{4}\right) \Lambda_\alpha \approx \Lambda_\alpha, \quad \Lambda_- \approx \left(\frac{4}{\varkappa^2}\right) \Lambda_\alpha \gg \Lambda_\alpha.$$

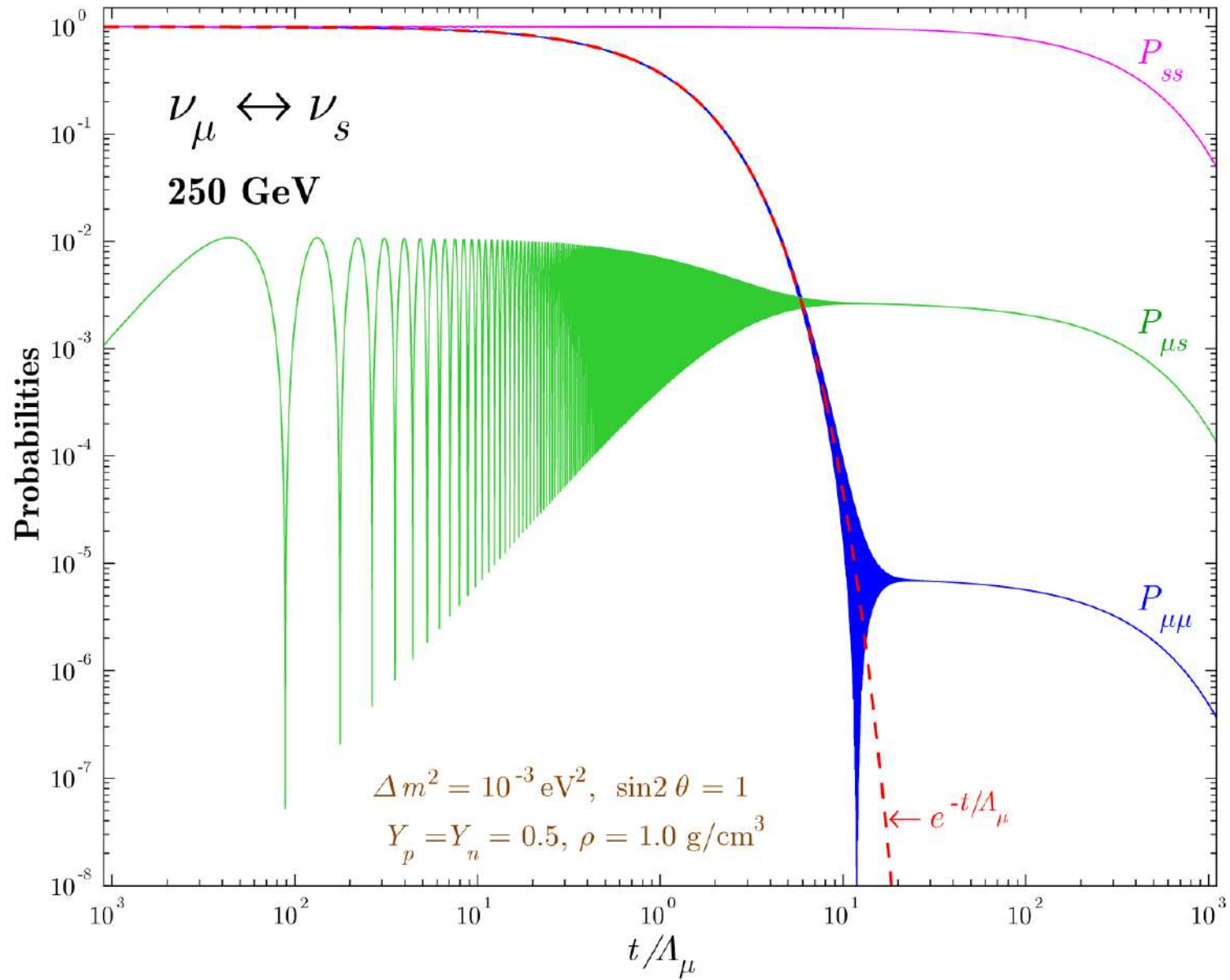
Due to the wide spread among the length/time scales Λ_\pm , Λ and L as well as among the amplitudes I_\pm and I , the regimes of neutrino oscillations are quite diverse for different ranges of variable t .

With reference to Figures in pp.130–133, one can see a regular gradation from slow (for $t \lesssim \Lambda_\mu$) to very fast (for $t \gtrsim \Lambda_\mu$) neutrino oscillations followed by the asymptotic nonoscillatory behavior:

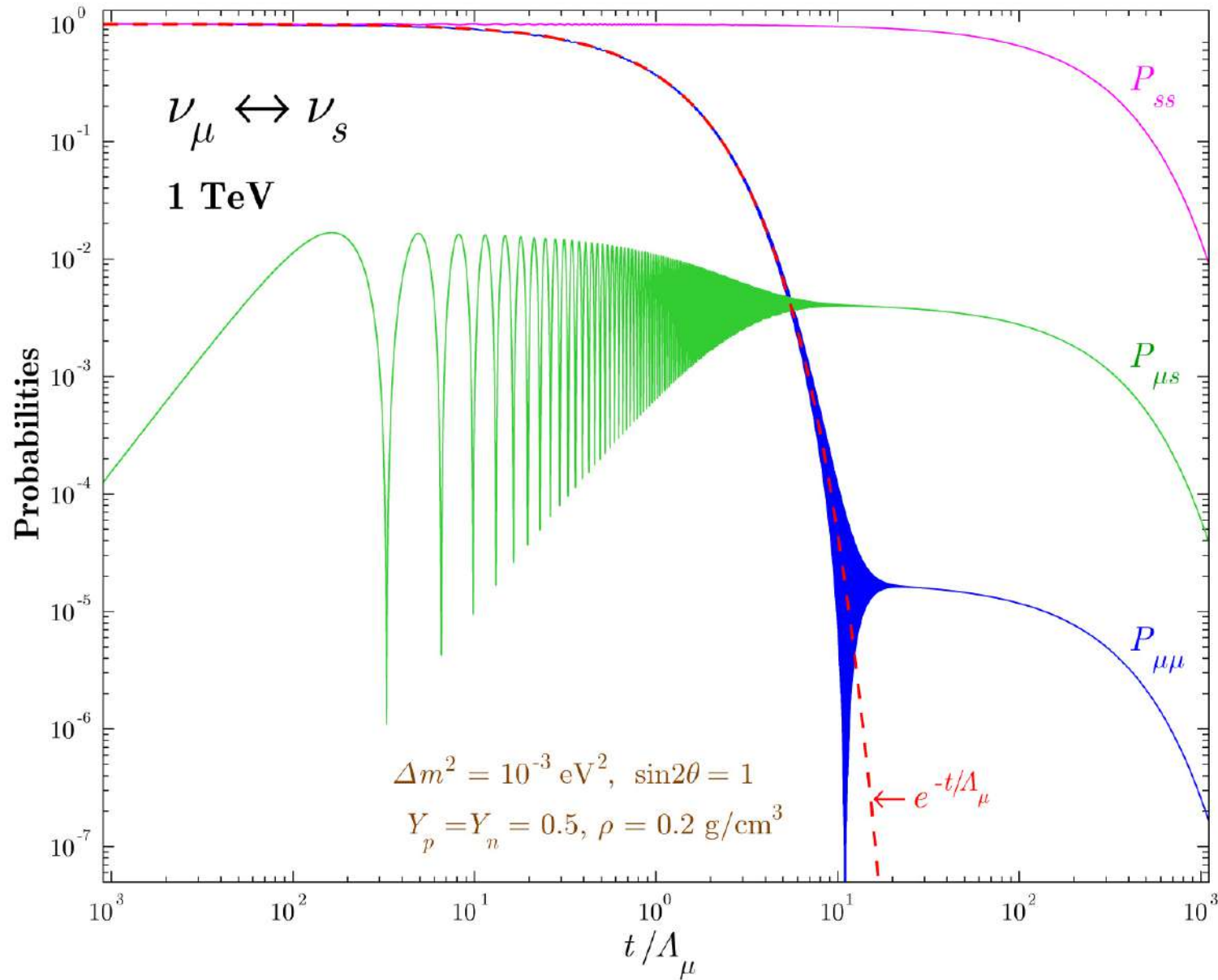
$$P_{\mu\mu}(t) \simeq \frac{\varkappa^4}{16} e^{-t/\Lambda_-},$$

