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Introduction to the Physics of Massive and Mixed Neutrinos

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Dedicated to the memory of the great neutrino physicist Bruno Pontecorvo

Preface

For many years neutrino was considered a massless particle. The theory of a two-component neutrino, which played a crucial role in the creation of the theory of the weak interaction, is based on the assumption that the neutrino mass is equal to zero.

We now know that neutrinos have nonzero, small masses. In numerous experiments with solar, atmospheric, reactor and accelerator neutrinos a new phenomenon, neutrino oscillations, was observed. Neutrino oscillations (periodic transitions between different flavor neutrinos ν_e, ν_μ, ν_τ) are possible only if neutrino mass-squared differences are different from zero and small and flavor neutrinos are "mixed".

The discovery of neutrino oscillations opened a new era in neutrino physics: an era of investigation of neutrino masses, mixing, magnetic moments and other neutrino properties. After the establishment of the Standard Model of the electroweak interaction at the end of the seventies, the discovery of neutrino masses was the most important discovery in particle physics. Small neutrino masses cannot be explained by the standard Higgs mechanism of mass generation. For their explanation a new mechanism is needed. Thus, small neutrino masses is the first signature in particle physics of a new beyond the Standard Model physics.

It took many years of heroic efforts by many physicists to discover neutrino oscillations. After the first period of investigation of neutrino oscillations, many challenging problems remained unsolved. One of the most important is the problem of the nature of neutrinos with definite masses. Are they Dirac neutrinos possessing a conserved lepton number which distinguish neutrinos and antineutrinos or Majorana neutrinos with identical neutrinos and antineutrinos? Many experiments of the next generation and new neutrino facilities are now under preparation and investigation. There is no doubt that exciting results are ahead.

This book is intended as an introduction to the physics of massive and mixed neutrinos. It is based on numerous lectures which I gave at different Universities and at CERN and other schools. I have tried to explain how many of the main results were derived. The details of the derivation can be easily followed by the reader.

I hope that this book will be useful for the physicists working in neutrino physics, for students and young physicists who plan to enter into this exciting field and to many scientists who are interested in the history of neutrino physics and its present status.

Dubna, Russia, Vancouver, Canada Samoil Bilenky
October 2009

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Chapter 1
Introduction

The idea of neutrino was put forward by W. Pauli in 1930. This was a dramatic time in physics. After it was established in the Ellis and Wooster experiment that the average energy of the electrons produced in the β-decay is significantly smaller than the total released energy, only the existence of neutrino, a neutral particle with a small mass and a large penetration length (much larger then the penetration length of the photon), which is emitted in the β-decay together with the electron, could save the fundamental law of the conservation of energy.

At the time when the neutrino hypothesis was proposed the only known elementary particles were electron and proton. In this sense neutrino (more exactly electron neutrino) is one of the "oldest" elementary particles. However, the existence of the neutrino was established only in the middle of the fifties when neutron, muon, pions, kaons, Λ and other strange particles were discovered.

We know at present that the twelve fundamental fermions exist in nature: six quarks u, d, c, s, t, b, three charged leptons e, μ, τ and three neutrinos ν_e, ν_μ, ν_τ. They are grouped in the three families, which differ in masses of particles but have universal electroweak interaction with photons and vector $W^\pm$, Z bosons. In the Lagrangian of the electroweak interaction, neutrinos enter on the same footings as the quarks and charged leptons. In spite of this similarity of the electroweak interaction *neutrinos are special particles*. There are two basic differences between neutrinos and other fundamental fermions.

- At all available at present energies, cross section of the interaction of neutrinos with matter is many order of magnitude smaller than the cross section of the electromagnetic interaction of leptons with matter (via the exchange of the virtual γ-quanta). This is connected with the fact that neutrinos interact with matter via the exchange of the heavy virtual $W^\pm$ and Z bosons.
- Neutrino masses are many order of the magnitude smaller than the masses of leptons and quarks.

Because of the extreme smallness of the neutrino cross section, special methods of the detection of neutrino processes must be developed. However, after such methods were developed the observation of neutrino processes allows us to obtain unique information:

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1. The measurement of the cross section of the deep inelastic processes $\nu_\mu(\bar{\nu}_\mu) + N \to \mu^-(\mu^+) + X)$ led to the establishment of the quark structure of the nucleon.
2. The detection of the solar neutrinos allowed us to establish the thermonuclear origin of solar energy and to obtain information about the central invisible part of the sun where the energy is produced.
3. The detection of neutrinos from the Supernova explosion allows us to investigate a mechanism of the gravitational collapse, etc.

The measurement of small neutrino masses is a difficult and challenging problem. This problem is still not solved. The observation of a new phenomenon-neutrino oscillations (the transitions between different flavor neutrinos) led to the determination of two neutrino mass-squared differences. From neutrino oscillation data and data of the β-decay experiments on the direct measurement of the neutrino mass it is possible to conclude that

- Neutrino masses are different from zero.
- Neutrino masses are smaller than $\sim 2\,\text{eV}$, i.e. many order of magnitude smaller than masses of leptons and quarks.

The unified theory of the weak and electromagnetic interactions (the Standard Model (SM)) describes all existing experimental data. However, existence of the Dark matter and internal problems of the Standard Model (the hierarchy problem) tell us that a more general, beyond the SM theory must exist. After the establishment of the SM at the end of the seventies, many experiments on the search for beyond the SM effects were done.

The first signature of a new beyond the SM physics was obtained in the neutrino oscillation experiments. In the framework of the SM small neutrino masses cannot be explained in a natural way. It is a common opinion that a new mechanism of the mass generation is required.

Discovery of neutrino oscillations signifies not only that neutrino mass-squared differences are different from zero but also that the states of flavor neutrino are superpositions ("mixture") of states of neutrinos with definite masses. Flavor neutrino states are connected with states of neutrinos with definite masses by the unitary mixing matrix which is characterized by three mixing angles and one phase.

The phenomenon of the neutrino mixing is similar to the well established quark mixing. However, the quark mixing angles are small and satisfy a hierarchy. The neutrino mixing angles are completely different: two angles are large and one is small (only upper bound is known at the moment). This is also an indication that quark and neutrino mixing have different origin.

The most common general explanation of the smallness of the neutrino mass is based on the assumption that the total lepton number is violated at a large scale (about $(10^{15} - 10^{16})\,\text{GeV}$). If this assumption is correct, neutrinos with definite masses are truly neutral *Majorana particles*. The leptons (and quarks) are Dirac particles. The leptons and antileptons (quarks and antiquarks) are different particles. They have the same masses but their electric charges differ in sign. If neutrino are Majorana particles in this case neutrinos and antineutrinos are identical. Observation of the neutrinoless double β-decay $(A, Z) \rightarrow (A, Z+2) + e^- + e^-$ would be a proof that neutrinos are Majorana particle.

The first rather long period of the investigation of neutrino masses, mixing and oscillations is practically finished. Neutrino oscillations were discovered. Four neutrino oscillation parameters (two mass-squared differences and two mixing angles) are determined with accuracies of about 10% or better. Strong bounds on the half-lives of the neutrinoless double β-decay of different nuclei were obtained. In 2009–2010 a new precision era of the investigation of the problem of neutrino masses, mixing and nature started. The main problems which will be addressed are the following

1. Are neutrinos with definite masses Majorana or Dirac particles?
2. What is the value of the third mixing angle θ_{13}?
3. Is the CP invariance violated in the lepton sector? What is the value of the CP phase?
4. What is the character of the neutrino mass spectrum? Is it normal with smaller mass-squared difference between lighter neutrinos or inverted with smaller mass-squared difference between heavier neutrinos?
5. What are the absolute values of the neutrino masses?
6. Is the number of massive neutrinos equal to the number of the flavor neutrinos (three) or larger than three? In other words do so called sterile neutrinos exist?
7. ...

Many neutrino experiments of the next generation with neutrinos from accelerators and reactors have started or will be started in the near future. New large detectors of atmospheric, solar and supernova neutrinos are under development. Technologies for new neutrino facilities (Super-beam, β-beam, Neutrino Factory) are being developed. *There is no doubt that a new exciting era of neutrino physics is ahead.*

In this book, I intend to give an introduction to the physics of massive and mixed neutrinos. I start with a brief review of the development of the phenomenological $V - A$ current$\times$current theory of the weak interaction starting from Pauli's hypothesis of the neutrino and Fermi's theory of the β-decay. In the next chapter we will consider the Standard Model. The Higgs mechanism of the generation of masses or quarks and leptons is discussed in some details. Then we consider mass terms for neutrinos in the Dirac and Majorana cases. We will describe in detail the procedure of the diagonalization of the mass terms in both these cases. Next chapter is devoted to the detailed consideration of the general properties of the neutrino mixing matrix. We obtain here the standard parametrization of the 3×3 mixing matrix. Then we present the theory of neutrino oscillations in vacuum. The three-neutrino oscillations are considered in detail. In the next chapter flavor neutrino transitions in matter are discussed. First, we derive Wolfenstein equation for neutrino evolution in matter. Then we consider in some detail the adiabatic solution of this equation and the resonance MSW effect. The next chapter is dedicated to the neutrinoless double β-decay of even–even nuclei. Basic elements of the theory of the decay are presented. Then we briefly discuss neutrino oscillation experiments and the data obtained. In the next chapter we discuss β-decay experiments on the measurement of the neutrino mass. In the last chapter we will consider neutrino in cosmology.

It is impossible in a book to give a full list of references. Taking into account a limited number of pages available for this book, I give here references only to some pioneer neutrino papers, relevant reviews and books. I would further recommend the web site by C. Giunti and M. Laveder (the Neutrino Unbound, http://www.nu.to.infn.it/) where it is possible to find many references to the neutrino literature (theory and experiment).

I conclusion I would like to enumerate some principal neutrino events.

1930. W. Pauli suggested in a letter addressed to the participants of the nuclear conference in Tuebingen that there exists a new neutral, spin 1/2, weakly interacting particle which is produced together with the electron in the β-decay of nuclei. Pauli called the new particle "neutron". Later E. Fermi proposed the name *neutrino* for this particle.

1934. E. Fermi proposed the first theory of the β-decay. Fermi considered the β-decay as four-fermion process in which a neutron is transformed into a proton with the emission of a electron-neutrino pair. He proposed the following Hamiltonian of the β-decay

$$\mathcal{H}_I = G_F \, \bar{p}\gamma^{\alpha} n \, \bar{e}\gamma_{\alpha}\nu + \text{h.c.}, \tag{1.1}$$

where G_F is the Fermi constant.

1934. Bethe and Peierls estimated the interaction cross section of neutrino with nuclei. The estimated value of the cross section was so small that for many years the neutrino was considered as an "undetectable particle".

1946. B. Pontecorvo proposed the radiochemical method of neutrino detection which is based on the observation of

$$\nu +^{37}\text{Cl} \to e^- +^{37}\text{Ar}$$

and other similar processes. As possible intensive sources of neutrinos Pontecorvo suggested reactors and the sun.

1947–1948. Pontecorvo, Puppi, Klein, Tiomno and Wheeler proposed the idea of the $\mu - e$ universality of the weak interaction.

1956. In the Reines and Cowen experiments the (anti)neutrino was discovered. In these experiments antineutrinos from a reactor were detected in a large scintillator counter via the observation of the reaction

$$\bar{\nu} + p \to e^+ + n.$$

1957. In the Davis experiment no production of ^{37}Ar in the process

$$\bar{\nu} +^{37}\text{Cl} \to e^- +^{37}\text{Ar}$$

with antineutrinos from a reactor was observed. This was the first indication in favor of the existence of the conserved lepton number which distinguishes the neutrino and the antineutrino.

1957. In the Wu et al. experiment with polarized ^{60}Co a large effect of the parity violation in the β-decay was discovered.

1957. Landau, Lee and Yang and Salam proposed the theory of the massless two-component neutrino. According to this theory the neutrino is the left-handed (or right-handed) particle and the antineutrino is the right-handed (or left-handed) particle. Under the inversion of the coordinate system the state of the left-handed (right-handed) particle is transformed into the state of the right-handed (left-handed) particle. In the two-component theory in which the neutrino is a particle with definite helicity (projection of the spin on the direction of momentum) parity is violated maximally.

1958. The helicity of the neutrino was determined in the Goldhaber, Grodzins and Sunyar experiment from the measurement of the circular polarization of γ-quanta in the chain of the reactions

$$\begin{array}{lll} e^- + \mathrm{Eu} \to \nu + & \mathrm{Sm}^* & \\ & \downarrow & \\ & \mathrm{Sm} & + \gamma \end{array}$$

It was established that the neutrino is the left-handed particle.

1958. Pontecorvo suggested that neutrinos have small masses, the total lepton number is violated and neutrino oscillations similar to $K^0 \leftrightarrows \bar{K}^0$ oscillations could take place. He considered effects of neutrino oscillations in experiments with reactor antineutrinos.

1958. Feynman and Gell-Mann, Marshak and Sudarshan proposed the current×current theory of the weak interaction. The Hamiltonian of this theory has the form

$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}} j^{\alpha} j^{\dagger}_{\alpha}. \tag{1.2}$$

Here

$$j_\alpha = 2\,(\bar{p}_L \gamma_\alpha n_L + \bar{\nu}_L \gamma_\alpha e_L + \bar{\nu}_L \gamma_\alpha \mu_L) \tag{1.3}$$

is the $\mu - e$ *universal* weak charged current (CC).

1962. In the Brookhaven neutrino experiment, the first experiment with accelerator high-energy neutrinos, it was established that neutrino which take part in CC weak interaction together with electron and neutrino which take part in CC weak interaction together with muon are *different particles*. They were called electron neutrino ν_e and muon neutrino ν_μ. In order to explain the data of the Brookhaven and other experiments it was necessary to introduce two separately conserved lepton numbers: the electron L_e and the muon L_μ. The weak charged current take the form

$$j_\alpha = 2\,(\bar{p}_L\gamma_\alpha n_L + \bar{\nu}_{eL}\gamma_\alpha e_L + \bar{\nu}_{\mu L}\gamma_\alpha \mu_L). \tag{1.4}$$

1962. Maki, Nakagawa and Sakata assumed that neutrinos have small masses and the fields of electron and muon neutrinos are connected with the fields of massive neutrinos ν_1 and ν_2 by the mixing relations

$$\begin{aligned} \nu_{eL} &= \cos\theta\ \nu_{1L} + \sin\theta\ \nu_{2L} \\ \nu_{\mu L} &= -\sin\theta\ \nu_{1L} + \cos\theta\ \nu_{2L}, \end{aligned} \tag{1.5}$$

where θ is a mixing angle.

1967. Glashow (1961), S. Winberg and A. Salam (1967) proposed the unified theory of the electromagnetic and weak interactions (The Standard Model)

1970. In the pioneer experiment by Davis et al solar neutrinos were detected. In this experiment solar ν_e's were detected by the Pontecorvo radiochemical Cl − Ar method via the observation of the reaction

$$\nu_e +^{37}\mathrm{Cl} \to e^- +^{37}\mathrm{Ar}.$$

The threshold of this reaction is 0.81 MeV. Thus, only high-energy solar neutrinos, mainly from the decay $^8\mathrm{B} \to {}^8\mathrm{Be} + e^+ + \nu_e$, were detected in the Davis experiment. The observed rate was 2–3 times smaller that the rate predicted by the Standard Solar Model (SSM). This discrepancy was called *solar neutrino problem.*

1973. In the experiment with high energy accelerator neutrinos at CERN a new class of the weak interaction, the so called neutral currents (NC), was discovered. In addition to CC deep inelastic processes

$$\nu_\mu(\bar{\nu}_\mu) + N \to \mu^-(\mu^+) + X \tag{1.6}$$

also NC processes

$$\nu_\mu(\bar{\nu}_\mu) + N \to \nu_\mu(\bar{\nu}_\mu) + X \tag{1.7}$$

were observed in the bubble chamber "Gargamelle". The discovery of the neutral currents was the decisive confirmation of the Glashow-Weinberg-Salam unified theory of the electroweak interaction.

1980s. In CDHS and CHARM experiments on the study of the deep inelastic scattering of neutrinos and antineutrinos on nucleons (CERN) the quark structure of nucleons was investigated in detail.

1978 Wolfenstein and Mikheev and Smirnov (**1986**) showed that for solar neutrinos, which are born in the central region of the sun and pass a large amount of matter on the way to the earth, matter effects due to *the mixing and coherent scattering on electrons* are important.

1987. Neutrinos from the gravitational collapse of a star were observed for the first time. Neutrinos from the supernova SN1987A in the Large Magellanic Cloud were detected in the Kamiokande, IMB and Baksan detectors.

1988. Solar neutrinos were detected in the experiment Kamiokande. In this experiment the flux of the solar B^8 neutrinos was determined through the detection of the recoil electrons in the scattering process

$$\nu + e \to \nu + e \tag{1.8}$$

The existence of the solar neutrino problem was confirmed.

1988. L. Lederman, M. Schwartz and J. Steinberger were awarded the Nobel Prize for "the discovery of the muon neutrino leading to classification of particles in families".

1991. In the GALLEX and SAGE experiments solar electron neutrinos were detected by the radiochemical method via the observation of the process

$$\nu_e +^{71}\mathrm{Ga} \to e^- +^{71}\mathrm{Ge}. \tag{1.9}$$

Because of the law threshold (0.23 MeV), in the GALLEX and SAGE experiments neutrinos from all reactions of the pp-cicle, including the main reaction $p + p \to d + e^+ + \nu_e$, were detected. The flux of solar neutrinos measured in these experiments was about two times smaller that the flux predicted by the SSM.

1990s. It was proven by experiments on the measurement of the width of the decay $Z \to \nu + \bar{\nu}$ at LEP (CERN) that three flavor neutrinos exist in Nature. This measurement allowed to establish that only three families of leptons and quarks exist.

1995. For the detection of the (anti)neutrino F. Reines was awarded the Nobel Prize.

1998. In the Super-Kamiokande experiment a large azimuth angle asymmetry of high-energy atmospheric muon neutrino events was observed. This was the first model-independent evidence for neutrino oscillations driven by a neutrino mass-squared difference $\Delta m^2_{23} \simeq 2.5 \cdot 10^{-3}\ \mathrm{eV}^2$.

2000. In the experiment DONUT at Fermilab the first direct evidence of the existence of the third neutrino ν_τ was obtained.

2002. In the solar neutrino experiment SNO solar neutrinos were detected through the observation of not only the CC reaction

$$\nu_e + d \to e^- + p + p \tag{1.10}$$

but also the NC reaction

$$\nu + d \to \nu + n + p. \tag{1.11}$$

This experiment solved the solar neutrino problem in a model-independent way: it was proven that solar ν_e's on the way from the cental part of the sun to the earth are transformed into other types of neutrinos.

2002 R. Davis and M. Koshiba were awarded the Nobel Prize for "pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos".

2002. In the reactor experiment KamLAND, $\bar{\nu}_e$'s from 57 reactors in Japan were detected through the observation of the reaction

$$\bar{\nu}_e + p \to e^+ + n \tag{1.12}$$

The average distance between reactors and detector was about 180 km. In this experiment, a model-independent evidence for neutrino oscillations driven by a neutrino mass-squared difference $\Delta m^2_{12} \simeq 8 \cdot 10^{-5}\,\mathrm{eV}^2$ was obtained.

2004. In the long-baseline accelerator neutrino experiment K2K the evidence for neutrino oscillations obtained in the atmospheric neutrino experiment Super-Kamiokande was confirmed. In this experiment, neutrinos from the accelerator at KEK were detected by the Super-Kamiokande detector at a distance of about 250 km.

2006. In the long-baseline neutrino experiment MINOS, the Super-Kamiokande atmospheric neutrino evidence for neutrino oscillations was additionally confirmed. In the MINOS experiment, neutrinos from the accelerator at Fermilab were detected by the detector in the Soudan mine at a distance of 735 km. An accuracy of about 10% in the measurement of the mass-squared difference Δm^2_{23} was achieved.

2007. A new solar neutrino experiment BOREXino started. In this experiment monochromatic ^{7}Be solar neutrinos with the energy 0.86 MeV were detected in real time.

2010. The observation of neutrino oscillations in the atmospheric Super-Kamiokande, solar SNO, reactor KamLAND and other neutrino oscillation experiments is *the most important recent discovery in particle physics*. Small neutrino masses cannot be naturally explained by the standard Higgs mechanism of mass generation. New physics and a new mechanism of neutrino mass generation beyond the Standard Model are required. To reveal new physics various high-precision experiments (T2K, Double CHOOZ, Daya Bay, RENO, NOvA and others) started or will be started during the next years.

Chapter 2
Weak Interaction Before the Standard Model

All existing present data are in perfect agreement with the unified theory of the electromagnetic and weak interactions (Standard Model). Before this theory was created, there was a long phenomenological period of the development of the theory of the weak interaction. In this introductory chapter we will briefly consider this period.

2.1 Pauli Hypothesis of Neutrino

The only weak process which was known in the twenties and thirties was the β-decay of nuclei. In 1914 Chadwick discovered that the energy spectrum of electrons from β-decay is continuous. If β-decay is a process of the transition of a nucleus (A,Z) into a nucleus (A,Z+1) and the electron (as it was believed at that time), from conservation of energy and momentum follows that the electron must have a fixed kinetic energy approximately equal to $Q \simeq m_{A,Z} - m_{A,Z+1} - m_e$ (where $m_{A,Z}$ ($m_{A,Z+1}$) is the mass of the initial (final) nucleus and m_e is the mass of the electron).

For many years continuous β spectra were interpreted as the result of the loss of energy of electrons in the target. However, in 1927 Ellis and Wooster performed a crucial calorimetric β-decay experiment. They measured the total energy released in a RaE (^{210}Bi) source which was put inside of a calorimeter. For the β-decay of ^{210}Bi the total energy release is $Q = 1.05$ MeV. In the Ellis and Wooster experiment it was found that the average energy per one β-decay is equal to (344 $\pm$ 34) KeV which is in an agreement with the average energy of the electrons (390 KeV). Thus, it was demonstrated that continuous β spectra cannot be explained by the energy loss of electrons in a target.

There were two possibilities to explain this experimental data

1. To assume that in β-decay together with the electron a neutral penetrating particle, which is not detected in experiments, is produced. The total released energy is shared between the electron and the new particle. As a result, electrons produced in β-decay, will have a continuous spectrum
2. To assume that in β-decay the energy is not conserved.

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The idea of new particle was proposed by W. Pauli. The second point of view was advocated by N. Bohr.

Pauli wrote about his idea in a letter to Geiger and Meitner who participated in a nuclear conference at Tübingen (December 4, 1930). Pauli asked Geiger and Meitner to inform the participants of the conference on his proposal.

Pauli called the new particle "neutron". He assumed that the "neutron" has spin 1/2, small mass (of the same order of magnitude as the mass of the electron) and large penetration length. Pauli assumed that the "neutron" is emitted together with the electron in the β-decay of nuclei. Later E. Fermi and E. Amaldi proposed to call the Pauli particle neutrino (from Italian, *neutral, small*).

Below there is Pauli's letter translated into English.

Dear Radioactive Ladies and Gentlemen,

As the bearer of these lines, to whom I graciously ask you to listen, will explain to you in more detail, how because of the "wrong" statistics of the N and Li6 nuclei and the continuous beta spectrum, I have hit upon a desperate remedy to save the "exchange theorem" of statistics and the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin 1/2 and obey the exclusion principle and which further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. The continuous beta spectrum would then become understandable by the assumption that in beta decay a neutron is emitted in addition to the electron such that the sum of the energies of the neutron and the electron is constant.

I agree that my remedy could seem incredible because one should have seen those neutrons very earlier if they really exist. But only the one who dare can win and the difficult situation, due to the continuous structure of the beta spectrum, is lighted by a remark of my honored predecessor, Mr Debye, who told me recently in Brussels: "Oh, It's well better not to think to this at all, like new taxes". From now on, every solution to the issue must be discussed. Thus, dear radioactive people, look and judge. Unfortunately, I cannot appear in Tübingen personally since I am indispensable here in Zürich because of a ball on the night of 6/7 December. With my best regards to you, and also to Mr Back.
Your humble servant W. Pauli

At the time when Pauli proposed the idea of the existence of the "neutron", nuclei were considered as bound states of protons and electrons. As it is seen from Pauli's letter he assumed that his new particle "exists in the nuclei". This assumption allowed him to solve the problem of the spin-statistic theorem for some nuclei. Let us consider the nucleus $^{7}N_{14}$. According to the proton-electron model this nucleus is a bound state of 14 protons and 7 electrons. Because spins of protons and electrons are equal to 1/2 the spin of $^{7}N_{14}$ must be half-integer. However, from the analysis of the spectrum of $^{7}N_{14}$ molecules it was found that nucleus $^{7}N_{14}$ satisfies Bose–Einstein statistics and, according to the spin-statistic theorem, the spin of the this nucleus must be integer. An odd number of "neutrons" in $^{7}N_{14}$ would make its spin integer.

After the discovery of the neutron (Chadwick, 1932) E. Majorana, W. Heisenberg and D. Ivanenko assumed that the constituents of nuclei are protons and neutrons. This assumption (which, as we know today, is the correct one) explained all nuclear data.

The problem of the spin of $^7N_{14}$ and other nuclei disappeared. What about β-decay and continuous β-spectrum? This problem was solved by quantitatively E. Fermi in 1934 on the basis of Pauli's hypothesis of the neutrino in the framework of the proton-neutron model of nuclei.

2.2 Fermi Theory of β-Decay

Fermi proposed the first Hamiltonian of the β-decay. He assumed that electron-neutrino pair *is produced in the transition of a neutron into a proton*

$$n \to p + e + \nu. \tag{2.1}$$

Fermi built the Hamiltonian of the process (2.1) in analogy with the Hamiltonian of the electromagnetic interaction.

The Hamiltonian of the electromagnetic interaction has the form of a scalar product of the electromagnetic current and the electromagnetic field $A^{\alpha}(x)$. For the Hamiltonian of the electromagnetic interaction of protons we have

$$\mathcal{H}_I^{\mathrm{EM}}(x) = e j_{\alpha}^{\mathrm{EM}} A^{\alpha}(x), \tag{2.2}$$

where e is the electric charge of the proton and the electromagnetic current is given by the expression

$$j_{\alpha}^{\mathrm{EM}}(x) = \bar{p}(x)\gamma_{\alpha} p(x) \tag{2.3}$$

where $p(x)$ is the proton field, $\bar{p}(x) = p^{\dagger}(x)\gamma^0$ and γ_{α} are the Dirac matrices.

Fermi suggested that the Hamiltonian of the process (2.1) is the product of the neutron-proton current

$$j_{\alpha}^{\mathrm{CC}}(x) = \bar{p}(x)\gamma_{\alpha} n(x), \tag{2.4}$$

which provides the transition $n \to p$ and a vector which provides the emission of the electron-neutrino pair. Assuming that there are no derivatives of the fields in the Hamiltonian, Fermi came to the following expression for the Hamiltonian of the β-decay

$$\mathcal{H}_I^{\beta}(x) = G_F\ \bar{p}(x)\gamma_{\alpha} n(x)\ \bar{e}(x)\gamma^{\alpha}\nu(x) + \mathrm{h.c.}, \tag{2.5}$$

where G_F is an interaction constant.

Let us stress an important difference between the Hamiltonians (2.2) and (2.5). The Hamiltonian (2.2) describes the interaction of two fermions and a boson while the Hamiltonian (2.5) describes the interaction of four fermions. As a consequence of that, the Fermi constant G_F and the electromagnetic charge e have *different dimensions*. In the system of the units $\hbar = c = 1$, which we are using, e is a dimensionless quantity whereas the Fermi constant G_F has the dimension $[M]^{-2}$. We will return to a discussion of this point later.

The Fermi Hamiltonian (2.5) allowed to describe only such β-decays, in which spins and parities of the initial and final nuclei are the same (Fermi selection rule)

$$\Delta I = 0 \quad \pi_i = \pi_f$$

However, it was also observed such β-decays of nuclei which satisfy the Gamov-Teller selection rule:

$$\Delta I = \pm 1,\ 0 \quad \pi_i = \pi_f,$$

The observation of such decays meant that in addition to the Fermi Hamiltonian the total Hamiltonian of the β-decay must include additional terms.

2.3 Fermi-Gamov-Teller Hamiltonian of β-Decay

The Fermi Hamiltonian is the product of vector×vector term. If we assume that in the Hamiltonian of the β-decay there are no derivatives of the fields, for *the most general four-fermion Hamiltonian* we obtain the sum of the products of scalar (S), vector (V), tensor (T), pseudovector (A) and pseudoscalar (P) terms:

$$\mathcal{H}_I^{\beta}(x) = \sum_{i=S,V,T,A,P} G_i\ \bar{p}(x)\ O_i n(x)\ \bar{e}(x)\ O^i \nu(x) + \text{h.c.} \tag{2.6}$$

Here

$$O \to 1,\ \gamma_\alpha,\ \sigma_{\alpha\beta},\ \gamma_\alpha\gamma_5,\ \gamma_5\,. \tag{2.7}$$

and G_i are coupling constants, which have the dimensions $[M]^{-2}$. This Hamiltonian was proposed by Gamov and Teller in 1936. The Hamiltonian (2.6) could describe all β-decay data. Transitions, which satisfy the Fermi selection rules, are due to V and S terms and transitions, which satisfy the Gamov-Teller selection rules, are due to A and T terms.

The Hamiltonian (2.6) contains five arbitrary coupling constants G_i. It was, however, general belief that the number of the fundamental constants in the Hamiltonian of the β-decay must be smaller. For many years the aim of experiments on the investigation of the β-decay was to find the dominant terms in the Hamiltonian (2.6). The situation was, however, uncertain until 1957. The data on the measurements of the β-spectra were in favor of the combination of S and T terms or V and A terms. However, the measurements of the electron-neutrino angular correlation, which could distinguish these two possibilities, gave contradictory results. In 1957–1958 understanding of the β-decay and other weak processes drastically changed. This was connected with the discovery of the nonconservation of parity in the weak interaction.

The Fermi and Gamov-Teller Hamiltonians are invariant under space inversion, i.e. these Hamiltonians conserve parity. There was a general belief at that time that parity is conserved in all interactions. However, from the study of the weak decays of kaons in the fifties, there were indications that this assumption was incorrect.

2.4 Violation of Parity in β-Decay

In 1956 Lee and Yang analyzed existing experimental data and came to the conclusion that there were no data contradicting the assumption that parity is not conserved in the weak interaction. They proposed different experiments that would check this possibility.

The first experiment, in which violation of parity was discovered was performed by Wu et al. in 1957. In this experiment the dependence of the probability of the β-decay of the polarized nuclei Co^{60} on the angle between the (pseudo)vector of the polarization and the vector of the momentum of the electron was measured. From the invariance under rotation (conservation of the angular momentum) for the probability of the emission of the electron with momentum $\mathbf{p}$ by a nucleus with the polarization $\mathbf{P}$ we have the following general expression

$$w_{\mathbf{P}}(\mathbf{k}) = w_0(1 + \alpha\, \mathbf{P} \cdot \mathbf{k}) = w_0(1 + \alpha P \cos\theta), \tag{2.8}$$

where $\mathbf{k} = \frac{\mathbf{p}}{p}$ is the unit vector in the direction of the momentum of the electron and α is the asymmetry parameter. If the parity is conserved we have

$$w_{\mathbf{P}}(\mathbf{k}) = w_{\mathbf{P}}(-\mathbf{k}). \tag{2.9}$$

Therefore, in this case the pseudoscalar term in the probability (2.8) must be equal to zero ($\alpha = 0$). In the experiment Wu et al. was found that $|\alpha|$ is not equal to zero and large ($\alpha \simeq -0.7$). Thus, it was proven that in the β-decay parity is not conserved.

From the discovery of the nonconservation of parity followed that the Hamiltonian of the β-decay is the sum of scalar and pseudoscalar terms. The most general four-fermion Hamiltonian which does not conserve parity was proposed by Lee and Yang. It has the form

$$\mathcal{H}_I^{\beta}(x) = \sum_{i=S,V,T,A,P} \bar{p}(x)\; O_i n(x)\; \bar{e}(x)\; O^i (G_i - G_i^{'}\gamma_5)\nu(x) + \text{h.c.} \tag{2.10}$$

The constants G_i and $G_i^{'}$ characterize the scalar and pseudoscalar terms of the Hamiltonian. The Wu et al. experiment suggested that the constants G_i and G_i' are of the same order.

The interaction (2.10) is characterized by 10 (!) coupling constants. *In 1957–1958 there were two fundamental steps which brought us to the modern effective Hamiltonian of the β-decay and other weak processes.*

2.5 Two-Component Neutrino Theory

The first step was the theory of the massless two-component neutrino, proposed by Landau, Lee and Yang and Salam.

A method of the measurement of the neutrino mass was proposed by Fermi and Perrin in 1934 . This method is based on the measurement of the high-energy part of β-spectrum in which the neutrino has a small energy. At the time of the discovery of the violation of parity, from experiments on the measurement of the neutrino mass m was found the following upper bound: $m \lesssim 200$ eV. Thus, it was found that neutrino mass is much smaller than the mass of the electron. The authors of the two-component neutrino theory assumed that neutrino mass is equal to zero.

Any fermion field can be presented in the form of the sum of left-handed and right-handed components. We have

$$\nu(x) = \nu_L(x) + \nu_R(x), \tag{2.11}$$

where

$$\nu_{L,R}(x) = \frac{1 \mp \gamma_5}{2} \nu(x) \tag{2.12}$$

are left-handed and right-handed components of the field $\nu(x)$. Notice that $\nu_L(x)$ and $\nu_R(x)$ have the same Lorenz-transformation properties as $\nu(x)$.

The authors of the two-component neutrino theory assumed that

$$\text{neutrino field is } \ \nu_L(x) \ \ (\text{or } \ \nu_R(x)).$$

This is possible if the neutrino mass is equal to zero. In fact, the mass term of the neutrino with mass m has the form

$$\mathcal{L}^m(x) = -m \ \bar{\nu}(x)\nu(x) = -m \ (\bar{\nu}_L(x)\nu_R(x) + \bar{\nu}_R(x)\nu_L(x)) \tag{2.13}$$

If the neutrino field is $\nu_L(x)$ (or $\nu_R(x)$) in this case the mass term (2.13) cannot be built (and, consequently, $m = 0$)[1]

If we assume that the neutrino field is $\nu_L(x)$ (or $\nu_R(x)$) in this case

[1] This is correct for the Dirac neutrino. As we will see later for the Majorana neutrino the mass term can be built also in the case of a $\nu_L(x)$ (or $\nu_R(x)$) field.

1. $G_i = G'_i$ (or $G_i = -G'_i$) and the parity in the β-decay is violated maximally. This corresponds to the results of the Wu et al. experiment.
2. The neutrino helicity is equal to -1 $(+1)$ and the antineutrino helicity is equal to $+1$ (-1).

In fact, for the neutrino field $\nu(x)$ we have the following expansion

$$\nu(x) = \int N_p \left(\sum_{r=\pm 1} u^r(p) c_r(p)\, e^{-ipx} + \sum_{r=\pm 1} u^r(-p) d_r^\dagger(p)\, e^{ipx} \right) d^3 p. \quad (2.14)$$

Here $c_r(p)$ $(d_r^\dagger(p)$ is the operator of the absorption of a neutrino (creation of an antineutrino) with momentum p and helicity r and $N_p = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}}$ is the normalization factor.

The Dirac equation for the massless neutrino has the form

$$\not{p}\, u^r(p) = 0, \quad (2.15)$$

where $\not{p} = \gamma_\alpha p^\alpha$. The spinor $u^r(p)$ describes a particle with helicity equal to $r(r = \pm 1)$. We have

$$\Sigma \cdot \mathbf{k}\, u^r(p) = r\ u^r(p), \quad (2.16)$$

where Σ is the operator of the spin and $\mathbf{k}$ is the unit vector in the direction of the momentum $\mathbf{p}$. For the operator of the spin we have

$$\Sigma = \gamma_5 \alpha = \gamma_5 \gamma^0 \gamma. \quad (2.17)$$

From (2.15) and (2.17) we have

$$\Sigma \cdot \mathbf{k}\, u^r(p) = \gamma_5\, u^r(p). \quad (2.18)$$

Thus, for a massless particle operator γ_5 is the operator of the helicity. From (2.16) we find

$$\gamma_5\, u^r(p) = r u^r(p). \quad (2.19)$$

Similarly, for the spinor $u^r(-p)$ which describes the state with negative energy $-p^0$ and momentum $-\mathbf{p}$ we have

$$\gamma_5\ u^r(-p) = -r\ u^r(-p). \quad (2.20)$$

From (2.19) we find that $\frac{1-\gamma_5}{2}$ is the projection operator:

$$\frac{1-\gamma_5}{2}\,u^{-1}(p) = u^{-1}(p), \qquad \frac{1-\gamma_5}{2}\,u^{1}(p) = 0. \tag{2.21}$$

From (2.20) we have

$$\frac{1-\gamma_5}{2}\,u^{1}(-p) = u^{1}(-p), \qquad \frac{1-\gamma_5}{2}\,u^{-1}(-p) = 0. \tag{2.22}$$

From these relations for the left-handed neutrino field we find

$$\nu_L(x) = \int N_p \left(u^{-1}(p)\,c_{-1}(p)\,e^{-ipx} + u^{1}(-p)\,d_1^{\dagger}(p)\,e^{ipx}\right) d^3p. \tag{2.23}$$

Analogously, for the right-handed neutrino field we have

$$\nu_R(x) = \int N_p \left(u^{1}(p)\,c_{1}(p)\,e^{-ipx} + u^{-1}(-p)\,d_{-1}^{\dagger}(p)\,e^{ipx}\right) d^3p. \tag{2.24}$$

The neutrino helicity was measured in 1958 in a spectacular experiment by Goldhaber, Grodzins and Sunyar. In this experiment the helicity of the neutrino was determined from the measurement of the circular polarization of γ-quanta in the chain of reactions

$$\begin{array}{l} e^- + \mathrm{Eu} \to \nu + \mathrm{Sm}^* \\ \qquad\qquad\qquad\quad\ \downarrow \\ \qquad\qquad\qquad\quad \mathrm{Sm} \ + \gamma \end{array} \tag{2.25}$$

It was found that the helicity of neutrino is negative:

$$h = -1 \pm 0.3 \tag{2.26}$$

The experiment by Goldhaber et al. confirmed the theory of the two-component neutrino. It was established that the neutrino is the left-handed particle and the neutrino field is $\nu_L(x)$.[2]

2.6 μ-e Universal Charged Current. Current × Current Theory

The next decisive step in the construction of the Hamiltonian of the β-decay and other weak processes was done by Feynman and Gell-Mann, Marshak and Sudarshan in 1957–1958. Generalizing the theory of the two-component neutrino,

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[2] Let us stress that the experiment by Goldhaber et al. does not exclude that the neutrino has a small mass. In fact, if in the Hamiltonian of the β-decay enters $\nu_L(x)$ and the neutrino mass is not equal to zero in this case the longitudinal polarization of the neutrino for $m \ll E$ is equal to $P_{\parallel} \simeq -1 + \frac{m^2}{2E^2} \simeq -1$.

Feynman and Gell-Mann, Marshak and Sudarshan assumed that *in the Hamiltonian of the weak interaction enter only left-handed components of fields*. In this case the most general four-fermion Hamiltonian of the β-decay has the form

$$\mathcal{H}_I^{\beta} = \sum_{i=S,V,T,A,P} G_i \; \bar{p}_L O_i n_L \; \bar{e}_L O^i \nu_L + \text{h.c.}, \tag{2.27}$$

where O_i are Dirac matrices (see (2.7)).

We have

$$\bar{e}_L O_i \nu_L = \bar{e} \frac{1+\gamma_5}{2} O_i \frac{1-\gamma_5}{2} \nu . \tag{2.28}$$

It is obvious that

$$\frac{1+\gamma_5}{2} \left(1;\ \sigma_{\alpha\beta};\ \gamma_5\right) \frac{1-\gamma_5}{2} = 0 . \tag{2.29}$$

Therefore, S, T and P terms do not enter into the Hamiltonian (2.27). Moreover A and V terms are connected by the relation:

$$\frac{1+\gamma_5}{2} \gamma_\alpha \gamma_5 \frac{1-\gamma_5}{2} = -\frac{1+\gamma_5}{2} \gamma_\alpha \frac{1-\gamma_5}{2} . \tag{2.30}$$

Thus, if we assume that only left-handed components of the fields enter into the four-fermion Hamiltonian , we come to the unique expression for the Hamiltonian of the β-decay

$$\begin{aligned} \mathcal{H}_I^{\beta} &= \frac{G_F}{\sqrt{2}} 4 \, \bar{p}_L \gamma_\alpha n_L \; \bar{e}_L \gamma^\alpha \nu_L + \text{h.c.} \\ &= \frac{G_F}{\sqrt{2}} \, \bar{p} \gamma_\alpha (1-\gamma_5) n \, \bar{e} \gamma^\alpha (1-\gamma_5) \nu + \text{h.c.} \end{aligned} \tag{2.31}$$

The Hamiltonian (2.31) is the simplest possible four-fermion Hamiltonian of the β-decay which takes into account large violation of parity. Like the Fermi Hamiltonian (2.5), it is characterized by only one interaction constant.[3]

The theory proposed by Feynman and Gell-Mann, Marshak and Sudarshan was a very successful one: the Hamiltonian (2.31) allowed to describe all existing β-decay data. We know today that (2.31) is the correct effective Hamiltonian of the β-decay, of the process $\bar{\nu} + p \to n + e^+$, and other connected processes.

Until now we have only considered the Hamiltonian of the β-decay. At the time when parity violation was discovered other weak processes involving a muon-neutrino pair were known:

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[3] In order to keep the numerical value of the Fermi constant the coefficient $\frac{1}{\sqrt{2}}$ was introduced in (2.31).

$$\mu^- + (A, Z) \to \nu + (A, Z-1) \quad (\mu - \text{capture}) \tag{2.32}$$

$$\mu^+ \to e^+ + \nu + \bar{\nu} \quad (\mu - \text{decay}). \tag{2.33}$$

In 1947 B. Pontecorvo suggested the existence of a *$\mu - e$ universal weak interaction*, which is characterized by the same Fermi constant G_F. He compared the probability of the μ-capture (2.32) with the probability of the K-capture

$$e^- + (A, Z) \to \nu + (A, Z-1) \tag{2.34}$$

and found that the constant of the interaction of the muon-neutrino pair with nucleons is of the same order as the Fermi constant G_F. The idea of a $\mu - e$ universal weak interaction was also proposed by Puppi, Klein, Tiomno and Wheeler.

In order to build a $\mu - e$ universal theory of the weak interaction, Feynman and Gell-Mann introduced the notion of *the charged weak current*[4]

$$j^\alpha = 2\,(\bar{p}_L \gamma^\alpha n_L \; + \; \bar{\nu}_{\mu L} \gamma^\alpha e_L \; + \; \bar{\nu}_{eL} \gamma^\alpha \mu_L) \tag{2.35}$$

and assumed that the Hamiltonian of the weak interaction has the current$\times$current form[5]

$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}}\, j^\alpha \; j^+_\alpha, \tag{2.36}$$

where G_F is the Fermi constant.

There are two types of terms in the Hamiltonian (2.36): nondiagonal and diagonal. The nondiagonal terms are given by

$$\begin{aligned} \mathcal{H}_I^{nd} = \frac{G_F}{\sqrt{2}} 4\,\{&[(\bar{p}_L \gamma^\alpha n_L)(\bar{e}_L \gamma_\alpha \nu_{eL}) \; + \text{h.c.}] \; + \\ &[(\bar{p}_L \gamma^\alpha n_L)(\bar{\mu}_L \gamma_\alpha \nu_{\mu L}) \; + \text{h.c.}] \; + \\ &\quad [(\bar{e}_L \gamma^\alpha \nu_{eL})(\bar{\nu}_{\mu L} \gamma_\alpha \mu_L) \; + \text{h.c.}]\} \end{aligned} \tag{2.37}$$

The first term of this expression is the Hamiltonian of β-decay of the neutron (2.1), of the process $\bar{\nu}_e + p \to e^+ + n$ and other connected processes. The second term of (2.37) is the Hamiltonian of the process $\mu^- + p \to \nu_\mu + n$ and other connected processes. Finally the third term of (2.37) is the Hamiltonian of the μ-decay (2.33) and other processes.

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[4] We denoted the fields of neutrinos which enter into the current together with the fields of electron and muon, correspondingly, by ν_e and ν_μ. At this point, this is simply a notation. We will see later that in fact ν_e and ν_μ are different particles.

[5] The current j^α changes the charge by one. This is the reason, why this current is called a charged current. Notice also that the hadron current has the form $j^\alpha = v^\alpha - a^\alpha$, where $v^\alpha = \bar{p}\gamma^\alpha n$ and $a^\alpha = \bar{p}\gamma^\alpha \gamma_5 n$ are vector and axial currents. The Feynman-Gell-Mann theory is called the $V - A$ theory.

