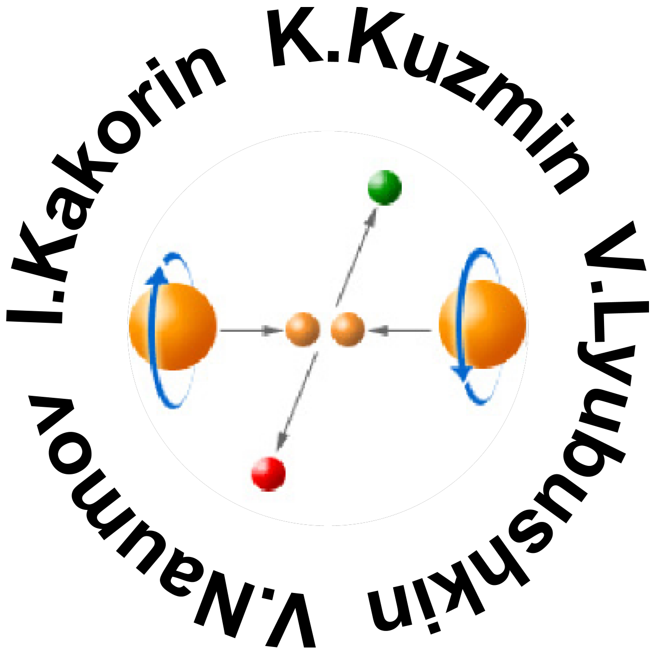
Kinematic formulas for GENIE implementation of Smith-Monitz model



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Here we provide all formulas with minimum comments (for details see *Notes to polarization studies: living document for internal use* by K. S. Kuzmin, V. A. Naumov, V. V. Lyubushkin, and I. D. Kakorin[1]). This document is referred to below as [**PolarizationNotes**].

1 General neutrino energy thresholds

he neutrino energy threshold of scattering on nucleus:

$$E_\nu^{\text{th}} = \frac{s^{\text{th}} - M_{\text{tar}}^2}{2M_{\text{tar}}} = \frac{(M_{\text{rnu}} + m + M_f)^2 - M_{\text{tar}}^2}{2M_{\text{tar}}}, \tag{1}$$

where E_ν, s^{th}, M_{tar}, M_{rnu}, M_f, and m are, respectively, the neutrino energy, minimal value of s-variable, mass of target nucleus, mass of remnant nucleus, mass of final nucleon, and final lepton mass.

As is explaned in [**PolarizationNotes**] energy threshold on bound nucleon,

$$E_\nu^{\text{th}} = \frac{(M_f + m)^2 + \text{p}^2 - (E_{\mathbf{p}} - E_b)^2}{2\,(E_{\mathbf{p}} - E_b - \text{p}\cos\theta)}. \tag{2}$$

The minimum of this expression occurs when $\cos\theta = -1$ and $\text{p} = p_F$.

2 Minimal and maximal values of Q^2

he minimum and maximum of Q^2-values for neutrino scattering on nucleus are given by:

$$\begin{aligned} Q^2_{\text{min}} &= 2E_\nu^* \left[E_\ell^*(W_{\text{min}}) - P_\ell^*(W_{\text{min}})\right] - m^2 \\ Q^2_{\text{max}} &= 2E_\nu^* \left[E_\ell^*(W_{\text{min}}) + P_\ell^*(W_{\text{min}})\right] - m^2, \end{aligned} \tag{3}$$

where E_ν^* – neutrino energy in CMS:

$$E_\nu^* = \frac{s - M_i^2}{2\sqrt{s}}; \tag{4}$$

and $E_\ell^*(W)$, $P_\ell^*(W)$ – final lepton energy and momentum in CMS as function of invariant hadronic mass W:

$$E_\ell^* \equiv E_\ell^*(W) = \frac{s + m^2 - W^2}{2\sqrt{s}}, \quad P_\ell^*(W) = \sqrt{\left[E_\ell^*(W)\right]^2 - m^2} \tag{5}$$

The minimal value of hadronic mass W_{min} at which E_ℓ^* and P_ℓ^* achieved their maximums is

$$W_{\text{min}} = M_{\text{rnu}} + M_f. \tag{6}$$

3 Minimal and maximal values of energy transfer

Minimal and maximal values of energy transfer ν for neutrino scattering on nucleus can be found from:

$$W^2 = (p+q)^2 = M_{\text{tar}}^2 - Q^2 + 2M_{\text{tar}}(E_\nu - E_\ell),$$

wherefrom

$$\nu(W) = \frac{W^2 + Q^2 - M_{\text{tar}}^2}{2M_{\text{tar}}}.$$

Minimum of ν is achieved at the minimal invariant hadronic mass (6):

$$\nu_{\text{min}} = \frac{(M_{\text{rnu}} + M_f)^2 + Q^2 - M_{\text{tar}}^2}{2M_{\text{tar}}} \tag{7}$$

and maximum values of ν is achieved at maximal hadronic mass value:

$$W_{\text{max}} = \sqrt{s} - m.$$

However, the maximal energy transfer ν also can be found from only leptonic variables. Using definition of Q^2 one obtains following equality

$$Q^2 = -q^2 = -(p_\nu - p_\ell)^2 = -m^2 + 2\left(E_\ell E_\nu - P_\ell P_\nu \cos\theta\right),$$