$$P_{ss}(t) \simeq e^{-t/\Lambda_-},$$

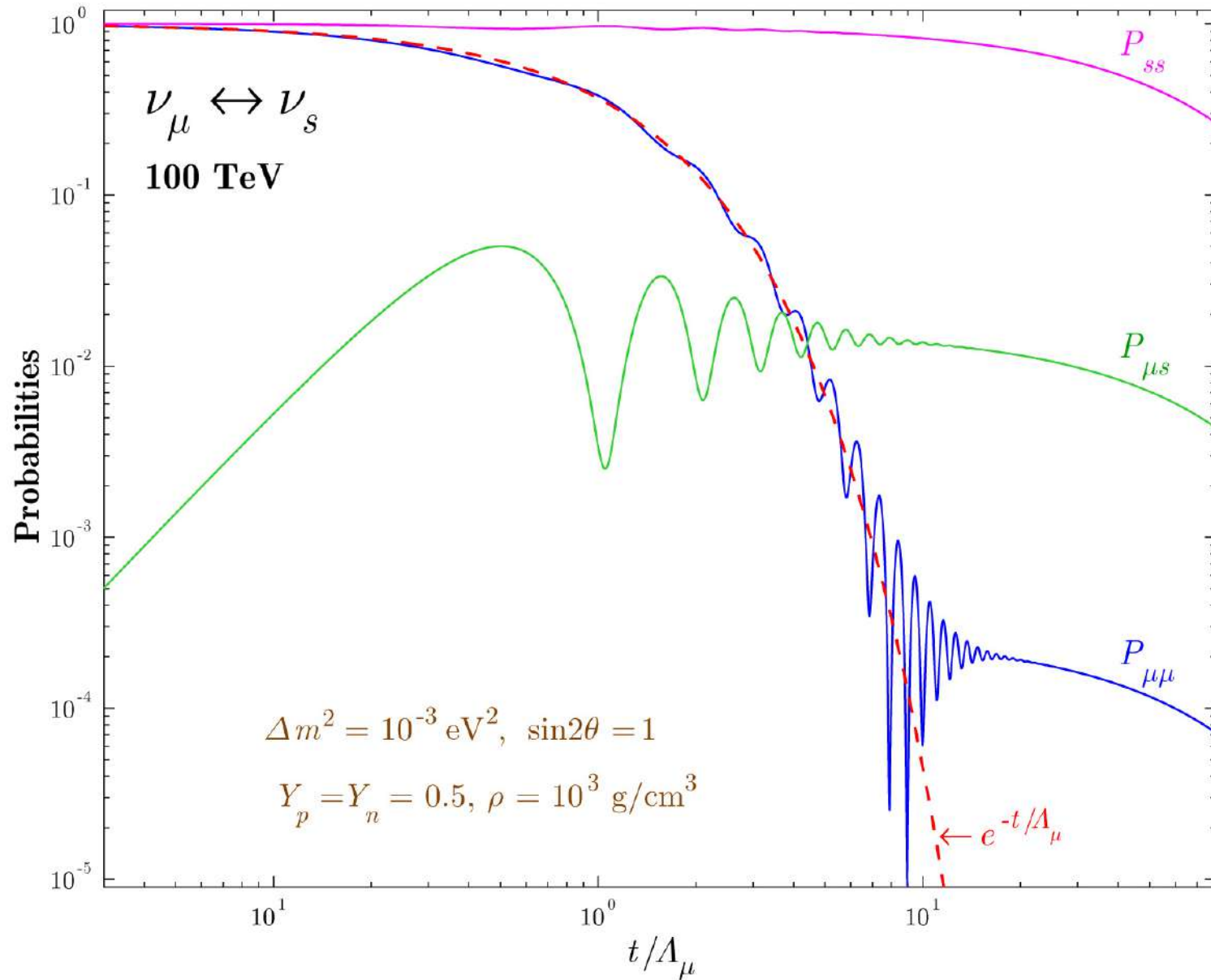
$$P_{\mu s}(t) = P_{s\mu}(t) \simeq \frac{\varkappa^2}{4} e^{-t/\Lambda_-}.$$