The diagonal terms of the Hamiltonian (2.36) are given by

$$\mathcal{H}^d = \frac{G_F}{\sqrt{2}}\, 4[(\bar{\nu}_{eL}\gamma^{\alpha} e_L)(\bar{e}_L\gamma_{\alpha}\nu_{eL}) + (\bar{\nu}_{\mu L}\gamma^{\alpha}\mu_L)(\bar{\mu}_L\gamma_{\alpha}\nu_{\mu L}) + (\bar{p}_L\gamma^{\alpha} n_L)(\bar{n}_L\gamma_{\alpha} p_L)] \tag{2.38}$$

The first term of the (2.38) is the Hamiltonian of the processes of elastic scattering of neutrino and antineutrino on an electron

$$\nu_e + e \to \nu_e + e \tag{2.39}$$

and

$$\bar{\nu}_e + e \to \bar{\nu}_e + e, \tag{2.40}$$

of the process $e^+ + e^- \to \bar{\nu}_e + \nu_e$ and other processes. Such processes were not known in the fifties. Their existence was predicted by the current×current theory.

The cross sections of the processes (2.39) and (2.40) are very small. The observation of such processes was a challenge. After many years of efforts, the cross section of the process (2.40) was measured by F. Reines et al. in an experiment with antineutrinos from a reactor. At that time the Standard Model already existed. According to the Standard Model, to the matrix elements of the processes (2.39) and (2.40) contributes also an additional (neutral current) Hamiltonian. The result of the experiment by F. Reines et al. was in agreement with the Standard Model.

2.7 Theory with Vector *W* Boson

Feynman and Gell-Mann considered a possible origin of the current×current interaction (2.36). Let us assume that a charged vector boson $W^{\pm}$ exists and the Lagrangian of the weak interaction (analogously to the Lagrangian of the electromagnetic interaction) has the form of a scalar product of the current and the vector field

$$\mathcal{L}_{\mathcal{I}} = -\frac{g}{2\sqrt{2}}\, j_{\alpha}\, W^{\alpha} + \text{h.c.} \tag{2.41}$$

where g is a dimensionless constant and the current j_{α} is given by Eq. (2.35).

In the theory with W-boson Feynman diagram of the β-decay of the neutron is presented in Fig. 2.1. If

$$Q^2 \ll m_W^2 \tag{2.42}$$

where Q is the momentum of the virtual W-boson and m_W is the mass of W-boson, the matrix element of the β-decay of the neutron can be obtained from the Hamiltonian (2.36) in which the Fermi constant is given by the relation

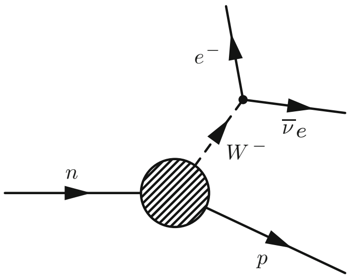

Fig. 2.1 Feynman diagram of the process $n \to p + e^- + \bar{\nu}$ in the theory with the $W^{\pm}$-boson

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\, m_W^2}. \tag{2.43}$$

It is easy to verify that in the theory with W-boson the effective Hamiltonian of all weak processes with the virtual W-boson and $Q^2 \ll m_W^2$ has current$\times$current form (2.36), in which the Fermi constant is given by the relation (2.43).

Thus, the theory with a charged vector W-boson could explain the current$\times$current structure of the weak interaction Hamiltonian and the fact that the Fermi constant has the dimension $[M]^{-2}$. As we will see later, (2.41) is a part of the total Lagrangian of the electroweak interaction of the Standard Model.[6]

2.8 First Observation of Neutrinos. Lepton Number Conservation

The first proof of the existence of neutrino was obtained in the mid-fifties in the experiment by F. Reines and C.L. Cowan. In this experiment (anti)neutrinos from the Savannah River reactor were detected through the observation of the process

$$\bar{\nu}_e + p \to e^+ + n. \tag{2.44}$$

Antineutrinos are produced in a reactor in a chain of β-decays of neutron-rich nuclei, products of the fission of uranium and plutonium. The energies of antineutrinos from a reactor are less than approximately 10 MeV. About $2.3 \cdot 10^{20}$ antineutrinos per second were emitted by the Savannah River reactor. The flux of $\bar{\nu}_e$'s in the Reines and Cowan experiment was about $10^{13}\,\mathrm{cm}^{-2}\, s^{-1}$.

In the theory of the two-component neutrino, the cross section of the process (2.44) is connected with the life-time τ_n of the neutron by the relation

[6] It is interesting to note that the idea of the charged vector W-boson was proposed by O. Klein at the end of the thirties.

$$\sigma(\bar{\nu}_e p \to e^+ n) = \frac{2\pi^2}{m_e^5 f \tau_n} p_e E_e, \tag{2.45}$$

where $E_e \simeq E_{\bar{\nu}} - (m_n - m_p)$ is the energy of the positron, p_e is the positron momenta, f=1.686 is the phase-space factor, m_n, m_p, m_e are masses of the neutron, proton and electron, respectively. From (2.45) for the cross section of the process (2.44), averaged over antineutrino spectrum, the value

$$\bar{\sigma}(\bar{\nu}_e p \to e^+ n) \simeq 9.5 \cdot 10^{-44} \text{ cm}^2 \tag{2.46}$$

was found.

A liquid scintillator ($1.4 \cdot 10^3$ l) loaded with $CdCl_2$ was used as a target in the experiment. Positron, produced in the process (2.44), slowed down in the scintillator and annihilated with electron, producing two γ- quanta with energies $\simeq$ 0.51 MeV and opposite momenta. A neutron, produced in the process was captured by Cd within about 5 μs, producing γ-quantum. The γ-quanta were detected by 110 photomultipliers. Thus, the signature of the $\bar{\nu}$-event in the Reines and Cowan experiment was two γ-quanta from the $e^+ - e^-$-annihilation in coincidence with a delayed γ-quantum from the neutron capture by cadmium. For the cross section of the process (2.44) the value

$$\sigma_\nu = (11 \pm 2.6)\ 10^{-44} \text{ cm}^2 \tag{2.47}$$

was obtained in the experiment. This value is in agreement with the predicted value (2.46).

The particle which is produced in the β-decay together with electron is called antineutrino. It is a direct consequence of the quantum field theory that antineutrino can produce a positron in the inverse β-decay (2.44) and other similar processes. Can antineutrinos produce electrons in weak processes of interaction with nucleons? The answer to this question was obtained from an experiment which was performed in 1956 by Davis et al. with antineutrinos from the Savannah River reactor. In this experiment ^{37}Ar from the process

$$\bar{\nu} +^{37} \text{Cl} \to e^- +^{37} \text{Ar} \tag{2.48}$$

was searched for. The process (2.48) was not observed in the experiment. It was shown that the ^{37}Ar production rate was about five times smaller than the rate expected if antineutrinos could produce electrons via the weak interaction.

Thus, it was established that antineutrinos from a reactor can produce positrons (the Reines-Cowan experiment) but cannot produce electrons. In order to explain this fact we assume that exist *conserving lepton number (charge)* L, the same for $\bar{\nu}$ and e^+. Let us put $L(\bar{\nu}) = L(e^+) = -1$. According to the quantum field theory the lepton charges of the corresponding antiparticles are opposite: $L(\nu) = L(e^-) = 1$. We also assume that the lepton numbers of proton, neutron and other hadrons are equal to zero. Conservation of the lepton number explain the negative result of the

Davis experiment. According to the law of conservation of the lepton number a neutrino is produced together with e^+ in the β^+-decay

$$(A, Z) \to (A, Z-1) + e^+ + \nu \tag{2.49}$$

2.9 Discovery of Muon Neutrino. Electron and Muon Lepton Numbers

In the expression (2.35) the fields of neutrinos, which enter into the charged current together with electron and muon fields, were denoted by ν_e and ν_μ, correspondingly. *Are ν_e and ν_μ the same or different particles?* The answer to this fundamental question was obtained in the famous Brookhaven neutrino experiment in 1962.

The first indication that ν_e and ν_μ are different particles was obtained from an analysis of the $\mu \to e\gamma$ data. The probability of the decay $\mu \to e\gamma$ was calculated by Feinberg in the theory with W-boson and a cutoff. It was found that if ν_e and ν_μ are identical particles and the cut-off is given by the mass of the W-boson the ratio R of the probability of the decay $\mu \to e\gamma$ to the probability of the decay $\mu \to e\,\nu\,\bar{\nu}$ is given by

$$R \simeq \frac{\alpha}{24\pi} \simeq 10^{-4} \tag{2.50}$$

The decay $\mu \to e\gamma$ was not observed in experiment. At the time of the Brookhaven experiment, for the upper bound of the ratio R was found the value

$$R < 10^{-8}, \tag{2.51}$$

which is much smaller than (2.50).

A direct proof of the existence of the second (muon) type of neutrino was obtained by L.M. Lederman, M. Schwartz, J. Steinberger et al. in the first experiment with accelerator neutrinos in 1962. The idea of the experiment was proposed by B.Pontecorvo in 1959.

A beam of π^+'s in the Brookhaven experiment was obtained by the bombardment of Be target by protons with an average energy of about 15 GeV. In the decay channel (about 21 m long) practically all π^+'s decay. After the channel there was shielding (13.5 m of iron), in which charged particles were absorbed. After the shielding there was the neutrino detector (aluminium spark chamber, 10 tons) in which the production of charged leptons was observed.

The dominant decay channel of the π^+-meson is

$$\pi^+ \to \mu^+ + \nu_\mu. \tag{2.52}$$

According to the universal $V - A$ theory, the ratio R of the width of the decay

$$\pi^+ \to e^+ + \nu_e \tag{2.53}$$

to the width of the decay (2.52) is equal to

$$R = \frac{m_e^2}{m_\mu^2}\,\frac{(1-\frac{m_e^2}{m_\pi^2})^2}{(1-\frac{m_\mu^2}{m_\pi^2})^2} \simeq 1.2 \cdot 10^{-4}. \tag{2.54}$$

Thus, the decay $\pi^+ \to e^+ \nu_e$ is strongly suppressed with respect to the decay $\pi^+ \to \mu^+ + \nu_\mu$.[7] From (2.54) follows that the neutrino beam in the Brookhaven experiment was practically a pure ν_μ beam (with a small about 1% admixture of ν_e from decays of muons and kaons).

Neutrinos, emitted in the decay (2.52), produce μ^- in the process

$$\nu_\mu + N \to \mu^- + X. \tag{2.55}$$

If ν_μ and ν_e would be the same particles, neutrinos from the decay (2.52) would produce also e^- in the reaction

$$\nu_\mu + N \to e^- + X. \tag{2.56}$$

Due to the $\mu - e$ universality of the weak interaction one could expect to observe in the detector practically equal number of muons and electrons.

In the Brookhaven experiment 29 muon events were detected. The observed six electron candidates could be explained by the background. The measured cross section was in agreement with the $V - A$ theory. Thus, it was proved that *ν_μ and ν_e are different particles.*[8]

The results of the Brookhaven and other experiments suggested that the total electron L_e and muon L_μ lepton numbers are conserved:

$$\sum_i L_e^{(i)} = \text{const}; \quad \sum_i L_\mu^{(i)} = \text{const} \tag{2.57}$$

[7] The reason for this suppression can be easily understood. Indeed, let us consider the decay (2.53) in the rest frame of the pion. The helicity of the neutrino is equal to -1. If we neglect the mass of the e^+, the helicity of the positron will be equal to $+1$ (the helicity of the positron will be the same in this case as the helicity of the antineutrino). Thus, the projection of the total angular momentum on the neutrino momentum will be equal to -1. The spin of the pion is equal to zero and consequently the process (2.53) in the limit $m_e \to 0$ is forbidden. These arguments explain the appearance of the small factor $(\frac{m_e}{m_\mu})^2$ in (2.54).

[8] In 1963 in the CERN with the invention of the magnetic horn the intensity and purity of neutrino beams were greatly improved. In the more precise 45 tons spark-chamber experiment and in the large bubble chamber experiment the Brookhaven result was confirmed.

Table 2.1 Lepton numbers of particles

Lepton number	ν_e e^-	ν_μ μ^-	Hadrons, γ
L_e	1	0	0
L_μ	0	1	0

The lepton numbers of particles are given in Table 2.1. The lepton numbers of antiparticles are opposite to the lepton numbers of the corresponding particles.

For many years all experimental data were in an agreement with (2.57). At present it is established that (2.57) is an approximate phenomenological rule. It is violated in neutrino oscillations due to small neutrino masses and neutrino mixing. Later we will discuss neutrino oscillations in details.

2.10 Strange Particles. Quarks. Cabibbo Current

The current×current Hamiltonian (2.36) with CC current (2.35) is the effective Hamiltonian of such processes in which leptons, neutrinos and nonstrange hadrons are participating. The first strange particles were discovered in cosmic rays in the fifties. Decays of strange particles were studied in details in accelerator experiments. From the investigation of the semi-leptonic decays

$$K^+ \to \mu^+ + \nu_\mu, \quad \Lambda \to n + e^- + \bar{\nu}_e,$$
$$\Sigma^- \to n + e^- + \bar{\nu}_e, \quad \Xi^- \to \Lambda + \mu^- + \bar{\nu}_\mu$$

and others the following *three phenomenological rules* were formulated.

I. The strangeness S in the decays of strange particles is changed by one

$$|\Delta S| = 1.$$

II. In the decays of the strange particles the rule

$$\Delta Q = \Delta S$$

is satisfied. Here $\Delta Q = Q_f - Q_i$ and $\Delta S = S_f - S_i$, where S_i and S_f are the initial and final total strangeness of the hadrons and Q_i and Q_f are the initial and final total electric charges of hadrons (in the unit of the proton charge).

III. The decays of strange particles are suppressed with respect to the decays of non strange particles.

In 1964 Gell-Mann and Zweig proposed the idea of three quarks u, d , s, constituents of strange and nonstrange hadrons. The quantum numbers of the quarks are presented in Table 2.2 Let us build the hadronic charged currents from the quark

Table 2.2 Quantum numbers of quarks (Q is the charge, S is the strangeness, B is the baryon number)

Quark	Q	S	B
u	2/3	0	1/3
d	$-1/3$	0	1/3
s	$-1/3$	-1	1/3

fields. The current (2.35) changes the charge by one. If we accept the Feynman-Gell-Mann, Marshak-Sudarshan prescription (into the weak current enter only left-handed components of the fermion fields) there are only two possibilities to build such currents from the fields of u, d and s quarks:

$$\bar{u}_L \gamma_\alpha d_L \quad \text{and} \quad \bar{u}_L \gamma_\alpha s_L. \tag{2.58}$$

The first current changes the charge by one and does not change the strangeness ($\Delta Q = 1$, $\Delta S = 0$). The second current changes the charge by one and the strangeness by one ($\Delta Q = 1$, $\Delta S = 1$). The matrix elements of these currents automatically satisfy rules I and II. Notice that this was one of the first arguments in favor of quark structure of the hadron current.

The weak interaction of the strange particles was included into the current$\times$current theory by N. Cabibbo in 1962. He assumed that the charged current which does not change strangeness and the charged current which changes the strangeness by one are, correspondingly, the $1 + i2$ and $4 + i5$ components of the $SU(3)$ octet current. In order to take into account the suppression of the decays with the change of the strangeness with respect to the decays in which the strangeness is not changed (the rule III) Cabibbo introduced a parameter which is called the Cabibbo angle θ_C. From the analysis of the experimental data on the investigation of the decays of strange particles he found that $\sin\theta_C \simeq 0.2$.

In terms of quark currents (2.58) the Cabibbo current has the form

$$j_\alpha^{\text{Cabibbo}}(x) = 2\,(\cos\theta_C\ \bar{u}_L(x)\gamma_\alpha\ d_L(x) + \sin\theta_C\ \bar{u}_L(x)\gamma_\alpha\ s_L(x) \tag{2.59}$$

The total weak charged current takes the form

$$j_\alpha(x) = 2\,(\bar{\nu}_{eL}(x)\gamma_\alpha\ e_L(x) + \bar{\nu}_{\mu L}(x)\gamma_\alpha\ \mu_L(x) + \bar{u}_L(x)\gamma_\alpha\ d_L^{'}(x), \tag{2.60}$$

where

$$d_L^{'}(x) = \cos\theta_C\ d_L(x) + \sin\theta_C\ s_L(x). \tag{2.61}$$

Notice that there is an asymmetry between quark and lepton terms in (2.60). Namely, in this expression there are two lepton terms and one quark term.

2.11 Charmed Quark. Quark and Neutrino Mixing

It was shown in 1970 by Glashow, Illiopulos and Maiani (GIM) that the charged current (2.59) induces a neutral current which does not change electric charge ($\Delta Q = 0$) and change the strangeness by one ($|\Delta S| = 1$). As a result, the decays like

$$K^{+} \to \pi^{+} + \nu + \bar{\nu}. \tag{2.62}$$

become possible in such a theory. In the theory with the current (2.60) the width of the decay (2.62) is many orders of magnitude larger than *the upper bound* of the width of the decay obtained in experiments.

Glashow, Illiopulos and Maiani assumed that there exists a fourth "charmed" quark c with charge 2/3 and that there is an additional term in the weak current into which enters the field of the new quark c_L and the combination of d_L and s_L fields orthogonal to the Cabibbo combination (2.61). The weak currents took the form

$$j_{\alpha}(x) = 2\,(\bar{\nu}_{eL}(x)\gamma_{\alpha}\, e_L(x) + \bar{\nu}_{\mu L}(x)\gamma_{\alpha}\, \mu_L(x) + \bar{u}_L(x)\gamma_{\alpha}\, d^{'}_L(x) + \bar{c}_L(x)\gamma_{\alpha}\, s^{'}_L(x)), \tag{2.63}$$

where

$$\begin{aligned} d^{'}_L(x) &= \cos\theta_C d_L(x) + \sin\theta_C s_L(x) \\ s^{'}_L(x) &= -\sin\theta_C d_L(x) + \cos\theta_C s_L(x)\,. \end{aligned} \tag{2.64}$$

As we will see in the next section, in the theory with the charged current (2.63) neutral current which changes the strangeness does not appear.

The relations (2.64) mean that *the fields of d and s quarks enter into the charged current in the mixed form*. The phenomenon of mixing is perfectly confirmed by experiment.

We make the following remark. In the current (2.63) lepton and quark terms enter symmetrically. It will be, however, full lepton-quark symmetry of the current if the neutrino masses are different from zero and the fields of neutrinos with definite masses, like the fields of quarks, enter into the CC in the mixed form

$$\begin{aligned} \nu_{\mu L}(x) &= \cos\theta\ \nu_{1L}(x) + \sin\theta\ \nu_{2L}(x) \\ \nu_{eL}(x) &= -\sin\theta\ \nu_{1L}(x) + \cos\theta\ \nu_{2L}(x), \end{aligned} \tag{2.65}$$

where $\nu_1(x)$ and $\nu_2(x)$ are the fields of the neutrinos with masses m_1 and m_2, correspondingly.

The existence of the c-quark means the existence of a new family of "charmed" particles. This prediction of the theory was perfectly confirmed by experiment. In 1974 the J/Ψ particles, bound states of $c - \bar{c}$, were discovered. In 1976 the $D^{\pm,0}$ mesons, bound states of charmed and nonstrange quarks, were discovered, etc. All data on the investigation of the weak decays and neutrino reactions were in agreement with the current×current theory with the current given by (2.63).

In 1975 the third charged lepton τ was discovered in experiments at $e^+ - e^-$ colliders. In the framework of the Standard Model, which we will consider in the next chapter, the existence of the third charged lepton requires the existence of the corresponding third type of neutrino ν_τ and an additional pair of quarks: the t (top) quark with electric charge 2/3 and the b (bottom) quark with electric charge $-1/3$. All these predictions of the SM were perfectly confirmed by numerous experiments.

The modern charged current has the form

$$j_\alpha^{\mathrm{CC}}(x) = 2(\bar{\nu}_{eL}(x)\gamma_\alpha\, e_L(x) + \bar{\nu}_{\mu L}(x)\gamma_\alpha\, \mu_L(x) + \bar{\nu}_{\tau L}(x)\gamma_\alpha\, \tau_L(x) \\ + \bar{u}_L(x)\gamma_\alpha\, d'_L(x) + \bar{c}_L(x)\gamma_\alpha\, s'_L(x) + \bar{t}_L(x)\gamma_\alpha\, b'_L(x)). \qquad (2.66)$$

Here

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\, \nu_{iL}(x) \quad l = e.\mu, \tau \qquad (2.67)$$

and

$$d'_L(x) = \sum_{q=u,s,b} V_{uq}\, q_L(x), \;\; s'_L(x) = \sum_{q=u,s,b} V_{cq} q_L(x), \;\; b'_L(x) = \sum_{q=u,s,b} V_{tq}\, q_L(x). \qquad (2.68)$$

Here U is an unitary 3×3 neutrino mixing matrix and V is an unitary 3×3 quark mixing matrix.

We know today that the vector $W^{\pm}$-boson exists and that the Lagrangian of the CC weak interaction has the form

$$\mathcal{L}_{\mathcal{I}}^{\mathrm{CC}}(x) = -\frac{g}{2\sqrt{2}}\, j_\alpha^{\mathrm{CC}}(x)\; W^\alpha(x) + \mathrm{h.c.} \qquad (2.69)$$

2.12 Summary and Outlook

The theory of the weak interaction started with the famous Fermi paper "An attempt of a theory of beta radiation". The Fermi theory was based on (1) The Pauli hypothesis of the existence of the neutrino. (2) The proton-neutron structure of nuclei. (3) The assumption that an electron-neutrino pair is produced in the process of transition of a neutron into a proton. (4) The assumption that in analogy with electromagnetic interaction the weak interaction is the vector one. Later in accordance with experimental data this last assumption was generalized and other terms (scalar, tensor, axial and pseudoscalar) were included.

The discovery of the parity violation in the β-decay and other weak processes played a revolutionary role in the development of the theory of the weak interaction. Soon after this discovery the two-component theory of massless neutrino was proposed. According to this theory in the Hamiltonian of the weak interaction the left-handed (or right-handed) component of the neutrino field enters. In less than one year this theory was confirmed by experiment. It was proved that neutrino is a left-handed particle.

The next fundamental step was the current$\times$current, V-A theory of the weak interaction which was based on the assumption that only left-handed components of the fields enter into charged current.

The electron neutrino was discovered in the fifties in the first reactor neutrino experiment. A few years later in the first accelerator neutrino experiment the muon neutrino was discovered.

After the hypothesis of quarks was proposed, the weak charged current started to be considered as quark and lepton current. One of the fundamental ideas which was put forward in the process of the phenomenological development of the theory was the idea of the quark mixing. At the very early stage of the development of the theory the idea of the existence of the charged heavy vector intermediate $W^{\pm}$ boson was proposed.

It was a long (about 40 years) extremely important period of the development of the physics of the weak interaction with a lot of bright, courageous ideas.[9] The theory which was finally proposed allowed to describe data of a huge number of experiments. The Standard Model of the weak interaction could not appear without the phenomenological V-A theory.

[9] And also many wrong ideas which we did not discussed here.

Chapter 3
The Standard Model of the Electroweak Interaction

3.1 Introduction

We will consider here the Glashow-Weinberg-Salam theory of the weak and electromagnetic interactions, which usually is called the Standard Model (SM). This theory is one of the greatest achievements of particle physics of the twentieth century. This theory predicted the existence of families of new particles (charmed, bottom, top), a new class of the weak interaction (Neutral currents), $W^{\pm}$ and Z^0 vector bosons and masses of these particles, the existence of the third type of neutrino (ν_τ), the existence of the scalar Higgs boson, etc. All predictions of the Standard Model are in perfect agreement with existing experimental data. The search for the Higgs boson is one of the major aim of experiments at the LHC accelerator in CERN.

The current$\times$current theory of the weak interaction, which we considered in the previous chapter, was a very successful theory. In the lowest order of the perturbation theory this theory allowed to describe all experimental data existing at the sixties. However, the current$\times$current theory and also the theory with the $W^{\pm}$ vector boson were unrenormalizable theories: the infinities of the higher orders of the perturbation theory could not be excluded in these theories by the renormalization of the masses and other physical constants.

This was the main reason why, in spite of big phenomenological success, these theories for many years were not considered as satisfactory ones. The Standard Model was born in the end of the sixties in an attempt to build a renormalizable theory of the weak interaction. The only renormalizable physical theory, that was known at that time, was quantum electrodynamics. The renormalizable theory of the weak interaction was build in the framework of *the unification of the weak and electromagnetic (electroweak) interactions*. This theory was proposed by Glashow, Weinberg and Salam. It was proved by t'Hooft and Veltman that the SM is a renormalizable theory.

The Standard Model is built in two stages. The first stage is based on the local gauge invariance of massless fields. Mass terms of all fields, except the electromagnetic field, appear as a result of the spontaneous violation of the symmetry (the second stage).

In the first two introductory sections we will consider two major ingredients of the Standard Model

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1. $SU(2)$ Yang-Mills local gauge invariance.
2. Spontaneous violation of the symmetry and the Higgs mechanism.

3.2 $SU(2)$ Yang-Mills Local Gauge Invariance

Let us assume that

$$\psi(x) = \begin{pmatrix} \psi^{+1}(x) \\ \psi^{-1}(x) \end{pmatrix}, \tag{3.1}$$

where $\psi^{\pm 1}(x)$ are spin 1/2 fields, is the doublet of a $SU(2)$ group. If the masses of $\psi^{\pm 1}(x)$ fields are the same and equal to m, the free Lagrangian of the field $\psi(x)$ is given by the expression

$$\mathcal{L}_0(x) = \bar{\psi}(x)\,(i\,\gamma^\alpha \partial_\alpha + m)\,\psi(x). \tag{3.2}$$

The Lagrangian (3.2) is invariant under *the global gauge* $SU(2)$ *transformations*

$$\psi'(x) = U\,\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\,U^+. \tag{3.3}$$

Here

$$U = e^{i\,\frac{1}{2}\,\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}}, \tag{3.4}$$

where $\boldsymbol{\tau}\cdot\boldsymbol{\Lambda} = \sum_{i=1}^{3} \tau_i\,\Lambda_i$, τ_i are the Pauli matrices and Λ_i are *arbitrary constants*. Let us stress that the Lagrangian $\mathcal{L}_0(x)$ is invariant under the transformation (3.3) because the derivative $\partial_\alpha\,\psi(x)$ is transformed in the same way as the field $\psi(x)$.

From the invariance under the transformation (3.3) follows that the isovector current

$$j_i^\alpha(x) = \bar{\psi}(x)\,\gamma^\alpha\,\frac{1}{2}\,\tau_i\,\psi(x) \tag{3.5}$$

satisfies the equation

$$\partial_\alpha j_i^\alpha(x) = 0 \tag{3.6}$$

and the total isotopic spin $T_i = \int j_i^0(x) d^3x$ is conserved.

We will build now the theory which is invariant under the *local gauge* $SU(2)$ *transformations*

$$\psi'(x) = U(x)\,\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\,U^+(x). \tag{3.7}$$

Here

$$U(x) = e^{i\,\frac{1}{2}\,\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}(x)}, \tag{3.8}$$

where $\Lambda_i(x)$ are *arbitrary functions of* x.[1]

For the derivative $\partial_\alpha\, \psi(x)$ we have

$$\begin{aligned}\partial_\alpha\, \psi(x) = U^+(x)\, U(x)\, \partial_\alpha\, U^+(x)\psi'(x) = \\ U^+(x)\, (\partial_\alpha + U(x)\, \partial_\alpha\, U^+(x))\, \psi'(x).\end{aligned} \tag{3.9}$$

It is obvious from (3.9) that the free Lagrangian (3.2) is not invariant under the transformation (3.7).

Let us consider an infinitesimal $SU(2)$ transformations, i.e. we will assume that Λ_i are small and in all expansions over Λ_i we keep only linear terms. We have

$$U(x) \simeq 1 + i\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \boldsymbol{\Lambda}(x), \quad U^+(x) \simeq 1 - i\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \boldsymbol{\Lambda}(x). \tag{3.10}$$

From (3.9) and (3.10) we find

$$\partial_\alpha\, \psi(x) = U^+(x) \left(\partial_\alpha - i\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \partial_\alpha\, \boldsymbol{\Lambda}(x)\right) \psi'(x). \tag{3.11}$$

From (3.7) follows that

$$\psi(x) = U^+(x)\, \psi'(x). \tag{3.12}$$

Comparing (3.11) and (3.12) we conclude that the field $\psi(x)$ and the derivative $\partial_\alpha \psi(x)$ are *transformed differently*. This is the reason why the free Lagrangian (3.2) in not invariant under local gauge transformations (3.7).

In order to build a theory which is invariant under local gauge transformations me must assume that the field $\psi(x)$ interacts with a vector field. In fact, let us consider the covariant derivative

$$D_\alpha\, \psi(x) = \left(\partial_\alpha + ig\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x)\right) \psi(x), \tag{3.13}$$

where g is a dimensionless constant and $A^i_\alpha(x)$ is a vector field.

It is obvious that the following equality is valid

$$D_\alpha\, \psi(x) = U^+(x)\, U(x)\, D_\alpha\, U^+(x)\, \psi'(x). \tag{3.14}$$

Let us consider the term $U(x)\, D_\alpha\, U^+(x)$. Using (3.10) we find

[1] The operator U is the operator of rotation in the three-dimensional isotopic space around the vector Λ by the angle $|\Lambda|$. Thus, global gauge invariance is the invariance under rotations which are the same in all space-time points. The local gauge invariance is the invariance under the rotations in the isotopic space which are different at different space-time points.

$$U(x)\, D_\alpha\, U^+(x) = \partial_\alpha - i\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \partial_\alpha\, \boldsymbol{\Lambda}(x) + ig\, U(x)\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x)\, U^+(x). \quad (3.15)$$

For the last term of (3.15) we have

$$\begin{aligned} U(x)\, \tfrac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x)\, U^+(x) &= \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x) + i \left[\frac{1}{2}\, \boldsymbol{\tau} \cdot \boldsymbol{\Lambda}(x)\, ,\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x) \right] \\ &= \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x) - \frac{1}{2}\, \boldsymbol{\tau} \cdot (\boldsymbol{\Lambda} \times \mathbf{A}_\alpha(x)), \end{aligned} \quad (3.16)$$

where we take into account that $[\frac{1}{2}\tau_i, \frac{1}{2}\tau_k] = i e_{ikl}\, \frac{1}{2}\tau_l$. From (3.15) and (3.16) we obtain

$$U(x)\, D_\alpha\, U^+(x) = \partial_\alpha + ig\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}'_\alpha(x) = D'_\alpha. \quad (3.17)$$

Here

$$\mathbf{A}'_\alpha(x) = \mathbf{A}_\alpha(x) - \frac{1}{g}\, \partial_\alpha\, \boldsymbol{\Lambda}(x) - \boldsymbol{\Lambda}(x) \times \mathbf{A}_\alpha(x). \quad (3.18)$$

Thus, from (3.14) and (3.17) we have

$$D_\alpha\, \psi(x) = U^+(x)\, D'_\alpha\, \psi'(x) \quad (3.19)$$

Comparing (3.12) and (3.19) we conclude that under the local gauge $SU(2)$ transformations, which include the phase transformation (3.7) of the spinor field $\psi(x)$ and the gauge transformation (3.18) of the vector field $\mathbf{A}_\alpha(x)$, the covariant derivative $D_\alpha\, \psi(x)$ and the field $\psi(x)$ are *transformed in the same way*.

Thus, if in the free Lagrangian (3.2) we will make the change

$$\partial_\alpha\, \psi(x) \to D_\alpha\, \psi(x) \quad (3.20)$$

we obtain the Lagrangian

$$\mathcal{L}_1(x) = \bar{\psi}(x)\, (i\, \gamma^\alpha D_\alpha + m)\, \psi(x) \quad (3.21)$$

which is invariant under the local gauge transformations (3.7) and (3.18).

The Lagrangian $\mathcal{L}_1(x)$ is the sum of the free Lagrangian of the field $\psi(x)$ and the Lagrangian of the interaction of the field $\psi(x)$ and the vector field $\mathbf{A}_\alpha(x)$. The total Lagrangian must include also the free Lagrangian of the field $\mathbf{A}_\alpha(x)$, which is invariant under the gauge transformation (3.18). In order to build the free Lagrangian of the field $\mathbf{A}_\alpha(x)$ we will consider the commutator $[D_\alpha, D_\beta]$. We have

$$[D_\alpha, D_\beta] = ig\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{F}_{\alpha\beta}(x), \quad (3.22)$$

where

$$\mathbf{F}_{\alpha\beta}(x) = \partial_\alpha \, \mathbf{A}_\beta(x) - \partial_\beta \, \mathbf{A}_\alpha(x) - g \, \mathbf{A}_\alpha(x) \times \mathbf{A}_\beta(x) \tag{3.23}$$

is the generalized stress tensor. From (3.17) we find the following relation

$$U \, [D_\alpha, D_\beta] \, U^+ = [D'_\alpha, D'_\beta]. \tag{3.24}$$

Further, from (3.22) and (3.24) we have

$$U(x) \, \frac{1}{2} \, \boldsymbol{\tau} \cdot \mathbf{F}_{\alpha\beta}(x) \, U^+(x) = \frac{1}{2} \, \boldsymbol{\tau} \cdot \mathbf{F}'_{\alpha\beta}(x), \tag{3.25}$$

where

$$\mathbf{F}'_{\alpha\beta}(x) = \partial_\alpha \, \mathbf{A}'_\beta(x) - \partial_\beta \, \mathbf{A}'_\alpha(x) - g \, \mathbf{A}'_\alpha(x) \times \mathbf{A}'_\beta(x). \tag{3.26}$$

Finally, from (3.10) from (3.25) we find that the tensor $\mathbf{F}_{\alpha\beta}(x)$ is transformed as an isotopic vector:

$$\mathbf{F}'_{\alpha\beta}(x) = \mathbf{F}_{\alpha\beta}(x) - \boldsymbol{\Lambda}(x) \times \mathbf{F}_{\alpha\beta}(x). \tag{3.27}$$

Thus, the scalar product $\mathbf{F}_{\alpha\beta} \cdot \mathbf{F}^{\alpha\beta}$ is invariant.

The free Lagrangian of the vector field $\mathbf{A}_\alpha(x)$, which is invariant under the transformation (3.18), can be chosen in the form

$$\mathcal{L}_0(x) = -\frac{1}{4} \, \mathbf{F}_{\alpha\beta}(x) \cdot \mathbf{F}^{\alpha\beta}(x). \tag{3.28}$$

The total Lagrangian of the spinor field $\psi(x)$ and the vector field $\mathbf{A}_\alpha(x)$, which is invariant under the transformations (3.7) and (3.18), is given by the following expression

$$\mathcal{L}(x) = \bar{\psi}(x) \left(i \, \gamma^\alpha (\partial_\alpha + i g \, \frac{1}{2} \, \boldsymbol{\tau} \cdot \mathbf{A}_\alpha(x)) + m \right) \psi(x) - \frac{1}{4} \, \mathbf{F}_{\alpha\beta}(x) \mathbf{F}^{\alpha\beta}(x). \tag{3.29}$$

Thus, the requirement of the local gauge $SU(2)$ invariance can be satisfied *if the spinor doublet field $\psi(x)$ interacts with the vector field $A^i_\alpha(x)$*. The vector field $A^i_\alpha(x)$ is called *gauge field*. Under the $SU(2)$ global gauge transformation ($\Lambda =$ const) the field $\mathbf{A}_\alpha(x)$ is transformed as an isotopic vector:

$$\mathbf{A}'_\alpha(x) = \mathbf{A}_\alpha(x) - \boldsymbol{\Lambda} \times \mathbf{A}_\alpha(x). \tag{3.30}$$

From (3.29) we obtain the following expression for the Lagrangian of the interaction of the field $\psi(x)$ and the gauge vector field $\mathbf{A}_\alpha(x)$

$$\mathcal{L}_I(x) = -g\,\mathbf{j}_\alpha(x)\cdot\mathbf{A}^\alpha(x) = -g\sum_{i=1}^{3} j^i_\alpha(x)\,A^{\alpha i}(x), \tag{3.31}$$

where

$$j^i_\alpha(x) = \bar{\psi}(x)\,\gamma_\alpha\,\frac{1}{2}\,\tau_i\,\psi(x) \tag{3.32}$$

is the isovector current.

The Lagrangian of the interaction $\mathcal{L}_I(x)$ can be written in the form

$$\mathcal{L}_I(x) = \left(-\frac{g}{2\sqrt{2}}\,j_\alpha(x)\,W^\alpha(x) + \text{h.c.}\right) - g\,j^3_\alpha(x)\,A^{\alpha 3}(x). \tag{3.33}$$

Here

$$j_\alpha(x) = 2\,j^{1+i2}_\alpha(x),\quad W_\alpha(x) = \frac{1}{\sqrt{2}}\,A^{1-i2}_\alpha(x), \tag{3.34}$$

where

$$j^{1\pm i2}_\alpha = j^1_\alpha \pm i j^2_\alpha,\quad A^{1\pm i2}_\alpha = A^1_\alpha \pm i A^2_\alpha. \tag{3.35}$$

For the current $j_\alpha(x)$ we have

$$j_\alpha(x) = 2\,\bar{\psi}(x)\,\gamma_\alpha\,\frac{1}{2}\,(\tau_1 + i\tau_2)\,\psi(x) = 2\,\bar{\psi}^{+1}(x)\,\gamma_\alpha\,\psi^{-1}(x), \tag{3.36}$$

where $\psi^{+1}(x)$ and $\psi^{-1}(x)$ are components of the isotopic doublet with the third projections of the isotopic spin I_3 equal to $\frac{1}{2}$ and $-\frac{1}{2}$, correspondingly. According to the Gell-Mann and Nishijima I_3 and the electric charge Q are connected by the relation

$$Q = I_3 + \frac{1}{2}\,Y. \tag{3.37}$$

Here Q is the electric charge in the unit of the proton charge and Y is the hypercharge. From (3.36) and (3.37) follows that the current $j_\alpha(x)$ changes the charges of particles by one ($\Delta Q = 1$). Thus, the field $W^\alpha(x)$ (due to the conservation of the total electric charge) is the field of the particles $W^\pm$ with electric charges equal to ± 1.

For the current $j^3_\alpha(x)$ we have

$$j^3_\alpha(x) = \bar{\psi}(x)\,\gamma_\alpha\,\frac{1}{2}\,\tau_3\,\psi(x) = \frac{1}{2}(\bar{\psi}^{+1}(x)\,\gamma_\alpha\,\psi^{+1}(x) - \bar{\psi}^{-1}(x)\,\gamma_\alpha\,\psi^{-1}(x)). \tag{3.38}$$

The current $j^3_\alpha(x)$ does not change the electric charges of particles and hence $A^3_\alpha(x)$ is the field of neutral, vector particles.

Thus, we have built the $SU(2)$ local gauge invariant Yang-Mills theory with gauge fields which include charged as well as neutral vector fields. Such a theory will be used as a basis for the theory of the weak and electromagnetic interactions (The Standard Model).

In conclusion we make the following remarks.

1. After the change

$$\partial_\alpha \psi(x) \to \left(\partial_\alpha + ig\,\frac{1}{2}\,\boldsymbol{\tau}\cdot\mathbf{A}_\alpha(x)\right)\psi(x) \tag{3.39}$$

in the free Lagrangian of the spinor field $\psi(x)$ we came to the interaction Lagrangian (3.31), which and is characterized by the interaction constant g and has the form of the product of the isovector current $\mathbf{j}_\alpha(x)$ and the isovector field $\mathbf{A}^\alpha(x)$. It is necessary, however, to stress that the requirements of the local gauge invariance do not fix the form of the interaction Lagrangian. For example, to the expression (3.31) we can add a tensor term

$$\mathcal{L}_I^T = \mu\,\bar\psi(x)\,\sigma_{\alpha\beta}\,\frac{1}{2}\vec{\boldsymbol{\tau}}\,\psi(x)\,\mathbf{F}_{\alpha\beta}, \tag{3.40}$$

which is invariant under the transformations (3.7) and (3.18). However, in order to "absorb" the term $\partial_\alpha\Lambda(x)$ in the transformation of the derivative of the fermion field (see (3.11)) and to provide the local gauge $SU(2)$ invariance we must make the change (3.39) which induces the interaction (3.31). In this sense the interaction (3.31) is *the minimal gauge invariant interaction* of the spinor and vector fields.
2. The mass term of the vector field $-\frac{1}{2}\,m_A^2\,\mathbf{A}_\alpha(x)\,\mathbf{A}^\alpha(x)$ is not invariant under the transformation (3.18). Thus, local gauge invariance requires that $W_\alpha(x)$ and $A^3_\alpha(x)$ are fields of massless particles. This is the reason why in a realistic theory the local $SU(2)$ gauge symmetry is violated. In the next section we will discuss the mechanism of the spontaneous violation of the gauge symmetry.

3.3 Spontaneous Symmetry Breaking. Higgs Mechanism

We will consider the scalar complex field $\phi(x)$ and assume that the Lagrangian of the field is given by the expression

$$\mathcal{L} = \partial_\alpha\phi^\dagger\,\partial^\alpha\phi - V(\phi^\dagger\,\phi). \tag{3.41}$$

Here

$$V(\phi^\dagger\,\phi) = -\mu^2\,\phi^\dagger\,\phi + \lambda\,(\phi^\dagger\,\phi)^2, \tag{3.42}$$

where μ^2 and λ are positive constants.

The first term of the Lagrangian (3.41) is the kinetic term. The second term $V(\phi^\dagger \phi)$ is the so called potential. Let us notice that the term $-\mu^2 \phi^\dagger \phi$ is not a mass term (it differs from a mass term by the sign).

The Lagrangian (3.41) is invariant under the global gauge transformation

$$\phi'(x) = e^{i\Lambda} \phi(x), \tag{3.43}$$

where Λ is a constant arbitrary phase.

Let us find the minimum of the potential. Equation (3.42) can be written in the form

$$V(\phi^\dagger \phi) = \lambda \left(\phi^\dagger \phi - \frac{\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}. \tag{3.44}$$

From this expression it is evident that the potential reaches its minimum at the value of the field which satisfies the condition

$$\phi_0^\dagger \phi_0 = \frac{\mu^2}{2\lambda}. \tag{3.45}$$

Thus, the potential reaches the minimum $\left(V(\phi_0^\dagger \phi_0) = -\frac{\mu^4}{4\lambda}\right)$ at

$$\phi = \phi_0 = \frac{\mathrm{v}}{\sqrt{2}} e^{i\alpha} = \text{const.} \tag{3.46}$$

Here α is a real, constant phase and we have used the standard notation

$$\mathrm{v}^2 = \frac{\mu^2}{\lambda}. \tag{3.47}$$

The Hamiltonian of the system is given by the following expression

$$\mathcal{H} = \partial_0 \phi^\dagger \partial_0 \phi + \sum_{i=1}^{3} \partial_i \phi^\dagger \partial_i \phi + V(\phi^\dagger \phi). \tag{3.48}$$

From this expression follows that at $\phi = \phi_0$ the energy of the field is minimal.

Thus, in the case of the complex scalar field with potential (3.42):

- The energy is minimal at *different from zero constant (vacuum) values of the field*.
- The minimum of the Hamiltonian is reached at an infinite number of vacuum values given by (3.47).

Let us put $\alpha = 0$. We have

$$\phi_0 = \frac{\mathrm{v}}{\sqrt{2}}. \tag{3.49}$$

With this choice we violate the symmetry of the Lagrangian. Such a violation is called spontaneous.[2]

We introduce now two real fields χ_1 and χ_1 which are connected with the field $\phi(x)$ by the following relation

$$\phi(x) = \frac{\mathrm{v}}{\sqrt{2}} + \frac{\chi_1 + i\chi_2}{\sqrt{2}}. \tag{3.50}$$

The fields $\chi_{1,2}$ are determined in such a way that their vacuum values are equal to zero. From (3.41) and (3.50) we obtain the following expression for the Lagrangian of the system

$$\mathcal{L} = \frac{1}{2}\sum_i \partial_\alpha \chi_i\, \partial^\alpha \chi_i - \frac{\lambda}{4}\left((\mathrm{v}+\chi_1)^2 + \chi_2^2 - \mathrm{v}^2\right)^2. \tag{3.51}$$

This expression can be rewritten in the form

$$\mathcal{L} = \frac{1}{2}\sum_i \partial_\alpha \chi_i\, \partial^\alpha \chi_i - \frac{1}{2}\, 2\,\mu^2\, \chi_1^2 - \lambda\, \mathrm{v}\, \chi_1\, (\chi_1^2 + \chi_2^2) - \frac{1}{4}\lambda\, (\chi_1^2 + \chi_2^2)^2. \tag{3.52}$$

Lagrangian (3.52) is the Lagrangian of the two interacting real fields χ_1 and χ_2. The second term in (3.52) is the mass term of the field χ_1. There are no mass term of the field χ_2 in (3.52).

In quantum theory the quanta of the field χ_1 are neutral particles with mass $m_{\chi_1} = \sqrt{2}\,\mu$ and the quanta of the field χ_2 are neutral massless particles ($m_{\chi_2} = 0$). These massless particles are called Goldstone bosons. Their appearance is a general consequence of the spontaneous violation of a continuous symmetry.