[1] `http://theor.jinr.ru/NeutrinoOscillations/Papers/PolarizationNotes.pdf`

which has solution relative to variable E_ℓ is

$$E_\ell^\pm = E_\ell^\pm(Q^2,\theta) = \frac{Q^2+m^2}{2E_\nu \sin^2\theta}\left[1 \pm \cos\theta\sqrt{1-\left(\frac{2mE_\nu\sin\theta}{Q^2+m^2}\right)^2}\right]. \tag{8}$$

From Eq. (8) it can be found:

$$E_\ell^-(Q^2,0) = \frac{m^2 E_\nu}{Q^2+m^2} + \frac{Q^2+m^2}{4E_\nu},$$

which provides the minimum of $E_\ell^-(Q^2,\theta)$ as $\partial E_\ell^-(Q^2,\theta)/\partial\theta\big|_{\theta=0} = 0$ and $\partial^2 E_\ell^-(Q^2,\theta)/\partial\theta^2\big|_{\theta=0} > 0$, and consequently:

$$\nu_{\text{max}} = E_\nu - \left(\frac{m^2E_\nu}{Q^2+m^2} + \frac{Q^2+m^2}{4E_\nu}\right). \tag{9}$$

It can be proved that the following inequalities always holds: $\nu_{\text{min}} \le \nu_{\text{max}} \le \nu(W_{\text{max}})$, therefore ν_{max} is used as upper limit of energy transfer.

The minimal and maximal values of energy transfer for bound nucleon can be found from

$$\nu = \frac{pq^*}{M_i^*},$$

where $p = (E_i - E_b, \mathbf{p}) = (\sqrt{M_i^2+\mathbf{p}^2} - E_b, \mathbf{p})$ is the 4-momentum of initial bound nucleon, $q^* = (\nu^*, \mathbf{q}^*)$ is the four-momentum transfer to bound nucleon with effective mass $M_i^* = \sqrt{p^2} = \sqrt{(E_i-E_b)^2-\mathbf{p}^2}$, $|\mathbf{q}^*| = \sqrt{Q^2+\nu^{*2}}$, and

$$\nu^* = \frac{M_f^2 + Q^2 - M_i^{*2}}{2M_i^*}.$$

$$\nu = \frac{\nu^*(E_i-E_b) - \mathbf{q}^*\mathbf{p}}{M_i^*} = \frac{\nu^*(E_i-E_b) - |\mathbf{q}^*||\mathbf{p}|\cos\phi}{M_i^*}, \tag{10}$$

where ϕ is the angle between $\mathbf{p}$ and $\mathbf{q}^*$. The minimum and maximum of Eq. (10) are achieved at $|\mathbf{p}|\cos\phi = \pm p_F$, therefore

$$\begin{aligned}\nu_{\text{min}} &= \frac{\nu^*(E_i-E_b) - p_F\sqrt{\nu^{*2}+Q^2}}{M_i^*}\\ \nu_{\text{max}} &= \frac{\nu^*(E_i-E_b) + p_F\sqrt{\nu^{*2}+Q^2}}{M_i^*}.\end{aligned} \tag{11}$$

4 Minimal and maximal values of initial nucleon energy

The four-momentum of the final lepton is $p_f = p + q$, where $q = (\nu, \mathbf{q})$ is the four-momentum transfer. Since $p_f^2 = M_f^2$ one obtains:

$$(E_i+\nu)^2 - (\mathbf{p}+\mathbf{q})^2 = M_f^2$$

or

$$(E_i+\nu)^2 = M_f^2 + \mathrm{p}^2 + \mathrm{q}^2 + 2\mathrm{pq}\cos\alpha,$$

where q=$|\mathbf{q}|$ and α is the angle between $\mathbf{p}$ and $\mathbf{q}$. Consequently

$$-2\mathrm{pq} \le (E_i+\nu)^2 - (E_i+E_b)^2 + M_i^2 - \mathrm{q}^2 - M_f^2 \le 2\mathrm{pq},$$

where we used the substitution $\mathrm{p}^2 = (E_i+E_b)^2 - M_i^2$. Squaring the last relation gives us:

$$\left[2E_i(\nu-E_b) + M_i^2 - M_f^2 - Q^2 - E_b^2\right]^2 - 4\mathrm{q}^2(E_i^2 + 2E_iE_b + E_b^2 - M_i^2) \le 0, \tag{12}$$

where the identity $\nu^2 - \mathrm{q}^2 = -Q^2$ was used. Rewriting the left part of inequality (12) in terms of new variables

$$c = \frac{\nu - E_b}{\mathrm{q}} \quad \text{and} \quad d = \frac{M_i^2 - M_f^2 - Q^2 + E_b^2 - 2E_b\nu}{2M_i\mathrm{q}},$$

and equating it to zero, yields

$$(1-c^2)(E_i+E_b)^2 - 2cdM_i(E_i+E_b) - (1+d^2)M_i^2 = 0.$$

The roots of this equation are

$$E_i^\pm = \frac{M_i(dc \pm \sqrt{1-c^2+d^2})}{1-c^2} - E_b. \tag{13}$$

However we also must take into account the Pauli-blocking condition: $E_i + \nu > E_F$, where E_F is the Fermi energy of final bound nucleon. Taking into account the Pauli-blocking we find that initial nucleon energy should be greater than $\max(E_i^+ + E_b, E_F - \nu)$, because $E_i^- < 0$.

Clearly, the maximum energy of the initial nucleon is just $\sqrt{p_F^2 + M_i^2} - E_b$.