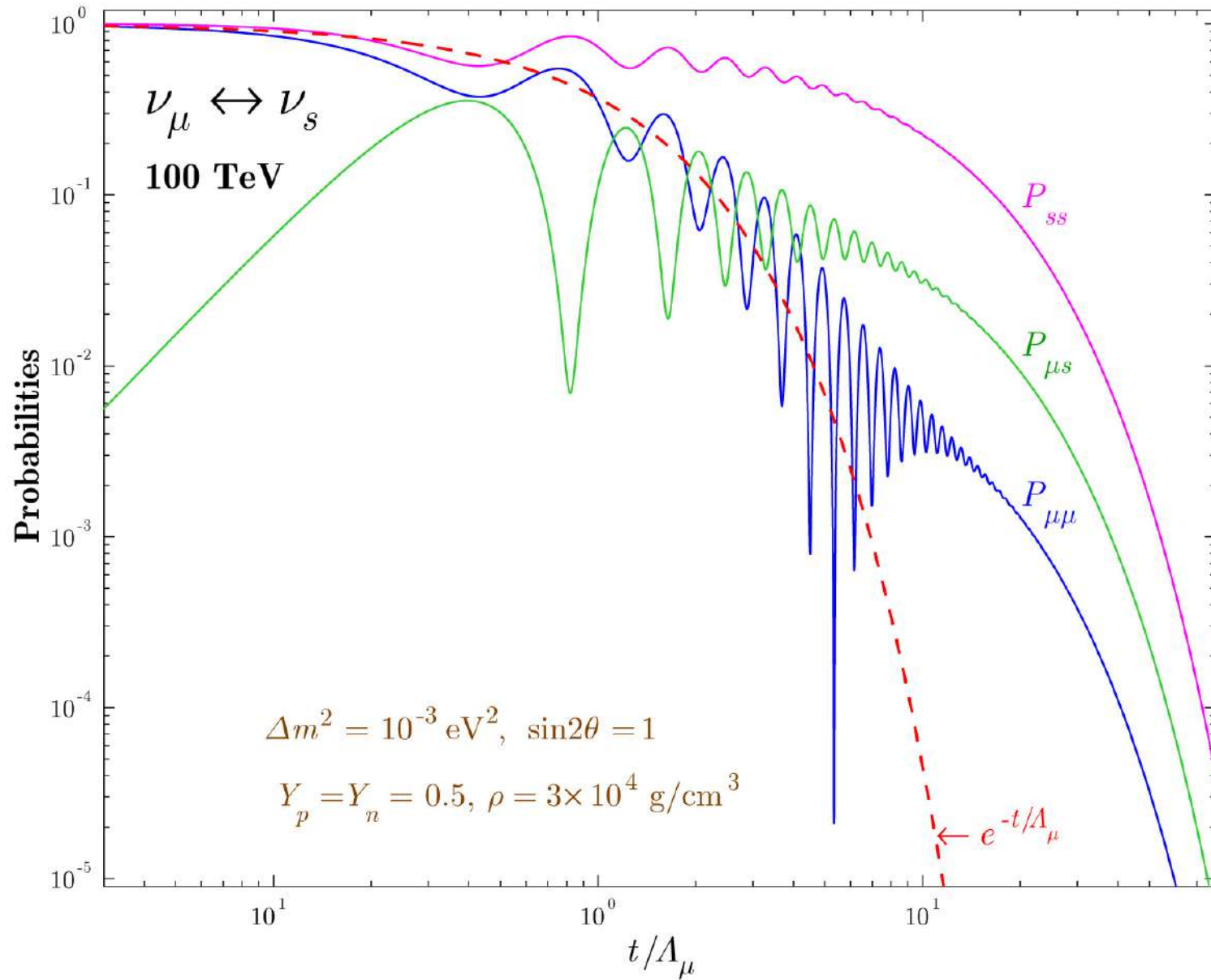
Survival and transition probabilities for $\nu_{\mu} \leftrightarrow \nu_s$ oscillations ($E_{\nu} = 250 \text{ GeV}$, $\rho = 1 \text{ g/cm}^3$).



Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 1000 \text{ GeV}$, $\rho = 0.2 \text{ g/cm}^3$).

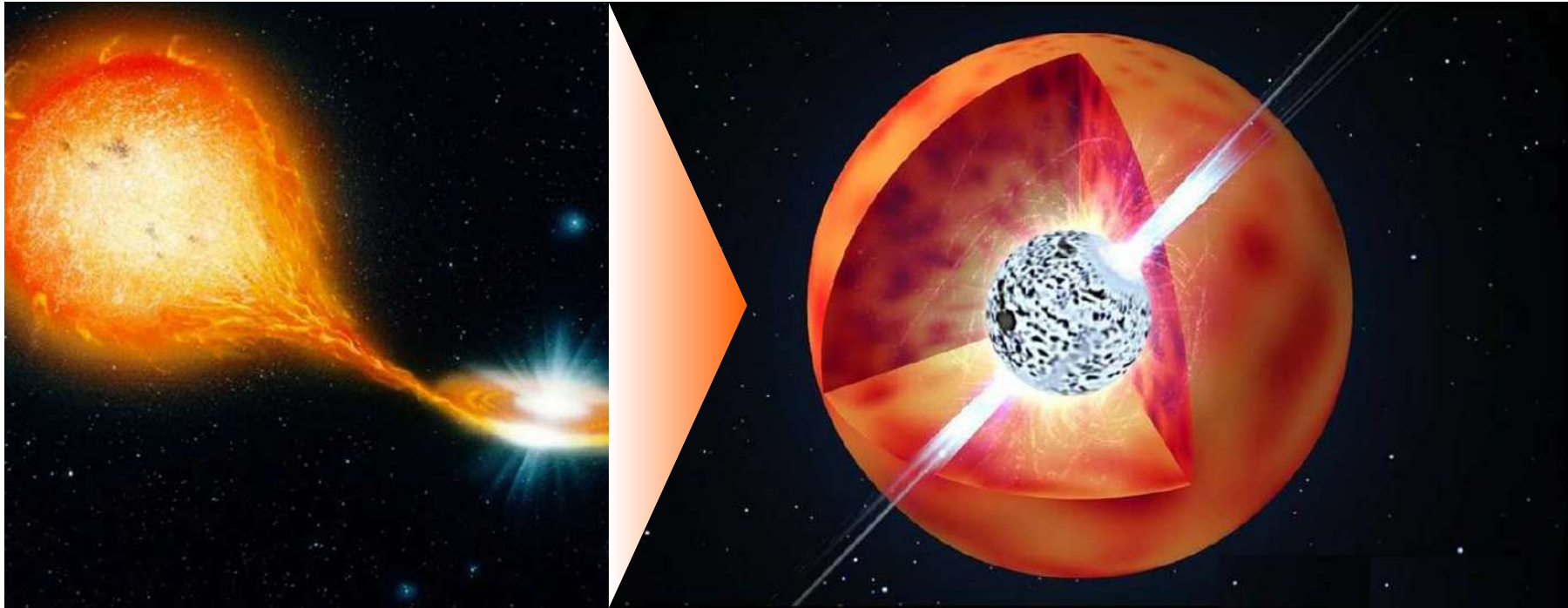


Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 100 \text{ TeV}$, $\rho = 10^{-3} \text{ g/cm}^3$).



Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 100 \text{ TeV}$, $\rho = 3 \times 10^{-4} \text{ g/cm}^3$).

The mechanism under discussion may be released in the Thorne–Żytkow objects (TŻO) – binaries with a neutron star submerged into a red supergiant core. Figure shows an artistic view of how a TŻO could be formed.



[See, e.g., URLs: <http://astrofishki.net/universe/hv-2112-neveroyatnyj-obekt-torna-zhitkov/> and <http://www.decifrandoastronomia.com.br/2017/01/uma-estrela-dentro-de-outra-conheca-hv.html>.]