The appearance of Goldstone bosons is a problem for theories with spontaneous symmetry violation: massless scalar bosons were not observed in experiments. In local gauge invariant theories, discussed in the previous section, all vector gauge bosons are massless and this is another problem for a realistic theory. Higgs showed that in the theory based on local gauge invariance and spontaneous violation of the symmetry Goldstone scalar bosons will not appear and gauge bosons are massive.

We will consider now the scalar complex field $\phi(x)$ and the real vector gauge field $A_\alpha(x)$ and assume that the Lagrangian of the system has the form

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[2] In quantum theory $\phi_0 = \langle 0|\phi|0\rangle$, where $\langle 0|\phi|0\rangle$ is the vacuum expectation value (vev) of the field ϕ. If $\langle 0|\phi(0)|0\rangle = \frac{v}{\sqrt{2}}$ the symmetry is spontaneously broken. Spontaneous symmetry breaking is a well known phenomenon in physics. A typical example is ferromagnetism. The Hamiltonian of a system of electron spins above the magnetic ordering temperature is invariant under space rotation. This invariance means that all directions in space are equivalent. In the ground state of a ferromagnet the electron spins are aligned, however, in one direction which means that rotational symmetry was spontaneously broken. After the spontaneous symmetry breaking, the system is invariant under the subgroup of the rotations around the magnetic moment.

$$\mathcal{L}(x) = ((\partial_\alpha + ig\,A_\alpha(x))\,\phi(x))^\dagger \left((\partial^\alpha + ig\,A^\alpha(x))\,\phi(x)\right)$$
$$- V(\phi^\dagger(x)\,\phi(x)) - \frac{1}{4}\,F_{\alpha\beta}(x)\,F^{\alpha\beta}(x). \tag{3.53}$$

Here

$$F_{\alpha\beta}(x) = \partial_\alpha\,A_\beta(x) - \partial_\beta\,A_\alpha(x), \tag{3.54}$$

g is a real dimensionless constant and the potential $V(\phi^\dagger\,\phi)$ is given by (3.42).

The Lagrangian (3.53) is invariant under the local gauge transformations

$$\phi'(x) = e^{i\,\Lambda(x)}\,\phi(x), \quad A'_\alpha(x) = A_\alpha(x) - \frac{1}{g}\,\partial_\alpha\,\Lambda(x), \tag{3.55}$$

where $\Lambda(x)$ is an arbitrary function of x.

The potential (3.44) has the minimum at the value of the scalar field which satisfies the condition

$$\phi_0^\dagger\,\phi_0 = \frac{\mathrm{v}^2}{2}, \tag{3.56}$$

where the constant v is given by (3.47). Thus, we have

$$\phi_0 = \frac{\mathrm{v}}{\sqrt{2}}\,e^{i\,\alpha}, \tag{3.57}$$

where α is an arbitrary phase. If we choose

$$\phi_0 = \frac{\mathrm{v}}{\sqrt{2}} \tag{3.58}$$

we will spontaneously violate the symmetry.

The complex field $\phi(x)$ can be presented in the form

$$\phi(x) = \frac{\mathrm{v} + \chi(x)}{\sqrt{2}}\,e^{i\,\theta(x)}, \tag{3.59}$$

where $\chi(x)$ and $\theta(x)$ are real functions. The vacuum values of these functions are equal to zero.

Due to the local gauge invariance of the Lagrangian (3.53) the phase $\theta(x)$ has no physical meaning. It can be removed by the choice of the gauge $\Lambda(x)$. In this case we have (unitary gauge)

$$\phi(x) = \frac{\mathrm{v} + \chi(x)}{\sqrt{2}}. \tag{3.60}$$

From (3.53) and (3.60) for the Lagrangian of the real scalar field $\chi(x)$ and the real vector field $A_\alpha(x)$ we obtain the following expression

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha\,\chi\,\partial^\alpha\,\chi + \frac{1}{2}\,g^2\,\mathrm{v}^2\,A_\alpha\,A^\alpha - \frac{1}{2}\,2\,\mu^2\,\chi^2 + \frac{1}{2}\,g^2(2\mathrm{v}\chi + \chi^2)\,A_\alpha\,A^\alpha - \frac{1}{4}\lambda(4\mathrm{v}\chi^3 + \chi^4) - \frac{1}{4}\,F_{\alpha\beta}\,F^{\alpha\beta}. \qquad (3.61)$$

The first and the last terms of (3.61) are the kinetic terms of the scalar and vector fields. The second and the third terms of this expression are mass terms of the vector and the scalar fields. These terms are generated by the spontaneous violation of the symmetry and come from the covariant derivative and the potential, correspondingly. Other terms of (3.61) describe interactions of the fields.

The masses of the vector and scalar particles, the quanta of the fields $A_\alpha(x)$ and $\chi(x)$, are given, correspondingly, by the relations

$$m_A = g\mathrm{v}, \quad m_\chi = \sqrt{2}\mu. \qquad (3.62)$$

This mechanism of the generation of the mass of the vector field is called *Higgs mechanism*. The field $\phi(x)$ is called Higgs field.

Thus, from the local gauge invariant theory of the interacting complex massless scalar field and the real massless vector gauge field, after spontaneous violation of symmetry we came to the theory of a massive neutral vector field and a massive scalar Higgs field. The massless vector field is characterized by two degrees of freedom (two projections of the spin) while the massive vector field is characterized by three degrees of freedom (three projections of the spin). Hence, as a result of the spontaneous violation of the symmetry, the Goldston degree of freedom of the complex scalar field became an additional degree of freedom of the vector field (2+2 degrees of freedom became 1+3).

3.4 The Standard Model for Quarks

We will now consider the unified theory of the weak and electromagnetic (electroweak) interactions (The Standard Model). The Standard Model is based on the following principles:

1. The local gauge $SU(2) \times U(1)$ invariance of the Lagrangian of massless fields.
2. The unification of the weak and electromagnetic interactions.
3. The Higgs mechanism of the generation of masses of particles.

The SM is the theory of spin 1/2 quarks and leptons, spin 1 gauge vector bosons and spin 0 Higgs bosons. The Lagrangian of the theory is built in such a way to include the Lagrangian of the $V-A$ charged current interaction, which describes the β-decay of nuclei, μ-decay, π-decay, decay of strange particles, neutrino processes and many other processes. In order to insure local gauge invariance we must assume that $W^{\pm}$-bosons exist. In this case the Lagrangian of the charged current interaction has the form (2.69).

In this section we will consider the Standard Model for the quarks. As we have seen in the first chapter, into the charged current enter left-handed components of the fields. Let us assume that[3]

$$\psi_{1L} = \begin{pmatrix} u'_L \\ d'_L \end{pmatrix} \quad \psi_{2L} = \begin{pmatrix} c'_L \\ s'_L \end{pmatrix} \quad \psi_{3L} = \begin{pmatrix} t'_L \\ b'_L \end{pmatrix} \tag{3.63}$$

are doublets of the $SU(2)$ group and the right-handed components of the quark fields $q'_R(x)$ $(q = u, d, \dots)$ are singlets of the group. The kinetic term of the free Lagrangian of the quark fields can be presented in the form

$$\mathcal{L}_0 = \sum_{a=1}^{3} \bar{\psi}_{aL}\, i\, \gamma^{\alpha} \partial_{\alpha}\, \psi_{aL} + \sum_{u_1=u,c,t} \bar{u}'_{1R}\, i\, \gamma^{\alpha} \partial_{\alpha}\, u'_{1R} + \sum_{d_1=d,s,b} \bar{d}'_{1R}\, i\, \gamma^{\alpha} \partial_{\alpha}\, d'_{1R}. \tag{3.64}$$

In order to ensure the invariance of the Lagrangian under the local gauge $SU(2)$ transformations we will make the change

$$\partial_{\alpha}\, \psi_{aL} \to (\partial_{\alpha} + i g\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_{\alpha})\, \psi_{aL}, \tag{3.65}$$

where $\mathbf{A}^{\alpha}(x)$ is the vector gauge field.

For the Lagrangian of the interaction of the quark and vector fields we obtain the following expression

$$\mathcal{L}_I(x) = -g\, \mathbf{j}_{\alpha}(x)\, \mathbf{A}^{\alpha}(x). \tag{3.66}$$

Here

$$\mathbf{j}_{\alpha} = \sum_{a=1}^{3} \bar{\psi}_{aL}(x)\, \gamma_{\alpha} \frac{1}{2} \boldsymbol{\tau}\, \psi_{aL}(x). \tag{3.67}$$

Let us separate the charged and neutral parts in the Lagrangian (3.66). We have

$$\mathcal{L}_I = \Big(-\frac{g}{2\sqrt{2}}\, j^{\mathrm{CC}}_{\alpha}\, W^{\alpha} + \mathrm{h.c.} \Big) - g\, \mathrm{j}^{3}_{\alpha} A^{3\alpha}. \tag{3.68}$$

Here

$$j^{\mathrm{CC}}_{\alpha} = 2\, j^{1+i2}_{\alpha} = 2\, (\bar{u}'_L\, \gamma_{\alpha}\, d'_L + \bar{c}'_L\, \gamma_{\alpha}\, s'_L + \bar{t}'_L\, \gamma_{\alpha}\, b'_L) \tag{3.69}$$

[3] The meaning of primes will be clear later.

is the charged current and

$$W_\alpha = \frac{1}{\sqrt{2}}\, A_\alpha^{1-i2} \tag{3.70}$$

is the field of the charged $W^{\pm}$-bosons. The current j^3_α is given by the following expression

$$j^3_\alpha = \sum_{a=1}^{3} \bar{\psi}_{aL}\, \gamma_\alpha \frac{1}{2}\tau_3\, \psi_{aL} = \frac{1}{2} \sum_{u_1=u,c,t} \bar{u}'_{1L}\, \gamma_\alpha\, u'_{1L} - \frac{1}{2} \sum_{d_1=d,s,b} \bar{d}'_{1L}\, \gamma_\alpha\, d'_{1L}. \tag{3.71}$$

The first term of the expression (3.68) is the CC Lagrangian. The last term of this expression cannot be identified, however, with the Lagrangian of the electromagnetic interaction: in the current j^3_α enter only left-handed components of the quark fields while in the electromagnetic quark current enter quark fields which are sum of left-handed and right-handed components . Thus, the CC weak interaction which violates a parity and the electromagnetic interaction which conserve a parity cannot be unified on the basis of the local $SU(2)$ group.

In order to build an unified theory of the weak and electromagnetic interactions it is necessary to enlarge the symmetry group. A new interaction Lagrangian must include the CC Lagrangian and the Lagrangian of the electromagnetic interaction. The minimal group of this type is the direct product

$$SU(2) \times U(1),$$

where $U(1)$ is the group of the hypercharge. We will assume that the hypercharge Y is connected to the electric charge Q by the Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{1}{2}\, Y. \tag{3.72}$$

Here Q is the electric charge (in the units of the proton charge) and I_3 is the third projection of the isotopic spin. For the doublets ψ_{aL} we have

$$Q^{\mathrm{up}} = \frac{2}{3} = \frac{1}{2} + \frac{1}{2}\, Y_L^{\mathrm{doub}} \quad Q^{\mathrm{down}} = -\frac{1}{3} = -\frac{1}{2} + \frac{1}{2}\, Y_L^{\mathrm{doub}}. \tag{3.73}$$

Thus, for the hypercharge of the left-handed doublets we find

$$Y_L^{\mathrm{doub}} = \frac{1}{3}. \tag{3.74}$$

For the right-handed singlets we have

$$Y_R^{\mathrm{up}} = 2\, Q^{\mathrm{up}} = 4/3, \quad Y_R^{\mathrm{down}} = 2\, Q^{\mathrm{down}} = -2/3. \tag{3.75}$$

In order to build the theory invariant under the local gauge $SU(2) \times U(1)$ transformations in the free Lagrangian of the quark fields (3.64) we have to change the derivatives of the doublet and singlet fields by the covariant derivatives

$$\partial_\alpha \psi_{aL} \to \left(\partial_\alpha + ig\,\frac{1}{2}\,\boldsymbol{\tau}\cdot\mathbf{A}_\alpha + ig'\,\frac{1}{2}\,Y_L^{\text{doub}}\,B_\alpha\right)\psi_{aL},$$
$$\partial_\alpha u'_{1R} \to \left(\partial_\alpha + ig'\,\frac{1}{2}\,Y_R^{\text{up}}\,B_\alpha\right)u'_{1R} \quad (u'_{1R} = u'_R, c'_R, t'_R),$$
$$\partial_\alpha d'_{1R} \to \left(\partial_\alpha + ig'\,\frac{1}{2}\,Y_R^{\text{down}}\,B_\alpha\right)d'_R \quad (d'_{1R} = d'_R, s'_R, b'_R), \tag{3.76}$$

where $\mathbf{A}_\alpha$ and B_α are $SU(2)$ and $U(1)$ vector gauge fields and g and g' are the corresponding coupling constants.

For the Lagrangian of the (minimal) interaction of the quarks and vector fields we will find the following expression

$$\mathcal{L}_I = -g\,\mathbf{j}_\alpha\,\mathbf{A}^\alpha - g'\,\frac{1}{2}\,j^Y_\alpha\,B^\alpha, \tag{3.77}$$

where the isovector current $\mathbf{j}_\alpha$ is given by (3.67) and for the hypercurrent j^Y_α we have

$$\frac{1}{2}\,j^Y_\alpha = \frac{1}{2}\,Y_L^{\text{doub}}\sum_{a=1}^{3}\bar{\psi}_{aL}\,\gamma_\alpha\,\psi_{aL} + \frac{1}{2}\,Y_R^{\text{up}}\sum_{u_1=u,c,t}\bar{u}'_{1R}\,\gamma_\alpha\,u'_{1R}$$
$$+\frac{1}{2}\,Y_R^{\text{down}}\sum_{d_1=d,s,b}\bar{d}'_{1R}\,\gamma_\alpha\,d'_{1R}. \tag{3.78}$$

It is a direct consequence of the Gell-Man-Nishijima relation (3.72) that the hypercurrent j^Y_α is connected to the electromagnetic current and the third component of the isovector current by the following relation

$$\frac{1}{2}\,j^Y_\alpha = j^{\text{EM}}_\alpha - j^3_\alpha. \tag{3.79}$$

Here

$$j^{\text{EM}}_\alpha = \frac{2}{3}\sum_{u_1=u,c,t}\bar{u}'_1\,\gamma_\alpha\,u'_1 + \left(-\frac{1}{3}\right)\sum_{d_1=d,s,b}\bar{d}'_1\,\gamma_\alpha\,d'_1 \tag{3.80}$$

is the electromagnetic current of the quarks. The Lagrangian of the interaction (3.77) takes the form

$$\mathcal{L}_I = -g\,\mathbf{j}_\alpha\,\mathbf{A}^\alpha - g'\,(j^{\text{EM}}_\alpha - j^3_\alpha)\,B^\alpha. \tag{3.81}$$

The local gauge $SU(2) \times U(1)$ symmetry can be satisfied only if $\mathbf{A}^{\alpha}$ and B^{α} are massless gauge fields. *The Standard Model is based on the Higgs mechanism of the spontaneous symmetry breaking.* Thus, we must include scalar Higgs fields in the system. In order for the theory to be invariant under $SU(2) \times U(1)$ transformations the Higgs fields must have definite $SU(2) \times U(1)$ transformation properties.

Let us assume that the Higgs field $\phi(x)$ is the $SU(2)$ doublet

$$\phi = \begin{pmatrix} \phi_{+} \\ \phi_{0} \end{pmatrix}, \tag{3.82}$$

where ϕ_{+} is the scalar complex field of particles with electric charges equal to ± 1 and $\phi_0(x)$ is the complex field of particles with electric charge equal to zero. From the Gell-Mann-Nishijima relation (3.72) follows that the hypercharge of the field $\phi(x)$ is equal to one ($Y_{\phi} = 1 + 0 = 1$).

For the free Lagrangian of the Higgs doublet we have

$$\mathcal{L}_0 = \partial_{\alpha}\phi^{\dagger}\, \partial^{\alpha}\phi - V(\phi^{\dagger}\, \phi). \tag{3.83}$$

Here the potential $V(\phi^{\dagger}\, \phi)$ is given by the expression

$$V(\phi^{\dagger}\, \phi) = -\mu^2\, \phi^{\dagger}\, \phi + \lambda\, (\phi^{\dagger}\, \phi)^2 = \lambda\, \left(\phi^{\dagger}\, \phi - \frac{\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}, \tag{3.84}$$

where μ^2 and λ are positive constants.

In order to ensure $SU(2) \times U(1)$ invariance of the theory in (3.83) we must change the derivative $\partial_{\alpha}\phi$ by the covariant derivative:

$$\partial_{\alpha}\phi \to \left(\partial_{\alpha} + ig\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}_{\alpha} + ig'\, \frac{1}{2}\, B_{\alpha}\right) \phi. \tag{3.85}$$

where $\mathbf{A}_{\alpha}$ and B_{α} are $SU(2)$ and $U(1)$ gauge fields. The Lagrangian takes the form

$$\mathcal{L} = \left(\left(\partial_{\alpha} + ig\, \frac{1}{2}\, \vec{\boldsymbol{\tau}} \cdot \mathbf{A}_{\alpha} + ig'\, \frac{1}{2}\, B_{\alpha}\right) \phi\right)^{\dagger} \left(\left(\partial^{\alpha} + ig\, \frac{1}{2}\, \boldsymbol{\tau} \cdot \mathbf{A}^{\alpha} + ig'\, \frac{1}{2}\, B^{\alpha}\right) \phi\right) \\ - V(\phi^{\dagger}\, \phi). \tag{3.86}$$

The potential (3.84) reaches a minimum at such values of the Higgs field which satisfy the relation

$$(\phi^{\dagger}\, \phi)_0 = \frac{\mathrm{v}^2}{2}, \tag{3.87}$$

where

$$\mathrm{v}^2 = \frac{\mu^2}{\lambda}. \tag{3.88}$$

Because of the conservation of the electric charge the vacuum expectation value of the charged field ϕ_+ is equal to zero. From (3.87) follows that we can choose ϕ_0 in the form

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{\mathrm{v}}{\sqrt{2}} \end{pmatrix}. \tag{3.89}$$

The complex scalar doublet $\phi(x)$ can be presented as follows

$$\phi(x) = e^{i\,\frac{1}{2}\,\boldsymbol{\tau}\cdot\boldsymbol{\theta}(x)} \begin{pmatrix} 0 \\ \frac{\mathrm{v}+H(x)}{\sqrt{2}} \end{pmatrix}, \tag{3.90}$$

where $\theta_i(x)$ $(i = 1, 2, 3)$ and $H(x)$ are real functions. The parametrization (3.90) was chosen in such a way that the vacuum values of the functions $\theta_i(x)$ and $H(x)$ are equal to zero.

The Lagrangian of the system we are considering, is invariant under the $SU(2) \times U(1)$ local gauge transformations. Let us choose the gauge in such a way that

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{\mathrm{v}+H(x)}{\sqrt{2}} \end{pmatrix}. \tag{3.91}$$

With this choice of the gauge (which is called unitary gauge) we find the following expression for the Lagrangian (3.86)

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha H\,\partial^\alpha H + \phi^\dagger \left(\frac{g}{2}\,\boldsymbol{\tau}\cdot\mathbf{A}_\alpha + \frac{g'}{2}\,B_\alpha\right)\left(\frac{g}{2}\,\boldsymbol{\tau}\cdot\mathbf{A}^\alpha + \frac{g'}{2}\,B^\alpha\right)\phi - V. \tag{3.92}$$

Let us consider the different terms of this expression. Taking into account that

$$\tau_i\,\tau_k = \delta_{ik} + i\,e_{ikl}\,\tau_l \tag{3.93}$$

we have

$$\boldsymbol{\tau}\cdot\mathbf{A}_\alpha\,\boldsymbol{\tau}\cdot\mathbf{A}^\alpha = \mathbf{A}_\alpha\,\mathbf{A}^\alpha = 2\,W^\dagger_\alpha\,W^\alpha + A^3_\alpha\,A^{3\alpha}, \tag{3.94}$$

where W^α is the field of the charged $W^\pm$ bosons given by Eq. (3.70).

Further, from (3.91) we find

$$\phi^\dagger\,\boldsymbol{\tau}\cdot\mathbf{A}_\alpha\,\phi = -\frac{1}{2}\,(\mathrm{v}+H)^2\,A^3_\alpha. \tag{3.95}$$

From (3.84), (3.91), (3.94) and (3.95) we obtain the following expression for the Lagrangian (3.92)

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha H\,\partial^\alpha H + \frac{g^2}{4}\,(\mathrm{v}+H)^2\,W^\dagger_\alpha\,W^\alpha + \frac{g^2+g'^2}{8}\,(\mathrm{v}+H)^2\,Z_\alpha\,Z^\alpha - \frac{\lambda}{4}\,(2\mathrm{v}H + H^2)^2, \tag{3.96}$$

where

$$Z_\alpha = \frac{g}{\sqrt{g^2+g'^2}} A^3_\alpha - \frac{g'}{\sqrt{g^2+g'^2}} B_\alpha . \tag{3.97}$$

The combination of the fields A^3_α and B_α, which is orthogonal to Z_α, we will call A_α. We have

$$A_\alpha = \frac{g'}{\sqrt{g^2+g'^2}} A^3_\alpha + \frac{g}{\sqrt{g^2+g'^2}} B_\alpha . \tag{3.98}$$

The Lagrangian (3.96) includes the mass terms of the vector W^α and Z^α fields and the mass term of the scalar field H:

$$\mathcal{L}^m = m_W^2 \, W^\dagger_\alpha \, W^\alpha + \frac{1}{2} m_Z^2 \, Z_\alpha \, Z^\alpha - \frac{1}{2} m_H^2 \, H^2 . \tag{3.99}$$

Here

$$m_W^2 = \frac{1}{4} g^2 \, \mathrm{v}^2, \quad m_Z^2 = \frac{1}{4} (g^2 + g'^2) \, \mathrm{v}^2, \quad m_H^2 = 2 \, \lambda \, \mathrm{v}^2 = 2 \, \mu^2 . \tag{3.100}$$

Thus, after spontaneous symmetry breaking $W^\alpha(x)$ became the field of the charged vector $W^\pm$ bosons with the mass $m_W = \frac{1}{2} g \mathrm{v}$, $Z^\alpha(x)$ became the field of neutral vector Z^0 bosons with the mass $m_Z = \frac{1}{2}\sqrt{g^2 + g'^2}\mathrm{v}$. The field $A_\alpha(x)$ remained massless. Three Goldston degrees of freedom of the Higgs doublet provided the masses of the $W^\pm$ and Z^0 bosons. The fourth degree of freedom is the neutral scalar field $H(x)$ of particles with the mass $m_H = \sqrt{2}\mu$ and spin equal to zero.

Let us now consider the Lagrangian (3.81) of the interaction of the quark and vector fields. This Lagrangian can be written in the form

$$\mathcal{L}_I = \left(- \frac{g}{2\sqrt{2}} \, j^{\mathrm{CC}}_\alpha \, W^\alpha + \mathrm{h.c.} \right) - \mathcal{L}^0_I , \tag{3.101}$$

where the first term is the Lagrangian of the CC interaction of the quarks and $W^\pm$ bosons and the second term

$$\mathcal{L}^0_I = - g j^3_\alpha \, A^{3\alpha} - g' \, \frac{1}{2} \, j^Y_\alpha \, B^\alpha \tag{3.102}$$

is the Lagrangian of the interaction of the quarks and neutral vector fields. Taking into account (3.97) and (3.98), for the Lagrangian $\mathcal{L}^0_I$ we find the following expression

$$\mathcal{L}^0_I = -\frac{1}{2}\sqrt{g^2 + g'^2} \, j^{\mathrm{NC}}_\alpha \, Z^\alpha - \frac{g g'}{\sqrt{g^2+g'^2}} \, j^{\mathrm{EM}}_\alpha \, A^\alpha , \tag{3.103}$$

where

$$j_{\alpha}^{\mathrm{NC}} = 2\, j_{\alpha}^{3} - 2\, \frac{g'^2}{g^2 + g'^2}\, j_{\alpha}^{\mathrm{EM}}. \tag{3.104}$$

and j_{α}^{EM} is the electromagnetic current of the quarks (see (3.80)).

The first term of (3.103) is the Lagrangian of the interaction of quarks and massive neutral vector Z^0 bosons. The weak current (3.104), which does not change the electric charges of quarks, is called the neutral current (NC). Before the SM appeared, the NC interaction was unknown. The unification of the CC weak and electromagnetic interactions on the basis of the local gauge $SU(2) \times U(1)$ group allowed *to predict the existence of the massive neutral vector Z^0-boson and a new type of the weak interaction (NC).* All predictions of the Standard Model were perfectly confirmed by numerous experiments.

The second term of the Lagrangian (3.103) is a product of the electromagnetic current and the massless vector field $A^{\alpha}(x)$. It can be identified with the Lagrangian of the electromagnetic interaction if the constant g and g' satisfy the following condition

$$\frac{g g'}{\sqrt{g^2 + g'^2}} = e, \tag{3.105}$$

where e is the charge of the proton. The massless vector field A^{α} is the electromagnetic field in this case. After the spontaneous breaking of the $SU(2) \times U(1)$ symmetry the Lagrangian of the system is invariant under the transformations of the local $U_{\mathrm{EM}}(1)$ group.

It is convenient to introduce the weak angle θ_W. We have

$$\frac{g'}{g} = \tan\theta_W. \tag{3.106}$$

From (3.97) and (3.98) we find that

$$Z_{\alpha} = \cos\theta_W\, A_{\alpha}^{3} - \sin\theta_W\, B_{\alpha}, \quad A_{\alpha} = \sin\theta_W\, A_{\alpha}^{3} + \cos\theta_W\, B_{\alpha}. \tag{3.107}$$

The condition of the unification of the weak and electromagnetic interactions (3.105) takes the form

$$g\, \sin\theta_W = e. \tag{3.108}$$

From (3.101), (3.102) and (3.103) follows that the Lagrangian of the interaction of the quarks and vector bosons is the sum of the CC Lagrangian, the NC Lagrangian and the electromagnetic Lagrangian:

$$\mathcal{L}_I = \left(-\frac{g}{2\sqrt{2}}\, j^{\rm CC}_{\alpha}\, W^{\alpha} + {\rm h.c.}\right) - \frac{g}{2\,\cos\theta_W}\, j^{\rm NC}_{\alpha}\, Z^{\alpha} - e\, j^{\rm EM}_{\alpha}\, A^{\alpha}. \tag{3.109}$$

Here

$$j^{\rm CC}_{\alpha} = 2\, j^{1+i2}_{\alpha} = 2 \sum_{u'_1=u'..;d'_1=d'..} \bar{u}'_{1L}\, \gamma_{\alpha}\, d'_{1L} \tag{3.110}$$

is the charged current,

$$j^{\rm EM}_{\alpha} = \sum_{q'=d',..u'..} e_{q'}\, \bar{q}'\, \gamma_{\alpha}\, q' \tag{3.111}$$

is the electromagnetic current ($e_{u'_1} = 2/3, \;\; e_{d'_1} = -1/3$) and

$$j^{\rm NC}_{\alpha} = 2\, j^3_{\alpha} - 2\, \sin^2\theta_W\, j^{\rm EM}_{\alpha} \tag{3.112}$$

is the neutral current.

The relations (3.100) are based on the assumption that the Higgs field is transformed as the $SU(2)$ doublet. We will show now that in the SM with the Higgs doublet *masses of* $W^{\pm}$ *and* Z^0 *bosons can be predicted*. The SM also allows to calculate the vacuum expectation value v. In fact, the Fermi constant G_F, which characterizes the effective four-fermion weak interaction induced by the exchange of the virtual W-boson at $Q^2 \ll m^2_W$, is given by the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\, m^2_W}. \tag{3.113}$$

The value of the Fermi constant is well known. The most precise value

$$G_F = 1.16637(1) \cdot 10^{-5}\ {\rm GeV}^{-2}$$

was obtained from the investigation of the μ-decay.

From (3.100) and (3.113) we easily find

$${\rm v} = (\sqrt{2}\, G_F)^{-1/2} \simeq 246\,{\rm GeV}. \tag{3.114}$$

This value characterizes the scale of the breaking of the electroweak symmetry.

From (3.100) and (3.106) we obtain the following relation between the masses of the W and Z bosons

$$\frac{m_W}{m_Z} = \cos\theta_W. \tag{3.115}$$

Finally, taking into account the unification condition (3.108), we find the following expressions for the masses of the W and Z bosons

$$m_W = \left(\frac{\pi\,\alpha}{\sqrt{2}\,G_F}\right)^{1/2} \frac{1}{\sin\theta_W}, \quad m_Z = \left(\frac{\pi\,\alpha}{\sqrt{2}\,G_F}\right)^{1/2} \frac{1}{\sin\theta_W \cos\theta_W}, \tag{3.116}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. We have

$$\left(\frac{\pi\,\alpha}{\sqrt{2}\,G_F}\right)^{1/2} = 37.2895(2) \cdot \mathrm{GeV} \tag{3.117}$$

The parameter $\sin^2\theta_W$ characterizes the neutral current (see (3.112)). The value of this parameter was determined from the data of numerous experiments on the investigation of NC processes. From the existing data it was found that

$$\sin^2\theta_W = 0.23122(15) \tag{3.118}$$

If we take into account radiative corrections, the relation (3.116) for the mass of the W boson is modified. We have in this case

$$m_W = \left(\frac{\pi\,\alpha}{\sqrt{2}\,G_F}\right)^{1/2} \frac{1}{\sin\theta_W (1 - \Delta r)}, \tag{3.119}$$

where the term Δr is due to the radiative corrections. For this term the value $\Delta r = 0.06969 \pm 0.00004 \pm 0.00014$ was obtained. From existing data for the masses of the $W^{\pm}$ and Z^0 bosons the following values were found

$$m_Z = (91.1874 \pm 0.0021) \cdot \mathrm{GeV} \qquad m_W = (89.403 \pm 0.029) \cdot \mathrm{GeV}. \tag{3.120}$$

These values are in a perfect agreement with the prediction of the Standard Model.[4]

We will now turn to the consideration of the generation of the quark masses. The mass term of the quark field $q(x)$ has the form

$$\mathcal{L}_m = -m_q\,\bar{q}\,q = -m_q\,\bar{q}_L\,q_R + \mathrm{h.c.}, \tag{3.121}$$

where m_q is the mass of the q-quark. In the SM left-handed fields are components of $SU(2)$ doublets and right-handed fields are $SU(2)$ singlets. Thus, the quark mass terms are not invariant under the $SU(2) \times U(1)$ transformations.

Masses (and mixing) of the quarks are generated in the SM via the mechanism of the spontaneous symmetry breaking. We will see, however, that unlike the case of

[4] The precise values of the fundamental parameters of the Standard Model can be found in the review "Electroweak model and constraints on new physics" published by the Particle Data Group in C. Amsler et al. [168].

the masses of W and Z bosons the standard Higgs mechanism does not put any constraints on the masses of quarks. In the SM, masses of quarks are free parameters.

Let us assume that in the total Lagrangian of the Standard Model there is the following Lagrangian of the Yukawa interaction of the quark and Higgs fields

$$\mathcal{L}_Y^{\rm down} = -\frac{\sqrt{2}}{\rm v} \sum_{a,q} \bar{\psi}_{aL} \, M_{aq'}^{\rm down} \, q'_R \, \phi + \text{h.c.}, \tag{3.122}$$

where $M_{aq'}^{\rm down}$ is a complex 3×3 matrix. Because ψ_{aL} and ϕ are the $SU(2)$ doublets and q'_R are singlets, it is obvious that the Lagrangian $\mathcal{L}_Y^{\rm down}$ is the $SU(2)$ scalar. Let us also require $U(1)$ invariance, i.e. the conservation of the hypercharge. The hypercharges of the quark and Higgs doublets are equal to 1/3 and 1, correspondingly. The Lagrangian (3.122) conserves the hypercharge if $2e_{q'_R} + 1 = 1/3$. Thus, $e_{q'_R} = -1/3$, i.e. the right-handed fields q'_R in (3.122) are the fields of the "down" quarks d'_R, s'_R, b'_R.

From (3.89) and (3.122) after the spontaneous symmetry breaking we find

$$\mathcal{L}_Y^{\rm down} = -\overline{D}'_L \, M^{\rm down} \, D'_R \, \left(1 + \frac{H}{\rm v}\right) + \text{h.c.} \tag{3.123}$$

Here

$$D'_{L,R} = \begin{pmatrix} d'_{L,R} \\ s'_{L,R} \\ b'_{L,R} \end{pmatrix} \tag{3.124}$$

The first term of (3.123) is the mass term of the down quarks and the second term is the Lagrangian of the interaction of the down quark and the Higgs boson field.

In order to obtain the mass term of up quarks we will use the conjugated Higgs doublet

$$\tilde{\phi} = i \, \tau_2 \phi^*. \tag{3.125}$$

The hypercharge of the doublet $\tilde{\phi}$ is equal to -1. From (3.89) and (3.125) we have

$$\tilde{\phi} = \begin{pmatrix} \frac{v+H}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{3.126}$$

We will assume that in addition to (3.122) the following Lagrangian of the Yukawa interaction of quarks and Higgs bosons enters in the total Lagrangian

$$\mathcal{L}_Y^{\rm up} = -\frac{\sqrt{2}}{\rm v} \sum_{a,q} \overline{\psi}_{aL} \;\; M_{aq'}^{\rm up} \, q'_R \;\, \tilde{\phi} + \text{h.c.}, \tag{3.127}$$

where $M^{\rm up}$ is a complex 3×3 matrix. From the conservation of the hypercharge we have $2e_{q'_R} + (-1) = 1/3$. Thus, $e_{q'_R} = 2/3$ and the index q' in (3.127) runs over u'_R, c'_R, t'_R. After the spontaneous symmetry breaking we find from (3.126) and (3.127)

$$\mathcal{L}_Y^{\rm up} = -\overline{U}'_L\, M^{\rm up}\, U'_R\, (1 + \frac{H}{\rm v}) + {\rm h.c.}, \tag{3.128}$$

where

$$U'_{L,R} = \begin{pmatrix} u'_{L,R} \\ c'_{L,R} \\ t'_{L,R} \end{pmatrix}. \tag{3.129}$$

The first term of (3.128) is the mass term of the up quarks and the second term is the Lagrangian of the interaction of the up quarks and the scalar Higgs field.

Let us now bring the mass terms of up and down quarks to the diagonal form. The complex matrices $M^{\rm up}$ and $M^{\rm down}$ can be diagonalized by the biunitary transformations (see Appendix B). We have

$$M^{\rm up} = V_L^{\rm up}\, m^{\rm up}\, V_R^{\rm up\dagger} \quad M^{\rm down} = V_L^{\rm down}\, m^{\rm down}\, V_R^{\rm down\dagger}. \tag{3.130}$$

Here $V_{L,R}^{\rm up}$ and $V_{L,R}^{\rm down}$ are unitary 3×3 matrices and $m^{\rm up}$ and $m^{\rm down}$ are diagonal matrices with positive diagonal elements.

From (3.123), (3.128) and (3.130) we find the following expressions for the quark mass terms

$$\mathcal{L}_m^{\rm up} = -\bar{U}\, m^{\rm up}\, U, \quad \mathcal{L}_m^{\rm down} = -\bar{D}\, m^{\rm down}\, D. \tag{3.131}$$

Here

$$U = U_L + U_R = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad D = D_L + D_R = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \tag{3.132}$$

$$m^{\rm up} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad m^{\rm down} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \tag{3.133}$$

and

$$U_{L,R} = V_{L,R}^{\rm up\dagger}\, U'_{L,R}, \quad D_{L,R} = V_{L,R}^{\rm down\dagger}\, D'_{L,R}. \tag{3.134}$$

From (3.131), (3.132) and (3.133) we obtain the standard mass terms of up and down quarks

$$\mathcal{L}_m^{\rm up}(x) = -\sum_{u_1=u,c,t} m_{u_1}\, \bar{u}_1(x)\, u_1(x), \quad \mathcal{L}_m^{\rm down}(x) = -\sum_{d_1=d,s,b} m_{d_1}\, \bar{d}_1(x)\, d_1(x). \tag{3.135}$$

Thus, $q(x)$ is the field of the q-quark with the mass m_q $(q = u, d, c, s, t, b)$.

The left-handed and right-handed fields of quarks with definite masses and primed quark fields, which have definite $SU(2) \times U(1)$ transformation properties, are connected by the unitary transformations (3.134).

Let us consider now the charged current of the quarks. From (3.110), (3.124) and (3.129) we find

$$j_\alpha^{\mathrm{CC}}(x) = 2\,\bar{U}'_L(x)\,\gamma_\alpha\,D'_L(x) \tag{3.136}$$

We will write down the charged current in terms of the fields of quarks with definite masses. From (3.134) and (3.136) we find

$$j_\alpha^{\mathrm{CC}}(x) = 2\,\bar{U}_L(x)\,\gamma_\alpha\,V\,D_L(x). \tag{3.137}$$

Here

$$V = (V_L^{\mathrm{up}})^\dagger\,V_L^{\mathrm{down}} \tag{3.138}$$

From (3.132) and (3.137) follows that the CC can be presented in the following form

$$j_\alpha^{\mathrm{CC}}(x) = 2\,[\bar{u}_L(x)\,\gamma_\alpha\,d_L^{\mathrm{mix}}(x) + \bar{c}_L(x)\,\gamma_\alpha\,s_L^{\mathrm{mix}}(x) + \bar{t}_L(x)\,\gamma_\alpha\,b_L^{\mathrm{mix}}(x)], \tag{3.139}$$

where

$$\begin{aligned} d_L^{\mathrm{mix}}(x) &= \sum_{d_1=d,s,b} V_{ud_1}\,d_{1L}(x) \\ s_L^{\mathrm{mix}}(x) &= \sum_{d_1=d,s,b} V_{cd_1}\,d_{1L}(x) \\ b_L^{\mathrm{mix}}(x) &= \sum_{d_1=d,s,b} V_{td_1}\,d_{1L}(x). \end{aligned} \tag{3.140}$$

From (3.138) follows that V is a unitary matrix

$$V^\dagger V = 1. \tag{3.141}$$

We came to an important conclusion: *the left-handed components of fields of the down quarks $d_L(x), s_L(x), b_L(x)$ enter into the CC of the SM in "mixed form" $d_L^{\mathrm{mix}}(x)$, $s_L^{\mathrm{mix}}(x)$, $b_L^{\mathrm{mix}}(x)$*. The unitary 3×3 mixing matrix V is called the Cabibbo -Kobayashi-Maskawa (CKM) mixing matrix. Let us stress that the mixing of quarks is due to the fact that the unitary matrices V_L^{up} and V_L^{down}, which connect left-handed primed and physical fields of up and down quarks, are *different*. It follows from (3.139) and (3.140) that the charged current changes the flavor of quarks ($d \to u$, $s \to u$, $c \to s$, etc).

Let us now express the electromagnetic current through the fields of physical quarks. From (3.111) we have

$$j_\alpha^{\mathrm{EM}} = \frac{2}{3}\,(\bar{U}'_L\,\gamma_\alpha\,U'_L + \bar{U}'_R\,\gamma_\alpha\,U'_R) - \frac{1}{3}\,(\bar{D}'_L\,\gamma_\alpha\,D'_L + \bar{D}'_R\,\gamma_\alpha\,D'_R). \tag{3.142}$$

Taking into account the unitarity of the matrices $V^{\mathrm{up}}_{L,R}$ and $V^{\mathrm{down}}_{L,R}$ we find

$$j_\alpha^{\mathrm{EM}} = \frac{2}{3}\,\bar{U}\,\gamma_\alpha\,U - \frac{1}{3}\,D\,\gamma_\alpha\,D. \tag{3.143}$$

From (3.132) and (3.143) we have

$$j_\alpha^{\mathrm{EM}}(x) = \sum_{q=u,d,c,\ldots} e_q\,\bar{q}(x)\,\gamma_\alpha\,q(x), \tag{3.144}$$

where $e_{u,c,t} = \frac{2}{3}$ and $e_{d,s,b} = -\frac{1}{3}$. Thus, we come to the standard expression for the electromagnetic current of quarks, which is diagonal in quark flavors.

Let us consider the neutral current of the Standard Model. From (3.112) we find

$$j_\alpha^{\mathrm{NC}} = \bar{U}'_L\,\gamma_\alpha\,U'_L - D'_L\,\gamma_\alpha\,D'_L - 2\,\sin^2\theta_W\,j_\alpha^{\mathrm{EM}} \tag{3.145}$$

In order to come to the fields of the physical quarks we will use the relations (3.134). Taking into account that V^{up}_L and V^{down}_L are unitary matrices we find

$$j_\alpha^{\mathrm{NC}} = \bar{U}_L\,\gamma_\alpha\,U_L - D_L\,\gamma_\alpha\,D_L - 2\,\sin^2\theta_W\,j_\alpha^{\mathrm{EM}}. \tag{3.146}$$

Finally from (3.146) we find that the neutral current is given by the following expression

$$j_\alpha^{\mathrm{NC}}(x) = \sum_{u_1=u,c,t} \bar{u}_1(x)\,\gamma_\alpha\,u_1(x) - \sum_{d_1=d,s,b} \bar{d}_1(x)\,\gamma_\alpha\,d_1(x) - 2\,\sin^2\theta_W\,j_\alpha^{\mathrm{EM}}(x). \tag{3.147}$$

From this expression we conclude that the *neutral current of the SM is diagonal in quark flavors.*

3.5 The Standard Model for Leptons

The Standard Model for neutrinos and charged leptons is based on the same $SU(2) \times U(1)$ local gauge group as the SM for the quarks we have considered in the previous section. We assume that the left-handed fields are transformed as doublets of the $SU(2)$ group

$$\psi_{eL} = \begin{pmatrix} \nu'_{eL} \\ e'_L \end{pmatrix}, \; \psi_{\mu L} = \begin{pmatrix} \nu'_{\mu L} \\ \mu'_L \end{pmatrix}, \; \psi_{\tau L} = \begin{pmatrix} \nu'_{\tau L} \\ \tau'_L \end{pmatrix} \tag{3.148}$$

and the right-handed lepton and neutrino fields l'_R and ν'_{lR} ($l = e, \mu, \tau$) are singlets of the group.

In order to obtain the Lagrangian of the interaction of the leptons and gauge vector bosons which satisfy the requirements of the local $SU(2) \times U(1)$ invariance we will change in the free Lagrangian the derivatives of the fields by the covariant derivatives. In the case of the lepton doublets (3.148) the following change must be performed

$$\partial_\alpha \psi_{lL} \to \left(\partial_\alpha + ig\,\frac{1}{2}\,\tau \cdot \mathbf{A}_\alpha + ig'\,\frac{1}{2}\,Y_L^{\mathrm{lep}} B_\alpha(x)\right) \psi_{lL} \quad l = e, \mu, \tau. \tag{3.149}$$

Let us stress that the coupling constant g must be the same in (3.76) and (3.149). This is connected with the fact that $SU(2)$ is a nonabelian group and the constant g enters into the field strength (see (3.23)). There is no such requirement for the abelian $U(1)$ group. This allow us to choose the hypercharges of all fields in such a way that the Gell-Mann-Nishigima relation (3.37) is satisfied. According to this relation

$$Y_L^{\mathrm{lep}} = -1, \quad Y_R^{\mathrm{lep}} = -2, \quad Y_L^{\nu} = 0, \tag{3.150}$$

where Y_L^{lep} is the hypercharge of the lepton doublets and Y_R^{lep} and Y_R^{ν} are hypercharges of the right-handed lepton and neutrino fields, respectively. Thus, in the right-handed part of the kinetic term of the free Lagrangian we must make the following change

$$\partial_\alpha\, l'_R \to \left(\partial_\alpha + ig'\,\frac{1}{2}\,(-2)\,B_\alpha\right) l'_R. \tag{3.151}$$

The $SU(2) \times U(1)$ invariant Lagrangian of the minimal interaction of lepton and vector boson fields is given by

$$\mathcal{L}_I^{\mathrm{lep}} = -g\,\mathbf{j}_\alpha\,\mathbf{A}^\alpha - g'\,\frac{1}{2}\,j_\alpha^Y\,B^\alpha. \tag{3.152}$$

Here

$$\mathbf{j}_{\vec{\alpha}} = \sum_{l=e,\mu,\tau} \bar{\psi}_{lL}\,\gamma_\alpha\,\frac{1}{2}\,\vec{\tau}\,\psi_{lL}, \quad \frac{1}{2}\,j_\alpha^Y = j_\alpha^{\mathrm{EM}} - j_\alpha^3 \tag{3.153}$$

where

$$j_\alpha^{\mathrm{EM}} = \sum_l (-1)\,\bar{l}'\,\gamma_\alpha\,l' \tag{3.154}$$

is the electromagnetic current of the charged leptons.

