«Избранные вопросы теоретической физики и астрофизики». Дубна: ОИЯИ, 2003. С. 151–152.

Real-axis integral representation for the two-body Coulomb scattering wave function

S. B. Levin [a,b*†], E. O. Alt [a‡], S. L. Yakovlev [b§]

[a] *Institut für Physik, Universität Mainz, D-55099 Mainz, Germany*
[b] *Department of Mathematical and Computational Physics, Institute for Physics, St. Petersburg State University, 198904 St. Petersburg, Russia*

A useful new integral representation for the two-body Coulomb scattering wave function is obtained where the integration is along the real axis.

We dedicate this paper to Vladimir B. Belyaev on occasion of his 70th birthday.

The two-body Coulomb scattering wave function appears in many problems of physical interest. Though, the exact solution of Schrödinger equation is explicitly known in terms of the confluent hypergeometric function $\Phi(a,c;x) \equiv {}_1F_1(a,c;x)$,

$$\begin{aligned}\psi^+(\mathbf{k},\mathbf{r}) &= (2\pi)^{-3/2} e^{-\pi\gamma/2}\Gamma(1+i\gamma) e^{i\mathbf{k}\cdot\mathbf{r}} \Phi(-i\gamma,1;i(kr-\mathbf{k}\cdot\mathbf{r})) \\ &= (2\pi)^{-3/2} e^{-\pi\gamma/2}\Gamma(1+i\gamma) e^{ikr} \Phi(1+i\gamma,1;i(\mathbf{k}\cdot\mathbf{r}-kr)),\end{aligned} \tag{1}$$

where the second form follows from the first by a Kummer transformation ([1], Eq. 13.1.27). As usual, $\Gamma(\cdot)$ is the Gamma-function, $\gamma = e_1 e_2 \mu$ the Sommerfeld parameter, and μ the reduced mass of two particles with the charges e_1 and e_2.

Nevertheless, for various purposes it proves advantageous not to work with the explicit representation (1) but, instead, to use an integral representation. But, regrettably, the known integral representations of the confluent hypergeometric function either require integration in the complex plane which prevents application to the evaluation of more complicated quantities for which the analytical behavior is difficult or even impossible to extract. Or, if the integration is along the real axis as in [2]

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 du\, e^{xu} u^{a-1} (1-u)^{c-a-1},\ \Re(c) > \Re(a) > 0, \tag{2}$$

* Supported by Deutscher Akademischer Austauschdienst
† Email: Sergey.Levin@pobox.spbu.ru
‡ Email: Erwin.Alt@uni-mainz.de
§ Email: Yakovlev@cph10.phys.spbu.ru

it is not applicable for the particular values of the indices $c = 1$ and $a = 1 + i\gamma$ needed here.

In fact, we encountered this problem in our investigation of the kernels of three-body equations in the presence of Coulomb interactions which will be the content of a forthcoming publication. The quantities occurring there turn out to be too complicated to allow study of their analytical properties. Thus, use of a real-axis integral representation of the Coulomb wave function was mandatory. The latter has also proved its value for the asymptotic analysis of the two-body Coulomb Green's function [3]. Since such a wave function representation may also benefit further analytical work, we deem a separate publication justified.

A real-axis integral representation of the confluent hypergeometric function which is applicable even for the set of indices needed can be obtained as follows. Consider the recurrence relation between $\Phi(a,c;x)$ and its derivative with higher index ([1], Eq. 13.4.13)

$$\Phi(a,c;x) = \frac{x}{c}\Phi'(a,c+1;x) + \Phi(a,c+1;x).$$

For the indices of interest the integral representation (2) is applicable to both functions on the r. h. side, leading to the following expression for $\Phi(1+i\gamma,1;-i\zeta)$:

$$\Phi(1+i\gamma,1;-i\zeta) = Q(\gamma)e^{-i\zeta}\int_0^1 dt\; e^{i\zeta t - i\gamma \ln\left(\frac{t}{1-t}\right)}(1 - i\zeta + i\zeta t). \tag{3}$$

Here, $Q(\gamma) = \sinh \pi\gamma/\pi\gamma$ is the Gamov factor and $\zeta = kr - \mathbf{k}\cdot\mathbf{r}$.

Another convenient real-axis integral representation is

$$\Phi(1+i\gamma,1;-i\zeta) = e^{-i\zeta}\left[1 + i\gamma Q(\gamma)\int_0^1 dt\; e^{-i\gamma\ln\left(\frac{t}{1-t}\right)}\left(\frac{1-e^{i\zeta t}}{t}\right)\right]. \tag{4}$$

The equivalence with (3) can be shown as follows. Making use of the identity

$$\frac{d}{dx}e^{-i\gamma\ln\frac{x}{1-x}} = -\frac{i\gamma}{x(1-x)}e^{-i\gamma\ln\frac{x}{1-x}}$$

and integrating the integral in (4) by parts, it can be rewritten as

$$\begin{aligned} J \;&:=\; \int_0^1 dt\; e^{-i\gamma\ln\left(\frac{t}{1-t}\right)}\left(\frac{1-e^{i\zeta t}}{t}\right) \\ &= -\frac{1}{i\gamma}\int_0^1 dx\, e^{-i\gamma\ln\frac{x}{1-x}} + \frac{1}{i\gamma}\int_0^1 dx\, e^{i\zeta x}e^{-i\gamma\ln\frac{x}{1-x}}(1 - i\zeta + i\zeta x). \end{aligned} \tag{5}$$

The first term on the r. h. side of (5) gives $i/(\gamma Q(\gamma))$. Hence, insertion of this expression for J in (4) leads to the representation (3) for $\Phi(1+i\gamma,1;-i\zeta)$.

Thus, Eqs. (3) or (4) yield two equivalent real-axis integral representations for the two-body Coulomb scattering wave function. They are valid in all configuration space.

---

[1] Handbook of Mathematical Functions, edited by M. Abramovitz and I. A. Stegun (Dover, New York, 1986).

[2] H. Bateman, *Higher Transcendental functions*, edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. 1

[3] S. B. Levin, E. O. Alt, S. L. Yakovlev, and N. Elander, unpublished.