The very bright red star HV 2112 in the Small Magellanic Cloud (see next slide) could be a massive supergiant-like star with a degenerate neutron core (TŻO). With its luminosity of over $10^5 L_{\odot}$, it could also be a super asymptotic giant branch star (SAGB), a star with an oxygen/neon core supported by electron degeneracy and undergoing thermal pulses with third dredge up.



Both TŻO and SAGB stars are expected to be rare. Calculations performed by Ch. A. Tout *et al.*^a indicate that HV 2112 is likely a genuine TŻO. But a much more likely explanation is that HV 2112 is an intermediate mass ($\sim 5M_{\odot}$) AGB star; a new TŻO candidate (HV 11417) is recently suggested.^b

^aCh. A. Tout, A. N. Żytkow, R. P. Church, & H. H. B. Lau, “HV 2112, a Thorne–Żytkow object or a super asymptotic giant branch star”, *Mon. Not. Roy. Astron. Soc.* **445** (2014) L36–L40, arXiv:1406.6064 [astro-ph.HE].

^bE. R. Beasor, B. Davies, I. Cabrera-Ziri, & G. Hurst, “A critical re-evaluation of the Thorne–Żytkow object candidate HV 2112”, arXiv:1806.07399 [astro-ph.SR].

7.8.2 Degenerate case.

The consideration must be completed for the case of degeneracy. Due to the condition $q_I < 0$, the density and composition of the “degenerate environment” are fine-tuned in such a way that

$$q = q_{-\zeta} = \Delta_c - i |\Delta_s|.$$

The simplest way is in coming back to the master equation. Indeed, in the limit of $q = q_{-\zeta}$, the Hamiltonian reduces to

$$\mathbf{H} = |\Delta_s| \begin{pmatrix} -i & \zeta \\ \zeta & i \end{pmatrix} \equiv |\Delta_s| \mathbf{h}_\zeta.$$

Considering that $\mathbf{h}_\zeta^2 = \mathbf{0}$, we promptly arrive at the solution of ME:

$$\tilde{\mathbf{S}}(t) = \mathbf{1} - it |\Delta_s| \mathbf{h}_\zeta$$

and thus

$$\begin{aligned} P_{\alpha\alpha}(t) &= (1 - |\Delta_s|t)^2 e^{-t/\Lambda}, \\ P_{\beta\beta}(t) &= (1 + |\Delta_s|t)^2 e^{-t/\Lambda}, \\ P_{\alpha\beta}(t) &= P_{\beta\alpha}(t) = (\Delta_s t)^2 e^{-t/\Lambda}. \end{aligned}$$

Since $1/\Lambda_\beta = 1/\Lambda_\alpha - 4|\Delta_s|$, the necessary condition for the total degeneration is

$$4\Lambda_\alpha |\Delta_s| \leq 1$$

and thus

$$1/\Lambda = 1/\Lambda_\alpha - 2|\Delta_s| \geq 2|\Delta_s|.$$

The equality only occurs when ν_β is sterile.

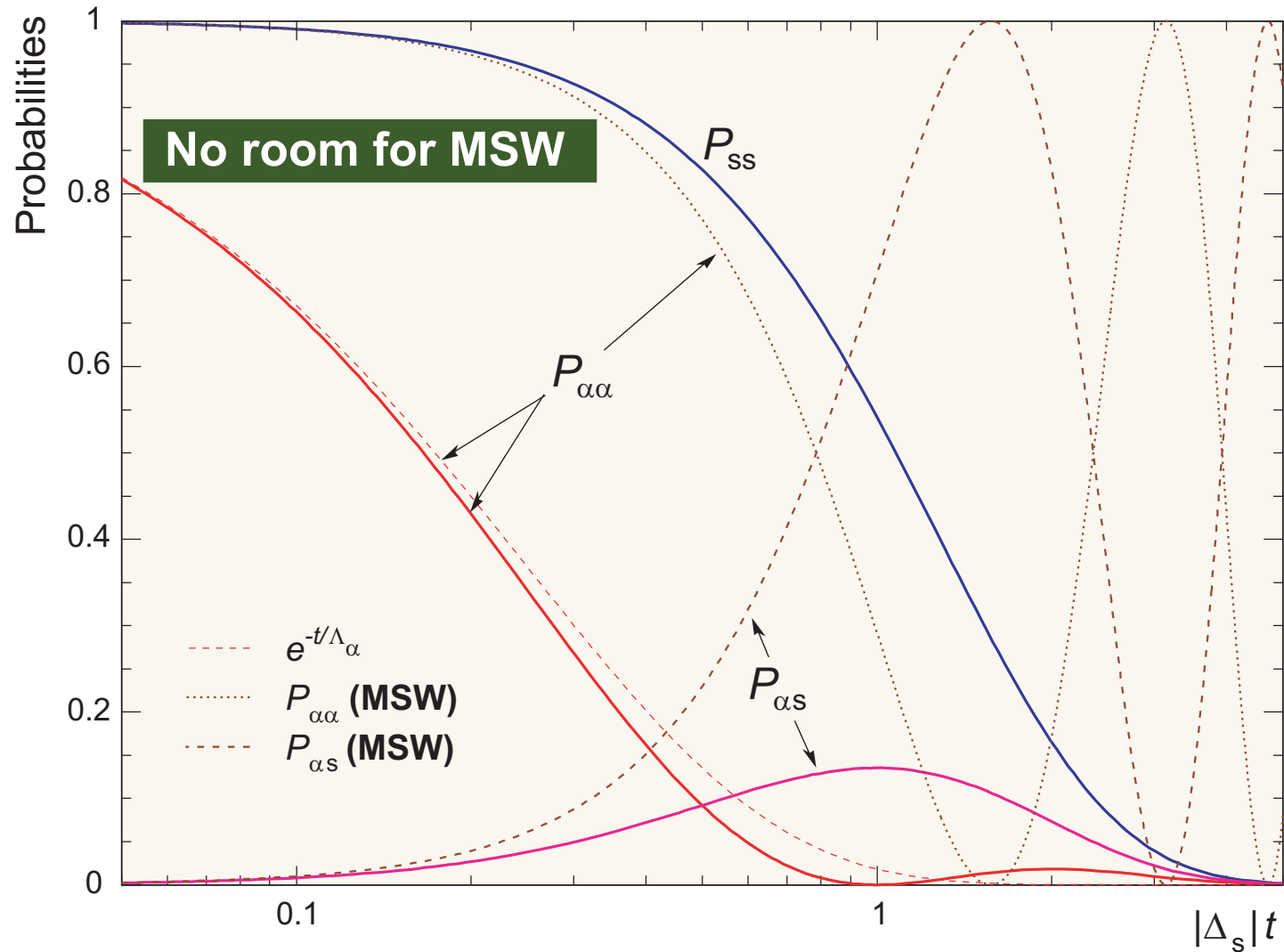
The degenerate solution must be compared with the standard MSW solution

$$\left. \begin{aligned} P_{\alpha\alpha}(t) = P_{ss}(t) &= \frac{1}{2} [1 + \cos(2\Delta_s t)], \\ P_{\alpha s}(t) = P_{s\alpha}(t) &= \frac{1}{2} [1 - \cos(2\Delta_s t)], \end{aligned} \right\} \text{(MSW)}$$

and with the classical penetration coefficient

$$\exp(-t/\Lambda_\alpha)$$

(with $1/\Lambda_\alpha$ numerically equal to $4|\Delta_s|$) relevant to the transport of unmixed active neutrinos through the same environment.