Repeating now transformations that were performed in the previous section we come to the following Lagrangian of the interaction of the charged leptons and neutrinos with the $W^{\pm}$ and Z^0 bosons and photons

$$\mathcal{L}_I^{\mathrm{lep}} = \left(-\frac{g}{2\sqrt{2}}\, j_\alpha^{\mathrm{CC}}\, W^\alpha + \mathrm{h.c.}\right) - \frac{g}{2\cos\theta_W}\, j_\alpha^{\mathrm{NC}}\, Z^\alpha - e\, j_\alpha^{\mathrm{EM}}\, A^\alpha. \tag{3.155}$$

Here

$$j_\alpha^{\mathrm{CC}} = 2 j_\alpha^{1+i2} = 2 \sum_{l=e,\mu,\tau} \bar{\nu}'_{lL}\, \gamma_\alpha\, l'_L \tag{3.156}$$

is the leptonic charged current,

$$j_\alpha^{\mathrm{NC}} = 2\, j_\alpha^3 - 2\, \sin^2\theta_W\, j_\alpha^{\mathrm{EM}} = \sum_{l=e,\mu,\tau} \bar{\nu}'_{lL}\, \gamma_\alpha\, \nu'_{lL} - \sum_{l=e,\mu,\tau} \bar{l}'_{lL}\, \gamma_\alpha\, l'_{lL} - 2\, \sin^2\theta_W\, j_\alpha^{\mathrm{EM}} \tag{3.157}$$

is the neutral current. The leptonic electromagnetic current is given by expression (3.154).

We will now come to the spontaneous violation of symmetry. Let us consider first charged leptons. The $SU(2) \times U(1)$ invariant Lagrangian of the Yukawa interaction of lepton and Higgs fields has the form

$$\mathcal{L}_Y^{\mathrm{lep}} = -\frac{\sqrt{2}}{\mathrm{v}} \sum_{l,\, l_1} \overline{\psi}_{lL}\, M_{ll'_1}^{\mathrm{lep}}\, l'_{1R}\, \phi + \mathrm{h.c.}, \tag{3.158}$$

where M^{lep} is a 3×3 complex matrix and ϕ is the doublet of Higgs fields. If we choose for the field $\phi(x)$ the expression (3.91) the symmetry will be spontaneously broken and for the Lagrangian $\mathcal{L}_Y^{\mathrm{lep}}$ we find the following expression

$$\mathcal{L}_Y^{\mathrm{lep}} = -\overline{L}'_L\, M^{\mathrm{lep}}\, L'_R \left(1 + \frac{H}{\mathrm{v}}\right) + \mathrm{h.c.} \tag{3.159}$$

Here

$$L'_{L,R} = \begin{pmatrix} e'_{L,R} \\ \mu'_{L,R} \\ \tau'_{L,R} \end{pmatrix}. \tag{3.160}$$

Let us now diagonalize the matrix M^{lep}. We have (see Appendix B)

$$M^{\mathrm{lep}} = U_L\, m^{\mathrm{lep}}\, U_R^\dagger. \tag{3.161}$$

where $U_{L,R}$ are unitary matrices and m^{lep} is the diagonal matrix with positive diagonal elements. From (3.159) and (3.161) we find

$$\mathcal{L}_Y^{\text{lep}} = -\overline{L}_L\, m^{\text{lep}}\, L_R \left(1 + \frac{H}{\text{v}}\right) + \text{h.c.}, \tag{3.162}$$

where

$$L_L = U_L^\dagger\ L'_L, \quad L_R = U_R^\dagger\ L'_R. \tag{3.163}$$

From (3.162) we obtain the following expression for the Lagrangian

$$L_Y^{\text{lep}} = -\overline{L}\, m^{\text{lep}}\, L \left(1 + \frac{H}{v}\right). \tag{3.164}$$

Here

$$L = L_L + L_R = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \quad m^{\text{lep}} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{3.165}$$

The Lagrangian $L_Y^{\text{lep}}(x)$ has the form

$$L_Y^{\text{lep}}(x) = \sum_{l=e,\mu,\tau} m_l\, \bar{l}(x)\, l(x) \left(1 + \frac{H(x)}{\text{v}}\right). \tag{3.166}$$

The first term of the Lagrangian (3.166) is the standard mass term of the charged leptons. The field $l(x)$ is the field of the leptons $l^\pm$ with the mass m_l ($l = e, \mu, \tau$). The second term of (3.166) is the Lagrangian of the interaction of the lepton and Higgs fields.

Let us express now the lepton charged current through the fields of physical leptons. The expression (3.156) can be written in the following matrix form

$$j_\alpha^{\text{CC}}(x) = 2\, \bar{\nu}'_L(x)\, \gamma_\alpha\, L'_L(x), \tag{3.167}$$

where

$$\nu'_L = \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}. \tag{3.168}$$

Taking into account (3.163), we have for the leptonic charged current

$$j_\alpha^{\text{CC}}(x) = 2\, \bar{\nu}_L^f(x)\, \gamma_\alpha\, L_L(x) = 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\, \gamma_\alpha\, l_L(x). \tag{3.169}$$

Here $l(x)$ is the field of lepton l with the mass m_l and

$$\nu_L^f = U_L^\dagger \, \nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \tag{3.170}$$

Let us consider now the electromagnetic current of leptons. Taking into account the unitarity of the matrices $U_{L,R}$ we find from (3.154)

$$j_\alpha^{\mathrm{EM}}(x) = -\bar{L}'_L(x)\,\gamma_\alpha\, L'_L(x) - \bar{L}'_R(x)\,\gamma_\alpha\, L'_R(x) = -\bar{L}(x)\,\gamma_\alpha\, L(x). \tag{3.171}$$

From (3.171) and (3.165) we obtain the following standard expression for the electromagnetic current of the leptons

$$j_\alpha^{\mathrm{EM}}(x) = \sum_{l=e,\mu,\tau} (-1)\,\bar{l}(x)\,\gamma_\alpha\, l(x). \tag{3.172}$$

For the lepton neutral current we have from (3.157)

$$j_\alpha^{\mathrm{NC}}(x) = \bar{\nu}'_L(x)\,\gamma_\alpha\,\nu'_L(x) - \bar{L}'_L(x)\,\gamma_\alpha\, L'_L(x) - 2\,\sin^2\theta_W\, j_\alpha^{\mathrm{EM}}(x). \tag{3.173}$$

In terms of the flavor neutrino fields and fields of physical leptons from (3.173) we find the following expression for the neutral current

$$j_\alpha^{\mathrm{NC}}(x) = \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\,\gamma_\alpha\,\nu_{lL}(x) - \sum_{l=e,\mu,\tau} \bar{l}_L(x)\,\gamma_\alpha\, l_L(x) - 2\,\sin^2\theta_W\, j_\alpha^{\mathrm{EM}}(x), \tag{3.174}$$

where $j_\alpha^{\mathrm{EM}}(x)$ is given by (3.172).

Finally it is easy to see that the kinetic term of the Lagrangian of the charged leptons takes the standard form

$$\mathcal{L}_0^{\mathrm{lep}}(x) = \bar{L}'(x)\, i\,\gamma^\alpha\partial_\alpha\, L'(x) = \sum_{1=e,\mu,\tau} \bar{l}(x)\, i\,\gamma^\alpha\partial_\alpha\, l(x). \tag{3.175}$$

For the kinetic term of the Lagrangian of the neutrino fields we find

$$\mathcal{L}_0(x) = \bar{\nu}'_L(x)\, i\,\gamma^\alpha\partial_\alpha\,\nu'_L(x) = \bar{\nu}_L^f(x)\, i\,\gamma^\alpha\partial_\alpha\,\nu_L^f(x) = \sum_{1=e,\mu,\tau} \bar{\nu}_{lL}(x)\, i\,\gamma^\alpha\partial_\alpha\,\nu_{lL}(x). \tag{3.176}$$

The Standard Model proposed by Glashow, Weinberg and Salam was based on the two-component neutrino theory. Thus, only left-handed neutrino fields enter into the Lagrangian of the original SM. If the Higgs field is transformed as doublet it is impossible to generate neutrino masses. Thus, *the original SM was built for massless neutrinos*.

For the massless neutrinos the total Lagrangian is invariant under the global gauge transformations

$$\nu'_{lL}(x) = e^{i\,\Lambda_l}\,\nu_{lL}(x), \quad l'(x) = e^{i\,\Lambda_l}\,l(x), \quad q'(x) \to q(x), \quad (l = e, \mu, \tau) \tag{3.177}$$

where Λ_l are arbitrary constant phases. From the invariance under the transformations (3.177) follows that the total electron L_e, muon L_μ and tau L_τ lepton numbers are conserved :

$$\sum_i L_e^i = \text{const} \quad \sum_i L_\mu^i = \text{const} \quad \sum_i L_\tau^i = \text{const}. \tag{3.178}$$

The lepton numbers of the particles are presented in Table 3.1. The lepton numbers of the antiparticles are opposite to the lepton numbers of the corresponding particles. The conservation of the total lepton numbers means that in the CC decays together with a μ^+ a muon neutrino ν_μ is produced, in the process of the CC interaction of an electron antineutrino $\bar{\nu}_e$ with a nucleon a e^+ is produced, etc. We know at present (see below) that the law of the conservation of electron, muon and tau lepton numbers is an approximate one. It is violated in neutrino oscillations due to the small neutrino masses and the neutrino mixing.

In the original (minimal) Standard Model, neutrino masses are equal to zero. We will now show that formally neutrino masses can be generated by the Standard Higgs mechanism. However, as we will see later, it is very unlikely that the SM mechanism of the mass generation is responsible for the observed neutrino masses. A new mechanism of neutrino mass generation is needed.

In order to generate neutrino masses we assume that in the total Lagrangian enters the following $SU(2) \times U(1)$ invariant Lagrangian of the Yukawa interaction of lepton and Higgs fields

$$\mathcal{L}_Y^\nu = -\frac{\sqrt{2}}{\mathrm{v}} \sum_{l',l} \overline{\psi}_{l'L}\, M'_{l'\,l}\, \nu'_{lR}\, \tilde{\phi} + \text{h.c.}, \tag{3.179}$$

where the right-handed fields ν'_{lR} are singlets of the $SU(2)$ group, M' is a complex 3×3 matrix and $\tilde{\phi}$ is the conjugated Higgs doublet given by (3.125).

After the spontaneous symmetry braking we find from (3.126) and (3.179)

$$\mathcal{L}_Y^\nu = -\sum_{l',l} \bar{\nu}'_{l'L}\, M'_{l'\,l}\, \nu'_{lR} \left(1+\frac{H}{\mathrm{v}}\right) + \text{h.c.} = -\bar{\nu}'_L\, M'\, \nu'_R \left(1+\frac{H}{\mathrm{v}}\right) + \text{h.c.}, \tag{3.180}$$

Table 3.1 Lepton numbers of the particles

	$\nu_e,\ e^-$	$\nu_\mu,\ \mu^-$	$\nu_\tau,\ \tau^-$	Quarks, W, Z, γ
L_e	1	0	0	0
L_μ	0	1	0	0
L_τ	0	0	1	0

where

$$\nu'_L = \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \quad \nu'_R = \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}. \tag{3.181}$$

With the help of (3.170) we find

$$\mathcal{L}_Y^{\nu} == -\bar{\nu}_L^f \, M \, \nu'_R \left(1 + \frac{H}{v}\right) + \text{h.c.}, \tag{3.182}$$

where $M = U_L^{\dagger} \, M'$. The first term of (3.182) is the neutrino mass term

$$\mathcal{L}^m = -\bar{\nu}_L^f \, M \, \nu'_R + \text{h.c.} = -\sum_{l,l'} \bar{\nu}_{l'L} \, M_{l';l} \, \nu'_{lR} + \text{h.c.} \tag{3.183}$$

For the complex matrix M we have

$$M = U \, m \, V^{\dagger}, \tag{3.184}$$

where U and V are unitary matrices and $m_{ik} = m_i \, \delta_{ik}, \;\; m_i > 0$.

For the neutrino mass term from (3.183) and (3.184) we find the following expression

$$\mathcal{L}^m(x) = -\bar{\nu}(x) \, m \, \nu(x) = -\sum_{i=1}^{3} m_i \; \bar{\nu}_i(x)\nu_i(x). \tag{3.185}$$

Here

$$U^{\dagger}\nu_L^f = \nu_L, \quad V^{\dagger}\nu'_R = \nu_R, \quad \nu = \nu_L + \nu_R. \tag{3.186}$$

and

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{3.187}$$

Thus, $\nu_i(x)$ is the field of the neutrino with the mass m_i. From (3.186) follows that the flavor neutrino fields $\nu_{lL}(x)$, which enter into the leptonic charged and neutral currents (3.169) and (3.174), are connected with the left-handed components of the massive neutrino fields $\nu_{iL}(x)$ by the *mixing relation*

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li} \; \nu_{iL}(x), \quad (l = e, \mu, \tau) \tag{3.188}$$

where U is the unitary mixing matrix. For the right-handed fields we have

$$\nu'_{lR}(x) = \sum_{i=1}^{3} V_{li}\ \nu_{iR}(x). \quad (l = e, \mu, \tau) \tag{3.189}$$

We have seen that in the case the massless neutrinos the electron L_e, muon L_μ and tau L_τ lepton numbers are conserved. In the case of the massive and mixed neutrinos the total Lagrangian is not invariant under the transformations (3.177) and L_e, L_μ and L_τ are not conserved. However, from (3.166), (3.169), (3.174) and (3.183) follows that the total Lagrangian is invariant under the following global gauge transformation

$$\nu_{lL}(x) \to e^{i\,\Lambda}\,\nu_{lL}(x), \quad \nu'_{lR}(x) \to e^{i\,\Lambda}\,\nu'_{lR}(x), \quad l(x) \to e^{i\,\Lambda}\,l(x) \quad (l = e.\mu, \tau), \tag{3.190}$$

where Λ is an arbitrary constant phase, the same for all lepton flavors. The invariance under the transformation (3.190) means that the total lepton number

$$L = L_e + L_\mu + L_\tau \tag{3.191}$$

is conserved

$$\sum_i L^i = \text{const.} \tag{3.192}$$

The mass term (3.183) is called *Dirac mass term*. The fields of the massive neutrinos $\nu_i(x)$ are *fields of the neutrinos ν_i and antineutrinos $\bar{\nu}_i$*. Neutrinos and antineutrinos have the same mass and differ by the conserved total lepton number: $L(\nu_i) = 1, \quad L(\bar{\nu}_i) = -1$.

Thus, we have shown that the Dirac neutrino masses can be generated by the Standard Higgs mechanism. There are, however, no theoretical constraints on the Yukawa couplings in the Lagrangians (3.122), (3.123), (3.158), (3.179). Thus, the Standard Model cannot predict the masses of the fermions.

In Table 3.2 we presented the masses of quarks and leptons. As is seen from Table 3.2, the mass of the electron is comparable with the masses of the u and d quarks. The mass of the muon (tau) differs from the masses of the quarks of the second (third) family by one (two) orders of magnitude. The absolute values of the neutrino masses are unknown at present. From the data of the tritium experiments, which we will discuss later, only the upper bound ($m_i < 2.2$ eV) was obtained. From Table 3.2 we see that the masses of the neutrinos are many orders of magnitude smaller than the masses of the quarks and leptons. For example, in the third family $m_3 \lesssim 10^{-9}\ m_\tau$, $m_3 \lesssim 10^{-9}\ m_b$, $m_3 \lesssim 10^{-11}\ m_t$.

It is very unnatural to assume that the same standard Higgs mechanism is responsible for the generation of the masses of the charged leptons, quarks *and the neutrinos*. It is common opinion that the neutrino masses are generated by a new, beyond the SM mechanism. We will consider such mechanisms later.

Table 3.2 Masses of quarks and leptons

I	$m_u = (1.5 - 3.3)$ MeV	$m_d = (3.5 - 6.0)$ MeV	$m_e = 0.511$ MeV	$\nu_1 \leq 2.2$ eV
II	$m_c = (1.27^{+0.07}_{-0.11})$ GeV	$m_s = (105^{+25}_{-35})$ MeV	$m_\mu = 105.658$ MeV	$\nu_2 \leq 2.2$ eV
III	$m_t = (171.3 \pm 1.1 \pm 1.2)$ GeV	$m_b = (4.20^{+0.17}_{-0.07})$ GeV	$m_\tau = 1.777$ GeV	$\nu_3 \leq 2.2$ eV

3.6 Summary and Outlook

The unified theory of the weak and electromagnetic interactions (The Standard Model) is based on the following basic principles

1. Local $SU(2) \times U(1)$ invariance of the Lagrangian of massless quark, lepton and vector fields with left-handed doublets and right-handed singlets.
2. Higgs mechanism (with Higgs doublet) of the generations of the masses of $W^{\pm}$ and Z^0 bosons, the quarks and the charged leptons.
3. Minimal interaction.

The SM Lagrangian contains as a low-energy limit the effective classical current$\times$current Lagrangian of the CC weak interaction. The SM *predicts* a new type of the weak interaction (NC), the existence of the vector $W^{\pm}$ and Z^0 bosons and the values of the masses of these particles.

In order that the Standard Model will be a renormalizable theory it is necessary that the sum of the electric charges of the particles, the fields of which are components of the doublets, is equal to zero:

$$3\left(\frac{2}{3} + \left(-\frac{1}{3}\right)\right) N_f^{\text{quarks}} + (0 + (-1))\, N_f^{\text{leptons}} = 0. \tag{3.193}$$

Here N_f^{quarks} and N_f^{leptons} are the numbers of the quark and lepton families. We took into account in (3.193) that there exist three colored quarks of each type. Thus, we have

$$N_f^{\text{quarks}} = N_f^{\text{leptons}}. \tag{3.194}$$

After the τ-lepton was discovered (1975) , the SM allowed to predict the existence of ν_τ-$SU(2)$ partner of τ and in accordance with (3.194) to predict the existence of the third family of quarks (b and t). All predictions of the Standard Model were perfectly confirmed by numerous experiments.

The minimal SM can be built for massless neutrinos. We have seen that Dirac neutrino masses can be generated in the framework of the SM with the Higgs doublet. However, it is highly unlikely that the standard Higgs mechanism is the mechanism of the generation of small neutrino masses. It is a common point of view that small neutrino masses are an evidence in favor of a beyond the SM mechanism of neutrino mass generation.

Chapter 4
Neutrino Mass Terms

4.1 Introduction

The neutrino mass term is the central object of the theory of massive and mixed neutrinos. It determines neutrino masses, neutrino mixture and neutrino nature (Dirac and Majorana). The possibility of the existence of so-called sterile neutrinos is also determined by the neutrino mass term.

In modern theories mass terms of fermions appear in the Lagrangian as a result of the braking of underlying symmetries. We have seen in the previous chapter how the neutrino mass term could appear in the SM with right-handed neutrino singlets.

Here we will consider all possible types of neutrino mass terms. Our discussion will be general, based only on Lorentz invariance. We will only use the fact that a mass term of any spin-1/2 field is a sum of Lorentz-invariant products of left-handed and right-handed components of the field.[1] It was established by the LEP experiments at CERN that three flavor neutrinos ν_e, ν_μ, ν_τ exist in nature. These flavor neutrinos take part in CC and NC weak processes due to the electroweak interaction via the standard leptonic charged and neutral currents[2]

$$j_\alpha^{\mathrm{CC}}(x) = 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\, \gamma_\alpha\, l_L(x), \quad j_\alpha^{\mathrm{NC}} = \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\, \gamma_\alpha\, \nu_{lL}(x). \tag{4.1}$$

Here $l(x)$ is the operator of physical charged leptons $l^{\pm}$ with mass m_l.

The fields $\nu_{lL}(x)$ $(l = e, \mu, \tau)$ must enter into the neutrino mass term. The structure of mass term depends on

- other fields (if any) which enter into the mass term,
- the conservation of the total lepton number $L = L_e + L_\mu + L_\tau$.

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[1] We remind that a fermion field $\psi(x)$ can be presented in the form $\psi(x) = \psi_L(x) + \psi_R(x)$, where left-handed $\psi_L(x)$ and right-handed $\psi_R(x)$ components of the field $\psi(x)$ are determined by the relations $\psi_{L,R}(x) = \frac{1 \mp \gamma_5}{2}\, \psi(x)$. From these relations follows that $\gamma_5\, \psi_{L,R}(x) = \mp \psi_{L,R}(x)$.

[2] The left-handed flavor fields ν_{lL} $(l = e, \mu, \tau)$ are often called active fields. Right-handed fields are called sterile: they do not enter into the standard charged and neutral currents.

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4.2 Dirac Mass Term

Let us assume that in addition to the flavor left-handed fields $\nu_{lL}(x)$ three right-handed neutrino fields $\nu_{lR}(x)$ enter into the mass term. In this case the most general neutrino mass term will have the form

$$\mathcal{L}^{\mathrm{D}}(x) = -\sum_{l',l} \bar{\nu}_{l'L}(x)\, M^{\mathrm{D}}_{l'l}\, \nu_{lR}(x) + \mathrm{h.c.} \tag{4.2}$$

Here the indexes l and l' run over e, μ, τ and M^{D} is a 3×3 complex matrix.

It is easy to check that if the mass term $\mathcal{L}^{\mathrm{D}}$ enters into the total Lagrangian, invariance under the following global gauge transformations

$$\begin{aligned} \nu'_{lL}(x) = e^{i\Lambda}\nu_{lL}(x), \quad \nu'_{lR}(x) = e^{i\Lambda}\nu_{lR}(x) \\ l'(x) = e^{i\Lambda}\, l(x), \quad q'(x) = q(x), \end{aligned} \tag{4.3}$$

holds. Here Λ is an arbitrary constant phase. From the invariance under the transformations (4.3) follows that the total lepton number L, which is the same for all charged leptons and all flavor neutrinos, is conserved.

The procedure of diagonalization of the mass term (4.2) was performed in details in Chap. 3. The complex matrix M^{D} can be diagonalized by the biunitary transformation (see Appendix B)

$$M^{\mathrm{D}} = U^{\dagger}\, m\, V. \tag{4.4}$$

Here U and V are unitary matrices and $m_{ik} = m_i\delta_{ik}, m_i > 0$. From (4.2) and (4.4) we find

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\ \nu_{iL}(x) \quad (l = e, \mu, \tau) \tag{4.5}$$

and

$$\nu_{lR}(x) = \sum_{i=1}^{3} V_{li}\ \nu_{iR}(x) \quad (l = e, \mu, \tau). \tag{4.6}$$

Here U and V are unitary matrices.

The mass term (4.2) takes the form

$$\mathcal{L}^{\mathrm{D}}(x) = -\sum_{i=1}^{3} m_i\ \bar{\nu}_i(x)\nu_i(x). \tag{4.7}$$

From (4.7) and (4.5) we conclude that

- $\nu_i(x)$ is the field of the neutrino with mass m_i (i=1,2,3).
- The left-handed flavor fields $\nu_{lL}(x)$, which enter into the standard charged and neutral currents, are "mixed" fields.

The unitary 3×3 mixing matrix U in (4.5) is called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix.

From (4.3), (4.5) and (4.6) follows that the total Lagrangian is invariant under the transformations

$$\nu_i'(x) = e^{i\Lambda}\,\nu_i(x), \quad l'(x) = e^{i\Lambda}\,l(x), \quad q'(x) = q(x). \tag{4.8}$$

This invariance means that $\nu_i(x)$ is the Dirac field of neutrinos and antineutrinos, particles with the same mass m_i. Lepton numbers of neutrino and antineutrinos are opposite: $L(\nu_i) = 1, \quad L(\bar\nu_i) = -1$. The mass term (4.2) is called *the Dirac mass term*.

4.3 Majorana Mass Term

A mass term is a sum of Lorentz-invariant products of left-handed and right-handed components of fields. We will show now that the conjugated fields

$$(\nu_{lL})^c = C\,\bar\nu_{lL}^T \quad \text{and} \quad (\nu_{lR})^c = C\,\bar\nu_{lR}^T \tag{4.9}$$

are, correspondingly, right-handed and left-handed fields. Here C is the unitary matrix of the charge conjugation, which satisfies the relations

$$C\,\gamma_\alpha^T\,C^{-1} = -\gamma_\alpha, \quad C^T = -C. \tag{4.10}$$

We have

$$\gamma_5\,\nu_{lL} = -\nu_{lL}, \quad \gamma_5\,\nu_{lR} = \nu_{lR}. \tag{4.11}$$

From these relations by hermitian conjugation and multiplication by γ^0 from the right we find

$$\bar\nu_{lL}\,\gamma_5 = \nu_{lL}, \quad \bar\nu_{lR}\,\gamma_5 = -\nu_{lR}. \tag{4.12}$$

Further, from (4.12) by transposition and multiplication from the left by the matrix C we obtain

$$\gamma_5\,(\nu_{lL})^c = (\nu_{lL})^c, \quad \gamma_5\,(\nu_{lR})^c = -(\nu_{lR})^c. \tag{4.13}$$

In (4.13) we took into account that

$$C\,\gamma_5^T\,C^{-1} = \gamma_5. \tag{4.14}$$

From (4.13) follows that $(\nu_{lL})^c$ is the right-handed and $(\nu_{lR})^c$ the left-handed component. From (4.9) and (4.10) we find

$$\overline{(\nu_{lL})^c} = -\nu_{lL}^T\,C^{-1}, \quad \overline{(\nu_{lR})^c} = -\nu_{lR}^T\,C^{-1}. \tag{4.15}$$

Taking into account that $(\nu_{lL})^c$ is the right-handed component we can easily build a neutrino mass term in which only flavor (active) left-handed neutrino fields enter. The most general mass term of this type has the form

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\sum_{l',l=e,\mu,\tau} \bar{\nu}_{l'L}\,M^{\mathrm{M}}_{l'l}\,(\nu_{lL})^c + \mathrm{h.c.} = -\frac{1}{2}\sum_{l',l=e,\mu,\tau} \bar{\nu}_{l'L}\,M^{\mathrm{M}}_{l'l}\,C\,\nu_{lL}^T + \mathrm{h.c.}, \tag{4.16}$$

where M^{M} is a complex non diagonal matrix.

The mass term (4.16) can be written in the following matrix form

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\,\bar{\nu}_L\,M^{\mathrm{M}}\,(\nu_L)^c + \mathrm{h.c.}, \tag{4.17}$$

where M^{M} is a $3{\times}3$ matrix and

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \tag{4.18}$$

We will show now that M^{M} is a symmetric matrix. In fact, taking into account (4.9) and (4.10), we find

$$\bar{\nu}_L\,M^{\mathrm{M}}\,(\nu_L)^c = \bar{\nu}_L\,M^{\mathrm{M}}\,C\,\bar{\nu}_L^T = -\bar{\nu}_L\,(M^{\mathrm{M}})^T\,C^T\,\bar{\nu}_L^T = \bar{\nu}_L\,(M^{\mathrm{M}})^T\,(\nu_L)^c. \tag{4.19}$$

Notice that the minus sign in the third term is due to the anticommutation properties of fermion fields. From (4.19) we have

$$M^{\mathrm{M}} = (M^{\mathrm{M}})^T. \tag{4.20}$$

We will now present the mass term (4.17) in the diagonal form. The symmetric complex matrix M^M can be presented in the form (see Appendix C)

$$M^{\mathrm{M}} = U\,m\,U^T, \tag{4.21}$$

where U is an unitary matrix and $m_{ik} = m_i\,\delta_{ik}, \quad m_i > 0$. From (4.17) and (4.21) we find

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\,\bar{\nu}_L\, U\, m\, U^T\, C\, \bar{\nu}_L^T + \mathrm{h.c.} = -\frac{1}{2}\,\overline{U^\dagger \nu_L}\, m\, (U^\dagger \nu_L)^c - \frac{1}{2}\,\overline{(U^\dagger \nu_L)^c}\, m\, U^\dagger \nu_L. \tag{4.22}$$

From (4.22) we find the following expression for the mass term

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\,\bar{\nu}^M\, m\, \nu^M. \tag{4.23}$$

Here

$$\nu^M = U^\dagger \nu_L + (U^\dagger \nu_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{4.24}$$

From (4.23) and (4.24) we have

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\sum_{i=1}^{3} m_i\, \bar{\nu}_i\, \nu_i. \tag{4.25}$$

Thus, $\nu_i(x)$ is the field of the neutrino with mass m_i. From (4.24) we obviously have

$$(\nu^M(x))^c = \nu^M(x). \tag{4.26}$$

Thus, the field of neutrinos with definite mass $\nu_i(x)$ satisfy the condition

$$\nu_i^c(x) = \nu_i(x). \tag{4.27}$$

The condition (4.27) is called *Majorana condition*. In the general case we have the following expansion for the neutrino field $\nu(x)$

$$\nu(x) = \int N_p \left(c_r(p)\, u^r(p)\, e^{-ipx} + d_r^\dagger(p)\, u^r(-p)\, e^{ipx} \right) d^3 p. \tag{4.28}$$

Here $c_r(p)$ ($d_r^\dagger(p)$) is the operator of the absorption of a neutrino (creation of a antineutrino) with momentum p and helicity r, the spinor $u^r(p)$ describes the state with momentum p and helicity r, $u^r(-p) = C\,(\bar{u}^r(p))^T$ and $N_p = \frac{1}{(2\pi)^{3/2}\sqrt{2p^0}}$ is the standard normalization factor. If the field $\nu(x)$ satisfies the Majorana condition (4.27) we have

$$c_r(p) = d_r(p). \tag{4.29}$$

Thus, if the neutrino field satisfies the Majorana condition in this case

$$\text{neutrino} \equiv \text{antineutrino}$$

and $\nu_i(x)$ is the field of purely neutral Majorana neutrinos with the mass m_i and helicities ± 1.

The field $\nu^M(x)$ is the sum of the left-handed and right-handed components:

$$\nu^M(x) = \nu^M_L(x) + \nu^M_R(x). \tag{4.30}$$

Comparing (4.24) and (4.30), we find

$$\nu^M_L(x) = U^\dagger \nu_L(x), \quad \nu^M_R(x) = (U^\dagger \nu_L(x))^c. \tag{4.31}$$

We conclude from (4.31) that *right-handed and left-handed components of the Majorana field are connected* by the relation

$$\nu^M_R(x) = (\nu^M_L(x))^c. \tag{4.32}$$

From (4.24) and (4.32) we have

$$\nu_{iR}(x) = (\nu_{iL}(x))^c. \tag{4.33}$$

This relation is a direct consequence of the Majorana condition. In fact we have

$$(\nu_{iL})^c = \left(\frac{1-\gamma_5}{2}\nu_i\right)^c = C\frac{1+\gamma_5^T}{2}\bar{\nu}_i^T = \frac{1+\gamma_5}{2}\nu_i^c = \frac{1+\gamma_5}{2}\nu_i = \nu_{iR} \tag{4.34}$$

Vice versa it is obvious that if left-handed and right-handed components are connected by the relation (4.33) the field ν_i satisfies the Majorana condition. Notice that the relation (4.33) represents the difference between the Dirac and Majorana fields: in the case of the Dirac field right-handed and left-handed components are independent while in case of the Majorana field they are connected by the relation (4.33). Let us consider the global gauge transformation

$$\nu'_L(x) = e^{i\Lambda}\, \nu_L(x), \tag{4.35}$$

where Λ is an arbitrary constant. For the conjugated field we have

$$(\nu'_L)^c(x) = e^{-i\Lambda}\, (\nu_L)^c(x). \tag{4.36}$$

It is obvious that the mass term (4.17) is not invariant under the gauge transformations (4.35). Thus, in the case of the Majorana mass term there is no global gauge invariance and there is no conserved lepton number which allow to distinguish neutrinos and antineutrinos. This is the reason why after the diagonalization of $\mathcal{L}^M$ we came to the fields of the Majorana neutrinos.

Finally from (4.24) we find

$$\nu_L(x) = U\, \nu^M_L(x). \tag{4.37}$$

From this relation we have

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\, \nu_{iL}(x), \tag{4.38}$$

where U is the unitary 3×3 mixing matrix. Thus, in the case of the Majorana mass term the left-handed flavor fields ν_{lL}, which enter into CC and NC of the Standard Model, are connected with the left-handed components of the Majorana fields ν_{iL} by the relation (4.38).[3]

In conclusion let us consider the neutrino kinetic term of the Lagrangian. We have

$$\mathcal{L}_0 = \sum_l \bar{\nu}_{lL}\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu_{lL} = \overline{\nu}_L\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu_L. \tag{4.39}$$

Taking into account the unitarity of the mixing matrix U we find from (4.31) and (4.39)

$$\mathcal{L}_0 = \overline{\nu}^M_L\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu^M_L = \sum_i \bar{\nu}_{iL}\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu_{1L}. \tag{4.40}$$

It is easy to see that

$$\bar{\nu}_{iL}\, i\,\gamma_{\alpha}\partial^{\alpha}\nu_{iL} = -\partial^{\alpha}\nu^T_{iL}\, i\,\gamma^T_{\alpha}\, \bar{\nu}^T_{iL} = -\partial^{\alpha}\overline{(\nu_{iL})^c}\, i\,\gamma_{\alpha}\,(\nu_{iL})^c. \tag{4.41}$$

Further we have

$$-\partial^{\alpha}\overline{(\nu_{iL})^c}\, i\,\gamma_{\alpha}\,(\nu_{iL})^c = -\partial^{\alpha}(\overline{(\nu_{iL})^c}\, i\,\gamma_{\alpha}\,(\nu_{iL})^c) + \overline{(\nu_{iL})^c}\, i\,\gamma_{\alpha}\,\partial^{\alpha}(\nu_{iL})^c. \tag{4.42}$$

A Lagrangian is determined up to a divergence of a vector. So, the first term of (4.42) can be omitted. For the kinetic term of the Lagrangian of the neutrino fields we find the following expression

$$\mathcal{L}_0 = \frac{1}{2}\sum_i \bar{\nu}_{iL}\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu_{iL}. + \frac{1}{2}\sum_i \overline{(\nu_{iL})^c}\, i\,\gamma_{\alpha}\,\partial^{\alpha}(\nu_{iL})^c = \frac{1}{2}\sum_i \overline{\nu}_i\, i\,\gamma^{\alpha}\partial_{\alpha}\, \nu_i, \tag{4.43}$$

where

$$\nu_i = \nu_{iL} + (\nu_{iL})^c \tag{4.44}$$

is the Majorana field.

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[3] Majorana mixing was considered for the first time by Gribov and Pontecorvo. Before this paper there was a belief that in the case of the left-handed neutrino fields neutrino masses are equal to zero. This is correct if the total lepton number L is conserved. As we have shown here, if only left-handed neutrino fields enter into the Lagrangian, neutrinos can have Majorana masses.

Finally from (4.25) and (4.43) we obtain the following expression for the total free Lagrangian of the neutrino fields

$$\mathcal{L}_0 = \frac{1}{2}\sum_{i=1}^{3} \overline{\nu}_i (i\,\gamma^\alpha \partial_\alpha - m_i)\,\nu_i \tag{4.45}$$

Notice that from (4.43) and (4.45) it is evident why we introduced the factor $\frac{1}{2}$ in (4.23) and other expressions.

Let us stress that the case of the Majorana mass term only active left-handed neutrino fields ν_{lL} enter into the total Lagrangian.

4.4 Dirac and Majorana Mass Term

The most general mass term, in which left-handed active flavor fields ν_{lL} and right-handed sterile fields ν_{lR} enter, has the form

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\,\bar{\nu}_L\, M_L^{\mathrm{M}} (\nu_L)^c - \bar{\nu}_L\, M^{\mathrm{D}}\,\nu_R - \frac{1}{2}\,\overline{(\nu_R)^c}\, M_R^{\mathrm{M}} \nu_R + \mathrm{h.c.}, \tag{4.46}$$

where M_L^{M} and M_R^{M} are complex non-diagonal symmetrical 3×3 matrices, M^{D} is a complex non-diagonal 3×3 matrix, column ν_L is given by (4.18) and

$$\nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix}. \tag{4.47}$$

The mass term (4.46) is the sum of the left-handed Majorana mass term, the Dirac mass term and right-handed Majorana mass term. It is called the Dirac and Majorana mass term. It is obvious that the mass term is not invariant under the global gauge transformations. Thus, in the theory with the Dirac and Majorana mass term the lepton number L is not conserved. We must expect, therefore, that the fields of neutrinos with definite masses are Majorana fields.

Let us perform the procedure of the diagonalization of the Dirac and Majorana mass term. The mass term $\mathcal{L}^{\mathrm{D+M}}$ can be written in the following matrix form

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\,\bar{n}_L\, M^{\mathrm{D+M}} (n_L)^c + \mathrm{h.c.} \tag{4.48}$$

Here

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} \tag{4.49}$$

and

$$M^{\mathrm{D+M}} = \begin{pmatrix} M_L^{\mathrm{M}} & M^{\mathrm{D}} \\ (M^{\mathrm{D}})^T & M_R^{\mathrm{M}} \end{pmatrix} \tag{4.50}$$

is a symmetrical 6×6 matrix. Notice that in (4.48) we took into account the following relation

$$\bar{\nu}_L\, M^{\mathrm{D}}\, \nu_R = -(\nu_R)^T\, (M^{\mathrm{D}})^T\, (\bar{\nu}_L)^T = \overline{(\nu_R)^c}\, (M^{\mathrm{D}})^T\, (\nu_L)^c. \tag{4.51}$$

The matrix $M^{\mathrm{D+M}}$ can be presented in the following diagonal form

$$M^{\mathrm{D+M}} = U\, m\, U^T, \tag{4.52}$$

where U is an unitary 6×6 matrix and $m_{ik} = m_i\, \delta_{ik}$ $(i, k = 1, \ldots 6)$.

From (4.48) and (4.52) we have

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\, \overline{U^\dagger n_L}\, m\, (U^\dagger\, n_L)^c + \mathrm{h.c.} = -\frac{1}{2}\, \bar{\nu}^M\, m\, \nu^M = -\frac{1}{2} \sum_{i=1}^{6} m_i\, \bar{\nu}_i\, \nu_i. \tag{4.53}$$

Here

$$\nu^M = \nu_L^M + (\nu_L^M)^c = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_6 \end{pmatrix}, \tag{4.54}$$

where

$$\nu_L^M = U^\dagger\, n_L. \tag{4.55}$$

From (4.54) we have

$$(\nu^M)^c = \nu^M \quad \text{and} \quad \nu_i^c = \nu_i \;\; (i = 1, 2, \ldots 6). \tag{4.56}$$

From (4.53) and (4.56) follow that $\nu_i(x)$ is the field of Majorana particles with mass m_i. It is obvious from (4.55) that ν_{lL} and $(\nu_{lR})^c$ are connected with left-handed components of the Majorana fields ν_{iL} by an unitary transformation. In fact, we have

$$n_L = U\, \nu_L^M. \tag{4.57}$$

From (4.57) we obtain the following relations

$$\nu_{lL}(x) = \sum_{i=1}^{6} U_{li}\, \nu_{iL}(x), \quad (\nu_{lR}(x))^c = \sum_{i=1}^{6} U_{\bar{l}i}\, \nu_{iL}(x), \tag{4.58}$$

where U is the unitary 6×6 mixing matrix. Thus, in the case of the Dirac and Majorana mass term, flavor field ν_{lL} is a "mixture" of the six left-handed fields of Majorana particles with mass m_i. The sterile field $(\nu_{lR})^c$ is a "mixture" of the same components.

Let us notice that the most popular seesaw mechanism of the neutrino mass generation, which allows to explain the smallness of neutrino masses, is based on the Dirac and Majorana mass term. We will discuss this mechanism later.

4.5 Neutrino Mass Term in the Simplest Case of Two Neutrino Fields

It is instructive to consider a neutrino mass term in the simplest case of two neutrino fields. Let us consider the Dirac and Majorana mass term in the case of one generation. We have

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\, m_L\, \bar{\nu}_L\, (\nu_L)^c - m_D\, \bar{\nu}_L\, \nu_R - \frac{1}{2}\, m_R\, \overline{(\nu_R)^c}\, \nu_R + \mathrm{h.c.} \tag{4.59}$$

We will assume CP invariance in the lepton sector. In this case m_L, m_D and m_R are real parameters.

The mass term $\mathcal{L}^{\mathrm{D+M}}$ can be presented in the following matrix form

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\, \bar{n}_L\, M^{\mathrm{D+M}} (n_L)^c + \mathrm{h.c.} \tag{4.60}$$

Here

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} \tag{4.61}$$

and

$$M^{\mathrm{D+M}} = \begin{pmatrix} m_L & m_D \\ m_D & m_R. \end{pmatrix} \tag{4.62}$$

It is convenient to present the matrix $M^{\mathrm{D+M}}$ in the form

$$M^{\mathrm{D+M}} = \frac{1}{2}\, \mathrm{Tr}\, M^{\mathrm{D+M}} + M, \tag{4.63}$$

where $\mathrm{Tr}\, M = 0$. We have

$$M = \begin{pmatrix} -\frac{1}{2}\,(m_R - m_L) & m_D \\ m_D & \frac{1}{2}\,(m_R - m_L) \end{pmatrix} \tag{4.64}$$

The matrix M can be easily diagonalized by orthogonal transformation. We have

$$M = O\,\bar{m}\,O^T. \tag{4.65}$$

Here

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.66}$$

is an orthogonal matrix and

$$\bar{m}_{1,2} = \mp\frac{1}{2}\sqrt{(m_R - m_L)^2 + 4\,m_D^2} \tag{4.67}$$

From (4.65), (4.66) and (4.67) we find

$$\tan 2\theta = \frac{2m_D}{m_R - m_L}, \quad \cos 2\theta = \frac{m_R - m_L}{\sqrt{(m_R - m_L)^2 + 4\,m_D^2}}. \tag{4.68}$$

For the matrix $M^{\mathrm{D+M}}$ from (4.63), (4.65) and (4.67) we have

$$M^{\mathrm{D+M}} = O\,m'\,O^T, \tag{4.69}$$

where

$$m'_{1,2} = \frac{1}{2}\,(m_R + m_L) \mp \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4\,m_D^2} \tag{4.70}$$

are eigenvalues of the matrix $M^{\mathrm{D+M}}$. These quantities can be positive or negative. Let us write down

$$m'_i = m_i\,\eta_i \tag{4.71}$$

where $m_i = |m'_i|$ and $\eta_i = \pm 1$. From (4.70) and (4.71) we have

$$M^{\mathrm{D+M}} = O\,m\,\eta\,O^T = U\,m\,U^T, \tag{4.72}$$

where $U = O\,\eta^{1/2}$ is an unitary matrix.

From (4.60) and (4.72) we obtain the following expression for the mass term

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\,\overline{\nu^M}\,m\,\nu^M = -\frac{1}{2}\sum_{i=1,2} m_i\,\bar{\nu}_i\,\nu_i. \tag{4.73}$$

Here

$$\nu^M = U^\dagger\, n_L + (U^\dagger\, n_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{4.74}$$

It is obvious from (4.74) that

$$\nu_i^c = \nu_i. \tag{4.75}$$

Thus, ν_1 and ν_2 are fields of Majorana neutrino with masses m_1 and m_2, respectively.

From (4.61), (4.66) and (4.74) we obtain the following mixing relations in the case of the Dirac and Majorana mass term for one neutrino family

$$\begin{aligned} \nu_L &= \cos\theta \sqrt{\eta_1}\, \nu_{1L} + \sin\theta \sqrt{\eta_2}\, \nu_{2L} \\ (\nu_R)^c &= -\sin\theta \sqrt{\eta_1}\, \nu_{1L} + \cos\theta \sqrt{\eta_2}\, \nu_{2L}. \end{aligned} \tag{4.76}$$

The neutrino masses m_1 and m_2 and the mixing angle θ are determined by three real parameters m_L, m_R and m_D (see relations (4.70) and (4.68)). The parameter η_i $(i = 1, 2)$ determines the CP parity of the Majorana neutrino ν_i (see next section).

Let us consider now the Majorana mass term in the case of two flavor fields (say, ν_μ and ν_τ). The mass term is given in this case by the following expression

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\, \bar{\nu}_L\, M^{\mathrm{M}} (\nu_L)^c + \mathrm{h.c.} \tag{4.77}$$

Here

$$\nu_L = \begin{pmatrix} \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \tag{4.78}$$

and

$$M^{\mathrm{M}} = \begin{pmatrix} m_{\mu\mu} & m_{\mu\tau} \\ m_{\mu\tau} & m_{\tau\tau}. \end{pmatrix}, \tag{4.79}$$

where $m_{\mu\mu}, m_{\mu\tau}, m_{\tau\tau}$ are real parameters (CP invariance is assumed).

It is obvious that if we change $m_{\mu\mu} \to m_L,\ m_{\tau\tau} \to m_R,\ m_{\mu\tau} \to m_D$ we can use the corresponding relations obtained for the Dirac and Majorana mass term. For the masses of the Majorana neutrinos ν_1 and ν_2 we have

$$m_{1,2} = \left| \frac{1}{2}\,(m_{\tau\tau} + m_{\mu\mu}) \mp \frac{1}{2} \sqrt{(m_{\tau\tau} - m_{\mu\mu})^2 + 4\, m_{\mu\tau}^2} \right|. \tag{4.80}$$

The flavor fields $\nu_{\mu L}$ and $\nu_{\tau L}$ are given by the relations

$$\begin{aligned}\nu_{\mu L} &= \cos\theta\sqrt{\eta_1}\,\nu_{1L} + \sin\theta\sqrt{\eta_2}\,\nu_{2L}\\ \nu_{\tau L} &= -\sin\theta\sqrt{\eta_1}\,\nu_{1L} + \cos\theta\sqrt{\eta_2}\,\nu_{2L},\end{aligned} \tag{4.81}$$

where for the mixing angle θ we have

$$\tan 2\theta = \frac{2m_{\mu\tau}}{m_{\tau\tau} - m_{\mu\mu}}, \quad \cos 2\theta = \frac{m_{\tau\tau} - m_{\mu\mu}}{\sqrt{(m_{\tau\tau} - m_{\mu\mu})^2 + 4\,m_{\mu\tau}^2}}. \tag{4.82}$$

4.6 Seesaw Mechanism of Neutrino Mass Generation

The seesaw mechanism, proposed at the end of the seventies, is based on the Dirac and Majorana mass term. It is apparently the most natural and viable mechanism of neutrino mass generation.

In order to expose the main idea of the mechanism let us consider the simplest case of one family. The Dirac and Majorana mass term is given in this case in the previous section (see expression (4.59)). The three parameters m_L, m_D and m_R characterize, correspondingly, left-handed Majorana, Dirac and right-handed Majorana mass terms. Particles with definite masses are Majorana particles. Their masses are given by the expressions (4.70) and the mixing angle is given by (4.68). We will now formulate the main assumptions of the seesaw mechanism.

1. We assume that there is no left-handed Majorana mass term, i.e. that

$$m_L = 0$$

2. We assume that the Dirac mass term is generated by the Standard Higgs mechanism, i.e. that m_D is of the order of a mass of quark or lepton.
3. The right-handed Majorana mass term breaks conservation of the lepton number. We assume that the lepton number is violated at a scale which is much larger than the electroweak scale,[4] i.e. that

$$m_R \equiv M_R \gg m_D. \tag{4.83}$$

From (4.70) and (4.83) follows that the masses of the Majorana particles are given by the expressions

$$m_1 \simeq \frac{m_D^2}{M_R} \ll m_D, \quad m_2 \simeq M_R \gg m_D. \tag{4.84}$$

[4] Majorana mass term is allowed only for particles with equal to zero electric charges such as neutrinos. For charged particles such as quarks and leptons the Majorana mass term is forbidden by the conservation of the electric charge. We also notice that because ν_R is $SU(2)$ singlet with equal to zero hypercharge, there are no constraints on m_R. It can be arbitrarily large.

For the mixing angle from (4.68) and (4.83) we find

$$\theta \simeq \frac{m_D}{M_R} \ll 1. \tag{4.85}$$

We also have $\eta_1 = -1$ and $\eta_2 = 1$. From (4.76) and (4.85) we obtain the following mixing relations

$$\begin{aligned} \nu_L &= i\, \nu_{1L} + \frac{m_D}{M_R}\, \nu_{2L} \\ (\nu_R)^c &= -i\, \frac{m_D}{M_R}\, \nu_{1L} + \nu_{2L}. \end{aligned} \tag{4.86}$$

In the framework of the seesaw mechanism, smallness of neutrino masses with respect to masses of quarks and leptons is connected with violation of the total lepton number at a large scale given by M_R. The suppression factor ($\frac{m_D}{M_R}$) is characterized by the ratio of the electroweak scale and the scale of the violation of the lepton number. Notice that if we put $m_D \simeq m_t \simeq 170$ GeV and $m_1 \simeq 5 \cdot 10^{-2}$ (the mass of the heaviest neutrino in the case of neutrino mass hierarchies) we find $M_R \simeq \frac{m_D^2}{m_1} \simeq 10^{15}$ GeV.

In the case of three families the seesaw mass matrix has the form

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}, \tag{4.87}$$

where m_D and M_R are 3×3 matrices and $M_R = M_R^T$. We will assume that $M_R \gg m_D$. Let us introduce the matrix m by the relation

$$U^T\, M\, U = m, \tag{4.88}$$

where U is a unitary matrix.

By analogy with the one family case we will choose the matrix U in the form

$$U = \begin{pmatrix} 1 & (m_D^T)^\dagger\, (M_R^{-1})^\dagger \\ -M_R^{-1}\, m_D^T & 1 \end{pmatrix} \tag{4.89}$$

It is easy to check that in the linear over $\frac{m_D}{M_R}$ approximation $U^\dagger U = 1$. From (4.88) and (4.89) follows that up to the terms linear in $\frac{m_D}{M_R}$ the matrix m takes a block-diagonal form

$$m \simeq \begin{pmatrix} -m_D M_R^{-1} m_D^T & 0 \\ 0 & M_R \end{pmatrix} \tag{4.90}$$

Thus, the Majorana neutrino mass matrix is given by

$$m_\nu = -m_D\, M_R^{-1}\, m_D^T. \tag{4.91}$$

The mass matrix of the heavy Majorana particles is M_R.

Values of neutrino masses and neutrino mixing angles are determined by the concrete form of matrices m_D and M_R. The structure of the relation (4.91) with large M_R in denominator ensure the smallness of neutrino masses with respect to masses of leptons and quarks.

Thus, if the seesaw mechanism is realized in nature then:

- Neutrinos are Majorana particles.
- Neutrino masses are much smaller than lepton and quark masses.
- Heavy Majorana particles, the seesaw partners of neutrinos, must exist.[5]

We have discussed the standard seesaw mechanism of the generation of small neutrino masses. Other approach which allows to explain the smallness of neutrino masses is based on the assumption that the total Lagrangian of the theory is the sum of the SM Lagrangian with massless neutrinos and non renormalizable effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{M}\sum_{l'l} y_{l'l}\,(\psi^T_{l'L}\,\sigma_2\,\phi)\,C^{-1}\,(\phi^T\,\sigma_2\,\psi^T_{lL}) + \mathrm{h.c.} \tag{4.92}$$

Here

$$\psi_{lL} = \begin{pmatrix} \nu_{lL} \\ l_L \end{pmatrix} \qquad (\ell = e, \mu, \tau)\,, \qquad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.93}$$

are lepton and Higgs doublets and σ_2 is the Pauli matrix. The Lagrangian $\mathcal{L}_{\mathrm{eff}}$ preserve the $SU(2) \times U(1)$ symmetry but violate the conservation of the lepton number L.

The field operator in Eq. (4.92) is a dimension-five operator. Because a Lagrangian is the dimension-four operator, the coefficient in (4.92) has a dimension of a mass. We will assume that $y_{l'l}$ are dimensionless coefficients. Thus, M has a dimension of a mass. The parameter M characterizes a large scale at which the Standard Model is violated.

After the spontaneous violation of the electroweak symmetry we have

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}}, \end{pmatrix} \tag{4.94}$$

[5] CP-violating decays of heavy Majorana particles in the early Universe is widely considered as a possible source of the barion asymmetry of the Universe. We will consider baryogenesis through leptogenesis in Chap. 11.

where $v \simeq 246\,\text{GeV}$ is the electroweak vacuum expectation value and $H(x)$ is the Higgs field.

From (4.92) and (4.94) we obtain the left-handed Majorana mass term

$$\mathcal{L}^{\rm M} = \frac{1}{2} \sum_{l'l} \nu^T_{l'L}\, C^{-1} M^M_{l'l}\, \nu_{lL} + \text{h.c.} \tag{4.95}$$

where the Majorana matrix M^M is given by the seesaw expression

$$M^M_{l'l} = \frac{y_{l'l}\, v^2}{M} \tag{4.96}$$

Let us notice that the effective Lagrangian $\mathcal{L}_{\rm eff}$ can be induced by three different beyond the SM interactions:

1. An interaction of lepton-Higgs pairs with a heavy Majorana singlet fermion N_R ($M_N \gg v$). The Lagrangian $\mathcal{L}_{\rm eff}$ is induced in this case by the diagrams with exchange of a virtual N_R between lepton-Higgs pairs.
2. An interaction of lepton pairs and Higgs pair with triplet heavy scalar boson Δ.[6] The effective Lagrangian $\mathcal{L}_{\rm eff}$ is induced in this case by the diagrams with exchange of a virtual Δ between lepton and Higgs pairs.
3. An interaction of lepton-Higgs pairs with heavy Majorana triplet fermion Σ_R. The diagrams with exchange of a virtual Σ_R between the lepton-Higgs pairs induce in this case the effective Lagrangian $\mathcal{L}_{\rm eff}$.

The standard seesaw mechanism is due to the lepton-Higgs-N_R interaction. It is called type I seesaw mechanism. Models with interactions 2 and 3 are called type II and type III seesaw models, respectively.

4.7 Summary and Outlook

We have considered all possible neutrino mass terms in the case of three flavor neutrino fields $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$ and three sterile fields $\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$.

Neutrinos with definite masses can be Majorana or Dirac particles. Neutrinos are the Majorana particles if the mass term is not invariant under the global gauge transformations and, therefore, there are no conserved lepton numbers. Neutrinos are the Dirac particles if the total lepton number $L = L_e + L_\mu + L_\tau$ is conserved.

If neutrino mass matrix is non-diagonal *the fields of the flavor neutrinos ν_{lL} are mixtures* of the left-handed components of the fields of neutrinos with definite masses. In the case of the Dirac or Majorana mass term we have

[6] Notice that because the projection of the isotopic spin of the neutrino field is equal to 1/2 a singlet scalar boson cannot be produced in neutrino–neutrino interaction.

$$\nu_{lL} = \sum_{i=1}^{3} U_{li}\, \nu_{iL} \tag{4.97}$$

where U is an unitary 3×3 mixing matrix. In the case of Dirac and Majorana mass term we have

$$\nu_{lL} = \sum_{i=1}^{6} U_{li}\, \nu_{iL}, \quad (\nu_{lR})^c = \sum_{i=1}^{6} U_{\bar{l}i}\, \nu_{iL}, \tag{4.98}$$

where U is an unitary 6×6 mixing matrix.

The minimal number of the massive neutrinos is equal to the number of the flavor neutrinos (three). If more than three neutrino masses are small, sterile neutrinos must exist. As we will see later, in this case left-handed flavor neutrinos ν_e, ν_μ and ν_τ will transfer into the sterile neutrinos which cannot be produced in weak processes and cannot be detected via the standard weak interaction. For the mixing in the most general case we have

$$\nu_{lL} = \sum_{i=1}^{3+n_s} U_{li}\, \nu_{iL}, \quad \nu_{sL} = \sum_{i=1}^{3+n_s} U_{si}\, \nu_{iL}. \tag{4.99}$$

Here U is $(3+n_s)\times(3+n_s)$ unitary mixing matrix, $l = e, \mu, \tau$, the index s takes the values $s_1, s_2, \ldots s_{n_s}$, n_s being the number of the sterile neutrinos.

Chapter 5
Neutrino Mixing Matrix

5.1 Introduction

In the previous section we have considered possible neutrino mass terms. We have shown that if in the total Lagrangian there is a neutrino mass term neutrinos are massive particles and the flavor neutrino fields $\nu_{eL}(x)$, $\nu_{\mu L}(x)$, $\nu_{\tau L}(x)$, which enter into the interaction Lagrangian of the Standard Model, are mixtures of left-handed components of the fields of neutrinos with definite masses $\nu_{iL}(x)$:

$$\nu_{lL}(x) = \sum_i U_{li}\ \nu_{iL}(x). \tag{5.1}$$

Here U is unitary mixing matrix and $\nu_i(x)$ is the field of the neutrino (Dirac or Majorana) with mass m_i.

The mechanism of the generation of the neutrino mass term at present is unknown. We also do not know if neutrinos with definite masses are Dirac or Majorana particles. However, as we will see later, it was established by all existing experimental data that neutrino mixing takes place. The unitary mixing matrix U is the object of central interest of theory and experiment. Here we will consider the general properties of the matrix U in the Dirac and Majorana cases. We will also introduce the standard parametrization of the 3×3 mixing matrix.

5.2 The Number of Angles and Phases in the Matrix U

Let us consider a unitary mixing matrix U in the general $n \times n$ case. The unitary matrix can be presented in the form $U = e^{iH}$, where H is a hermitian $n \times n$ matrix. Such a matrix is characterized by n(diagonal elements) +2 $(\frac{n^2-n}{2})$ (non-diagonal elements) $= n^2$ real parameters.

The number of angles which characterizes a unitary $n \times n$ matrix coincides with the number of parameters which characterizes a real orthogonal $n \times n$ matrix O which satisfies the condition

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$$O^T O = 1. \tag{5.2}$$

An orthogonal matrix can be presented in the form $O = e^A$, where $A^T = -A$. The diagonal elements of the matrix A are equal to zero. The number of real non diagonal elements is equal to $\frac{n(n-1)}{2}$. Thus, the number of angles which characterize a unitary matrix U is equal to

$$n_\theta = \frac{n(n-1)}{2}. \tag{5.3}$$

Other parameters of the matrix U are phases. The number of phases is equal to

$$n_\phi = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \tag{5.4}$$

The number of *physical phases, which characterize the mixing matrix, is smaller than* n_ϕ. This is connected with the fact that the mixing matrix enters into the charged current together with fields of charged leptons and neutrinos.

We will consider first the case of Dirac neutrinos. From the reasons which will be clear below we denote the field of neutrino with definite masse by $\nu'_i(x)$, the charged lepton field by $l'(x)$ and the mixing matrix by U'.

Taking into account neutrino mixing for the leptonic charged current $j_\alpha^{\mathrm{CC}\dagger}$ we have the following expression

$$j_\alpha^{\mathrm{CC}\dagger}(x) = 2\sum_l \bar{l}'_L(x)\,\gamma_\alpha\,\nu'_{lL}(x) = 2\sum_{l,i} \bar{l}'_L(x)\,\gamma_\alpha\,U'_{li}\,\nu'_{iL}(x). \tag{5.5}$$

The unitary matrix U' can be written in the form

$$U' = S^\dagger(\beta)\,U\,S(\alpha). \tag{5.6}$$

Here $S_{ll'}(\beta) = e^{i\beta_l}\,\delta_{ll'}$, $S_{ik}(\alpha) = e^{i\alpha_i}\,\delta_{ik}$ and

$$U = S(\beta)\,U'\,S^\dagger(\alpha). \tag{5.7}$$

The unitary phase matrix $S(\alpha)$ can be presented in the form

$$S(\alpha) = e^{i\alpha_n}\begin{pmatrix} e^{i(\alpha_1-\alpha_n)} & & \\ & \vdots & \\ & 1 & \end{pmatrix} \tag{5.8}$$

There are $n + (n-1) = 2n - 1$ free parameters $(\beta_l - \alpha_n, \alpha_i - \alpha_n)$ in the right-hand side of the relation (5.7). These parameters can be chosen in such a way to make $2n - 1$ phases in the matrix U equal to zero. Thus, the number of phases in the matrix U is equal to

$$n_{\phi}^{D} = \frac{n(n+1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2} \tag{5.9}$$

Let us introduce the fields

$$\nu_i(x) = e^{i\alpha_i}\,\nu'_i(x), \quad l(x) = e^{i\beta_l}\,l'(x). \tag{5.10}$$

It is obvious that the free Lagrangian of the neutrino and lepton fields and the Lagrangians of the electromagnetic and NC interactions are invariant under the transformation (5.10). The leptonic charged current takes the form

$$j_{\alpha}^{\mathrm{CC}\dagger}(x) = 2\sum_{l,i} \bar{l}_L(x)\,\gamma_\alpha\,U_{li}\,\nu_{iL}(x). \tag{5.11}$$

The phases of the fermion fields are arbitrary, unmeasurable quantities. Thus, the fields $\nu'_i(x)$ ($l'(x)$) and $\nu_i(x)$ ($l(x)$) are physically equivalent. The number of physical phases characterizing the mixing matrix in the case of the Dirac neutrinos is given by n_{ϕ}^{D}, which is the number of phases that cannot be removed from the mixing matrix by the transformation (5.7) with arbitrary $\beta_l - \alpha_n$ and $\alpha_i - \alpha_n$.

From (5.3) and (5.9) follows that in the simplest case of the mixing of two Dirac neutrinos the 2×2 mixing matrix is real. It is characterized by one mixing angle. In the case of the mixing of three Dirac neutrinos *the 3×3 mixing matrix is characterized by three mixing angles and one phase.*

Let us consider now the case of the Majorana neutrinos. We will denote the field of the Majorana neutrino with definite mass by $\nu_i(x)$, the lepton field by $l'(x)$ and the mixing matrix by U'. The lepton charged current has the form

$$j_{\alpha}^{\mathrm{CC}\dagger}(x) = 2\sum_{l,i} \bar{l}'_L(x)\,\gamma_\alpha\,U'_{li}\,\nu_{iL}(x), \tag{5.12}$$

where $\nu_i(x)$ satisfy the Majorana condition

$$\nu_i^c(x) = \nu_i(x) \tag{5.13}$$

and the matrix U' is given by the relation (5.6). Now, n phases which enter into the matrix $e^{-i\alpha_n}\,S(\beta)$, can be included into the fermion Dirac fields $l'(x)$. The Majorana condition (5.13), however, fixes phases of the neutrino fields. For the charged current we have in the Majorana case the following expression

$$j_{\alpha}^{\mathrm{CC}}(x) = 2\sum_{l,i} \bar{l}_L(x)\,\gamma_\alpha\,U^M_{li}\,\nu_{iL}(x). \tag{5.14}$$

Here

$$U^M = U\,S^M(\bar{\alpha}), \tag{5.15}$$

where

$$S^M(\bar{\alpha}) = e^{-i\alpha_n}\ S(\alpha). \tag{5.16}$$

All non-diagonal elements of the matrix $S^M(\bar{\alpha})$ are equal to zero. For the diagonal elements we have

$$S^M_{ii}(\bar{\alpha}) = e^{i\bar{\alpha}_i},\ i \neq n,\ S^M_{nn}(\bar{\alpha}) = 1,\ \ \bar{\alpha}_i = \alpha_i - \alpha_n. \tag{5.17}$$

The phase matrix $S^M(\bar{\alpha})$ is characterized by $n-1$ phases. The total number of phases which characterize the Majorana mixing matrix U_M is equal to

$$n^M_{\phi} = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2}. \tag{5.18}$$

We conclude from (5.3) and (5.18) that in the case of mixing of two Majorana neutrinos, the 2×2 mixing matrix is characterized by one mixing angle and one phase. In the case of mixing of three Majorana neutrinos the 3×3 mixing matrix is characterized by three mixing angles and three phases.

5.3 *CP* Conservation in the Lepton Sector

In this section we will obtain constraints on the unitary mixing matrix which follow from the assumption of the *CP* conservation in the lepton sector. Let us consider first the case of Dirac neutrinos with definite masses. From the assumption of the *CP* invariance follows that the CC Lagrangian satisfies the following condition

$$V_{CP}\ \mathcal{L}^{\mathrm{CC}}_I(x)\ V^{-1}_{CP} = \mathcal{L}^{\mathrm{CC}}_I(x'). \tag{5.19}$$

Here V_{CP} is the operator of the *CP* conjugation and $x' = (x^0, -\mathbf{x})$ and

$$\mathcal{L}^{\mathrm{CC}}_I(x) = -\frac{g}{\sqrt{2}}\sum_{l,i}\bar{l}_L(x)\ \gamma_\alpha\ U_{li}\ \nu_{iL}(x)\ W^{\alpha\dagger} - \frac{g}{\sqrt{2}}\sum_{l,i}\bar{\nu}_{lL}(x)\ \gamma_\alpha\ U^*_{li}\ l_L(x)\ W^{\alpha}. \tag{5.20}$$

Let us consider a lepton field $l'(x)$. Under the *CP* transformation the field $l'(x)$ is transformed as follows

$$V_{CP}\ l'(x)\ V^{-1}_{CP} = e^{-2i\beta_l}\ \gamma^0\ C\ \bar{l}'^T(x'). \tag{5.21}$$

Here β_l is an phase and C is the matrix of the charge conjugation, which satisfies the conditions

$$C\ \gamma^T_\alpha\ C^{-1} = -\gamma_\alpha,\ \ C^T = -C. \tag{5.22}$$

Taking into account that phases of fermion fields are arbitrary, we can include the phase factor $e^{i\beta_l}$ into the field $l'(x)$. We have

$$V_{CP}\, l(x)\, V_{CP}^{-1} = \gamma^0\, C\, \bar{l}^T(x'), \tag{5.23}$$

where $l(x) = e^{i\beta_l}\, l'(x)$. Thus, by a redefinition of arbitrary phases of the lepton fields we can put the *CP* phase factors of the fields equal to one. The *CP* phase factors of the Dirac neutrino fields can also be put equal to one:

$$V_{CP}\, \nu_i(x)\, V_{CP}^{-1} = \gamma^0\, C\, \bar{\nu}_i^T(x'). \tag{5.24}$$

Let us consider the left-handed components of the fields. If we multiply (5.23) by $\frac{1-\gamma_5}{2}$ from the left and take into account (5.22) we find

$$V_{CP}\, l_L(x)\, V_{CP}^{-1} = \gamma^0\, C\, \frac{1+\gamma_5^T}{2}\, \bar{l}^T(x') = \gamma^0\, C\, \bar{l}_L^T(x'). \tag{5.25}$$

Analogously, for the neutrino field we have

$$V_{CP}\, \nu_{iL}(x)\, V_{CP}^{-1} = \gamma^0\, C\, \bar{\nu}_{iL}^T(x'). \tag{5.26}$$

From (5.25) and (5.26) we easily find

$$V_{CP}\, \bar{l}_L(x)\, V_{CP}^{-1} = -l_L^T(x')\, C^{-1}\gamma^0, \quad V_{CP}\, \bar{\nu}_{iL}(x)\, V_{CP}^{-1} = -\nu_{iL}^T(x')\, C^{-1}\gamma^0. \tag{5.27}$$

We will consider now the interaction Lagrangian (5.20). Using relations (5.22), (5.25), (5.26) and (5.27), we have

$$\begin{aligned} V_{CP}\, \bar{l}_L(x)\, \gamma_\alpha\, U_{li}\, \nu_{iL}(x)\, V_{CP}^{-1} &= -l_L^T(x')\, C^{-1}\gamma^0\, \gamma_\alpha\, U_{li}\, \gamma^0\, C\, \bar{\nu}_{iL}^T(x') \\ &= -\delta_\alpha\, \bar{\nu}_{iL}(x')\, \gamma_\alpha\, U_{li}\, l_L(x'). \end{aligned} \tag{5.28}$$

Here $\delta = (1, -1, -1, -1)$ is the sign factor. Notice that in (5.28) we took into account the anticommutator properties of the fermion fields.

Under the *CP* transformation the field of the vector $W^{\pm}$ bosons is transformed as follows

$$V_{CP}\, W_\alpha(x)\, V_{CP}^{-1} = -e^{-2i\beta_W}\, \delta_\alpha\, W_\alpha^{\dagger}(x'), \tag{5.29}$$

where β_W is a phase. Taking into account that the phase of the non-hermitian $W_\alpha(x)$ field is arbitrary, we can include the phase factor $e^{i\beta_W}$ into the W field. In this case we have

$$V_{CP}\, W_\alpha(x)\, V_{CP}^{-1} = -\delta_\alpha\, W_\alpha^{\dagger}(x'). \tag{5.30}$$

With the help of (5.28) and (5.30) we find

$$V_{CP}\,\mathcal{L}_I^{\mathrm{CC}}(x)\,V_{CP}^{-1} = -\frac{g}{\sqrt{2}}\sum_{l,i}\bar{\nu}_{iL}(x')\,\gamma_\alpha\,U_{li}\,l_L(x')\,W^\alpha(x')$$
$$-\frac{g}{\sqrt{2}}\sum_{l,i}\bar{l}_L(x')\,\gamma_\alpha\,U_{li}^*\,\nu_{iL}(x')\,W^{\alpha\dagger}(x'). \qquad (5.31)$$

From (5.19), (5.20) and (5.31) we conclude that in the case of *CP* invariance the mixing matrix U for the Dirac neutrinos is real:

$$U_{li} = U_{li}^*. \qquad (5.32)$$

More exactly this result can be formulated as follows: if arbitrary *CP* phase factors of the lepton, Dirac neutrino and W fields are chosen to be equal to one, then from *CP* invariance follows that the neutrino mixing matrix is real.

Let us comment the condition (5.32). The second term of the CC Lagrangian (5.20) is responsible for the transition

$$l^- \to \nu_i + W^-. \qquad (5.33)$$

The amplitude of this transition is given by U_{li}^*. The first term of the Lagrangian (5.20) is responsible for the *CP*-conjugated transition

$$l^+ \to \bar{\nu}_i + W^+. \qquad (5.34)$$

The amplitude of this transition is given by U_{li}. In the case of *CP* invariance the amplitudes of the transitions (5.33) and (5.34) are equal.

We will consider now the Majorana mixing matrix. The *CP* transformation of the Majorana field has the form

$$V_{CP}\,\nu_i(x)\,V_{CP}^{-1} = \eta_i^*\,\gamma^0\,C\,\bar{\nu}_i^T(x') = \eta_i^*\,\gamma^0\,\nu_i(x'), \qquad (5.35)$$

where we take into account the Majorana condition

$$\nu_i^c(x) = C\,\bar{\nu}_i^T(x) = \nu_i(x). \qquad (5.36)$$

In the case of Majorana fields the *CP* phase factors are not arbitrary as in the case of the Dirac fields. We will show now that the Majorana *CP* phase factors can take the values $\pm i$. In fact, from (5.35) by hermitian conjugation and multiplication from the right by the matrix γ^0 we find

$$V_{CP}\,\bar{\nu}_i(x)\,V_{CP}^{-1} = \eta_i\,\bar{\nu}_i(x')\,\gamma^0. \qquad (5.37)$$

From this relation we have

$$V_{CP}\,C\,\bar{\nu}_i^T(x)\,V_{CP}^{-1} = \eta_i\,C\,\gamma^{0T}\,C^{-1}\,C\,\bar{\nu}_i^T(x'). \qquad (5.38)$$

Finally from (5.36) and (5.22) we obtain the following relation

$$V_{CP}\, \nu_i(x)\, V_{CP}^{-1} = -\eta_i\, \gamma^0\, \nu_i(x'). \tag{5.39}$$

If we now compare (5.35) and (5.39) we conclude that

$$\eta_i^* = -\eta_i. \tag{5.40}$$

Thus, we have

$$\eta_i^2 = -1, \quad \text{and} \quad \eta_i = \pm i. \tag{5.41}$$

From (5.35) we have

$$V_{CP}\, a_i(p_i)\, V_{CP}^{-1} = \eta_i^*\, a_i(p_i'). \tag{5.42}$$

Here $a_i(p_i)$ $(a_i(p_i'))$ is the operator of the absorption of the Majorana neutrino with mass m_i and momentum $p_i = (\sqrt{p^2 + m_i^2}, \mathbf{p})$ $(p_i' = (\sqrt{p^2 + m_i^2}, -\mathbf{p}))$. From (5.42) we find for the creation operators

$$V_{CP}\, a_i^\dagger(p_i)\, V_{CP}^{-1} = \eta_i\, a_i^\dagger(p_i'). \tag{5.43}$$

The vector $|p_i\rangle = a_i^\dagger(p_i)\, |0\rangle$ describes the state of the Majorana neutrino with momentum p_i ($|0\rangle$ is the vacuum state). From (5.43) we have

$$V_{CP}\, |p_i\rangle = \eta_i\, |p_i'\rangle \tag{5.44}$$

Thus, η_i is the *CP* parity of the Majorana neutrino with mass m_i.

We will find now conditions on the Majorana mixing matrix which can be obtained from the assumption of *CP* invariance in the lepton sector. It will be convenient to choose arbitrary *CP* phase factors of the lepton fields equal to $-i$:

$$V_{CP}\, l(x)\, V_{CP}^{-1} = -i\, \gamma^0\, C\, \bar{l}^T(x'), \tag{5.45}$$

From (5.27), (5.35), (5.22) and (5.45) we have

$$\begin{aligned} V_{CP}\, \bar{l}_L(x)\, \gamma_\alpha\, U_{li}^M\, \nu_{iL}(x)\, V_{CP}^{-1} &= -\eta_i^*\, l_L^T(x')\, C^{-1}\gamma^0\, \gamma_\alpha\, U_{li}^M\, \frac{1-\gamma_5}{2}\, \gamma^0 \nu_i(x') \\ &= -\delta_\alpha\, \rho_i\, \bar{\nu}_{iL}(x')\, \gamma_\alpha\, U_{li}^M\, l_L(x'), \end{aligned} \tag{5.46}$$

where the parameter ρ_i is determined as follows

$$\eta_i = i\, \rho_i, \quad \rho_i = \pm 1. \tag{5.47}$$

Let us notice that in (5.46) we take into account that

$$\bar{\nu}_i(x) = -\nu_i^T(x)\, C^{-1}. \tag{5.48}$$

Now, taking into account (5.30), we find

$$\begin{aligned} V_{CP}\, \mathcal{L}_I^{\mathrm{CC}}(x)\, V_{CP}^{-1} = &-\frac{g}{\sqrt{2}} \sum_{l,i} \rho_i\, \bar{\nu}_{iL}(x')\, \gamma_\alpha\, U_{li}^M\, l_L(x')\, W^\alpha(x') \\ &-\frac{g}{\sqrt{2}} \sum_{l,i} \rho_i\, \bar{l}_L(x')\, \gamma_\alpha\, U_{li}^{M*}\, \nu_{iL}(x')\, W^{\alpha\dagger}(x'). \end{aligned} \tag{5.49}$$

From (5.19), (5.20) and (5.49) we finally find the following condition of the *CP* invariance in the case of the mixing of the Majorana neutrinos

$$\rho_i\, U_{li}^M = U_{li}^{M*}. \tag{5.50}$$

Let us stress that we have chosen the arbitrary *CP* phase factors of the lepton fields equal to $-i$ and the W field equal to one.

5.4 Standard Parametrization of 3×3 Mixing Matrix

We will consider here the unitary 3×3 mixing matrix for Dirac neutrinos and introduce the standard parameters (three mixing angles and one phase) which characterizes it. Let us consider three orthogonal and normalized vectors

$$|i\rangle \quad (i = 1, 2, 3) \quad \langle i|k\rangle = \delta_{ik}. \tag{5.51}$$

In order to obtain three general "mixed" vectors we will perform three Euler rotations. The first rotation will be performed at the angle θ_{12} around the vector $|3\rangle$. The new orthogonal and normalized vectors are

$$\begin{aligned} |1\rangle^{(1)} &= c_{12}\, |1\rangle + s_{12}\, |2\rangle \\ |2\rangle^{(1)} &= -s_{12}\, |1\rangle + c_{12}\, |2\rangle \\ |3\rangle^{(1)} &= \phantom{-c_{12}\, |1\rangle + s_{12}} |3\rangle, \end{aligned} \tag{5.52}$$

where $c_{12} = \cos\theta_{12}$ and $s_{12} = \sin\theta_{12}$. In the matrix form, (5.52) can be written as follows

$$|\nu\rangle^{(1)} = U^{(1)}\, |\nu\rangle. \tag{5.53}$$

Here

$$|\nu\rangle^{(1)} = \begin{pmatrix} |1\rangle^{(1)} \\ |2\rangle^{(1)} \\ |3\rangle^{(1)} \end{pmatrix}, \quad |\nu\rangle = \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \end{pmatrix} \tag{5.54}$$

and

$$U^{(1)} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.55}$$

Let us perform now the second rotation at the angle θ_{13} around the vector $|2\rangle^{(1)}$. At this step we will introduce the *CP* phase δ, connected with the rotation of the vector of the third family $|3\rangle$. We will obtain the following three orthogonal vectors:

$$\begin{aligned} |1\rangle^{(2)} &= c_{13}\,|1\rangle^{(1)} + s_{13}e^{-i\delta}\,|3\rangle^{(1)} \\ |2\rangle^{(2)} &= |2\rangle^{(1)} \\ |3\rangle^{(2)} &= -s_{13}e^{i\delta}\,|1\rangle^{(1)} + c_{13}\,|3\rangle^{(1)}. \end{aligned} \tag{5.56}$$

In the matrix form we have

$$|\nu\rangle^{(2)} = U^{(2)}\,|\nu\rangle^{(1)}. \tag{5.57}$$

Here

$$U^{(2)} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}. \tag{5.58}$$

Finally, let us perform the rotation around the vector $|1\rangle^{(2)}$ at the angle θ_{23}. The new orthogonal vectors are

$$\begin{aligned} |1\rangle^{\text{mix}} &= |1\rangle^{(2)} \\ |2\rangle^{\text{mix}} &= c_{23}|\,2\rangle^{(2)} + s_{23}\,|3\rangle^{(2)} \\ |3\rangle^{\text{mix}} &= -s_{23}\,|2\rangle^{(2)} + c_{23}\,|3\rangle^{(2)}. \end{aligned} \tag{5.59}$$

We have

$$|\nu^{\text{mix}}\rangle = U^{(3)}\,|\nu\rangle^{(2)}. \tag{5.60}$$

Here

$$U^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}. \tag{5.61}$$

From (5.53), (5.57) and (5.60) we find

$$|\nu^{\text{mix}}\rangle = U\,|\nu\rangle, \tag{5.62}$$

where

$$U = U^{(3)}\, U^{(2)}\, U^{(1)}. \tag{5.63}$$

From (5.55), (5.58) and (5.61) we have

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.64}$$

From (5.64) we find that the unitary 3×3 Dirac mixing matrix has the following form

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix}. \tag{5.65}$$

This is the so called standard parametrization of the 3×3 mixing matrix. This matrix is characterized by three mixing angles θ_{12}, θ_{23} and θ_{13} and the phase δ.

As we have seen in the previous section, in the case of CP conservation in the lepton sector $U^* = U$. Thus, the phase δ is responsible for effects of the CP violation: if CP is conserved, $\delta = 0$. The mixing angles are parameters which can take values in the ranges $0 \leq \theta_{12} \leq \pi$, $0 \leq \theta_{13} \leq \pi$, $0 \leq \theta_{23} \leq \pi$[1] and the phase δ can take values in the range $0 \leq \delta \leq 2\pi$.

The 3×3 Majorana mixing matrix has the form (see previous section)

$$U^M = U S^M(\bar{\alpha}) \tag{5.66}$$

where the phase matrix $S^M(\bar{\alpha})$ is characterized by two Majorana phases and has the form

$$S^M(\bar{\alpha}) = \begin{pmatrix} e^{i\bar{\alpha}_1} & & \\ & e^{i\bar{\alpha}_2} & \\ & & 1 \end{pmatrix}. \tag{5.67}$$

and the matrix U is given by (5.65).

In the case of CP invariance the mixing matrix satisfies the condition (5.50). Let us consider the elements U^M_{ei} of the mixing matrix. From (5.50) and (5.65) we find

[1] Under the change $\theta \to \pi + \theta$, the 2×2 mixing matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

changes sign. Only squares of the mixing matrix enter into observable quantities. Thus, angles θ and $\pi + \theta$ cannot be distinguished. From (5.64) it is obvious that this argument is applicable to the general 3×3 matrix.

that in the case of *CP* invariance

$$e^{2i\bar{\alpha}_i} = \rho_i \quad (i = 1, 2), \;\; e^{-2i\delta} = \rho_3. \tag{5.68}$$

5.5 On Models of Neutrino Masses and Mixing

The discovery of neutrino oscillations and the measurement of the neutrino oscillation parameters Δm^2_{12}, Δm^2_{23}, $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ as well as the determination of the upper bound of the parameter $\sin^2\theta_{13}$ inspired a lot of models of neutrino masses and mixing. We will discuss existing neutrino oscillation data later. From their global analysis follows that the neutrino mixing angles are in the following 1σ ranges

$$0.29 \leq \sin^2\theta_{12} \leq 0.33, \quad 0.41 \leq \sin^2\theta_{23} \leq 0.54, \quad \sin^2\theta_{13} \leq 0.05. \tag{5.69}$$

Thus the following values of the mixing parameters

$$\sin^2\theta_{12} = \frac{1}{3}, \quad \sin^2\theta_{23} = \frac{1}{2}, \quad \sin^2\theta_{13} = 0. \tag{5.70}$$

are in agreement with experimental data. If we choose these values of the parameters we find from the general expression (5.65) that the neutrino mixing matrix has the form

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{5.71}$$

Notice that the elements of the second column of the matrix (5.71) are equal. We have

$$\nu_{2L} = \frac{1}{\sqrt{3}}(\nu_{eL} + \nu_{\mu L} + \nu_{\tau L}). \tag{5.72}$$

The matrix (5.71) is called *tri-bimaximal matrix*.

Let us consider the equation for the eigenstates and eigenvalues of a real symmetrical matrix M:

$$M\, u_i = m_i\, u_i. \tag{5.73}$$

Here u_i are normalized eigenstates and m_i are eigenvalues of the matrix M. We have

$$\sum_l u_i(l)\, u_k(l) = \delta_{ik}. \tag{5.74}$$

From (5.73) and (5.74) we find

$$M_{l'l} = \sum_i u_i(l')\, m_i\, u_i(l') = U_{l'i}\, m_i\, U^T_{l'i} = (U\, m\, U^T)_{l'l}, \tag{5.75}$$

where the mixing matrix U is given by the relation

$$U_{li} = u_i(l). \tag{5.76}$$

In the case of the tri-bimaximal matrix (5.71) we have

$$u_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \tag{5.77}$$

From (5.75), (5.71) and (5.76) we find

$$M_{TBM} = \frac{m_1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{m_2}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{5.78}$$

From (5.78) follows that the mass matrix in the tri-bimaximal case has the form

$$M_{TBM} = \begin{pmatrix} x & y & y \\ x & x+v & y-v \\ y & y-v & x+v \end{pmatrix}, \tag{5.79}$$

where

$$x = \frac{2}{3}m_1 + \frac{1}{3}m_2, \quad y = -\frac{1}{3}m_1 + \frac{1}{3}m_2, \quad v = -\frac{1}{2}m_1 + \frac{1}{2}m_3. \tag{5.80}$$

Many models of neutrino masses and mixing, which lead to the tri-bimaximal mixing, were proposed. They are based on different symmetries. We briefly discuss here a model based on the discrete alternating A4 group.

The group A4 is the group of even permutations of four objects. All elements of the group are products of two generators S and T which satisfy the relations

$$S^2 = T^3 = (ST)^3 = 1. \tag{5.81}$$

The number of elements in the A4 group is equal to $\frac{4!}{2} = 12$. Taking into account (5.81) we can easily see that all possible products of S and T (elements of the A4 group) are given by

$$1, T, S, ST, TS, T^2, T^2S, TST, ST^2, STS, T^2ST, TST^2. \tag{5.82}$$

The group A4 has four irreducible representations: one triplet and three singlets

$$3, 1, 1', 1''. \tag{5.83}$$

For one-dimensional unitary representations we can choose $S = 1, T = e^{i\alpha}$. From the relations (5.81) follows that $3\alpha = 2\pi n$, where n is an integer number. Thus, we have

$$1 : S = 1 \;\; T = 1; \quad 1' : S = 1 \;\; T = \omega^2 : \quad 1'' : S = 1 \;\; T = \omega, \tag{5.84}$$

where $\omega = e^{\frac{2\pi}{3}i}$. It is obvious that

$$\omega^3 = 1, \quad 1 + \omega + \omega^2 = 0. \tag{5.85}$$

In the basis in which the matrix T is diagonal, the 3×3 unitary matrix T is given by

$$T = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix}, \tag{5.86}$$

where α_i is a real phase. From (5.81) follows that $\alpha_i = \frac{2\pi}{3}n_i$. We can choose the three-dimensional unitary representation of the generator T in the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{5.87}$$

The real unitary matrix S satisfies the condition $S^T S = 1$. Taking into account that $S^2 = 1$ we have $S^T = S$. We can check that the three-dimensional unitary symmetrical matrix

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{5.88}$$

satisfies the relations (5.81).

We will assume that the lepton doublets

$$L_{lL} = \begin{pmatrix} \nu_{lL} \\ l_L \end{pmatrix} \quad l = e, \mu, \tau \tag{5.89}$$

are A4 triplets. Further, we will assume the existence of the scalar triplets ϕ_T, ϕ_S and scalar singlet ξ (flavon fields) which enter into the A4 invariant Yukawa interaction with lepton doublets. After spontaneous symmetry breaking, the charged-lepton and neutrino mass terms are generated.

With the vacuum alignment

$$< \phi_T >= (v_T, 0, 0) \tag{5.90}$$

we come to the diagonal charged lepton mass matrix and symmetrical neutrino mass matrix

$$M = \begin{pmatrix} x & y & z \\ y & x_1 & y - v \\ z & y - v & x_2 \end{pmatrix} \tag{5.91}$$

We assume that the matrix M is real. In this case it is characterized by six real parameters.

With the alignment

$$< \phi_S >= (v_S, v_S, v_S), \quad < \xi >= u \tag{5.92}$$

the A4 symmetry is broken down to the G_S symmetry generated by the operator S. The mass matrix M satisfies the relation

$$SMS = M, \tag{5.93}$$

where S is given by (5.88). From (5.93) we find that the mass matrix M is given by

$$M = \begin{pmatrix} x & y & z \\ y & x + v + y - z & w \\ z & w & x + v \end{pmatrix}. \tag{5.94}$$

This matrix is characterized by four real parameters. It is obvious that it does not have the tri-bimaximal form (5.79).

We will come to the tri-bimaximal mixing if we assume $\mu - \tau$ symmetry of the neutrino mixing matrix[2]

$$S_{\mu\tau} M S_{\mu\tau} = M, \tag{5.95}$$

where

$$S_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.96}$$

[2] It is obvious that from this relation follows: $V_{\mu e} = V_{e\mu} = V_{\tau e} = V_{e\tau}, \;\; V_{\mu\mu} = V_{\tau\tau}$.

In fact, from (5.95) and (5.96) we find that

$$z = y. \tag{5.97}$$

With the relation (5.97) we come to the tri-bimaximal neutrino mixing matrix(5.79). In order to implement $\mu - \tau$ symmetry we need to assume that there are no $1'$ and $1''$ scalar flavon fields in the Yukawa interaction.

Thus, spontaneously broken A4 symmetry with alignments (5.90) and (5.92) allows to obtain the tri-bimaximal mixing. This mixing is in good agreement with experimental data and can be considered as the leading approximation. Existing experimental data do not exclude, however, some corrections to the tri-bimaximal values (5.70) (see (5.69)). In the model, we are discussing, corrections from higher-dimensional operators to all mixing angles are of the same order. Taking into account that the corrections to the angle θ_{12} can be of the order of 5% we can predict that the departure of the angle θ_{23} from $\frac{\pi}{4}$ can also be of the same order and that the value of the angle θ_{13} can be close to the present CHOOZ bound ($\simeq 5 \cdot 10^{-2}$). These predictions will be checked in T2K, DOUBLE CHOOZ, Daya Bay, RENO and other experiments.

We have briefly discussed a model based on the spontaneously broken A4 group. The observed pattern of neutrino mixing and masses can be accommodated in different A4 models (with different alignments, in extra dimensions, with five-dimensional effective operators, etc.) and models based on many other groups. It is not excluded, however, that the tri-bimaximal mixing is accidental. An established theory of neutrino (and quark) masses and mixing does not exist at present. The creation of such a theory requires further experimental and theoretical efforts.

Chapter 6
Neutrino Oscillations in Vacuum

6.1 Introduction

The observation of neutrino oscillations in the atmospheric Super Kamiokande, solar SNO, reactor KamLAND, solar Homestake, GALLEX-GNO, SAGE and accelerator K2K and MINOS neutrino experiments was one of the most important recent discoveries in elementary particle physics. We will discuss experimental results later. In this section we will consider in some detail the theory of neutrino oscillations in vacuum. Neutrinos and antineutrinos are emitted in weak decays of pions and kaons, which are produced at accelerators and in the processes of interaction of cosmic ray protons in the atmosphere, in decays of muons, products of decays of pions and kaons, in β-decays of nuclei, products of the fission of uranium and plutonium in reactors, in nuclear reactions in the sun, etc. The first question which we will address here will be the following: *what are the states of neutrinos, produced in weak interaction processes, in the case of neutrino mixing.*

6.2 Flavor Neutrino States

Let us consider a charged current weak decay

$$a \to b + l^{+} + \nu_l \quad (l = e, \mu, \tau) \tag{6.1}$$

(a and b are some hadrons). The leptonic part of the standard Lagrangian of the CC interaction has the form

$$\mathcal{L}_I^{\mathrm{CC}} = -\frac{g}{2\sqrt{2}}\, j_\alpha^{\mathrm{CC}}\, W^\alpha + \mathrm{h.c.}, \tag{6.2}$$

where the leptonic CC is given by the expression

$$j_\alpha^{\mathrm{CC}} = 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}\, \gamma_\alpha\, l_L \tag{6.3}$$

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and ν_{lL} is *the mixed field*

$$\nu_{lL} = \sum_{i=1}^{3} U_{li}\,\nu_{iL}. \tag{6.4}$$

Here U is a unitary mixing matrix and ν_i is the field of neutrino (Dirac or Majorana) with mass m_i.