Survival and transition probabilities for $\nu_\alpha \leftrightarrow \nu_s$ oscillations in the case of degeneracy ($q = q_{-\zeta}$). The standard MSW probabilities (dotted and dash-dotted curves) together with the penetration coefficient for unmixed ν_α (dashed curve) are also shown.

7.9 Conclusions.

We have considered, on the basis of the MSW evolution equation with complex indices of refraction, the conjoint effects of neutrino mixing, refraction and absorption on high-energy neutrino propagation through matter. The adiabatic solution with correct asymptotics in the standard MSW and classical limits has been derived. In the general case the adiabatic behavior is very different from the conventional limiting cases.

A noteworthy example is given by the active-to-sterile neutrino mixing. It has been demonstrated that, under proper conditions, the survival probability of active neutrinos propagating through a very thick medium of constant density may become many orders of magnitude larger than it would be in the absence of mixing. The quantitative characteristics of this phenomenon are highly responsive to changes in density and composition of the medium as well as to neutrino energy and mixing parameters.

Considering a great variety of latent astrophysical sources of high-energy neutrinos, the effect may open a new window for observational neutrino astrophysics.

Backup



The standard $(\beta\beta)_{2\nu}$ is observed for a dozen isotopes with $T_{1/2}^{2\nu} \sim 10^{19-25}$ years. Some most recent averaged/recommended $T_{1/2}^{2\nu}$ are collected in Table and are compared with theoretical predictions.

Element	Isotope	$T_{1/2}^{2\nu}$ (years)	
		Measured	Calculated
Calcium	${}^{48}_{20}\text{Ca}$	$5.3^{+1.2}_{-0.8} \times 10^{19}$	$6 \times 10^{18} - 5 \times 10^{20}$
Germanium	${}^{76}_{32}\text{Ge}$	$(1.88 \pm 0.08) \times 10^{21}$	$7 \times 10^{19} - 6 \times 10^{22}$
Selenium	${}^{82}_{34}\text{Se}$	$8.7^{+0.2}_{-0.1} \times 10^{19}$	$3 \times 10^{18} - 6 \times 10^{21}$
Zirconium	${}^{96}_{40}\text{Zr}$	$(2.3 \pm 0.2) \times 10^{19}$	$3 \times 10^{17} - 6 \times 10^{20}$
Molybdenum	${}^{100}_{42}\text{Mo}$	$7.06^{+0.15}_{-0.12} \times 10^{18}$	$1 \times 10^{17} - 2 \times 10^{22}$
Molybdenum–Ruthenium	${}^{100}_{42}\text{Mo} - {}^{100}_{44}\text{Ru}(0_1^+)$	$6.7^{+0.5}_{-0.4} \times 10^{20}$	$5 \times 10^{19} - 2 \times 10^{21}$
Cadmium	${}^{116}_{48}\text{Cd}$	$(2.69 \pm 0.09) \times 10^{19}$	$3 \times 10^{18} - 2 \times 10^{21}$
Tellurium	${}^{128}_{52}\text{Te}$	$(2.25 \pm 0.09) \times 10^{24}$	$9 \times 10^{22} - 3 \times 10^{25}$
Tellurium	${}^{130}_{52}\text{Te}$	$(7.91 \pm 0.21) \times 10^{20}$	$2 \times 10^{19} - 7 \times 10^{20}$
Xenon	${}^{136}_{54}\text{Xe}$	$(2.18 \pm 0.05) \times 10^{21}$	—
Neodymium	${}^{150}_{60}\text{Nd}$	$(9.34 \pm 0.65) \times 10^{18}$	$6 \times 10^{16} - 4 \times 10^{20}$
Neodymium–Samarium	${}^{150}_{60}\text{Nd} - {}^{150}_{62}\text{Sm}(0_1^+)$	$1.2^{+0.3}_{-0.2} \times 10^{20}$	—
Uranium	${}^{238}_{92}\text{U}$	$(2.0 \pm 0.6) \times 10^{21}$	$2 \times 10^{19} - 2 \times 10^{23}$

[From A. S. Barabash, “Precise half-life values for two-neutrino double- β decay: 2020 review,” Universe 6 (2020) 159, arXiv:2009.14451 [nucl-ex] (experiment); E. Fiorini, “Experimental prospects of neutrinoless double beta decay,” Phys. Scripta T121 (2005) 86–93 (theory; of course these calculations are outdated, but I did not find a fresh review).]