In the case of neutrino mixing the flavor lepton numbers L_e, L_μ and L_τ are not conserved. The neutrino, which is produced together with the lepton l^+ in a CC weak process *is called flavor neutrino* ν_l $(l = e, \mu, \tau)$.[1]

The vector of the state of the final particles in the decay (6.1) is given by

$$|f\rangle = \sum_{i=1}^{3} |\nu_i\rangle\,|l^+\rangle\,|b\rangle\,\langle\nu_i\,l^+\,b\,|S|\,a\rangle. \tag{6.5}$$

Here $|\nu_i\rangle$ is the normalized state of the neutrino ν_i with momentum p_i, $|l^+\rangle$ is the state of l^+ with momentum p_l, $|a\rangle$ and $|b\rangle$ are the states of the initial and final hadrons with momenta p_a and p_b and $\langle\nu_i\,l^+\,b\,|S|\,a\rangle$ is the element of the S-matrix.

From existing experimental data on the measurement of the high-energy part of the β-spectrum of ^{3}H follows that $m_i \lesssim 2.2$ eV (we will discuss these data in the Chap. 9.) The neutrino energy E in neutrino experiments $\gtrsim 1$ Mev (in the solar and reactor experiments $E \gtrsim 1$ MeV, in atmospheric and accelerator experiments $E \gtrsim 1$ GeV, etc.)

We will assume that $\mathbf{p_i} = p_i\mathbf{k}$ ($\mathbf{k}$ is a unit vector) and $p_i \to p = E$ at $m_i \to 0$. We have

$$p_i \simeq p - \alpha\frac{m_i^2}{2E}, \tag{6.6}$$

where $\alpha \sim O(1)$ depends on the process in which neutrinos are produced.

Neglecting in the matrix element $\langle\nu_i\,l^+\,b\,|S|\,a\rangle$ terms of the order of[2]

$$\frac{m_i^2}{E^2} \leq 10^{-12} \tag{6.7}$$

we have

---

[1] Let us stress that this is a phenomenological definition of a flavor neutrino (ν_e, ν_μ, ν_τ) in the case of the nonconservation of the flavor lepton numbers.

[2] In the end-point part of the β-spectrum of ^{3}H the neutrino energies (a few eV) are comparable with the neutrino masses. Thus, in this part of the β-spectrum neutrino masses cannot be neglected. The measurement of the end-point part of the β-spectrum of ^{3}H allows to obtain an information about neutrino masses.

$$\langle \nu_i\, l^+\, b\, |S|\, a\rangle \simeq U_{li}^*\, \langle \nu_l\, l^+\, b\, |S|\, a\rangle_{SM}. \tag{6.8}$$

Here $\langle \nu_l\, l^+\, b\, |S|\, a\rangle_{SM}$ is the matrix element of the process (6.1) calculated in the SM with massless neutrino ν_l. We have

$$\langle \nu_l\, l^+\, b\, |S|\, a\rangle_{SM} = -i\, \frac{G_F}{\sqrt{2}}\, N\, 2\, \bar{u}_L(p)\, \gamma_\alpha\, u_L(-p_{\,l})\, \langle b|\, J^\alpha(0)\, |a\rangle\, (2\pi)^4\, \delta(P' - P). \tag{6.9}$$

Here P and P' are the total initial and final momenta, J^α is the hadronic charged current and N is the product of the standard normalization factors.

From (6.8) we find for the final state (6.5)

$$|\, f\rangle \simeq |\nu_l\rangle\;\; |l^+\rangle\;\; |b\rangle\;\; \langle \nu_l\, l^+\, b\, |S|\, a\rangle_{\mathrm{SM}}, \tag{6.10}$$

where the neutrino state $|\nu_l\rangle$ is given by the expression

$$|\nu_l\rangle = \sum_{i=1}^{3} U_{li}^*\;\; |\nu_i\rangle. \tag{6.11}$$

From the unitarity of the mixing matrix follows that the states $|\nu_l\rangle$ are orthogonal and normalized

$$\langle \nu_{l'}|\nu_l\rangle = \sum_{i=1}^{3} U_{l'i}\;\, U_{li}^* = \delta_{l'l} \tag{6.12}$$

We come to the following important conclusion. Flavor neutrinos ν_l, which are produced in a CC weak decays (together with leptons l^+), are described by the normalized states (6.11), which are *coherent superpositions of states of neutrinos with different masses*. If we assume that the interaction of neutrinos is given by the Lagrangian (6.2), the amplitudes of the production of flavor neutrinos ν_l (up to negligible terms of the order of $\frac{m_i^2}{E^2} \ll 1$) are given by the Standard Model. Notice that in this approximation the helicity of the flavor neutrino ν_l is equal to -1.

Analogously, the normalized state of the flavor antineutrino $\bar{\nu}_l$, which is produced in a CC weak process together with a lepton l^-

$$a' \to b' + l^- + \bar{\nu}_l \quad (l = e, \mu, \tau) \tag{6.13}$$

in the case of Dirac neutrinos is given by the expression

$$|\bar{\nu}_l\rangle = \sum_{i=1}^{3} U_{li}\;\; |\bar{\nu}_i\rangle. \tag{6.14}$$

Here $|\bar{\nu}_i\rangle$ is the state of an antineutrino with momentum p_i and positive helicity. In the case of Majorana neutrinos we have

$$|\bar{\nu}_l\rangle = \sum_{i=1}^{3} U_{li}\ |\nu_i\rangle, \tag{6.15}$$

where $|\nu_i\rangle$ is the state of a Majorana neutrino with momentum p_i and positive helicity.

The relations (6.11), (6.14) and (6.15) are basic relations of the theory of neutrino oscillations. We obtain them assuming that small neutrino masses can be neglected in matrix elements of weak processes in which flavor neutrinos are produced. This assumption can only be valid if it is impossible to distinguish the emission of neutrinos with different masses in weak processes. We will demonstrate now that this is really the case.

Let us consider as an example the decay $\pi^+ \to \mu^+ + \nu_i$. In the pion rest frame the neutrino momentum p_i is given by the relation

$$p_i \simeq E - \alpha\,\frac{m_i^2}{2E}, \tag{6.16}$$

where $\alpha = \frac{m_\pi^2 + m_\mu^2}{2m_\pi^2} \simeq 0.8$ and

$$E = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}. \tag{6.17}$$

For the difference of momenta of neutrinos with different masses we find

$$|\Delta p_{ik}| = |p_k - p_i| \simeq \frac{|\Delta m_{ik}^2|}{2E}. \tag{6.18}$$

where $\Delta m_{ik}^2 = m_k^2 - m_i^2$.

On the other side from the Heisenberg uncertainty relation for the quantum mechanical uncertainty of neutrino momenta we have

$$(\Delta p)_{\mathrm{QM}} \simeq \frac{1}{a}, \tag{6.19}$$

where a characterizes the size of the wave packet of the pion.

From analysis of the data of neutrino oscillation experiments, which we will discuss later, for the largest neutrino mass-squared difference the value

$$\Delta m_{23}^2 \simeq 2.4 \cdot 10^{-3}\ \mathrm{eV}^2 \tag{6.20}$$

was obtained. From (6.20) we find (for E $\simeq$ 1GeV)

$$\frac{2E}{\Delta m_{23}^2} \simeq 1.7 \cdot 10^2 \text{ km}. \tag{6.21}$$

This value is many orders of magnitude larger than the microscopic length of the wave packet: From (6.18), (6.19) and (6.21) we conclude that

$$\Delta p_{23} \ll (\Delta p)_{\text{QM}}. \tag{6.22}$$

Thus, due to the Heisenberg uncertainty principle it is impossible to distinguish the emission of neutrinos with different masses in the pion decay.

6.3 Oscillations of Flavor Neutrinos

Neutrino oscillations are periodic transitions between different flavor neutrinos in neutrino beams. We will consider a beam of neutrinos which was produced in CC weak decays ($\pi^+ \to \mu^+ + \nu_\mu, \quad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu, \quad (A, Z) \to (A, Z+1) + e^- + \bar{\nu}_e$, etc.) at an accelerator or at a reactor or at a Neutrino Factory or in decays of radioactive nuclei (β-beam), etc.

In the quantum field theory the dependence of states on the time is given by the Schrodinger equation

$$i\,\frac{\partial|\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle, \tag{6.23}$$

where H is the total Hamiltonian. We will consider here the evolution of states in vacuum. In this case H is the free Hamiltonian. The general solution of Eq. (6.23) has the form

$$|\Psi(t)\rangle = e^{-i\,Ht}\,|\Psi(0)\rangle, \tag{6.24}$$

where $|\Psi(0)\rangle$ is the state at the initial time ($t = 0$).

We will apply now this general formalism to a neutrino beam. The initial state is the state of the flavor neutrino ν_l ($l = e, \mu, \tau$) given by Eq. (6.11). Thus, we have

$$|\Psi(0)\rangle = |\nu_l\rangle. \tag{6.25}$$

Taking into account that

$$H\,|\nu_i\rangle = E_i\,|\nu_i\rangle, \tag{6.26}$$

where

$$E_i = \sqrt{p_i^2 + m_i^2} \tag{6.27}$$

from (6.11) and (6.24) we find for the state of the left-handed neutrino at the time $t \geq 0$ the following expression

$$|\nu_l\rangle_t = e^{-i\,Ht}\,|\nu_l\rangle = \sum_{i=1}^{3} e^{-i\,E_i\,t}\,U_{li}^{*}\,|\nu_i\rangle. \tag{6.28}$$

Similarly, for state of the right-handed antineutrino at the time t we have

$$|\bar{\nu}_l\rangle_t = e^{-i\,Ht}\,|\bar{\nu}_l\rangle = \sum_{i=1}^{3} e^{-i\,E_i\,t}\,U_{li}\,|\bar{\nu}_i\rangle. \tag{6.29}$$

In general the neutrino energies E_i ($i = 1, 2, 3$) are different. This means that $|\nu_l\rangle_t$ is *a nonstationary state*. In other words during the time t different mass-components of the coherent neutrino state acquire *different phases*.

Neutrinos are detected via the observation of weak CC and NC processes. At the time t the state of the neutrino is the superposition (6.28) of the states $|\nu_i\rangle$. Let us consider the production of a lepton l' in the CC inclusive process

$$\nu_i + N \to l^{'} + X. \tag{6.30}$$

Neglecting small neutrino masses in the matrix element of the process (6.30) we find

$$\langle l'\,X\,|S|\,\nu_i\;N\,\rangle \simeq \langle l'\,X\,|S|\,\nu_{l'}\;N\,\rangle_{SM}\;U_{li}, \tag{6.31}$$

where $\langle l'\,X\,|S|\,\nu_{l'}\;N\,\rangle_{SM}$ is the SM matrix element of the process

$$\nu_{l'} + N \to l' + X \tag{6.32}$$

calculated for massless neutrinos $\nu_{l'}$. We have

$$\langle l'\,X\,|S|\,\nu_{l'}\;N\,\rangle_{SM} = -i\,\frac{G_F}{\sqrt{2}}\,N\,2\,\bar{u}_L(p_{l'})\,\gamma_\alpha\,u_L(p)\,\langle X|\,J^\alpha(0)\,|N\rangle\,(2\pi)^4\,\delta(P'-P), \tag{6.33}$$

where $p_{l'}$ is the momentum of the lepton, p is the neutrino momentum, P and P' are the initial and final total momenta.

From (6.28) and (6.31) we obtain the following expression for the amplitude of the transition $\nu_l \to \nu_{l'}$ during the time t

$$A(\nu_l \to \nu_{l'}) = \sum_{i=1}^{3} U_{l'i}\,e^{-i\,E_i\,t}\,U_{li}^{*}. \tag{6.34}$$

Analogously, for the amplitude of the transition $\bar{\nu}_l \to \bar{\nu}_{l'}$ during the time t we find

$$A(\bar{\nu}_l \to \bar{\nu}_{l'}) = \sum_{i=1}^{3} U^*_{l'i}\, e^{-i\,E_i\,t}\, U_{li}. \tag{6.35}$$

It is useful to obtain these results in another way. Taking into account that flavor states $|\nu_l\rangle$ form a complete system, we have

$$|\nu_l\rangle_t = e^{-iHt}\,|\nu_l\rangle = \sum_{l'} |\nu_{l'}\rangle\langle\nu_{l'}|e^{-i\,Ht}\,|\nu_l\rangle, \tag{6.36}$$

where $\langle\nu_{l'}|e^{-i\,Ht}\,|\nu_l\rangle$ is the amplitude of the probability to find at the time t the flavor neutrino $\nu_{l'}$ in the beam of neutrinos which originally at $t = 0$ was a beam of ν_l. From (6.11) we find

$$\langle\nu_{l'}|e^{-i\,Ht}\,|\nu_l\rangle = \sum_{i,i'} U_{l'i'}\langle\nu_{i'}|e^{-i\,Ht}\,|\nu_i\rangle U^*_{li} = \sum_{i=1}^{3} U_{l'i}\, e^{-i\,E_i\,t}\, U^*_{li} \equiv \mathcal{A}(\nu_l \to \nu_{l'}). \tag{6.37}$$

Let us consider the matrix element of the process

$$\nu_{l'} + N \to l'' + X \tag{6.38}$$

Neglecting neutrino masses we have

$$\begin{aligned} \langle l''\,X\,|S|\,\nu_{l'}\,N\,\rangle &= \sum_i \langle l''\,X\,|S|\,\nu_i\,N\,\rangle\, U^*_{l'i} \\ \simeq\ \langle l'\,X\,|S|\,\nu_{l'}\,N\,\rangle_{SM} \sum_i U^*_{l''i}\, U_{l'i} &= \delta_{l'l''}\,\langle l'\,X\,|S|\,\nu_{l'}\,N\,\rangle_{SM} \end{aligned} \tag{6.39}$$

Thus, we have demonstrated that a flavor neutrino $\nu_{l'}$ interacting with a nucleon produces lepton l' (and X). We can conclude that in the approximation $m_i \to 0$ the lepton flavor numbers L_e, L_μ and L_τ are effectively conserved in *weak processes*. In this approximation the matrix elements of weak processes are given by the SM.

The amplitude $\mathcal{A}(\nu_l \to \nu_{l'})$, given by Eq. (6.34), has a simple meaning: the factor U^*_{li} is the amplitude of the transition from the initial flavor state $|\nu_l\rangle$ into the state of a neutrino with definite mass $|\nu_i\rangle$; the factor $e^{-iE_i\,t}$ describes the propagation in the state with definite mass; the factor $U_{l'i}$ is the amplitude of the transition from the state $|\nu_i\rangle$ into the state $|\nu_{l'}\rangle$. The coherent sum over all states with definite masses is performed.

It is obvious that if all E_i are the same, due to unitarity of the mixing matrix we have

$$\mathrm{A}(\nu_l \to \nu_{l'}) = \delta_{l'l}, \quad \mathrm{A}(\bar{\nu}_l \to \bar{\nu}_{l'}) = \delta_{l'l}. \tag{6.40}$$

Thus, transitions between different flavor neutrinos (antineutrinos) can take place only in the case if at least one phase difference is not equal to zero.

The probabilities of the transitions $\nu_l \to \nu_{l'}$ and $\bar\nu_l \to \bar\nu_{l'}$ are equal, respectively,

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \left|\sum_{i=1}^{3} U_{l'i}\, e^{-i\,E_i t}\, U_{li}^*\right|^2 \tag{6.41}$$

and

$$\mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) = \left|\sum_{i=1}^{3} U_{l'i}^*\, e^{-i\,E_i t}\, U_{li}\right|^2. \tag{6.42}$$

From (6.41) and (6.42) the following general relations between transition probabilities can easily be obtained:

1.

$$\begin{aligned} &\sum_{l'} \mathrm{P}(\nu_l \to \nu_{l'}) = 1, \quad \sum_{l'} \mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) = 1, \\ &\sum_{l} \mathrm{P}(\nu_l \to \nu_{l'}) = 1, \quad \sum_{l} \mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) = 1. \end{aligned} \tag{6.43}$$

These relations are a consequence of the unitarity of the mixing matrix. In fact, we have

$$\begin{aligned} &\sum_{l'} \mathrm{P}(\nu_l \to \nu_{l'}) = \sum_{l',i,k} U_{l'i}U_{l'k}^*\, e^{-i\,(E_i-E_k)\,t}\, U_{li}^*U_{lk} = \\ &\sum_{i,k} \delta_{ik}\, e^{-i\,(E_i-E_k)\,t}\, U_{li}^*U_{lk} = \sum_{i} U_{li}^*U_{li} = 1. \end{aligned} \tag{6.44}$$

2.

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \mathrm{P}(\bar\nu_{l'} \to \bar\nu_l). \tag{6.45}$$

This relation is a consequence of the CPT invariance of the theory and can easily be obtained by direct comparison of (6.41) and (6.42).
From (6.45) follows that the probabilities of the transitions $\nu_l \to \nu_l$ and $\bar\nu_l \to \bar\nu_l$ are equal

$$\mathrm{P}(\nu_l \to \nu_l) = \mathrm{P}(\bar\nu_l \to \bar\nu_l), \tag{6.46}$$

3.

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) \tag{6.47}$$

This relation is valid in the case of *CP* invariance in the lepton sector. In fact, we have in this case (see Sect. 5.3)

$$U_{li} = U_{li}^{*} \ \text{(Dirac neutrinos)}, \quad \rho_i \ U_{li} = U_{li}^{*} \ \text{(Majorana neutrinos)}, \tag{6.48}$$

where $\rho_i = \pm 1$. From (6.41), (6.42) and (6.48) we easily obtain relation (6.47). The probabilities of the transitions $\nu_l \to \nu_l$ and $\bar{\nu}_l \to \bar{\nu}_l$ are equal due to *CPT* invariance. Thus, in order to check whether *CP* is conserved in the lepton sector we need to test the relations (6.47) at $l' \neq l$. Let us also notice that in the case of *CP* invariance, from (6.45) and (6.47) follows that

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \mathrm{P}(\nu_{l'} \to \nu_l), \quad \mathrm{P}(\bar{\nu}_l \to \bar{\nu}_{l'}) = \mathrm{P}(\bar{\nu}_{l'} \to \bar{\nu}_l). \tag{6.49}$$

These relations are a consequence on the invariance under time reversal (T) which obviously holds if both *CP* invariance and *CPT* invariance take place.

We have considered the case of three flavor neutrinos (ν_e, ν_μ, ν_τ) and three flavor antineutrinos ($\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$). Let us discuss now the most general case of active and sterile neutrinos. For neutrino mixing we have in this case

$$\nu_{lL} = \sum_{i=1}^{3+n_s} U_{li} \nu_{iL} \ \ (l = e, \mu, \tau), \quad \nu_{sL} = \sum_{i=1}^{3+n_s} U_{si} \nu_{iL} \ \ (s = s_1, s_2, \ldots s_{n_s}). \tag{6.50}$$

Here ν_i is the field of neutrinos (Dirac or Majorana) with mass m_i. If all neutrino masses are much smaller than the neutrino energy, the neutrino masses can be neglected in the matrix elements of the processes in which neutrinos are produced and for the states of flavor left-handed neutrinos we will have

$$|\nu_l\rangle = \sum_{i=1}^{3+n_s} U_{li}^{*} \ |\nu_i\rangle, \tag{6.51}$$

where $|\nu_i\rangle$ is the state of neutrinos with mass m_i and momentum p_i. Sterile fields ν_{sL} do not enter into the SM Lagrangian of the weak interaction. We determine states of sterile left-handed neutrino ν_s as follows

$$|\nu_s\rangle = \sum_{i=1}^{3+n_s} U_{si}^{*} \ |\nu_i\rangle. \tag{6.52}$$

From the unitarity of the $(3 + n_s) \times (3 + n_s)$ mixing matrix U we find that

$$\langle \nu_s | \nu_l \rangle = 0. \tag{6.53}$$

The total system of active and sterile left-handed states

$$|\nu_\alpha\rangle = \sum_{i=1}^{3+n_s} U^*_{\alpha i}\ |\nu_i\rangle\ , \quad \alpha = e, \mu, \tau, s_1, s_2, \dots s_{n_s} \tag{6.54}$$

satisfies the conditions

$$\langle \nu_{\alpha'}|\nu_\alpha\rangle = \sum_{i=1}^{3+n_s} U_{\alpha' i}\ U^*_{\alpha i} = \delta_{\alpha'\alpha}. \tag{6.55}$$

If the initial neutrino state at $t = 0$ is $|\nu_\alpha\rangle$, for the neutrino state at the time $t \geq 0$ we have

$$|\nu_\alpha\rangle_t = e^{-iHt}\ |\nu_\alpha\rangle = \sum_{\alpha'} |\nu_{\alpha'}\rangle\ \langle \nu_{\alpha'}|\ e^{-iHt}\ |\nu_\alpha\rangle. \tag{6.56}$$

Here

$$\langle \nu_{\alpha'}|\ e^{-i\,Ht}\ |\nu_\alpha\rangle = \mathrm{A}(\nu_\alpha \to \nu_{\alpha'}) = \sum_{i=1}^{3+n_s} U_{\alpha' i}\ e^{-i\,E_i t}\ U^*_{\alpha i} \tag{6.57}$$

is the amplitude of the transition $\nu_\alpha \to \nu'_\alpha$ during the time t.

Analogously, for the amplitude of the transition $\bar\nu_\alpha \to \bar\nu_{\alpha'}$ we have

$$\mathrm{A}(\bar\nu_\alpha \to \bar\nu_{\alpha'}) = \langle \bar\nu_{\alpha'}|\ e^{-i\,Ht}\ |\bar\nu_\alpha\rangle = \sum_{i=1}^{3+n_s} U^*_{\alpha' i}\ e^{-i\,E_i t}\ U_{\alpha i}. \tag{6.58}$$

From (6.57) and (6.58) we find the following expressions for the probabilities of the transitions $\nu_\alpha \to \nu_{\alpha'}$ and $\bar\nu_\alpha \to \bar\nu_{\alpha'}$

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \left|\sum_{i=1}^{3+n_s} U_{\alpha' i}\ e^{-i\,E_i t}\ U^*_{\alpha i}\right|^2; \quad \mathrm{P}(\bar\nu_\alpha \to \bar\nu_{\alpha'}) = \left|\sum_{i=1}^{3+n_s} U^*_{\alpha' i}\ e^{-i\,E_i t}\ U_{\alpha i}\right|^2. \tag{6.59}$$

It is easy to check that the transition probabilities are correctly normalized. In fact, taking into account the unitarity of the $(3+n_s)\times(3+n_s)$ mixing matrix U, we find

$$\sum_{\alpha'} \mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \sum_{\alpha',i,k} U_{\alpha' i}\ U^*_{\alpha' k}\ e^{-i\,(E_i - E_k)\,t}\ U^*_{\alpha i}\ U_{\alpha k} = \sum_i |U_{\alpha i}|^2 = 1 \tag{6.60}$$

Sterile neutrinos cannot be detected in the standard weak processes. One of the possibilities to obtain an information about transitions into sterile states can be based on the unitarity relation (6.60). In fact, from (6.60) we have

$$\sum_{l'=e,\mu,\tau} \mathrm{P}(\nu_l \to \nu_{l'}) = 1 - \sum_{s=s_1,s_2,\dots s_{n_s}} \mathrm{P}(\nu_l \to \nu_s) \tag{6.61}$$

The left-handed part of this relation is the total probability of the transition of a flavor neutrino ν_l into all possible flavor active neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$. This probability can be measured if neutrinos are detected at some distance from the source by the observation of a NC process. If it would occur that the probability $\sum_{l'=e,\mu,\tau} \mathrm{P}(\nu_l \to \nu_{l'})$ is less than one it would be a proof of the transition of an active neutrino into sterile states.

Let us return back to the expression (6.59) for the neutrino transition probability. Taking into account that a common phase cannot be observed, the transition probability can be presented in the form

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \left| \sum_{i=1}^{3+n_s} U_{\alpha' i}\, e^{-i\,(E_i - E_j)\,t}\, U^*_{\alpha i} \right|^2 \tag{6.62}$$

(j is fixed). Thus, the transition probability depends on the elements of the neutrino mixing matrix and $2 + n_s$ phase differences.

In quantum field theory states of particles are characterized by their momenta, helicities, masses, barion numbers, etc. We will assume that a mixed neutrino state is characterized by momentum $\mathbf{p}$, masses m_i (and also elements of the mixing matrix $U_{\alpha i}$ and helicity equal to -1). In other words we will assume that

$$\mathbf{p_i} = \mathbf{p}. \tag{6.63}$$

Taking into account that $\frac{m_i^2}{p^2} \ll 1$, we have

$$E_i \simeq p + \frac{m_i^2}{2p}, \quad \text{and} \quad E_i - E_j \simeq \frac{\Delta m^2_{ji}}{2E}, \tag{6.64}$$

where $E \simeq p$ is the neutrino energy and

$$\Delta m^2_{ji} = m_i^2 - m_j^2. \tag{6.65}$$

In (6.62) t is the difference of neutrino production time and detection time. For the ultrarelativistic neutrino

$$t \simeq L, \tag{6.66}$$

where L is the distance between neutrino production and detection points. Thus, in the expression (6.62) for the transition probability the oscillation phase is equal to

$$(E_i - E_j)\,t \simeq \frac{\Delta m^2_{ji}}{2E}\, L. \tag{6.67}$$

Taking into account unitarity condition

$$\sum_i U_{\alpha' i}\, U^*_{\alpha i} = \delta_{\alpha'\alpha} \tag{6.68}$$

from (6.62) and (6.68) we find the following convenient expression for the neutrino transition probability

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \left| \delta_{\alpha'\alpha} + \sum_{i \neq j} U_{\alpha' i} \left(e^{-i \frac{\Delta m^2_{ji}}{2E} L} - 1 \right) U^*_{\alpha i} \right|^2 \tag{6.69}$$

Analogously, antineutrino transition probability is given by the expression

$$\mathrm{P}(\bar\nu_\alpha \to \bar\nu_{\alpha'}) = \left| \delta_{\alpha'\alpha} + \sum_{i \neq j} U^*_{\alpha' i} \left(e^{-i \frac{\Delta m^2_{ji}}{2E} L} - 1 \right) U_{\alpha i} \right|^2 \tag{6.70}$$

The expression (6.62) for the transition probability $\nu_\alpha \to \nu_{\alpha'}$ can be also presented in the form

$$\begin{aligned}\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) &= \sum_{i,k} U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\; U_{\alpha k} e^{-i \frac{\Delta m^2_{ki}}{2E} L} \\ &= \sum_i |U_{\alpha' i}|^2\, |U_{\alpha i}|^2 + 2\,\mathrm{Re} \sum_{i>k} U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\; U_{\alpha k} e^{-i \frac{\Delta m^2_{ki}}{2E} L}.\end{aligned} \tag{6.71}$$

Further, from the unitarity relation (6.68) we easily obtain the following relation

$$\sum_i |U_{\alpha' i}|^2\, |U_{\alpha i}|^2 = \delta_{\alpha'\alpha} - 2\,\mathrm{Re} \sum_{i>k} U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\; U_{\alpha k}. \tag{6.72}$$

Combining now (6.71) and (6.72) we have

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \delta_{\alpha'\alpha} - 2\,\mathrm{Re} \sum_{i>k} U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\; U_{\alpha k} \left(1 - e^{-i \frac{\Delta m^2_{ki}}{2E} L} \right). \tag{6.73}$$

Finally, taking into account that for any complex a and b

$$\mathrm{Re}\, a\, b = \mathrm{Re} a\, \mathrm{Re} b - \mathrm{Im} a\, \mathrm{Im} b$$

from (6.73) for the transition probability we obtain the following expression

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \delta_{\alpha'\alpha} - 2 \sum_{i>k} \mathrm{Re}\,(U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\, U_{\alpha k}) \left(1 - \cos \frac{\Delta m^2_{ki}}{2E}\, L\right)$$
$$+ 2 \sum_{i>k} \mathrm{Im}\,(U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\, U_{\alpha k})\, \sin \frac{\Delta m^2_{ki}}{2E}\, L. \qquad (6.74)$$

Analogously, for the probability of the transition $\bar{\nu}_\alpha \to \bar{\nu}_{\alpha'}$ we find

$$\mathrm{P}(\bar{\nu}_\alpha \to \bar{\nu}_{\alpha'}) = \delta_{\alpha'\alpha} - 2 \sum_{i>k} \mathrm{Re}\,(U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\, U_{\alpha k}) \left(1 - \cos \frac{\Delta m^2_{ki}}{2E}\, L\right)$$
$$- 2 \sum_{i>k} \mathrm{Im}\,(U_{\alpha' i}\, U^*_{\alpha' k}\, U^*_{\alpha i}\, U_{\alpha k})\, \sin \frac{\Delta m^2_{ki}}{2E}\, L \qquad (6.75)$$

From these expressions we conclude that neutrino and antineutrino transition probabilities depend on the parameter $\frac{L}{E}$. They are determined by the elements of the neutrino mixing matrix $U_{\alpha i}$ and $2+n_s$ independent mass-squared differences Δm^2_{ki}.

6.4 Two-Neutrino Oscillations

In this section we will consider the simplest case of the oscillations between two neutrinos

$$\nu_\alpha \rightleftarrows \nu_{\alpha'} \quad \alpha, \alpha' = e, \mu, \tau, s_1, \ldots \qquad (6.76)$$

which are driven by the two neutrino mixing

$$\nu_{\alpha L} = \sum_{i=1,2} U_{\alpha i}\, \nu_{iL}, \qquad (6.77)$$

where U is an unitary 2×2 matrix.

We will label neutrino masses in such a way that $m_1 < m_2$. From (6.69) and (6.70) for $j = 1$ we have

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) == |\delta_{\alpha'\alpha} + U_{\alpha' 2}\, (e^{-i\, \frac{\Delta m^2}{2E}\, L} - 1)\, U^*_{\alpha 2}|^2 \qquad (6.78)$$

and

$$\mathrm{P}(\bar{\nu}_\alpha \to \bar{\nu}_{\alpha'}) == |\delta_{\alpha'\alpha} + U^*_{\alpha' 2}\, (e^{-i\, \frac{\Delta m^2}{2E}\, L} - 1)\, U_{\alpha 2}|^2. \qquad (6.79)$$

Here $\Delta m^2 = \Delta m^2_{12} = m^2_2 - m^2_1$.

Let us consider the case $\alpha' \neq \alpha$. From (6.78) and (6.79) we find

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \mathrm{P}(\bar{\nu}_\alpha \to \bar{\nu}_{\alpha'}) = 2\,|U_{\alpha'2}|^2\,|U_{\alpha 2}|^2 \left(1 - \cos\frac{\Delta m^2}{2E}\,L\right) \quad (\alpha' \neq \alpha). \tag{6.80}$$

If ν_1 and ν_2 are Dirac particles, the neutrino mixing matrix is real (see Sect. 5.3). In this case the *CP* invariance in the lepton sector takes place automatically. If ν_1 and ν_2 are Majorana particles there is one *CP* phase in the mixing matrix. However, it is easy to see that the Majorana *CP* phase does not enter into transition probabilities. This is the reason why in the case of the transitions between two types of neutrinos the relation (6.47) is always valid.

A real orthogonal 2×2 mixing matrix has the form

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta, \end{pmatrix}, \tag{6.81}$$

where θ is a mixing angle. From (6.80) and (6.81) we obtain the following standard expression for the probability of the transition between two types of neutrinos

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \frac{1}{2}\,\sin^2 2\theta \left(1 - \cos\frac{\Delta m^2}{2E}\,L\right) = \sin^2 2\theta\ \sin^2\frac{m^2}{4\,E}\,L \quad (\alpha \neq \alpha'). \tag{6.82}$$

It is obvious from (6.80) that in the two-neutrino case probabilities of all possible non diagonal transitions are equal

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \mathrm{P}(\bar{\nu}_\alpha \to \bar{\nu}_{\alpha'}) = \mathrm{P}(\nu_{\alpha'} \to \nu_\alpha) = \mathrm{P}(\bar{\nu}_{\alpha'} \to \bar{\nu}_\alpha) \quad (\alpha \neq \alpha'). \tag{6.83}$$

From these relations and conditions of the conservation of the probability (6.44) for probabilities of neutrino and antineutrino to survive we obtain the following relations

$$\mathrm{P}(\nu_\alpha \to \nu_\alpha) = \mathrm{P}(\nu_{\alpha'} \to \nu_{\alpha'}) = \mathrm{P}(\bar{\nu}_\alpha \to \bar{\nu}_\alpha) = \mathrm{P}(\bar{\nu}_{\alpha'} \to \bar{\nu}_{\alpha'}) \quad (\alpha \neq \alpha'). \tag{6.84}$$

Finally, from (6.78) we find

$$\mathrm{P}(\nu_\alpha \to \nu_\alpha) = 1 - \mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = 1 - \frac{1}{2}\,\sin^2 2\theta \left(1 - \cos\frac{\Delta m^2}{2E}\,L\right). \tag{6.85}$$

Thus, all possible transitions between two types of neutrinos and antineutrinos are described by two oscillation parameters: $\sin^2 2\theta$ and Δm^2.

The expressions (6.82) and (6.78) are written in the standard units $\hbar = c = 1$. If we introduce $\hbar$ and c for the transition probability we will find the following expression

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \frac{1}{2}\,\sin^2 2\theta \left(1 - \cos\frac{\Delta m^2\,c^4}{2E\,\hbar\,c}\,L\right) = \sin^2 2\theta\,\sin^2\frac{\Delta m^2\,c^4}{4E\,\hbar\,c}\,L \quad (\alpha \neq \alpha'). \tag{6.86}$$

If we use $\hbar\,c \simeq 1.97 \cdot 10^{-11}$ MeV cm the expression for the transition probability takes the form

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \frac{1}{2}\,\sin^2 2\theta \left(1 - \cos 2.54\,\frac{\Delta m^2}{E}\,L\right) = \sin^2 2\theta\,\sin^2 1.27\,\frac{\Delta m^2}{E}\,L. \tag{6.87}$$

Here Δm^2 is the neutrino mass-squared difference in eV^2, E is the neutrino energy in MeV (GeV) and L is the distance between neutrino source and neutrino detector in m (km).

We will now introduce *the oscillation length*. The expression (6.82) can be written in the form

$$\mathrm{P}(\nu_\alpha \to \nu_{\alpha'}) = \frac{1}{2}\,\sin^2 2\theta \left(1 - \cos 2\,\pi\,\frac{L}{L_{\mathrm{osc}}}\right) \quad (\alpha \neq \alpha'), \tag{6.88}$$

where

$$L_{\mathrm{osc}} = 4\,\pi\,\frac{E}{\Delta m^2} \tag{6.89}$$

is the oscillation length. Introducing $\hbar$ and c, for the oscillation length we obtain the following expression

$$L_{\mathrm{osc}} = 4\,\pi\,\frac{E\,\hbar\,c}{\Delta m^2\,c^4}. \tag{6.90}$$

From this expression we find

$$L_{\mathrm{osc}} \simeq 2.47\,\frac{E}{\Delta m^2}\,\mathrm{m}. \tag{6.91}$$

where E is the neutrino energy in MeV and Δm^2 is the neutrino mass-squared difference in eV^2.

The two-neutrino transition probabilities given by the expressions (6.82) and (6.85) are standard formulas which are usually used in the analysis of experimental data. Below we will present arguments why these formulas are good approximations in the case of oscillations between three neutrinos.

In Fig. 6.1 the transition probability $\mathrm{P}(\nu_\mu \to \nu_\tau)$ as a function of $\frac{L}{E}$ (in units $\frac{\mathrm{m}}{\mathrm{MeV}}$) is plotted. We assumed that $\sin^2 2\theta = 1$ (as we will see later, this value is in agreement with experimental data). At the points

$$\frac{L}{E} = \frac{2\pi\,n}{2.54\,\Delta m^2} \quad \text{and} \quad \frac{L}{E} = \frac{2\pi\left(n + \frac{1}{2}\right)}{2.54\,\Delta m^2} \quad (n = 0, 1, 2, \ldots)$$

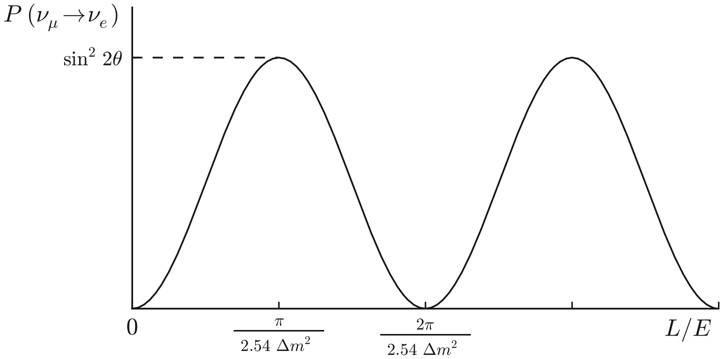

Fig. 6.1 Transition probability $P(\nu_\mu \to \nu_\tau)$ as a function of $\frac{L}{E}$

only ν_μ or, correspondingly, ν_τ can be observed. At other values of $\frac{L}{E}$ both ν_μ and ν_τ can be found. The probability to find ν_τ is given by $P(\nu_\mu \to \nu_\tau)$. A muon neutrino ν_μ can be found with the probability $1 - P(\nu_\mu \to \nu_\tau) = P(\nu_\mu \to \nu_\mu)$. From Fig. 6.1 it is clear why phenomenon, we are considering, is called neutrino oscillations.

6.5 Three-Neutrino Oscillations. *CP* Violation in the Lepton Sector

All existing data are well described if we assume that the number of massive neutrinos is equal to the measured number of the flavor neutrinos (three). In this section we will consider neutrino oscillations in vacuum in the case of three mixed neutrinos. In this case the flavor field $\nu_{lL}(x)$ is given by the expression

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\ \nu_{iL}(x) \quad l = e, \mu, \tau. \tag{6.92}$$

Here U is a unitary 3×3 Pontecorvo-Maki-Nakagava-Sakata (PMNS) matrix, and $\nu_i(x)$ is the field of neutrino with the mass m_i. The field $\nu_i(x)$ can be a Dirac field of neutrinos and antineutrinos or a Majorana field of neutrinos. We will show now that by the investigation of flavor neutrino oscillations it is impossible to distinguish these two fundamentally different possibilities. In fact, Dirac and Majorana mixing matrices differ by the number of *CP* phases (one in the Dirac case and three in the Majorana case). The Majorana mixing matrix has the form

$$U_{li}^{M} = U_{li}\ e^{i\bar{\alpha}_i}. \tag{6.93}$$

Here $\bar{\alpha}_1 = 0$, $\bar{\alpha}_{2,3}$ are Majorana phases and the matrix U has the form of the Dirac mixing matrix. Let us consider the amplitude of the transition $\nu_l \to \nu_{l'}$. Taking into account the relation (6.93), we have

$$\mathcal{A}^M(\nu_l \to \nu_{l'}) = \sum_{i=1}^{3} U^M_{l'i}\, e^{-iE_i t}\, U^{M*}_{l'i} = \sum_{i=1}^{3} U_{l'i}\, e^{-iE_i t}\, U^*_{l'i} = \mathcal{A}^D(\nu_l \to \nu_{l'}). \tag{6.94}$$

It is obvious that the same relation is valid for the amplitude of the transition $\bar{\nu}_l \to \bar{\nu}_{l'}$. Thus, additional Majorana phases do not enter into flavor neutrino and antineutrino transition amplitudes. This means that the study of flavor neutrino oscillations does not allow to reveal the nature of the massive neutrinos. For that, as we will discuss later, it is necessary to study processes, like neutrinoless double β decay of nuclei, in which the total lepton number is not conserved. When we will consider neutrino oscillations we will use the Dirac matrix U, which is characterized by three mixing angles and one *CP* phase.

In the previous section we have shown that the neutrino and antineutrino transition probabilities are given by the expressions

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \delta_{l'l} + \mathrm{R}_{l'l} + \frac{1}{2}\,\mathrm{A}^{CP}_{l'l}, \quad \mathrm{P}(\bar{\nu}_l \to \bar{\nu}_{l'}) = \delta_{l'l} + \mathrm{R}_{l'l} - \frac{1}{2}\,\mathrm{A}^{CP}_{l'l}, \tag{6.95}$$

where

$$\mathrm{R}_{l'l} = -2\sum_{i>k} \mathrm{Re}\,(U_{l'i}\, U^*_{l'k}\, U^*_{li}\, U_{lk}) \left(1 - \cos\frac{\Delta m^2_{ki}}{2E}\,L\right) \tag{6.96}$$

and

$$\mathrm{A}^{CP}_{l'l} = 4\sum_{i>k} \mathrm{Im}\,(U_{l'i}\, U^*_{l'k}\, U^*_{li}\, U_{lk})\, \sin\frac{\Delta m^2_{ki}}{2E}\,L. \tag{6.97}$$

From (6.95) we have

$$\mathrm{R}_{l'l} = \frac{1}{2}(\mathrm{P}(\nu_l \to \nu_{l'}) + \mathrm{P}(\bar{\nu}_l \to \bar{\nu}_{l'})) - \delta_{l'l} \tag{6.98}$$

and

$$\mathrm{A}^{CP}_{l'l} = \mathrm{P}(\nu_l \to \nu_{l'}) - \mathrm{P}(\bar{\nu}_l \to \bar{\nu}_{l'}). \tag{6.99}$$

Thus, $\mathrm{R}_{l'l}$ and $\mathrm{A}^{CP}_{l'l}$ are the *CP*-even and *CP*-odd parts of the transition probabilities. Let us consider $\mathrm{A}^{CP}_{l'l}$. The CPT invariance implies that

$$\mathrm{A}^{CP}_{l'l} = -\mathrm{A}^{CP}_{l\,l'}. \tag{6.100}$$

From this relation follows that $\mathrm{A}^{CP}_{ll} = 0$. If *CP* invariance holds, we have $\mathrm{A}^{CP}_{l'l} = 0 \ (l' \neq l)$. Thus, $\mathrm{A}^{CP}_{l'l}$ is the *CP* asymmetry. If the asymmetry is not equal to zero, it is a signature of the violation of the *CP* invariance in the lepton sector.

It follows from the relation (6.100) that in the case of three flavor neutrinos there are three different from zero CP asymmetries: $\mathrm{A}^{CP}_{e\mu}$, $\mathrm{A}^{CP}_{\tau\mu}$, $\mathrm{A}^{CP}_{\tau e}$. These three CP asymmetries are connected by the following cyclic relations

$$\mathrm{A}^{CP}_{e\mu} = \mathrm{A}^{CP}_{\tau e} = -\mathrm{A}^{CP}_{\tau\mu}, \tag{6.101}$$

In fact, from the conservation of the probabilities we have

$$\sum_{l'=e,\mu,\tau} \mathrm{P}(\nu_l \to \nu_{l'}) = 1, \qquad \sum_{l'=e,\mu,\tau} \mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) = 1. \tag{6.102}$$

If we now subtract from the first relation the second one, we obtain

$$\sum_{l'=e,\mu,\tau} \mathrm{A}^{CP}_{l'l} = 0. \tag{6.103}$$

From (6.103) we find the following relations

$$\mathrm{A}^{CP}_{\mu e} + \mathrm{A}^{CP}_{\tau e} = 0,\ \ \mathrm{A}^{CP}_{e\mu} + \mathrm{A}^{CP}_{\tau\mu} = 0,\ \ \mathrm{A}^{CP}_{e\tau} + \mathrm{A}^{CP}_{\mu\tau} = 0. \tag{6.104}$$

From (6.104) we easily obtain the relations (6.101). Thus, in the case of three-neutrino mixing there exists only one independent CP asymmetry.

We will consider now the expression (6.97) for the CP asymmetry. Let us introduce the quantity

$$\mathrm{J}^{ik}_{l'l} = \mathrm{Im}\,(U_{l'i}\, U^*_{l'k}\, U^*_{li}\, U_{lk}). \tag{6.105}$$

It is obvious that $\mathrm{J}^{ik}_{l'l} \neq 0$ only if $i \neq k$ and $l' \neq l$.

We will show now that

$$\mathrm{J}^{ik}_{l'l} = \pm\mathrm{J}, \quad l' \neq l,\ \ i \neq k, \tag{6.106}$$

where

$$\mathrm{J} = \mathrm{J}^{21}_{e\mu}. \tag{6.107}$$

In fact, from (6.105) we obtain the following symmetry relations

$$\mathrm{J}^{ik}_{l'l} = -\mathrm{J}^{ki}_{l'l}, \quad \mathrm{J}^{ik}_{l'l} = -\mathrm{J}^{ik}_{ll'}. \tag{6.108}$$

Further, from the unitarity of the 3×3 mixing matrix U we find

$$\sum_{i=1}^{3} \mathrm{J}^{ik}_{l'l} = \delta_{l'l}\ \mathrm{Im}\,(U^*_{l'k}\, U_{lk}) = 0, \quad \sum_{l'=e,\mu,\tau} \mathrm{J}^{ik}_{l'l} = \delta_{ik}\ \mathrm{Im}\,(U^*_{li}\, U_{lk}) = 0. \tag{6.109}$$

From (6.108) and the first Eq. (6.109) we obtain the following relations

$$\mathrm{J}^{21}_{l'l} + \mathrm{J}^{31}_{l'l} = 0, \quad \mathrm{J}^{12}_{l'l} + \mathrm{J}^{32}_{l'l} = 0, \quad \mathrm{J}^{13}_{l'l} + \mathrm{J}^{23}_{l'l} = 0. \tag{6.110}$$

From (6.110) and (6.108) we easily find the following cycle relations

$$\mathrm{J}^{21}_{l'l} = \mathrm{J}^{13}_{l'l} = \mathrm{J}^{32}_{l'l}. \tag{6.111}$$

Further, from (6.108) and the second Eq. (6.109) we have

$$\mathrm{J}^{ik}_{\mu e} + \mathrm{J}^{ik}_{\tau e} = 0, \quad \mathrm{J}^{ik}_{e\mu} + \mathrm{J}^{ik}_{\tau\mu} = 0 \quad \mathrm{J}^{ik}_{e\tau} + \mathrm{J}^{ik}_{\mu\tau} = 0. \tag{6.112}$$

From (6.112) and (6.108) we obtain the following cycle relations

$$\mathrm{J}^{ik}_{e\mu} = \mathrm{J}^{ik}_{\mu\tau} = \mathrm{J}^{ik}_{\tau e}. \tag{6.113}$$

It is evident from (6.111) and (6.113) that all nonzero elements of $\mathrm{J}^{ik}_{l'l}$ are equal to $\pm\mathrm{J}$.

From (6.97) and (6.111) for the *CP* asymmetry $\mathrm{A}^{CP}_{l'l}$ we find the following expression

$$\mathrm{A}^{CP}_{l'l} = 4J^{21}_{l'l}\left(\sin\frac{\Delta m^2_{12}}{2E}\,L + \sin\frac{\Delta m^2_{23}}{2E}\,L - \sin\frac{\Delta m^2_{13}}{2E}\,L\right). \tag{6.114}$$

This expression can be written in a more compact and convenient form. In fact, it is obvious that

$$\Delta m^2_{13} = \Delta m^2_{12} + \Delta m^2_{23}. \tag{6.115}$$

Further, for any a and b we have the following relation

$$\begin{aligned}\sin a + \sin b - \sin(a+b) &= 2\sin\frac{(a+b)}{2}\left(\cos\frac{(a-b)}{2} - \cos\frac{(a+b)}{2}\right)\\ &= 4\,\sin\frac{a}{2}\,\sin\frac{b}{2}\,\sin\frac{(a+b)}{2}.\end{aligned} \tag{6.116}$$

From (6.114), (6.115), (6.116) and (6.107) we find the following expression

$$\mathrm{A}^{CP}_{e\mu} = 16\,\mathrm{J}\,\sin\frac{\Delta m^2_{12}}{4E}\,L\,\sin\frac{\Delta m^2_{23}}{4E}\,L\,\sin\frac{(\Delta m^2_{12} + \Delta m^2_{23})}{4E}\,L. \tag{6.117}$$

Other asymmetries are connected with $\mathrm{A}^{CP}_{e\mu}$ by the relations (6.101). The quantity J, which determines the *CP* asymmetries in all channels, is usually called Jarlskog invariant. It is invariant under the transformation

$$U'_{li} = e^{-i\beta_l}\, U_{li}\, e^{i\alpha_l}, \tag{6.118}$$

where α_i and β_l are arbitrary constant phases. In fact, we have

$$\mathrm{J}^{ik}_{l'l} = \mathrm{Im}(U_{l'i}\, U^*_{l'k}\, U^*_{li}\, U_{lk}) = \mathrm{Im}(U'_{l'i}\, U'^*_{l'k}\, U'^*_{li}\, U'_{lk}) = (\mathrm{J}^{ik}_{l'l})' \tag{6.119}$$

It is easy to see that (for Dirac neutrinos) the mixing matrices U and U' are equivalent. In fact, let us consider the lepton charged current. We have

$$j^{\mathrm{CC}}_\alpha = 2\sum_{l,i} \bar{\nu}_{iL} U^*_{li} \gamma_\alpha l_L = 2\sum_{l,i} \bar{\nu}'_{iL} U'^*_{li} \gamma_\alpha l'_L, \tag{6.120}$$

where the primed fields are determined as follows

$$\nu'_i(x) = e^{-i\alpha_i}\nu_i(x), \quad l'(x) = e^{-i\beta_l} l(x) \tag{6.121}$$

The fields $\nu'_i(x)$ and $l'(x)$ cannot be distinguished from $\nu_i(x)$ and $l(x)$. Thus, the mixing matrix (in the Dirac case) is determined up to the phase transformation (6.118) and the transition probabilities must be invariant under this transformation.

In the standard parametrization of the mixing matrix U (see previous chapter) for the Jarlskog invariant we obtain the following expression

$$J = -c_{12}c_{23}c^2_{13}s_{12}s_{23}s_{13}\sin\delta. \tag{6.122}$$

6.6 Three-Neutrino Oscillations in the Leading Approximation

In the general case of the three-neutrino mixing the $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$ transition probabilities depend on six parameters (two mass-squared differences Δm^2_{12} and Δm^2_{23}, three mixing angles θ_{12}, θ_{23} and θ_{13} and the CP-phase δ) and have rather complicated form. However, from the analysis of the neutrino oscillation data it was found that one mass-squared difference is much smaller that the other one:

$$\Delta m^2_{23} \simeq 2.4\cdot 10^{-3}\mathrm{eV}^2\ \Delta m^2_{12} \simeq 3\cdot 10^{-2}\ \Delta m^2_{23} \simeq 8\cdot 10^{-5}\mathrm{eV}^2 \tag{6.123}$$

where $\Delta m^2_{ik} = m^2_k - m^2_i$. If we take into account (6.123), the transition probabilities are given by very simple two-neutrino type formulas, which are valid with accuracies of a $\sim$ a few %.

In fact, from the general expression (6.70), in the three-neutrino case, we have

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \left|\delta_{l'l} + \sum_{i\neq 2} U_{l'i}\left(e^{-i\,\frac{\Delta m^2_{2i}}{2E}\,L} - 1\right) U^*_{li}\right|^2 \tag{6.124}$$

Let us consider neutrino oscillations driven by the "large" neutrino mass-squared difference Δm^2_{23} (atmospheric, long baseline accelerator and reactor experiments). In such experiments the condition

$$\frac{\Delta m^2_{23}}{2E} L \gtrsim 1. \tag{6.125}$$

must be satisfied. For the values of the parameter $\frac{L}{E}$ which satisfy this condition the phase difference due to Δm^2_{12} is small

$$\frac{\Delta m^2_{12}}{2E} L = \frac{\Delta m^2_{12}}{\Delta m^2_{23}} \frac{\Delta m^2_{23}}{2E} L \simeq 3 \cdot 10^{-2} \frac{\Delta m^2_{23}}{2E} L \ll 1. \tag{6.126}$$

and can be neglected. In this approximation, we obtain the following expressions for the neutrino and antineutrino transition probabilities

$$\mathrm{P}(\nu_l \to \nu_{l'}) \simeq \left| \delta_{l'l} + U_{l'3} \left(e^{-i \frac{\Delta m^2_{23}}{2E} L} - 1 \right) U^*_{l3} \right|^2 \tag{6.127}$$

and

$$\mathrm{P}(\bar\nu_l \to \bar\nu_{l'}) \simeq \left| \delta_{l'l} + U^*_{l'3} \left(e^{-i \frac{\Delta m^2_{23}}{2E} L} - 1 \right) U_{l3} \right|^2 \tag{6.128}$$

It is obvious that only the modules of the elements of the neutrino mixing matrix enter into these expressions. Thus, if we neglect a few-% contribution of a small phase difference to the transition probabilities we have

$$\mathrm{P}(\nu_l \to \nu_{l'}) \simeq \mathrm{P}(\bar\nu_l \to \bar\nu_{l'}), \tag{6.129}$$

i.e. the relations (6.47) are satisfied in this case independently from the *CP* invariance in the lepton sector.

Let us consider the appearance probabilities $\mathrm{P}(\nu_l \to \nu_{l'})$ with $l' \neq l$. From (6.127) we have

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \frac{1}{2} \mathrm{A}_{l'l} \left(1 - \cos \Delta m^2_{23} \frac{L}{2E} \right) \quad (l' \neq l), \tag{6.130}$$

where

$$\mathrm{A}_{l'l} = 4 |U_{l'3}|^2 \, |U_{l3}|^2 \tag{6.131}$$

is the oscillation amplitude.

For the ν_l-survival probability we find

$$\mathrm{P}(\nu_l \to \nu_l) = 1 - \sum_{l' \neq l} \mathrm{P}(\nu_l \to \nu_{l'}) = 1 - \mathrm{B}_{ll} \left(1 - \cos \Delta m_{23}^2 \frac{L}{2E}\right), \tag{6.132}$$

where

$$\mathrm{B}_{ll} = \sum_{l' \neq l} \mathrm{A}_{l'l} = 4 \; |U_{l3}|^2 \; (1 - |U_{l3}|^2). \tag{6.133}$$

Let us consider the $\nu_\mu \to \nu_l$ transitions ($l = e, \mu, \tau$). For the oscillation amplitudes in the standard parametrization of the mixing matrix (see Sect. 5.4) we have the following expressions

$$\begin{aligned} \mathrm{A}_{e\mu} &= \sin^2\theta_{23} \; \sin^2 2\theta_{13}, \quad \mathrm{A}_{\tau\mu} = \sin^2 2\theta_{23} \; \cos^4\theta_{13}, \\ \mathrm{B}_{\mu\mu} &= 4\sin^2\theta_{23}\cos^2\theta_{13}(1 - \sin^2\theta_{23}\cos^2\theta_{13}). \end{aligned} \tag{6.134}$$

For the $\nu_e \to \nu_e$ survival probability from (6.132) and (6.133) we obtain the expression

$$\mathrm{P}(\nu_e \to \nu_e) = \mathrm{P}(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \frac{1}{2} \; \sin^2 2\theta_{13} \left(1 - \cos \Delta m_{23}^2 \frac{L}{2E}\right). \tag{6.135}$$

In the reactor CHOOZ experiment $\bar{\nu}_e \to \bar{\nu}_e$ neutrino oscillations, driven by Δm_{23}^2, were searched for. No positive signal for the $\bar{\nu}_e$ disappearance was found in the experiment. From the analysis of the data of the CHOOZ experiment the following upper bound was obtained for the parameter $\sin^2\theta_{13}$ (see Sect. 10.7)

$$\sin^2\theta_{13} \lesssim 5 \cdot 10^{-2}. \tag{6.136}$$

If we neglect the contribution to the transition probabilities not only the small phase difference $\frac{\Delta m_{12}^2}{2E} L$ but also the small parameter $\sin^2\theta_{13}$ we have in this approximations

$$\mathrm{A}_{e\mu} \simeq 0. \tag{6.137}$$

Thus, if we neglect contribution to the transition probability of the small parameters $\frac{\Delta m_{12}^2}{\Delta m_{23}^2}$ and $\sin^2\theta_{13}$, driven by Δm_{23}^2 the leading oscillations are the $\nu_\mu \rightleftarrows \nu_\tau$ oscillations. For the probability of ν_μ ($\bar{\nu}_\mu$) to survive we obtain, in this approximation, from (6.132) and (6.133) the following standard two-neutrino expression

$$\mathrm{P}(\nu_\mu \to \nu_\mu) = \mathrm{P}(\bar{\nu}_\mu \to \bar{\nu}_\mu) \simeq 1 - \frac{1}{2} \; \sin^2 2\theta_{23} \left(1 - \cos \Delta m_{23}^2 \frac{L}{2E}\right). \tag{6.138}$$

The existing data of atmospheric and long baseline accelerator neutrino oscillation experiments are perfectly described by (6.138) (see Chap. 10). Several future high precision neutrino experiments are in preparation in order to see beyond the leading approximation effects of the three-neutrino mixing (see Sect. 10.8).

We will now consider vacuum neutrino oscillations driven by the small neutrino mass difference Δm^2_{12}, i.e. neutrino oscillation experiments with the such values of the parameter $\frac{L}{E}$ which satisfy the inequality

$$\frac{\Delta m^2_{12}}{2E}\, L \gtrsim 1. \tag{6.139}$$

This condition is satisfied in the reactor KamLAND experiment (see Sect. 10.4).

If the condition (6.125) is satisfied we have

$$\frac{\Delta m^2_{23}}{2E}\, L = \frac{\Delta m^2_{23}}{\Delta m^2_{12}}\, \frac{\Delta m^2_{12}}{2E}\, L \gg 1 \tag{6.140}$$

Due to averaging over the energy resolution, over the distances between neutrino production and detection points etc. the effect of the large phase difference in the transition probabilities is averaged.

The $\bar{\nu}_e \to \bar{\nu}_e$ ($\nu_e \to \nu_e$) transition probability is given by the following general expression

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_e) = \mathrm{P}(\nu_e \to \nu_e) = \left| \sum_i^3 |U_{ei}|^2\, e^{i\, \frac{\Delta m^2_{i2}}{2E}\, L} \right|^2. \tag{6.141}$$

From (6.141) for the averaged probabilities we find the following expression

$$\overline{\mathrm{P}}(\bar{\nu}_e \to \bar{\nu}_e) = \overline{\mathrm{P}}(\nu_e \to \nu_e) = \left| \sum_{i=1,2} |U_{ei}|^2\, e^{i\, \frac{\Delta m^2_{i2}}{2E}\, L} \right|^2 + |U_{e3}|^4. \tag{6.142}$$

The first term of this expression can be presented in the form

$$\left| \sum_{i=1,2} |U_{ei}|^2\, e^{i\, \frac{\Delta m^2_{i2}}{2E}\, L} \right|^2 = \sum_{i=1,2} |U_{ei}|^4 + 2\, |U_{e1}|^2\, |U_{e2}|^2\, \cos \frac{\Delta m^2_{12}}{2E}\, L. \tag{6.143}$$

Further, from the unitarity of the mixing matrix we have

$$\sum_{i=1,2} |U_{ei}|^2 = 1 - |U_{e3}|^2. \tag{6.144}$$

From this relation we find

$$\sum_{i=1,2} |U_{ei}|^4 = (1 - |U_{e3}|^2)^2 - 2\,|U_{e1}|^2\,|U_{e2}|^2. \tag{6.145}$$

Combining (6.142), (6.143) and (6.145) the averaged survival probabilities can be presented in the form

$$\overline{\mathrm{P}}(\nu_e \to \nu_e) = \overline{\mathrm{P}}(\bar\nu_e \to \bar\nu_e) = |U_{e3}|^4 + (1 - |U_{e3}|^2)^2\,\mathrm{P}^{(12)}(\nu_e \to \nu_e). \tag{6.146}$$

Here

$$\mathrm{P}^{(12)}(\nu_e \to \nu_e) = 1 - \frac{1}{2}\,\sin^2 2\,\theta_{12}\left(1 - \cos\frac{\Delta m^2_{12}}{2E}\,L\right) \tag{6.147}$$

is the two-neutrino $\nu_e \to \nu_e$ survival probability driven by Δm^2_{12}.

If we neglect $|U_{e3}|^2 = \sin^2\theta_{13} \lesssim 5 \cdot 10^{-2}$, we obtain the following expression for the $\bar\nu_e$ survival probability in vacuum

$$\mathrm{P}(\bar\nu_e \to \bar\nu_e) = 1 - \frac{1}{2}\,\sin^2 2\,\theta_{12}\left(1 - \cos\frac{\Delta m^2_{12}}{2E}\,L\right). \tag{6.148}$$

This expression was used for the analysis of the data of the KamLAND reactor neutrino experiment in which neutrino oscillations, driven by Δm^2_{12}, were observed (see Sect. 10.4).

From the conservation of the probability we have

$$\mathrm{P}(\bar\nu_e \to \bar\nu_e) = 1 - \mathrm{P}(\bar\nu_e \to \bar\nu_\mu) - \mathrm{P}(\bar\nu_e \to \bar\nu_\tau). \tag{6.149}$$

We will show now that

$$\mathrm{P}(\bar\nu_e \to \bar\nu_\tau) \simeq \tan^2\,\theta_{23}\,\mathrm{P}(\bar\nu_e \to \bar\nu_\mu). \tag{6.150}$$

From (6.70), in the approximation $|U_{e3}|^2 \to 0$, we find the following expression for the probability of the transition $\mathrm{P}(\bar\nu_e \to \bar\nu_l)\;\;(l = e, \mu, \tau)$

$$\mathrm{P}(\bar\nu_e \to \bar\nu_l) \simeq |\delta_{le} + U^*_{l1}\,(e^{i\,\frac{\Delta m^2_{12}}{2E}\,L} - 1)\,U_{e1}|^2. \tag{6.151}$$

Thus, in the approximation $|U_{e3}|^2 \to 0$ only one neutrino mass-squared difference (Δm^2_{12}) enters into the transition probabilities of $\bar\nu_e \to \bar\nu_l$. It is easy to check that from (6.151) we obtain the expression (6.148) for $\bar\nu_e$ survival probability.

For the $\bar\nu_e \to \bar\nu_\mu$ and $\bar\nu_e \to \bar\nu_\tau$ transition probabilities we find from (6.151), respectively, the following expressions

$$\mathrm{P}(\bar\nu_e \to \bar\nu_\mu) = 2|U_{\mu 1}|^2\,|U_{e1}|^2)\left(1 - \cos\frac{\Delta m^2_{12}}{2E}\,L\right) \tag{6.152}$$

and

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\tau) = 2|U_{\tau 1}|^2\,|U_{e1}|^2)\left(1 - \cos\frac{\Delta m_{12}^2}{2E}\,L\right). \tag{6.153}$$

Further, in the approximation $|U_{e3}|^2 \to 0$ we have

$$U_{\mu 1} = -\sin\theta_{12}\,\cos\theta_{23}, \quad U_{\tau 1} = \sin\theta_{12}\,\sin\theta_{23}. \tag{6.154}$$

From (6.152), (6.153) and (6.154) we obtain the following expressions

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\mu) = \frac{1}{2}\,\sin^2 2\theta_{12}\,\cos^2\,\theta_{23}\left(1 - \cos\frac{\Delta m_{12}^2}{2E}\,L\right) \tag{6.155}$$

and

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\tau) = \frac{1}{2}\,\sin^2 2\theta_{12}\,\sin^2\,\theta_{23}\left(1 - \cos\frac{\Delta m_{12}^2}{2E}\,L\right). \tag{6.156}$$

From (6.155) and (6.156) follows that the $\bar{\nu}_e \to \bar{\nu}_\tau$ and $\bar{\nu}_e \to \bar{\nu}_\mu$ transition probabilities are connected by the relation (6.150). Notice also the relations

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\mu) = \cos^2\,\theta_{23}\,(1 - \mathrm{P}(\bar{\nu}_e \to \bar{\nu}_e)) \tag{6.157}$$

and

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\tau) = \sin^2\,\theta_{23}\,(1 - \mathrm{P}(\bar{\nu}_e \to \bar{\nu}_e)), \tag{6.158}$$

which can be easily obtained from (6.149) and (6.150). From the data of the atmospheric and accelerator long baseline experiments follows that $\theta_{23} \simeq \pi/4$ (see Chap. 10). We have

$$\mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\tau) \simeq \mathrm{P}(\bar{\nu}_e \to \bar{\nu}_\mu). \tag{6.159}$$

Thus, the disappearance of reactor antineutrinos, observed in the KamLAND experiment, is due to transitions the $\bar{\nu}_e \to \bar{\nu}_\tau$ and $\bar{\nu}_e \to \bar{\nu}_\mu$.

Let us summarize the results of this section. From the analysis of the data of neutrino oscillation experiments follows that one neutrino mass-squared difference is much smaller than the other one and the mixing angle θ_{13} is small:

$$\Delta m_{12}^2 \simeq 3 \cdot 10^{-2}\,\Delta m_{23}^2, \quad \sin^2\theta_{13} \leq 5 \cdot 10^{-2}. \tag{6.160}$$

If we neglect a few-% contribution of these small parameters to the transition probabilities a simple picture of neutrino oscillations emerges in this approximation:

1. In the atmospheric and accelerator long baseline experiments the leading oscillations are $\nu_\mu \rightleftarrows \nu_\tau$ ($\bar{\nu}_\mu \rightleftarrows \bar{\nu}_\tau$) oscillations; the ν_μ ($\bar{\nu}_\mu$) survival probability depends on the parameters Δm^2_{23} and $\sin^2 2\theta_{23}$ and is given by the expression (6.138).
2. The leading neutrino oscillations observed in the reactor KamLAND experiment are $\bar{\nu}_e \to \bar{\nu}_\tau$ and $\bar{\nu}_e \to \bar{\nu}_\mu$ oscillations; the $\bar{\nu}_e$ survival probability depends on the parameters Δm^2_{12} and $\sin^2 2\theta_{12}$ and has two-neutrino form (6.148).

Existing neutrino oscillation data are in a good agreement with this approximation. One of the major purpose of the future high precision neutrino oscillation experiments is to find beyond the leading approximation effects of the three-neutrino mixing.

Chapter 7
Neutrino in Matter

7.1 Introduction

Up to now we have considered the propagation of massive neutrinos in vacuum. We have seen that the flavor content of the neutrino beam in vacuum is determined by the neutrino mass-squared differences Δm_{i1}^2 and elements of the neutrino mixing matrix U_{li}. As was first shown by Wolfenstein , in the case of matter not only neutrino masses and mixing but also the coherent scattering of neutrinos in matter must be taken into account. The contribution of the coherent scattering into the Hamiltonian is determined by the electron number-density. If the electron density depends on the distance (as in the case of the sun) the transitions probabilities between different flavor neutrinos in matter can have resonance character (MSW effect). We will consider here in some detail the propagation of neutrino in matter.

7.2 Evolution Equation of Neutrino in Matter

We will first obtain the vacuum evolution equation for neutrino with momentum **p** in flavor representation. The state vector $|\Psi(t)\rangle$ of neutrino with momentum **p** satisfies the Schrödinger equation

$$i\,\frac{\partial\,|\Psi(t)\rangle}{\partial t} = H_0\,|\Psi(t)\rangle, \tag{7.1}$$

where H_0 is the free Hamiltonian. The state $|\Psi(t)\rangle$ can be expanded over the total system of states of the flavor neutrinos ν_l with momentum **p**. We have

$$|\Psi(t)\rangle = \sum_l a_l(t)\,|\nu_l\rangle. \tag{7.2}$$

Here

$$|\nu_l\rangle = \sum_i U_{li}^*\,|\nu_i\rangle, \tag{7.3}$$

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where

$$H_0 \, |\nu_i\rangle = E_i \, |\nu_i\rangle, \quad E_i = \sqrt{p^2 + m_i^2} \simeq p + \frac{m_i^2}{2\,E}. \tag{7.4}$$

and $a_l(t) = \langle \nu_l | \Psi(t) \rangle$ is the amplitude of the probability to find ν_l in the state which is described by $|\Psi(t)\rangle$.

From (7.1) and (7.2) we obtain the following equation for the function $a_l(t)$

$$i \, \frac{\partial \, a_{l'}(t)}{\partial t} = \sum_l \langle \nu_{l'} \, |H_0| \, \nu_l \rangle \, a_l(t). \tag{7.5}$$

From (7.3) it is obvious that

$$\langle \nu_l | \nu_i \rangle = U_{li}, \quad \langle \nu_i | \nu_l \rangle = U_{li}^*. \tag{7.6}$$

Taking into account this relation, for the free Hamiltonian in the flavor representation we have the following expression

$$\langle \nu_{l'} \, |H_0| \, \nu_l \rangle = \sum_i U_{l'i} \, E_i \, U_{li}^* \simeq p + \sum_i U_{l'i} \, \frac{m_i^2}{2\,E} \, U_{li}^*. \tag{7.7}$$

From (7.5) and (7.7) we obtain neutrino evolution equation in the flavor representation[1]:

$$i \, \frac{\partial \, a(t)}{\partial t} = U \, \frac{m^2}{2\,E} \, U^\dagger \, a(t). \tag{7.8}$$

This equation can be easily solved. In fact, let us introduce the function[2]

$$a'(t) = U^\dagger \, a(t). \tag{7.9}$$

From (7.8) we find that the function $a'(t)$ satisfies the following equation

$$i \, \frac{\partial \, a'(t)}{\partial t} = \frac{m^2}{2\,E} \, a'(t). \tag{7.10}$$

It is obvious that the solution of this equation has the form

$$a'(t) = e^{-i \, \frac{m^2}{2\,E}(t - t_0)} \, a'(t_0), \tag{7.11}$$

where $a'(t_0)$ is the wave function at the initial time t_0.

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[1] The unit matrix $p \cdot I$ can be excluded from the Hamiltonian by a redefinition of a common phase of the wave function.

[2] It is obvious that $a_i' = \sum_l U_{li}^* \, \langle \nu_l | \Psi(t) \rangle = \langle \nu_i | \Psi(t) \rangle$. Thus, a_i' is the amplitude of the probability to find a neutrino in the state with momentum $\mathbf{p}$, mass m_i and energy E_i.

From (7.9) and (7.11) we find that the solution of Eq. (7.8) is given by

$$a(t) = U\, e^{-i\,\frac{m^2}{2E}(t-t_0)}\, U^{\dagger}\, a(t_0). \tag{7.12}$$

Let us assume that the initial state $|\Psi(t_0)\rangle$ is the state of the flavor neutrino ν_l:

$$|\Psi(t_0)\rangle = |\nu_l\rangle. \tag{7.13}$$

In this case $a_{l_1}(t_0) = \delta_{l_1 l}$ and from (7.12) we obtain the standard expression for the $\nu_l \to \nu_{l'}$ transition probability in vacuum (see Sect. 6.3)

$$\mathrm{P}(\nu_l \to \nu_{l'}) = |\sum_i U_{l'i}\, e^{-i\Delta m^2_{1i}\frac{L}{2E}}\, U^*_{li}\,|^2, \tag{7.14}$$

where $\Delta m^2_{1i} = m^2_i - m^2_1$ and $L \simeq (t-t_0)$ is the distance between neutrino production and detection points.

Let us now consider the evolution equation of a neutrino with momentum $\mathbf{p}$ in matter. For the propagation of neutrinos in matter the most important effect is coherent elastic scattering of neutrinos in forward direction, which does not change the state of the matter. We will first consider the CC interaction. This interaction can give contribution only to the process of the elastic scattering of ν_e on electrons ($\nu_e + e \to \nu_e + e$), which is due to $W^{\pm}$ exchange. From the standard CC Lagrangian

$$\mathcal{L}_I^{\mathrm{CC}} = -\frac{g}{\sqrt{2}}\, \bar{\nu}_{eL}\, \gamma_\alpha\, e_L\, W^\alpha + \mathrm{h.c.} \tag{7.15}$$

for the low-energy effective Hamiltonian of the CC $\nu_e - e$ interaction we find the following expression[3]

$$\mathcal{H}_I^{\mathrm{CC}}(\mathbf{x}) = \frac{G_F}{\sqrt{2}} 2\, \bar{\nu}_{eL}(\mathbf{x})\, \gamma_\alpha\, \nu_{eL}(\mathbf{x})\, \bar{e}(\mathbf{x})\, \gamma^\alpha (1-\gamma_5)\, e(\mathbf{x}). \tag{7.16}$$

An effective Hamiltonian of the neutrino interaction, which is determined by the coherent $\nu_e - e$ scattering, can be obtained from the diagonal matrix element

$$\langle \mathbf{p}\ \mathrm{mat}|\, \mathcal{H}_I^{\mathrm{CC}}(\mathbf{x})\, |\mathbf{p}\ \mathrm{mat}\rangle, \tag{7.17}$$

where the vector

$$|\mathbf{p}\ \mathrm{mat}\rangle = |\mathbf{p}\rangle\, |\mathrm{mat}\rangle \tag{7.18}$$

describes the left-handed neutrino with momentum $\mathbf{p}$ and the matter.

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[3] We use here the Schrödinger representation.

It is obvious that the pseudovector $\langle \mathrm{mat}|\ \bar{e}(\mathbf{x})\ \gamma^{\alpha}\gamma_5\ e(\mathbf{x})\ |\mathrm{mat}\rangle$ is equal to zero for unpolarized matter. For the vector $\langle \mathrm{mat}|\ \bar{e}(\mathbf{x})\ \gamma^{\alpha}\ e(\mathbf{x})\ |\mathrm{mat}\rangle$ we have in the matter rest frame

$$\langle \mathrm{mat}|\ \bar{e}(\mathbf{x})\ \gamma^{\alpha}\ e(\mathbf{x})\ |\mathrm{mat}\rangle = \langle \mathrm{mat}|e^{\dagger}(\mathbf{x})\ e(\mathbf{x})\ |\mathrm{mat}\rangle\ \delta_{\alpha 0} = n_e(\mathbf{x})\ \delta_{\alpha 0}, \tag{7.19}$$

where $n_e(\mathbf{x})$ is the electron number density at the point $\mathbf{x}$. Further we have

$$\langle \mathbf{p}|\ \bar{\nu}_{eL}(\mathbf{x})\ \gamma_0\ \nu_{eL}(\mathbf{x})\ |\mathbf{p}\rangle = 1. \tag{7.20}$$

From (7.16), (7.19) and (7.20) we obtain the following expression for the effective CC interaction Hamiltonian of neutrinos in matter in the flavor representation

$$H_I^{\mathrm{mat}}(t) = \sqrt{2}\ G_F n_e(t)\ \beta. \tag{7.21}$$

Here $\beta_{\nu_e;\nu_e} = 1$, all other elements of the matrix β are equal to zero. Notice that we take into account that for ultra-relativistic neutrinos

$$x \simeq t, \tag{7.22}$$

where $x = \mathbf{x} \cdot \mathbf{k}$, $\mathbf{k}$ being the unit vector in the direction of the neutrino momentum.

Let us now consider the NC interaction. Induced by the Z^0 exchange, the Hamiltonian of the NC interaction of neutrinos with electrons and nucleons has the form

$$\mathcal{H}_I^{\mathrm{NC}}(\mathbf{x}) = \frac{G_F}{\sqrt{2}} 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(\mathbf{x})\ \gamma^{\alpha}\ \nu_{lL}(\mathbf{x})\ j_{\alpha}^{\mathrm{NC}}(\mathbf{x}), \tag{7.23}$$

where $j_{\alpha}^{\mathrm{NC}}(\mathbf{x})$ is the sum of the electron and quark (nucleon) neutral currents. Only the vector part of the NC can give a contribution to the effective Hamiltonian of neutrinos in matter. For the vector part of the electron neutral current we have

$$v_{\alpha}^{\mathrm{NC}(e)}(\mathbf{x}) = \left(-\frac{1}{2} + 2\sin^2\theta_W \right) \bar{e}(\mathbf{x})\ \gamma_{\alpha}\ e(\mathbf{x}), \tag{7.24}$$

where θ_W is the weak angle. For the vector part of the effective hadron neutral current we have

$$v_{\alpha}^{\mathrm{NC}(N)}(\mathbf{x}) = \frac{1}{2}\bar{N}(\mathbf{x})\ \gamma_{\alpha}\ \tau_3\ N(\mathbf{x}) - 2\sin^2\theta_W\ \bar{p}(\mathbf{x})\ \gamma_{\alpha}\ p(\mathbf{x}), \tag{7.25}$$

where

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \tag{7.26}$$

is a $SU(2)$ doublet (p and n are proton and neutron fields).

We will consider the propagation of neutrino in a neutral medium of electrons, protons and neutrons. From (7.25) and (7.26) we find the following expressions for the vector part of the proton and neutron NC

$$v_\alpha^{\mathrm{NC}(p)}(\mathbf{x}) = \left(\frac{1}{2} - 2\sin^2\theta_W\right) \bar{p}(\mathbf{x})\ \gamma_\alpha\ p(\mathbf{x}) \tag{7.27}$$

$$v_\alpha^{\mathrm{NC}(n)}(\mathbf{x}) = \left(-\frac{1}{2}\right) \bar{n}(\mathbf{x})\ \gamma_\alpha\ n(\mathbf{x}). \tag{7.28}$$

For the corresponding matter matrix elements we have

$$\langle \mathrm{mat}|\ v_\alpha^{\mathrm{NC}(e)}(\mathbf{x})\ |\mathrm{mat}\rangle = \left(-\frac{1}{2} + 2\sin^2\theta_W\right) n_e(\mathbf{x})\ \delta_{\alpha 0}, \tag{7.29}$$

$$\langle \mathrm{mat}|\ v_\alpha^{\mathrm{NC}(p)}(\mathbf{x})\ |\mathrm{mat}\rangle = \left(\frac{1}{2} - 2\sin^2\theta_W\right) \rho_p(\mathbf{x})\ \delta_{\alpha 0}, \tag{7.30}$$

and

$$\langle \mathrm{mat}|\ v_\alpha^{\mathrm{NC}(n)}(\mathbf{x})\ |\mathrm{mat}\rangle = -\frac{1}{2}\ \rho_n(\mathbf{x})\ \delta_{\alpha 0}. \tag{7.31}$$

Here $n_e(\mathbf{x})$, $n_p(\mathbf{x})$ and $n_n(\mathbf{x})$ are the number densities of electrons, protons and neutrons, respectively.

For the neutral matter $n_e(\mathbf{x}) = n_p(\mathbf{x})$. From (7.29) and (7.30) we conclude that the contributions of the electron and proton NC to the effective Hamiltonian cancel each other. Thus, we have

$$\langle \mathrm{mat}|\ j_\alpha^{\mathrm{NC}}(\mathbf{x})\ |\mathrm{mat}\rangle = -\frac{1}{2}\ \rho_n(\mathbf{x})\ \delta_{\alpha 0}. \tag{7.32}$$

The NC interaction (7.23) is $\nu_e - \nu_\mu - \nu_\tau$ universal. This means that the effective NC Hamiltonian in the flavor representation will be proportional to the unit matrix. Such an interaction cannot change the flavor content of the neutrino beam. In fact, proportional to the unit matrix interaction can be removed from the total Hamiltonian by the following transformation of the wave function

$$a'(t) = e^{i\alpha(t)} a(t), \tag{7.33}$$

where $\alpha(t)$ is an unphysical common phase.

Thus, in the case of three flavor neutrinos only the $\nu_e - e$ CC interaction gives a contribution to the effective Hamiltonian.[4] The evolution equation of neutrino in matter has the form

[4] Let us notice, however, that if sterile neutrinos exist, both CC and NC effective Hamiltonians of the neutrino interaction with matter must be taken into account.

$$i\,\frac{\partial\, a(t)}{\partial t} = \left(U\,\frac{m^2}{2\,E}\,U^{\dagger} + \sqrt{2}G_F n_e(t)\,\beta\right) a(t). \tag{7.34}$$

We have obtained the effective CC Hamiltonian of the interaction of the left-handed electron neutrinos and electrons. Let us consider right-handed electron antineutrinos. Taking into account the relations (5.22) we have

$$\bar{\nu}_{eL}\,\gamma_\alpha\,\nu_{eL} = -\nu_{eL}^T\,\gamma_\alpha^T\,\bar{\nu}_{eL}^T = -\nu_{eL}^T\,C^{-1}C\,\gamma_\alpha^T\,C^{-1}C\,\bar{\nu}_{eL}^T = -\bar{\nu}_{eR}^c\,\gamma_\alpha\,\nu_{eR}^c. \tag{7.35}$$

From (7.35) and (7.16) we conclude that the effective Hamiltonian of the $\bar{\nu}_e - e$ interaction differs in sign from the effective Hamiltonian of the $\nu_e - e$ interaction. It is given by the expression

$$\overline{H}_I^{\mathrm{mat}}(t) = -\sqrt{2}\,G_F n_e(t)\,\beta. \tag{7.36}$$

In order to see the connection of the effective Hamiltonian with coherent scattering of electron neutrinos on electrons it is instructive to obtain the expression (7.21) in another way.

The index of refraction of a particle with the momentum p is given by the following classical expression

$$n(x) = 1 + \frac{2\pi}{p^2}\sum_a f_a(0)\,n_a(x). \tag{7.37}$$

Here $f_a(0)$ is the amplitude of the elastic scattering on particles of the type a in the forward direction and n_a is the number density of particles a. The second term of the relation (7.37) is due to coherent scattering in matter.

If $p \gg m$, the energy of a particle in matter is connected with its momentum by the relation

$$E \simeq p\,n. \tag{7.38}$$

If we subtract from (7.38) the energy of the particle in vacuum $E \simeq p$ we will obtain, determined by the coherent scattering, an effective potential of the particle in matter. We have

$$V_{\mathrm{eff}}(x) \simeq (n(x) - 1)\,p = \frac{2\pi}{p}\sum_a f_a(0)\,n_a(x). \tag{7.39}$$

In the case of neutrinos the amplitude of the process $\nu_e e \to \nu_e e$ is different from the amplitudes of the processes $\nu_{\mu,\tau} e \to \nu_{\mu,\tau} e$. This is connected with the fact that to the matrix element of the process $\nu_e e \to \nu_e e$ the diagrams with the exchange of W and Z bosons give a contribution, while to the matrix elements of the processes $\nu_{\mu,\tau} e \to \nu_{\mu,\tau} e$ only the diagram with the exchange of the Z boson contributes.

Thus, *the refraction indices of* ν_e *and* $\nu_{\mu,\tau}$ *are different.* In the case of neutrino mixing, this difference of refraction indices leads to important effects for the neutrino transitions in matter.

Due to the $\nu_e - \nu_\mu - \nu_\tau$ universality of the NC we have

$$f^{\mathrm{NC}}_{\nu_e e\to\nu_e e}(0) = f^{\mathrm{NC}}_{\nu_\mu e\to\nu_\mu e}(0) = f^{\mathrm{NC}}_{\nu_\tau e\to\nu_\tau e}(0). \tag{7.40}$$

From (7.37) and (7.40) follows that the contribution of the NC to effective potential of the neutrinos in matter is proportional to the unit matrix. Thus, the NC interaction does not change the flavor content of the neutrino beam and can be excluded from the Hamiltonian.[5] For the contribution of the CC to the amplitude of the process $\nu_e e \to \nu_e e$ in the electron rest frame we have

$$f^{\mathrm{CC}}_{\nu_e e\to\nu_e e}(0) = \frac{G_F\ p}{\sqrt{2}\pi}. \tag{7.41}$$

From (7.37) and (7.41) we obtain the following expression for the effective interaction of neutrino with matter

$$H_I^{\mathrm{mat}} = \sqrt{2}\, G_F n_e\ \beta. \tag{7.42}$$

This expression coincides with (7.21).

7.3 Propagation of Neutrino in Matter with Constant Density

We will consider first the case of a constant electron number density. The total Hamiltonian of neutrino in matter $H = U\frac{m^2}{2E}U^\dagger + \sqrt{2}G_F n_e \beta$ is a hermitian matrix. It can be diagonalized by the unitary transformation:

$$H = U^{\mathrm{m}}\, E^{\mathrm{m}}\, U^{\mathrm{m}\dagger}. \tag{7.43}$$

Here E_i^{m} are eigenvalues of the matrix H and U^{m} is a unitary matrix. The evolution equation is given by

$$i\,\frac{\partial\, a(t)}{\partial t} = U^{\mathrm{m}}\, E^{\mathrm{m}}\, U^{\mathrm{m}\dagger}\, a(t). \tag{7.44}$$

In an analogy with the vacuum case considered before let us introduce the function

$$a'(t) = U^{\mathrm{m}\dagger} a(t). \tag{7.45}$$

From (7.44) and (7.45) we obtain

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[5] We assume that there are no sterile neutrinos.

$$i\,\frac{\partial a'(t)}{\partial t} = E^{\mathrm{m}}\,a'(t). \tag{7.46}$$

It is obvious that the solution of this equation has the form

$$a'(t) = e^{-i\,E^{\mathrm{m}}\,(t-t_0)}\,a'(t_0)\;, \tag{7.47}$$

where $a'(t_0)$ is the wave function at the initial time t_0. From (7.45) and (7.47)we obtain for the wave function in the flavor representation the following expression

$$a(t) = U^{\mathrm{m}}\,e^{-i\,E^{\mathrm{m}}\,(t-t_0)}\,U^{\mathrm{m}\dagger}\,a(t_0). \tag{7.48}$$

Let us assume that at t_0 the flavor neutrino ν_l was produced. From (7.48) we find that the probability of the $\nu_l \to \nu_{l'}$ transition in matter with the constant density is given by the expression

$$\mathrm{P}(\nu_l \to \nu_{l'}) = |\sum_i U^{\mathrm{m}}_{l'i}\,e^{-i\,E^{\mathrm{m}}_i\,(t-t_0)}\,U^{\mathrm{m}*}_{li}|^2. \tag{7.49}$$

If we compare (7.49) with (6.41) we come to the conclusion that the probability of the $\nu_l \to \nu_{l'}$ transition in matter with a constant density has the same form as the probability of the $\nu_l \to \nu_{l'}$ transition in vacuum. Equation (7.49) has a simple meaning: U^{m*}_{li} is the amplitude of the transition from the state of the initial neutrino ν_l to the state of the neutrino with energy E^m_i in matter, the factor $e^{-i\,E^m_i\,(t-t_0)}$ describes the propagation in matter of the state with the energy E^m_i and $U^m_{l'i}$ is the amplitude of the transition from the state of the neutrino with energy E^m_i to the state of the final flavor neutrino $\nu_{l'}$. A coherent sum over the states of neutrino with definite energies is performed in (7.49).[6] Let us now consider the simplest case of two flavor neutrinos: $\nu_e - \nu_{\mu,\tau}$. For the vacuum mixing we have

$$\begin{aligned}\nu_{eL} &= \cos\theta\;\nu_{1L} + \sin\theta\;\nu_{2L}\\ \nu_{xL} &= -\sin\theta\;\nu_{1L} + \cos\theta\;\nu_{2L},\end{aligned} \tag{7.50}$$

where $x = \mu$ or τ and ν_1 and ν_2 are the neutrino fields with masses m_1 and m_2. Thus, the vacuum mixing matrix U is a 2×2 real orthogonal matrix

[6] It is instructive to obtain the same result in an arbitrary representation. The state with definite energy in matter is the eigenstate of the total Hamiltonian: $H\;|\nu_i\rangle^{\mathrm{m}} = E^{\mathrm{m}}_i\;|\nu_i\rangle^{\mathrm{m}}$. Further, we have $\langle\nu_{l'}|H|\,\nu_l\rangle = \sum_i\langle\nu_{l'}|\nu_i\rangle^{\mathrm{m}}E^{\mathrm{m}}_i\;{}^{\mathrm{m}}\langle\nu_i|\nu_l\rangle$. Comparing this relation with (7.43) we conclude that $\langle\nu_l|\nu_i\rangle^{\mathrm{m}} = U^{\mathrm{m}}_{li}$. For the flavor neutrino state we obtain the following relation $|\nu_l\rangle = \sum_i U^{\mathrm{m}*}_{li}\;|\nu_i\rangle^{\mathrm{m}}$. If at the time t_0 a flavor neutrino ν_l was produced, at the time t we have $|\nu_l\rangle_t = \sum_i U^{\mathrm{m}*}_{li}\;e^{-iE^{\mathrm{m}}_i(t-t_0)}|\nu_i\rangle^{\mathrm{m}} = \sum_{l'}|\nu_{l'}\rangle(\sum_i U^{\mathrm{m}}_{l'i}e^{-iE^{\mathrm{m}}_i(t-t_0)}\;U^{\mathrm{m}*}_{li})$. Thus, $A(\nu_l \to \nu_{l'}) = \sum_i U^{\mathrm{m}}_{l'i}e^{-iE^{\mathrm{m}}_i(t-t_0)}\;U^{\mathrm{m}*}_{li}$ is the amplitude of the $\nu_l \to \nu_{l'}$ transition in matter.

$$U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{x1} & U_{x2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{7.51}$$

It is convenient to present the total effective Hamiltonian in the form

$$H = \frac{1}{2}\mathrm{Tr}\, H + H^{\mathrm{m}}. \tag{7.52}$$

Here

$$\frac{1}{2}\mathrm{Tr}\, H = \frac{m_1^2 + m_2^2}{4E} + \frac{1}{2}\sqrt{2}\, G_F\, n_e \tag{7.53}$$

and H^{m} is the traceless part of the Hamiltonian. We have[7]

$$H^{\mathrm{m}} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2 \cos 2\theta + A & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A \end{pmatrix}, \tag{7.54}$$

where

$$A = 2\sqrt{2}\, G_F\, n_e\, E \tag{7.55}$$

and $\Delta m^2 = m_2^2 - m_1^2$. We will label the neutrino masses in such a way that $m_2 > m_1$. Thus, $\Delta m^2 > 0$.

The first term of (7.52) is proportional to the unit matrix and we can exclude it from the Hamiltonian. The real symmetrical 2×2 matrix H^m can be diagonalized by the orthogonal transformation (see Appendix A). We have

$$H^{\mathrm{m}} = U^{\mathrm{m}}\, E^{\mathrm{m}}\, U^{\mathrm{m}\dagger}. \tag{7.56}$$

Here

$$U^{\mathrm{m}} = \begin{pmatrix} \cos\theta^{\mathrm{m}} & \sin\theta^{\mathrm{m}} \\ -\sin\theta^{\mathrm{m}} & \cos\theta^{\mathrm{m}} \end{pmatrix} \tag{7.57}$$

and

$$E^{\mathrm{m}} = \begin{pmatrix} E_1^{\mathrm{m}} & 0 \\ 0 & E_2^{\mathrm{m}} \end{pmatrix} \tag{7.58}$$

where

$$E_{1,2}^{\mathrm{m}} = \mp \frac{1}{4E}\sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2} \tag{7.59}$$

are eigenvalues of the matrix H^{mat}.