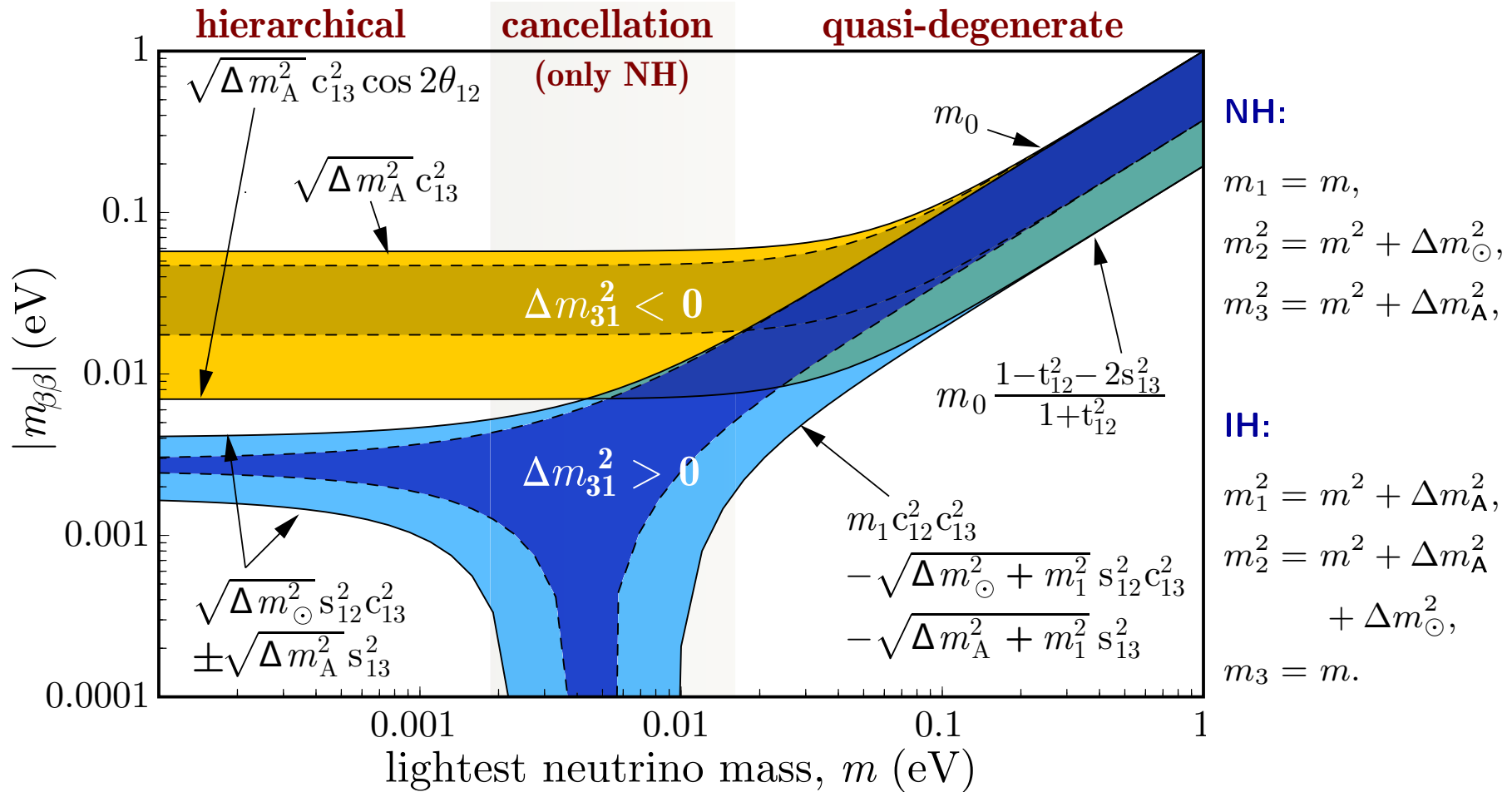
Best current results on $0\nu\beta\beta$ decay. The $T_{1/2}^{0\nu}$ and $\langle m_{\beta\beta} \rangle (\equiv \langle |m_{\beta\beta}| \rangle)$ limits are at 90% C.L.

Element	Isotope	$Q_{2\beta}$ (keV)	$T_{1/2}^{0\nu}$ (years)	$\langle m_{\beta\beta} \rangle$ (eV)	Experiment
Calcium	^{48}Ca	4267.98	$> 5.8 \times 10^{22}$	$< 3.5 - 22$	ELEGANT-IV
Germanium	^{76}Ge	2039.00	$> 8.0 \times 10^{25}$	$< 0.12 - 0.26$	GERDA
			$> 1.9 \times 10^{25}$	$< 0.24 - 0.52$	Majorana Demonstrator
Selenium	^{82}Se	2997.9	$> 3.6 \times 10^{23}$	$< 0.89 - 2.4$	NEMO-3
Zirconium	^{96}Zr	3355.85	$> 9.2 \times 10^{21}$	$< 7.2 - 19.5$	NEMO-3
Molybdenum	^{100}Mo	3034.40	$> 1.1 \times 10^{24}$	$< 0.33 - 0.62$	NEMO-3
Cadmium	^{116}Cd	2813.50	$> 2.2 \times 10^{23}$	$< 1.0 - 1.7$	AURORA
Tellurium	^{128}Te	866.6	$> 1.1 \times 10^{24}$	–	Geochemical
Tellurium	^{130}Te	2527.52	$> 1.5 \times 10^{25}$	$< 0.11 - 0.52$	CUORE
Xenon	^{136}Xe	2457.83	$> \mathbf{1.07} \times \mathbf{10^{26}}$	$< \mathbf{0.061} - \mathbf{0.165}$	KamLAND-Zen
			$> 1.8 \times 10^{25}$	$< 0.15 - 0.40$	EXO-200
Neodymium	^{150}Nd	3371.38	$> 2.0 \times 10^{22}$	$< 1.6 - 5.3$	NEMO-3

The $\langle m_{\beta\beta} \rangle$ limits are listed as reported in the original publications.^a

[M. J. Dolinski, A. W. P. Poon, & W. Rodejohann, “Neutrinoless double-beta decay: Status and prospects,” *Ann. Rev. Nucl. Part. Sci.* **69** (2019) 219–251, arXiv:1902.04097 [nucl-ex].]

^aFor a bit another approach, see A. S. Barabash, “Brief review of double beta decay experiments”, arXiv:1702.06340 [nucl-ex]; the Q values shown in the Table are borrowed from that paper.

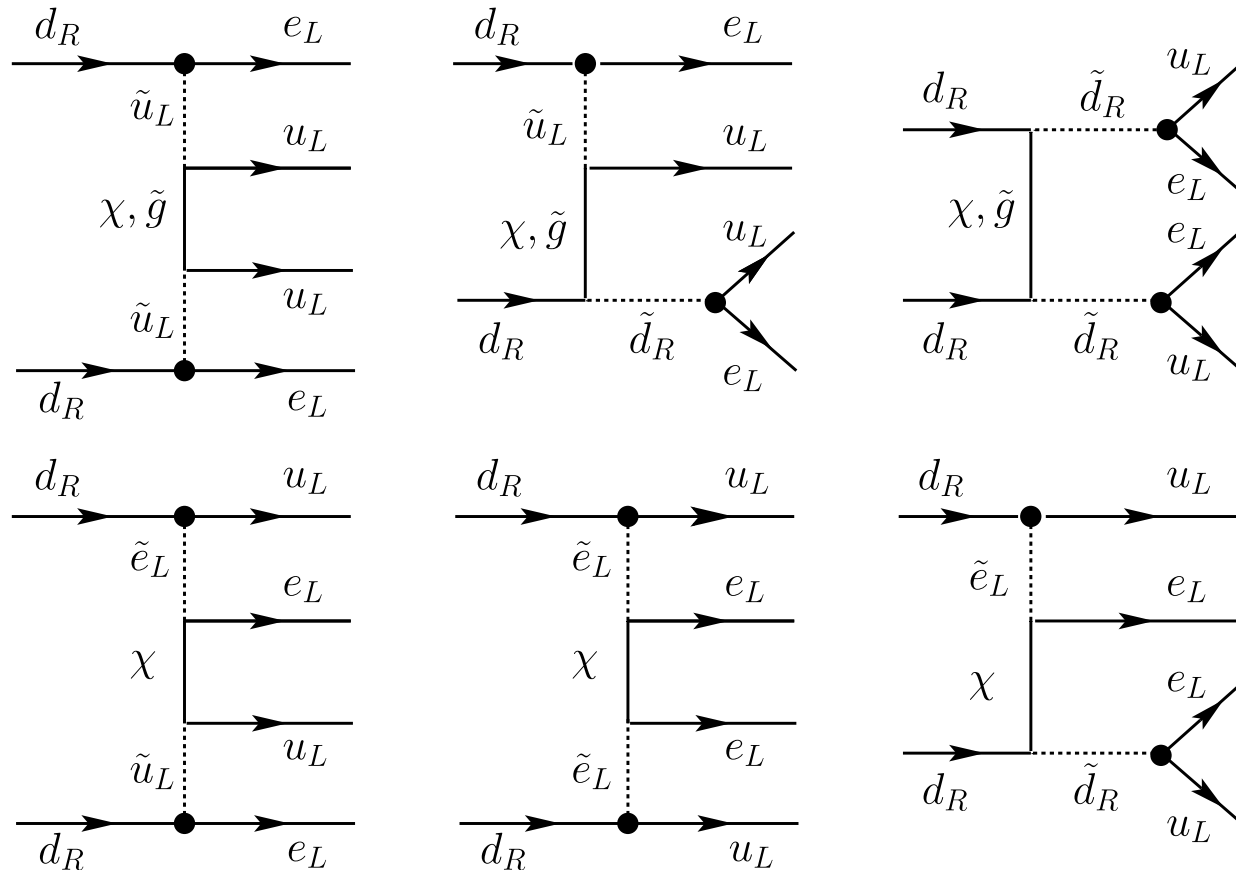


The main properties of $|m_{\beta\beta}|$ vs. smallest neutrino mass (m). The value of $\sin^2 2\theta_{13} = 0.02$ has been chosen, m_0 is the common mass scale (measurable in KATRIN or by cosmology via $\sum_i m_i/3$) for quasi-degenerate masses $m_1 \simeq m_2 \simeq m_3 \equiv m_0 \gg \sqrt{\Delta m_A^2}$ (corrections are small as $m \gtrsim 0.03$ eV).

[Taken from M. Lindner, A. Merle, and W. Rodejohann, "Improved limit on θ_{13} and implications for neutrino masses in neutrinoless double beta decay and cosmology," Phys. Rev. D **73** (2006) 053005, hep-ph/0512143.]

Schechter-Valle (black-box) theorem.

Current particle models (GUTs, R-parity violating SUSY, etc.) provide mechanisms, other than neutrino mass, which can contribute to or even dominate the $0\nu\beta\beta$ process (see example below).



Δ R-parity violating contribution to $0\nu\beta\beta$ decay mediated by sfermions and neutralinos (gluinos).

[Figure is borrowed from J. D. Vergados, H. Ejiri, and F. Simkovic, "Theory of neutrinoless double-beta decay," Rep. Prog. Phys. **75** (2012) 106301, arXiv:1205.0649 [hep-ph], where many other examples can be found.]

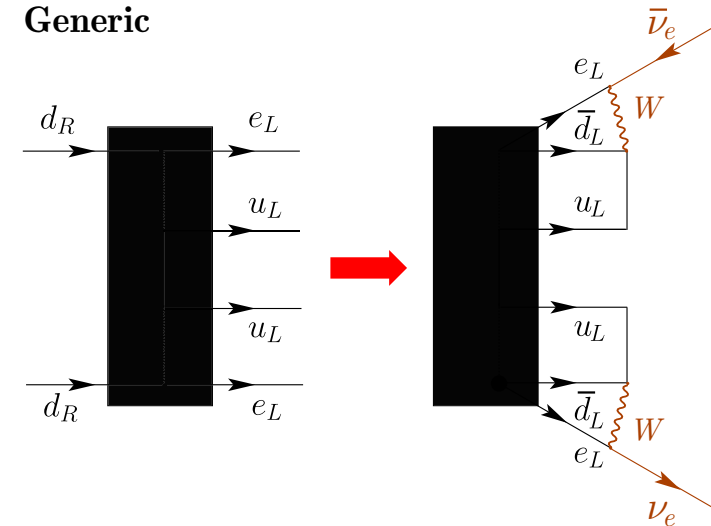
Schechter and Valle proved^a that

for any realistic gauge theory including the usual (SM) W -gauge-field interaction with left-handed e and ν_e and with u and d quarks, if $0\nu\beta\beta$ -decay takes place, regardless of the mechanism causing it, the neutrino is Majorana particle with nonzero mass.

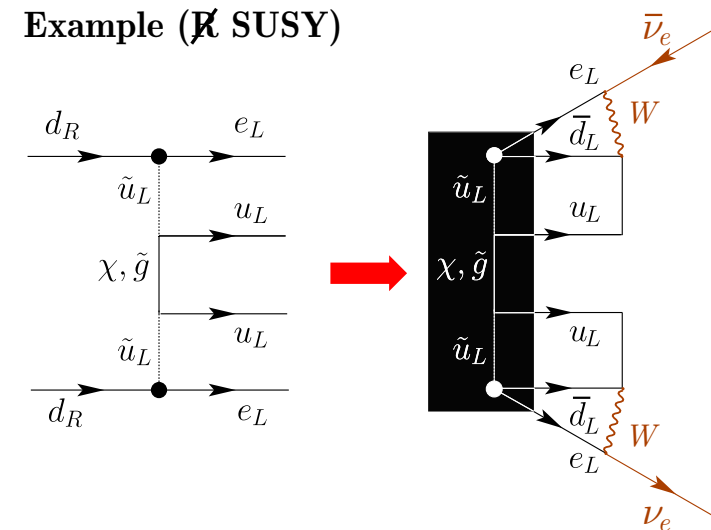
The reason is that one can consider the $0\nu\beta\beta$ elementary interaction process $dd \rightarrow uuee$ as generated by the black box, which can include any mechanism. Then the legs of the black box can be arranged to form a diagram which generates $\bar{\nu}_e \rightarrow \nu_e$ transitions. This diagram contributes to the Majorana mass of the electron neutrino through radiative corrections at some order of perturbation theory, even if there is no tree-level Majorana neutrino mass term.

It is however clear that the black-box amplitudes are strongly suppressed (at least by a factor $\propto G_F^2$) with respect to the standard tree-level $0\nu\beta\beta$ -decay amplitude. Model calculations show that the standard amplitude corresponding to a value of $|m_{\beta\beta}| = O(0.1)$ eV generates radiatively a Majorana mass $O(10^{-24})$ eV.

Generic



Example (NOT SUSY)



^aJ. Schechter and J. W. F. Valle, "Neutrinoless double- β decay in $SU(2) \times U(1)$ theories," Phys. Rev. D **25** (1982) 2951–2954. A generalization to 3ν (mixed) case was made by M. Hirsch, H. V. Klapdor-Kleingrothaus, and S. G. Kovalenko, "On the SUSY accompanied neutrino exchange mechanism of neutrinoless double beta decay," Phys. Lett. B **372** (1996) 181–186, Phys. Lett. B **381** (1996) 488 (erratum), hep-ph/9512237.

7.10 Double see-saw & inverse see-saw.

The see-saw can be implemented by introducing additional neutrino **singlets** beyond the three RH neutrinos involved into the see-saw type I. One have to distinguish between

- RH neutrinos ν_R , which carry $B - L$ and perhaps (not necessary) form $SU(2)_R$ doublets with RH charged leptons, and
- Neutrino singlets ν_S , which have no Yukawa couplings to the LH neutrinos but may couple to ν_R .

If the singlets have **nonzero** Majorana masses \mathbf{M}_{SS} while the RH neutrinos have a **zero** Majorana mass, $\mathbf{M}_{RR} = 0$, the see-saw mechanism may proceed via mass couplings of the singlets to RH neutrinos, \mathbf{M}_{RS} . In the basis (ν_L, ν_R, ν_S) , the 9×9 mass matrix is

$$\begin{pmatrix} \mathbf{0} & \mathbf{m}_{LR} & \mathbf{0} \\ \mathbf{m}_{LR} & \mathbf{0} & \mathbf{M}_{RS} \\ \mathbf{0} & \mathbf{M}_{RS}^T & \mathbf{M}_{SS} \end{pmatrix}.$$

Assuming that the eigenvalues of \mathbf{M}_{SS} are much smaller than the eigenvalues of \mathbf{M}_{RS} , the light physical LH Majorana neutrino masses are then **doubly suppressed**,

$$\mathbf{M}_1 \simeq \mathbf{m}_{LR} \mathbf{M}_{RS}^{-1} \mathbf{M}_{SS} (\mathbf{M}_{RS}^T)^{-1} \mathbf{m}_{LR}^T, \quad \mathbf{M}_2^2 \simeq \mathbf{M}_{RS}^2 + \mathbf{m}_{LR}^2.$$

This scenario is usually used in string inspired models [see, e.g., R.N.Mohapatra & J.W.Valle, Phys. Rev. D **34** (1986) 1642; M.C.Gonzalez-Garcia & J.W.F.Valle, Phys. Lett. B **216** (1989) 360].

7.11 Radiative see-saw.

An alternative mechanism relies on the radiative generation of neutrino masses [H.Georgi & S.L.Glashow, Phys. Rev. D 7 (1973) 2487; P.Cheng & L.-F.Li, Phys. Rev. D 17 (1978) 2375; Phys. Rev. D 22 (1980) 2860; A.Zee, Phys. Lett. B 93 (1980) 389;...] In this scheme, the neutrinos are massless at the tree level, but pick up small masses due to loop corrections.

In a typical model [K.S. Babu & V.S. Mathur, Phys. Rev. D 11 (1988) 3550] the see-saw formula is modified as

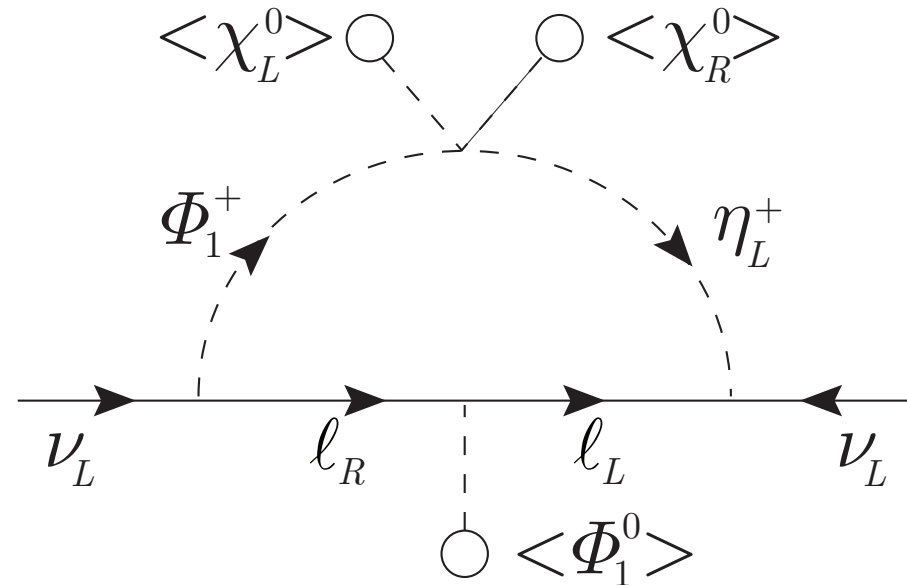
$$m_\nu \sim \left(\frac{\alpha}{\pi}\right) \frac{m_l^2}{M},$$

where the prefactor $\alpha/\pi \approx 2 \times 10^{-3}$ arises due to the loop structure of the neutrino mass diagram. Light neutrinos are now possible even for relatively “light” mass scale M of “new physics.”

The scalar sector consists of the multiplets

$$\chi_{L,R} = (\chi^+, \chi^0)_{L,R}, \quad \Phi = \begin{pmatrix} \Phi_1^0 & \Phi_2^+ \\ \Phi_1^- & \Phi_2^0 \end{pmatrix}, \quad \eta_{L,R}^+.$$

The diagram in the figures is responsible for generation of Majorana masses for ν_L . The analogous diagram is obtained by the replacement $L \rightarrow R$ and $\Phi_1^+ \rightarrow \Phi_2^+$.



7.12 TeV-scale gauged $B - L$ symmetry with Inverse see-saw.

Consider briefly one more inverse see-saw model [S.Khalil, Phys. Rev. D 82 (2010) 077702].

The model is based on the following:

- (i) The SM singlet Higgs boson, which breaks the $B - L$ gauge symmetry, has $B - L$ unit charge.
- (ii) The SM singlet fermion sector includes two singlet fermions S_{\pm} with $B - L$ charges ± 2 with opposite matter parity.

The Lagrangian of neutrino masses, in the flavor basis, is given by

$$\bar{\nu}_L \mathbf{m}_D \nu_R + \nu_R^c \mathbf{M}_N S_- + \mu_s \bar{S}_- S_-.$$

In the limit $\mu_s \rightarrow 0$, which corresponds to the unbroken $(-1)^{L+S}$ symmetry, the light neutrinos remain massless. Therefore, a small nonvanishing μ_s can be considered as a slight breaking of a this global symmetry and the smallness of μ_s is natural. Small μ_s can also be generated radiatively.

In the basis (ν_L, ν_R^c, S_-) , the 9×9 mass matrix is

$$\begin{pmatrix} \mathbf{0} & \mathbf{m}_D & \mathbf{0} \\ \mathbf{m}_D^T & \mathbf{0} & \mathbf{M}_N \\ \mathbf{0} & \mathbf{M}_N^T & \mu_s \end{pmatrix}.$$

So, up to the notation, it reproduces all the properties of the double see-saw.