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[7] Let us notice that for $\nu_e - \nu_{\mathrm{sterile}}$ case the effective Hamiltonian can be obtained from (7.54) by the change $A \to A^s = 2\sqrt{2}\, G_F\, (n_e - \frac{1}{2} n_n)\, E$.

From (7.54), (7.56), (7.57), (7.58) and (7.59) we find that the mixing angle θ^{m} is given by the following relations

$$\cos 2\theta^{\mathrm{m}} = \frac{\Delta m^2 \cos 2\theta - A}{\sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2}},$$
$$\sin 2\theta^{\mathrm{m}} = \frac{\Delta m^2 \sin 2\theta}{\sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2}} \tag{7.60}$$

For $\tan 2\theta^{\mathrm{m}}$ we find from these relations

$$\tan 2\theta^{\mathrm{m}} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A}. \tag{7.61}$$

From (7.49) and (7.57) for the probability of the $\nu_l \to \nu_{l'}$ ($\nu_{l'} \to \nu_l$) transition in matter we find the following expression

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \mathrm{P}(\nu_{l'} \to \nu_l) = \frac{1}{2} \sin^2 2\theta^{\mathrm{m}} \,(1 - \cos \Delta E^{\mathrm{m}} L),\ (e' \neq e). \tag{7.62}$$

Here $l' \neq l$, (l or l' is equal to e), $L \simeq (t - t_0)$ is the distance, which the neutrino passes in matter and

$$\Delta E^{\mathrm{m}} = E_2^{\mathrm{m}} - E_1^{\mathrm{m}} = \frac{1}{2E} \sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2}. \tag{7.63}$$

The probability of ν_l ($\nu_{l'}$) to survive can be obtained from the condition of the conservation of the probabilities. We have

$$\mathrm{P}(\nu_l \to \nu_l) = \mathrm{P}(\nu_{l'} \to \nu_{l'}) = 1 - \frac{1}{2} \sin^2 2\theta^{\mathrm{m}} \,(1 - \cos \Delta E^{\mathrm{m}} L). \tag{7.64}$$

The expression (7.62) can be written in the form

$$\mathrm{P}(\nu_l \to \nu_{l'}) = \mathrm{P}(\nu_{l'} \to \nu_l) = \frac{1}{2} \sin^2 2\theta^{\mathrm{m}} \left(1 - \cos 2\pi \frac{L}{L_{\mathrm{osc}}^{\mathrm{m}}}\right), \tag{7.65}$$

where

$$L_{\mathrm{osc}}^{\mathrm{m}} = \frac{4\pi\, E}{\sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2}} \tag{7.66}$$

is the oscillation length of neutrino in a matter with a constant density.

It is obvious from (7.59) and (7.63) that at $n_e = 0$ we have $\theta^{\mathrm{m}} = \theta$, $\Delta E^{\mathrm{m}} = \Delta m^2$ and expressions (7.62), (7.64) and (7.66) coincide with the vacuum two-neutrino transition probabilities and oscillation length, correspondingly.

If $n_e \neq 0$ the neutrino mixing angle can be significantly different from the vacuum value. Let us assume that at some energy E the following equality

$$\Delta m^2 \cos 2\theta = A = 2\sqrt{2}\, G_F\, \rho_e\, E \tag{7.67}$$

is satisfied. It follows from (7.67) that in this case $\theta^{\mathrm{m}} = \pi/4$ independently on the value of the vacuum angle θ. Thus, if the condition (7.67) is satisfied, the mixing angle and the oscillation amplitude in matter become maximal. The condition (7.67) is called MSW resonance condition. We will return to the discussion of this condition later.

If the resonance condition is satisfied, the oscillation length in matter can also be significantly different from the oscillation length in vacuum. In fact we have in this case

$$L^{\mathrm{m}}_{\mathrm{osc}} = \frac{L_{\mathrm{osc}}}{\sin 2\theta} \tag{7.68}$$

where $L_{\mathrm{osc}} = 4\pi \frac{E}{\Delta m^2}$ is the vacuum oscillation length.

7.4 Adiabatic Neutrino Transitions in Matter

We will consider here neutrino transitions in the sun's matter. The electron density in the sun is not constant. It is maximal in the center of the sun and decreases practically exponentially to its periphery. Let us consider the general evolution equation of neutrino in matter

$$i\,\frac{\partial a(t)}{\partial t} = H^{\mathrm{m}}(t)\, a(t), \tag{7.69}$$

where $H^{\mathrm{m}}(t) = H_0 + H_I(t)$ and $a(t)$ are the total effective Hamiltonian and neutrino wave function in the flavor representation.

The hermitian Hamiltonian $H^{\mathrm{m}}(t)$ can be diagonalized by the unitary transformation

$$H^{\mathrm{m}}(t) = U^{\mathrm{m}}(t)\, E^{\mathrm{m}}(t)\, U^{\mathrm{m}\dagger}(t), \tag{7.70}$$

where $U^{\mathrm{m}}(t)$ is an unitary matrix and $E^{\mathrm{m}}_i(t)$ is the eigenvalue of the Hamiltonian.

Let us introduce the function

$$a'(t) = U^{\mathrm{m}\dagger}(t)\, a(t)\,. \tag{7.71}$$

From (7.69), (7.70) and (7.71) we obtain the following equation for the function $a'(t)$

$$i\,\frac{\partial a'(t)}{\partial t} = \left(E^{\mathrm{m}}(t) - i\,U^{\mathrm{m}\dagger}(t)\,\frac{\partial U^{\mathrm{m}}(t)}{\partial t}\right) a'(t) \tag{7.72}$$

The derivative $\frac{\partial U^{\mathrm{m}}(t)}{\partial t}$ is determined by the derivative of the electron density $n_e(t)$. We will assume that the function $n_e(t)$ depends on t so weakly that the second term in Eq. (7.72) can be neglected. In this approximation, which is called the adiabatic approximation, the solution of the evolution equation can be easily found. We have

$$a'_i(t) = e^{-i\int_0^t E_i^{\mathrm{m}}(t)dt},\, a'_i(t_0), \tag{7.73}$$

where t_0 is the time of neutrino production. Thus, in the adiabatic approximation the neutrino remains on the same energy level during evolution.[8] From (7.71) and (7.73) we find the following adiabatic solution of the evolution equation in the flavor representation

$$a_{\nu_{l'}}(t) = \sum_{i,l''} U^{\mathrm{m}}_{l'i}(t)\, e^{-i\int_{t_0}^t E_i^{\mathrm{m}}(t)dt}\, U^{\mathrm{m}*}_{l''i}(t_0)\, a_{\nu_{l''}}(t_0). \tag{7.74}$$

From this expression follows that in the adiabatic approximation the probability of the transition $\nu_l \to \nu_{l'}$ during the time $t - t_0$ is given by the expression

$$\begin{aligned}\mathrm{P}^{\mathrm{m}}(\nu_l \to \nu_{l'}) &= |\sum_i U^{\mathrm{m}}_{l'i}(t)\, e^{-i\int_{t_0}^t E_i^{\mathrm{m}}(t)dt}\, U^{\mathrm{m}*}_{li}(t_0)|^2 = \sum_i |U^{\mathrm{m}}_{l'i}(t)|^2\, |U^{\mathrm{m}}_{li}(t_0)|^2 \\ &+ \sum_{i\neq k} U^{\mathrm{m}}_{l'i}(t)\, U^{\mathrm{m}*}_{l'k}(t) e^{-i\int_{t_0}^t (E_i^{\mathrm{m}}(t)-E_k^{\mathrm{m}}(t))dt}\, U^{\mathrm{m}*}_{li}(t_0)\, U^{\mathrm{m}}_{lk}(t_0).\end{aligned} \tag{7.75}$$

Because in the adiabatic approximation the neutrino remains on the same energy level, the transition amplitude has a very simple structure (similar to the structure of the transition amplitudes in the case of a matter with a constant density and vacuum): $U^{\mathrm{m}*}_{li}(t_0)$ is the amplitude of the transition from the state of the initial ν_l to the state with energy $E_i(t_0)$; the factor $e^{-i\int_{t_0}^t E_i^{\mathrm{m}}(t)dt}$ describes the propagation in the state with energy E_i^{m}; $U^{\mathrm{m}}_{l'i}(t)$ is the amplitude of the transition from the state with energy $E_i^{\mathrm{m}}(t)$ to the state of the final $\nu_{l'}$. The coherent sum over i is performed.

In the case of solar neutrinos, the transition probability must be averaged over the central region of the sun, in which solar ν_e are produced ($\sim 10^5$ km), over the energy resolution etc. After integration over many periods of oscillations, the

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[8] In fact, for the state of a neutrino with definite energy at the time t we have: $H(t)\;|\nu_i(t)\rangle^{\mathrm{m}} = E_i^{\mathrm{m}}(t)\;|\nu_i(t)\rangle^{\mathrm{m}}$. Further, we find $\langle\nu_{l'}\,|H(t)|\,\nu_l\rangle = \sum_i \langle\nu_{l'}|\nu_i(t)\rangle^{\mathrm{m}} E_i^{\mathrm{m}}(t)\;{}^{\mathrm{m}}\langle\nu_i(t)|\nu_l\rangle$. Comparing this relation with (7.70) we conclude that $\langle\nu_l|\nu_i(t)\rangle^{\mathrm{m}} = U^{\mathrm{m}}_{li}(t)$. Taking into account this relation we have $a_l(t) = \langle\nu_l|\Psi(t)\rangle = \sum_i \langle\nu_l|\nu_i(t)\rangle^{\mathrm{m}}\;{}^{\mathrm{m}}\langle\nu_i(t)|\Psi(t)\rangle = U^{\mathrm{m}}_{li}(t)\;a'_i(t)$. Thus, $a'_i(t) ={}^{\mathrm{m}}\;\langle\nu_i(t)|\Psi(t)\rangle$ is the amplitude of the probability to find the neutrino in the state with the energy $E_i(t)$ at the time t.

oscillatory terms in the transition probability disappear . From (7.75) we find the following expression for the probability of the solar ν_e to survive in this case

$$\mathrm{P}(\nu_e \to \nu_e) = \sum_i |U^{\mathrm{m}}_{ei}(x)|^2 \, |U^{m}_{ei}(x_0)|^2, \tag{7.76}$$

where $x_0 \simeq t_0$ is the point where neutrino was produced and $x \simeq t$ is the final point. Thus, in the adiabatic approximation the averaged transition probability is determined only by *the elements of the mixing matrix at the initial and final points*. Let us consider now the simplest case of two-neutrino $\nu_e \leftrightarrows \nu_{\mu,\tau}$ transitions in matter. In this case the matrix $U^{\mathrm{m}}(t)$ is real, orthogonal 2×2 matrix:

$$U^{\mathrm{m}}(t) = \begin{pmatrix} \cos\theta^{\mathrm{m}}(t) & \sin\theta^{\mathrm{m}}(t) \\ -\sin\theta^{\mathrm{m}}(t) & \cos\theta^{\mathrm{m}}(t) \end{pmatrix}. \tag{7.77}$$

It is obvious that the eigenvalues of the Hamiltonian $E^{\mathrm{m}}_{1,2}(t)$ and the mixing angle $\theta^{\mathrm{m}}(t)$ are given by Eqs. (7.59) and (7.60) in which A is equal to $A(t) = 2\sqrt{2}\, G_F\, n_e(t)\, E$.

Taking (7.77) into account from (7.72) for the function $a'(t)$ we find the following evolution equation

$$i\,\frac{\partial a'(t)}{\partial t} = H\, a'(t), \tag{7.78}$$

where the 2×2 matrix H is given by

$$H = \begin{pmatrix} E^{\mathrm{m}}_1(t) & -i\dot{\theta}^{\mathrm{m}}(t) \\ i\dot{\theta}^{\mathrm{m}}(t) & E^{\mathrm{m}}_2(t) \end{pmatrix}. \tag{7.79}$$

We will present the matrix H in the usual form

$$H = \frac{1}{2}(E^{\mathrm{m}}_1(t) + E^{\mathrm{m}}_2(t)) + \begin{pmatrix} -\frac{1}{2}(E^{\mathrm{m}}_2(t) - E^{\mathrm{m}}_1(t)) & -i\dot{\theta}^{\mathrm{m}}(t) \\ i\dot{\theta}^{\mathrm{m}}(t) & \frac{1}{2}(E^{\mathrm{m}}_2(t) - E^{\mathrm{m}}_1(t)). \end{pmatrix} \tag{7.80}$$

Let us introduce the parameter of adiabaticity

$$\gamma(x) = \frac{(E^{\mathrm{m}}_2(x) - E^{\mathrm{m}}_1(x))}{2|\frac{d\theta^{\mathrm{m}}(x)}{dx}|}, \tag{7.81}$$

where $x \simeq t$. The solution of the evolution Eq. (7.78) is adiabatic if nondiagonal elements of the second matrix in (7.80) are much smaller than the diagonal elements, i.e. if the parameter of adiabaticity is much larger than one:

$$\gamma(x) \gg 1\,. \tag{7.82}$$

From (7.59) and (7.60) we find

$$\gamma(x) = \frac{1}{2E}\,\frac{[(\Delta m^2 \cos 2\theta - A(x))^2 + (\Delta m^2 \sin 2\theta)^2]^{3/2}}{\Delta m^2 \sin 2\theta\, |\frac{dA(x)}{dx}|}. \tag{7.83}$$

When neutrino, produced in the central region of the sun, is traveling towards the surface it could pass through the point $x = x_R$ where the condition

$$\Delta m^2 \cos 2\theta = A(x_R) \tag{7.84}$$

is satisfied. This condition is called MSW resonance condition. At the point $x = x_R$, the mixing in matter is maximal and the difference of the neutrino energies is minimal

$$\theta^{\mathrm{m}}(x_R) = \frac{\pi}{4}, \quad \Delta E^{\mathrm{m}}(x_R) = \frac{\Delta m^2 \sin 2\theta}{2E}\,. \tag{7.85}$$

The resonance region is the most important one for neutrino transitions. Let us estimate the neutrino energies at which the resonance condition (7.84) is satisfied. We have

$$A(x) = 2\sqrt{2}\,G_F\,n_e(x)\,E \simeq 2\sqrt{2}\,G_F\,Y_e\,\rho(x)\,\frac{E}{M_p} \simeq 1.1\cdot 10^{-7}\mathrm{eV}^2\,\frac{\rho(x)}{\mathrm{g/cm}^3}\,\frac{E}{\mathrm{MeV}}. \tag{7.86}$$

Here ρ is the matter density and $Y_e = \frac{n_e}{n_N}$ is the ratio of the electron and nucleon number densities. In (7.86) we used the Standard Solar Model value $Y_e \simeq 2/3$.

The density $\rho(x)$ is well described by the exponential function

$$\rho_(x) \simeq \rho_0\, e^{-x/r_0}. \tag{7.87}$$

Here

$$r_0 = \frac{R}{10.54} \simeq 6.6\cdot 10^4\ \mathrm{km}, \tag{7.88}$$

where R is the solar radius and $\rho_0 \simeq 150\ \mathrm{g/cm}^3$.

From the global analysis of the data of the solar neutrino experiments and of the data of the KamLAND reactor experiment for the neutrino oscillation parameters the following best-fit values were found

$$\Delta m^2 \equiv \Delta m^2_{12} = 7.94\cdot 10^{-5}\ \mathrm{eV}^2, \quad \tan^2\theta \equiv \tan^2\theta_{12} = 0.447. \tag{7.89}$$

From these values we have

$$\Delta m^2 \cos 2\theta \simeq 3\cdot 10^{-5}\ \mathrm{eV}^2. \tag{7.90}$$

From (7.83) we find for the value of the parameter of the adiabaticity at the resonance point x_R the following expression

$$\gamma(x_R) = \frac{\Delta m^2 \sin^2 2\theta}{2E \cos 2\theta \, |\frac{d\, ln n_e}{dx}|_{x_R}} \,. \tag{7.91}$$

Taking into account (7.87) we have

$$\gamma(x_R) = \frac{\Delta m^2 \sin^2 2\theta \, r_0}{2E \cos 2\theta} \tag{7.92}$$

Now, using the best-fit values (7.89) of the neutrino oscillation parameters and the value (7.88) of the parameter r_0 for $\gamma(x_R)$ we obtain the following expression

$$\gamma(x_R) \simeq 3 \cdot 10^4 \left(\frac{\mathrm{MeV}}{E}\right) \,. \tag{7.93}$$

In the solar neutrino experiments neutrinos with energies in the range (0.2–16) MeV are detected. From (7.93) follows that in the whole interval of the detected solar neutrino energies $\gamma(x_R) \gg 1$. If the parameter of the adiabaticity is large at the resonance point it will also be large in all other points. We come to the conclusion that solar neutrino transitions are adiabatic. Taking into account that $\theta^{\mathrm{m}}(x) = \theta$ (θ is the vacuum mixing angle) we obtain from (7.57) and (7.76) the following expression for the ν_e survival probability

$$\begin{aligned} \mathrm{P}(\nu_e \to \nu_e) &= \cos^2\theta \, \cos^2\theta^{\mathrm{m}}(x_0) + \sin^2\theta \, \sin^2\theta^{\mathrm{m}}(x_0) \\ &= \frac{1}{2}(1 + \cos 2\theta \, \cos 2\theta^{\mathrm{m}}(x_0)) \,. \end{aligned} \tag{7.94}$$

Notice that in the general two-neutrino case for the averaged ν_e survival probability in matter we have

$$\mathrm{P}(\nu_e \to \nu_e) = \sum_{i,k} |U^{\mathrm{m}}_{ek}(x)|^2 \, \mathrm{P}_{ki} \, |U^{\mathrm{m}}_{ei}(x_0)|^2 \tag{7.95}$$

where P_{ki} is the probability of the transition from the state with energy E^{m}_i to the state with energy E^{m}_k. From the conservation of the total probability we have

$$\mathrm{P}_{11} = 1 - \mathrm{P}_{21}; \quad \mathrm{P}_{22} = 1 - \mathrm{P}_{12} \,. \tag{7.96}$$

Further, we have $\mathrm{P}_{21} = \mathrm{P}_{12}$. From (7.95) and (7.96) we easily find the following general expression for the $\nu_e \to \nu_e$ survival probability

$$\mathrm{P}(\nu_e \to \nu_e) = \frac{1}{2} + \left(\frac{1}{2} - P_{12}\right) \cos 2\theta \, \cos 2\theta^{m}(x_0) \tag{7.97}$$

In the adiabatic approximation $P_{12} = 0$ and the expression (7.97) coincides with (7.94).

In the literature exist different approximate expressions for the transition probability P_{12}. In Landau-Zenner approximation, which is based on the assumption that transitions occur mainly in the resonance region, we have

$$\mathrm{P}_{12} = e^{-\frac{\pi}{2}\,\gamma(x_R)\,F}\,, \tag{7.98}$$

where $\gamma(x_R)$ is the parameter of the adiabaticity at the resonance point, given by the expression (7.91), and the factor F depends on the electron number density. If the electron density is exponential, $F = 1 - \tan^2\theta$.

From (7.98) follows that for

$$\gamma(x_R) >> 1 \tag{7.99}$$

we have $P_{12} \simeq 0$. Thus, we come to the same conclusion as before: if the inequality (7.99) is satisfied, neutrino transitions in matter are adiabatic.

In the adiabatic approximation transition probability depends on vacuum parameters Δm^2 and θ and on the electron number density at the production point. Let us determine the neutrino energy at which $A(x_0) = \Delta m^2 \cos 2\theta$. Taking into account (7.86) and (7.90) we have

$$E_0 = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}\,G_F\,n_e(0)} \simeq 2\ \mathrm{MeV}. \tag{7.100}$$

In the region of neutrino energies significantly larger than E_0 from (7.60) we find that $\cos 2\theta^{\mathrm{m}}(x_0) \simeq -1$ ($\theta^{\mathrm{m}}(x_0) \simeq \pi/2$). Thus, in the high energy region the survival probability is given by the expression

$$\mathrm{P}(\nu_e \to \nu_e) \simeq \frac{1}{2}\,(1 - \cos 2\theta) = \sin^2\theta. \tag{7.101}$$

From (7.89) and (7.101) we obtain the following value of the ν_e survival probability in the high-energy region:

$$\mathrm{P}(\nu_e \to \nu_e) \simeq 0.31. \tag{7.102}$$

In the region of neutrino energies significantly smaller than E_0 from (7.60) we find that $\theta^{\mathrm{m}}(x_0) \simeq \theta$. Thus, in this region the ν_e survival probability is given by the vacuum expression

$$\mathrm{P}(\nu_e \to \nu_e) \simeq 1 - \frac{1}{2}\,\sin^2 2\theta. \tag{7.103}$$

Using (7.89) we find

$$\mathrm{P}(\nu_e \to \nu_e) \simeq 0.57. \tag{7.104}$$

In every of the two neutrino energy regions, we considered, the ν_e transition probabilities practically do not depend on energy and differ approximately by the factor two. In the transition region a strong energy dependence must be exhibited. Detailed calculations show that the transition region between high-energy and low-energy regimes lies in the interval (2–5) MeV.

In the recent BOREXINO experiment the ^{7}Be neutrinos with energy $E = 0.87$ Mev was detected. For the $\nu_e \to \nu_e$ transition probability the value

$$\mathrm{P}(\nu_e \to \nu_e) = 0.56 \pm 0.10 \tag{7.105}$$

was found. In this experiment the $\nu_e \to \nu_e$ transition probability in the high energy region at effective neutrino energy $E = 8.6$ MeV was also determined. It was found

$$\mathrm{P}(\nu_e \to \nu_e) = 0.35 \pm 0.10 \tag{7.106}$$

The values (7.105) and (7.106) are in a good agreement with the predictions (7.104) and (7.102).

Up to now we have considered two-neutrino transition probabilities. In the three-neutrino case the average probability for $\nu_e \to \nu_e$ transitions of solar neutrinos in matter is given by the following expression[9]

$$\mathrm{P}(\nu_e \to \nu_e) = |U_{e3}|^4 + (1 - |U_{e3}|^2)^2 \, \mathrm{P}^{(12)}_{\nu_e \to \nu_e}(\Delta m^2_{12}, \tan^2\theta_{12}) \;, \tag{7.107}$$

where $P^{(12)}_{\nu_e \to \nu_e}(\Delta m^2_{12}, \tan^2\theta_{12})$ is the two-neutrino transition probability driven by Δm^2_{12}. From the data of the CHOOZ reactor experiment it follows that $|U_{e3}|^2 \lesssim 5 \cdot 10^{-2}$. Thus, with an accuracy of a few % the $\nu_e \to \nu_e$ transition probability for solar neutrinos in the three-neutrino case is given by the two neutrino expression

$$\mathrm{P}(\nu_e \to \nu_e) \simeq \mathrm{P}^{(12)}_{\nu_e \to \nu_e}(\Delta m^2_{12}, \tan^2\theta_{12}). \tag{7.108}$$

The two-neutrino $\nu_e \to \nu_e$ transition probability, which we considered before, is usually used in the analysis of the solar neutrino data.

[9] The same expression is valid for the vacuum neutrino oscillations driven by Δm^2_{12} (such as neutrino oscillations observed in the KamLAND reactor experiment). We derived the corresponding vacuum three-neutrino ν_e survival probability in Sect. 6.6.

Chapter 8
Neutrinoless Double Beta-Decay

8.1 Introduction

As we have seen in the previous sections, there are two fundamentally different possibilities for the massive neutrinos ν_i: they can be Dirac particles, if the total lepton number $L = L_e + L_\mu + L_\tau$ is conserved, or truly neutral Majorana particles if there are no conserved leptons numbers. *The problem of the nature of massive neutrinos ν_i (Dirac or Majorana?) is one of the most fundamental problem of neutrino physics.* The solution of this problem will have an important impact on the understanding of the origin of neutrino masses and mixing. If it will be proved that ν_i are Majorana particles, it will be a strong argument in favor of the seesaw mechanism of neutrino mass generation which is commonly considered as the most natural explanation of the smallness of neutrino masses.

Neutrino oscillations is an interference phenomenon sensitive to very small values of neutrino mass-squared differences. However, as we have shown before, by the investigation of neutrino oscillations it is impossible to decide on the nature of neutrinos with definite masses: are they Dirac or Majorana particles.

In order to reveal the nature of neutrinos with definite masses it is necessary to study processes in which the total lepton number L is violated.

We will demonstrate first that for massless neutrinos and the left-handed SM neutrino interaction, theories with Dirac and Majorana neutrinos are equivalent. In the case of the Majorana field $\nu_i(x)$, right-handed and left-handed components are not independent. They are connected by the relation (see Sect. 4.3)

$$\nu_{iR}(x) = (\nu_{iL}(x))^c \tag{8.1}$$

In the Dirac case, right-handed and left-handed components are independent. If $m_i = 0$, the right-handed fields do not enter into the Lagrangian. Hence, there is no way to distinguish Dirac and Majorana fields in this case.

We will show now that in the case of massless neutrinos helicities play the role of the conserved lepton numbers. Let us consider the CC Lagrangian

$$\mathcal{L}_I^{\mathrm{CC}} = -\frac{g}{\sqrt{2}} \sum_l \bar{\nu}_{lL} \gamma_\alpha l_L \; W^\alpha - \frac{g}{\sqrt{2}} \sum_l \bar{l}_L \gamma_\alpha \nu_{lL} \; W^{\alpha\dagger} \tag{8.2}$$

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in the general case $m_i \neq 0$. Taking into account neutrino mixing we have

$$\mathcal{L}_I^{\mathrm{CC}} = -\frac{g}{\sqrt{2}} \sum_{i,l} U_{li}^* \, \bar{\nu}_{iL} \gamma_\alpha l_L \; W^\alpha - \frac{g}{\sqrt{2}} \sum_{i,l} U_{li} \, \bar{l}_L \gamma_\alpha \nu_{iL} \; W^{\alpha \dagger}. \tag{8.3}$$

In the matrix elements of the processes of production of l^+ and a neutrino (due to the first term of (8.3)) or absorption of neutrino with production of l^- (due to the second term of (8.3)) enters the spinor $\frac{1-\gamma_5}{2} \, u^r(p_i)$ (p_i is the neutrino momentum). For the spinor $u^r(p_i)$ we have

$$\gamma \cdot p_i \; u^r(p_i) = m_i \; u^r(p_i), \quad \Sigma \; u^r(p_i) = r u^r(p_i), \tag{8.4}$$

where Σ is the helicity operator. We have

$$\Sigma = \gamma_5 \gamma \cdot s, \tag{8.5}$$

where the vector s^α satisfies the condition $s \cdot p = 0$. In the rest frame we have $s_0^\alpha = (0, \; \mathbf{k})$, $\mathbf{k}$ being the unit vector in the momentum direction. In the system where the neutrino momentum is equal to $\mathbf{p}_i = p_i \mathbf{k}$, we have $s^\alpha = \frac{1}{m_i} \, (p_i, \; \mathbf{k} \; p_i^0)$. Taking into account terns linear in $\frac{m_i}{2E}$, we find from (8.4)

$$\frac{1-\gamma_5}{2} \, u^r(p_i) \simeq \frac{1-r}{2} \, u^r(p_i) + r \frac{m_i}{2E} \gamma^0 u^r(p_i) \; . \tag{8.6}$$

In the matrix elements of the processes of production of l^- and an (anti)neutrino (due to the second term of (8.3)) or absorption of an (anti)neutrino with the production of l^+ (due to the first term of (8.3)) enters the spinor $\frac{1-\gamma_5}{2} \, v^r(p_i)$ $(v^r(p_i) = u^r(-p_i) = C \, (\bar{u}^r(p_i))^T)$. Taking into account terms linear in $\frac{m_i}{2E}$ we have

$$\frac{1-\gamma_5}{2} \, v^r(p) = \frac{1+r}{2} \, v^r(p) + r \frac{m_i}{2E} \gamma^0 v^r(p) \; . \tag{8.7}$$

From (8.6) and (8.7) follows that in the case of massless neutrinos in weak processes together with l^+ a (strictly) left-handed neutrino ν_l is produced. This neutrino can produce only l^- (due to the second term of (8.3)). Analogously, due to the second term of (8.3) in weak processes together with l^- a right-handed (anti)neutrino $\bar{\nu}_l$ is produced. This (anti)neutrino in weak interaction processes can produce only l^+ (due to the first term of (8.3)). Thus, we came to the conclusion that for massless neutrinos the conserved helicity ensure the conservation of the lepton charges independently of the Dirac or Majorana nature of fields ν_i.

This conservation is a consequence of the global gauge invariance which holds in the case of massless neutrinos. In fact, if $m_i = 0$ the field $\nu_{lL}(x)$ satisfies the Dirac equation

$$i \gamma^\alpha \partial_\alpha \; \nu_{lL}(x) = 0 \tag{8.8}$$

and canonical commutation relations (independently on the nature of $\nu_{iL}(x)$). The total Lagrangian is invariant under the global gauge transformations

$$l'(x) = e^{i\Lambda_l}\, l(x), \quad \nu'_{lL}(x) = e^{i\Lambda_l}\, \nu_{lL}(x) \;\; (l = e, \mu, \tau). \tag{8.9}$$

From this invariance follows that flavor lepton numbers L_e, L_μ and L_τ are conserved.

We now will assume that $\nu_i(x)$ are Majorana fields with $m_i \neq 0$. In this case, processes in which the total lepton number is violated become possible.

The Majorana operator $\nu_{iL}(x)$ is the sum of the operators of absorption and creation of *neutrinos*. Thus, neutrinos which are produced in a weak decay together with, say, l^+ (due to the first term of the Lagrangian (8.3)) can produce l'^+ in a process of interaction with a nucleon (due to the *same* first term of the Lagrangian). For example, the chain of the processes $\pi^+ \to \mu^+ + \nu_i, \quad \nu_i + N \to \mu^+ + X$) becomes possible in the case of massive Majorana neutrinos. However, the chain of such processes is strongly suppressed with respect to a chain of usual processes in which the total lepton number is conserved (such as $\pi^+ \to \mu^+ + \nu_\mu$ due to the first term of the Lagrangian (8.3) and $\nu_\mu + N \to \mu^- + X$ due to the *second* term of the Lagrangian (8.3)).

In fact, from (8.6) follows that in the neutrino-production process together with l^+ mainly left-handed neutrinos are produced. From (8.7) we see that in the cross section of the absorption of such neutrinos with the production of l'^+ the small factors $(\frac{m_i}{E})^2$ enter. The probability of the production of right-handed neutrinos, which have a "large" weak absorption cross section, is also suppressed by the factors $(\frac{m_i}{E})^2$. Thus, a chain of the processes, of production and absorption of a neutrino, induced by the first term of the Lagrangian (8.3), in which l^+ and l'^+ are produced, is suppressed by the factors $(\frac{m_i}{E})^2$ with respect to the chain of usual neutrino processes, induced by the first and the second terms of the Lagrangian (8.3). Taking into account that neutrino energies in neutrino processes are larger than $\sim$1 MeV and neutrino masses are smaller than $\sim$1 eV we conclude that the suppression factor for neutrino processes in which the total lepton number is changed by two is extremely small:

$$\frac{m_i^2}{E^2} \lesssim 10^{-12}. \tag{8.10}$$

Thus, in a foreseeable future it does not seem possible to reveal the nature of massive neutrinos via the observation of the violation of the total lepton number in neutrino experiments.

We will consider weak interaction processes *with virtual Majorana neutrinos* in which the total lepton number is changed by two. Such processes are induced by the first (or second) term of the Lagrangian (8.3). Examples are

1. Neutrinoless double β-decay ($0\nu\beta\beta$-decay) of even-even nuclei

$$(A, Z) \to (A, Z+2) + e^- + e^- . \tag{8.11}$$

2. Decays of kaons

$$K^+ \to \pi^- + \mu^+ + \mu^+; \;\; K^+ \to \pi^- + e^+ + e^+, \;\; K^+ \to \pi^- + \mu^+ + e^+ \,. \tag{8.12}$$

3. The process

$$\mu - + (A, Z) \to (A, Z - 2) + e^+ \,. \tag{8.13}$$

4. Decays of τ-leptons

$$\tau^- \to e^+ + \pi^- + \pi^-, \;\; \tau^- \to \mu^+ + \pi^- + \pi^-, \;\; \tau^- \to e^+ + \pi^- + K^- \,. \tag{8.14}$$

etc.

Matrix elements of such processes contain propagator $\langle 0|\, T(\nu_{lL}(x_1)\, \nu^T_{l'L}(x_2))\, |0\rangle$. It will be shown later that neutrino masses enter into this propagator in the form

$$\sum_i U_{li}\, U_{l'k} \frac{1-\gamma_5}{2}\, \frac{(\gamma \cdot q + m_i)}{q^2 - m_i^2}\, \frac{1-\gamma_5}{2}\, C \simeq m_{ll'}\, \frac{1}{q^2}\, \frac{1-\gamma_5}{2}\, C\,, \tag{8.15}$$

where

$$m_{ll'} = \sum_i U_{li}\, U_{l'i}\, m_i \tag{8.16}$$

and we took into account that $m_i^2 \ll q^2$. Thus, the matrix element of a lepton number violating process with virtual Majorana neutrinos and the emission of two leptons l^- and l'^-, is proportional to $m_{ll'}$. It is easy to see that the matrix element of a process, in which l^+ and l'^+ are produced, is proportional to $\sum_i U^*_{li}\, U^*_{l'i}\, m_i = m^*_{ll'}$.

Let us consider $|m_{ll'}|$. Taking into account the Cauchy-Schwarz inequality and the unitarity of the mixing matrix, we find

$$|m_{ll'}| \le \sqrt{\sum_i |U_{li}|^2 m_i^2}\, \sqrt{\sum_i |U_{l'i}|^2} \le m_{\max}, \tag{8.17}$$

where $m_{\max}$ is the mass of the heaviest neutrino. From the data of experiments on the measurement of neutrino masses through the investigation of the high-energy part of the β-spectrum of tritium it was found: $m_{\max} \le 2.2$ eV (see Sect. 9.2). Thus, we have

$$|m_{ll'}| \le 2.2 \text{ eV}\,. \tag{8.18}$$

The probabilities of the processes (8.11), (8.12), (8.13) and others, in which the lepton number is changed by two and the Majorana neutrinos are virtual particles, are extremely small. First, they are processes of the second order in the Fermi constant

G_F. And, second, they are helicity-suppressed processes. In the probabilities of such processes enter very small helicity suppression factor $\frac{|m_{l'l}|^2}{<Q^2>}$, where $< Q^2 >$ is an average momentum-transfer squared (typically $\gtrsim 10$ MeV^2).

The sensitivities to the parameter $|m_{ll'}|$ of experiments on the search for the processes (8.12), (8.13), (8.14) and other similar processes are much worse than the upper bound (8.18). For example, from the experiment on the search for the process $\mu^- \mathrm{Ti} \to e^+ \mathrm{Ca}$ the following upper bound was obtained

$$\frac{\Gamma(\mu^- \mathrm{Ti} \to e^+ \mathrm{Ca})}{\Gamma(\mu^- \mathrm{Ti} \to \mathrm{all})} \le 1.7 \cdot 10^{-12}. \tag{8.19}$$

For the probability of the decay $K^+ \to \pi^- \mu^+ \mu^+$ it was found:

$$\frac{\Gamma(K^+ \to \pi^- \mu^+ \mu^+)}{\Gamma(K^+ \to \mathrm{all})} \le 3 \cdot 10^{-9}. \tag{8.20}$$

From these results the following bounds were obtained, correspondingly

$$|m_{\mu e}| \le 82\ \mathrm{MeV}, \quad |m_{\mu\mu}| \le 4 \cdot 10^4\ \mathrm{MeV}. \tag{8.21}$$

The exceptional process, which is sensitive to the expected Majorana neutrino masses, is neutrinoless double β-decay of some even-even nuclei. The possibilities to use large targets (in present-day experiments tens of kg, in future experiments about 1 ton and maybe more), to reach small background and high energy resolution make experiments on the search for this decay a unique source of information about the nature of massive neutrinos ν_i. In the next sections we will consider this process in detail.

8.2 Basic Elements of the Theory of $0\nu\beta\beta$-Decay

We will consider here the elements of the theory of neutrinoless double β-decay. Let us consider an even-even nucleus (A, Z) with mass $M_{A,Z}$. The mass of odd-odd nucleus $(A, Z+1)$ with the same atomic number is larger than $M_{A,Z}$. Thus, (A, Z) nucleus cannot have usual β-decay $(A, Z) \to (A, Z+1) + e^- + \bar{\nu}_e$. If, however, exist even-even nucleus $(A, Z+2)$ with mass smaller than $M_{A,Z}$, the nucleus (A, Z) can decay into $(A, Z+2)$ with emission of two electrons. An example is even-even nucleus ^{76}Ge. The decay $^{76}\mathrm{Ge} \to ^{76}\mathrm{As} + e^- + \bar{\nu}_e$ is forbidden because the ^{76}As nucleus is heavier than ^{76}Ge. However, the transition of ^{76}Ge into lighter even-even nucleus ^{76}Se and two electrons is allowed.

We present in Table 8.1 several even-even nuclei, the $\beta\beta$ decay of which has been searched for (or is planned to be searched for) in different experiments.

Two types of $\beta\beta$ decays are possible

1. Two-neutrino double β decay ($2\nu\beta\beta$-decay)

Table 8.1 $\beta\beta$ candidate nuclei. In the first column nuclei transitions are indicated; in the second column Q-values are shown; in the third column element abundances are presented

Transition	$T_0 = Q_{\beta\beta}$ (KeV)	Abundance (%)
$^{76}Ge \rightarrow ^{76}Se$	$2,039.6 \pm 0.9$	7.8
$^{100}Mo \rightarrow ^{100}Ru$	$3,934 \pm 6$	9.6
$^{130}Te \rightarrow ^{130}Xe$	$2,533 \pm 4$	34.5
$^{136}Xe \rightarrow ^{136}Ba$	$2,479 \pm 8$	8.9
$^{150}Nd \rightarrow ^{150}Sm$	$3,367.1 \pm 2.2$	5.6
$^{82}Se \rightarrow ^{82}Kr$	$2,995 \pm 6$	9.2
$^{48}Ca \rightarrow ^{48}Ti$	$4,271 \pm 4$	0.187

$$(A, Z) \rightarrow (A, Z+2) + e^- + e^- + \bar{\nu}_e + \bar{\nu}_e \, . \tag{8.22}$$

2. Neutrinoless double β decay ($0\nu\beta\beta$-decay)

$$(A, Z) \rightarrow (A, Z+2) + e^- + e^- . \tag{8.23}$$

Two-neutrino double β decay is a process of the second order in the Fermi constant G_F, which is governed by the standard CC Hamiltonian of the weak interaction. This decay was observed in more than ten different nuclei with half-lifes in the range $(10^{18} - 10^{20})$ years. The largest statistics was obtained in the NEMO-3 experiment. For half-life of the $2\nu\beta\beta$-decay of ^{100}Mo in this experiment was obtained

$$T^{2\nu}_{1/2} = (7.72 \pm 0.02 \text{ (stat)} \pm 0.54 \text{ (syst)}) \cdot 10^{18} \text{ y}. \tag{8.24}$$

The neutrinoless double β decay (8.23) is allowed only in the case if the total lepton number is not conserved. We will consider this process assuming that neutrinos have the SM interaction and the lepton number is violated due to a Majorana neutrino mass term.

For the effective Hamiltonian of the process we have

$$\mathcal{H}_I(x) = \frac{G_F}{\sqrt{2}} 2 \sum_i \bar{e}_L(x) \gamma_\alpha \, U_{li} \, \nu_{iL}(x) \, j^\alpha(x) + \text{h.c.} \tag{8.25}$$

Here G_F is the Fermi constant, $j^\alpha(x)$ is the hadronic charged current and the field $\nu_i(x)$ satisfies the condition

$$\nu_i^c(x) = C \bar{\nu}_i^T(x) = \nu_i(x). \tag{8.26}$$

The neutrinoless double β-decay is the second order in G_F process with the virtual neutrinos. The matrix element of the process is given by the following expression

$$\langle f|S^2|i\rangle = 4\frac{(-i)^2}{2\,!}\left(\frac{G_F}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}\int\sum_i \bar{u}_L(p_1)e^{ip_1x_1}\gamma_\alpha\, U_{ei}\langle 0|T(\nu_{iL}(x_1)\,\nu^T_{iL}(x_2)|0\rangle$$
$$\times\gamma^T_\beta\, U_{ei}\bar{u}^T_L(p_2)e^{ip_2x_2}\langle N_f|T(J^\alpha(x_1)J^\beta(x_2))|N_i\rangle\, d^4x_1d^4x_2-(p_1\rightleftarrows p_2). \quad (8.27)$$

Here p_1 and p_2 are electron momenta, $J^\alpha(x)$ is the weak charged current in the Heisenberg representation,[1] $|N_i\rangle$ and $|N_f\rangle$ are the states of the initial and the final nuclei with 4-momenta $P_i = (E_i, \mathbf{p_i})$ and $P_f = (E_f, \mathbf{p_f})$, respectively, and $N_p = \frac{1}{(2\pi)^{3/2}\sqrt{2p^0}}$ is the standard normalization factor. The diagram of $0\nu\beta\beta$-decay is presented in Fig. 8.1.

Let us consider the neutrino propagator. From the Majorana condition (8.26) we find

$$\langle 0|T(\nu_{iL}(x_1)\nu^T_{iL}(x_2)|0\rangle = -\frac{1-\gamma_5}{2}\langle 0|T(\nu_i(x_1)\bar{\nu}_i(x_2))|0\rangle\,\frac{1-\gamma_5}{2}\,C. \quad (8.28)$$

Further, we have

$$\langle 0|T(\nu_i(x_1)\bar{\nu}_i(x_2))|0\rangle = \frac{i}{(2\pi)^4}\int e^{-iq\,(x_1-x_2)}\frac{\gamma\cdot q+m_i}{q^2-m_i^2}d^4q \quad (8.29)$$

Thus, for the neutrino propagator we find the following expression[2]

$$\langle 0|T(\nu_{iL}(x_1)\bar{\nu}_{iL}(x_2))|0\rangle = -\frac{i}{(2\pi)^4}\int e^{-iq\,(x_1-x_2)}\frac{m_i}{q^2-m_i^2}\,d^4q\,\frac{1-\gamma_5}{2}\,C. \quad (8.30)$$

The neutrino propagator is proportional to m_i. It is obvious from (8.29) that this is connected with the fact that only left-handed neutrino fields enter into the Hamilto-

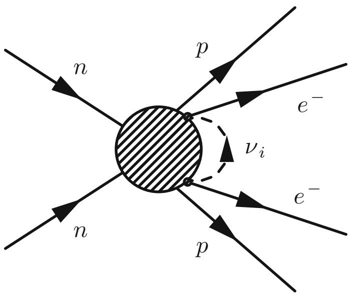

Fig. 8.1 Feynman diagram of the neutrinoless double β-decay

[1] Thus, in (8.27) the strong interaction are taking into account.

[2] Notice that in the case of the Dirac neutrinos $\langle 0|\nu_{iL}(x_1)\nu^T_{iL}(x_2)|0\rangle = \frac{1-\gamma_5}{2}\langle 0|\nu_i(x_1)\,\nu^T_i(x_2)|0\rangle\frac{1-\gamma^T_5}{2} = 0$. The neutrinoless double β-decay is obviously forbidden in the Dirac case.

nian of the weak interaction. In the case of massless neutrinos ($m_i = 0,\ i = 1, 2, 3$), in accordance with the theorem on the equivalence of the theories with massless Majorana and Dirac neutrinos, the matrix element of the neutrinoless double β-decay is equal to zero.

Let us consider the second term of the matrix element (8.27). It is easy to show that

$$\begin{aligned}\bar{u}_L(p_1)\gamma_\alpha(1-\gamma_5)\gamma_\beta C\bar{u}_L^T(p_2) &= \bar{u}_L(p_2)C^T\gamma_\beta^T(1-\gamma_5^T)\gamma_\alpha^T\bar{u}_L^T(p_1)\\ &= -\bar{u}_L(p_2)\gamma_\beta(1-\gamma_5)\gamma_\alpha C\bar{u}_L^T(p_1)\,. \end{aligned}\quad (8.31)$$

If we take into account (8.31) and the relation

$$T(J^\beta(x_2)J^\alpha(x_1)) = T(J^\alpha(x_1)J^\beta(x_2)) \quad (8.32)$$

we can show that the second term of the matrix element (8.27) is equal to the first one. For the matrix element we obtain the following expression

$$\begin{aligned}\langle f|S^2|i\rangle = -4\left(\frac{G_F}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}\int \bar{u}_L(p_1)e^{ip_1x_1}\gamma_\alpha\frac{i}{(2\pi)^4}\sum_i U_{ei}^2 m_i\int\frac{e^{-iq\ (x_1-x_2)}}{p^2-m_i^2}d^4q\\ \times\frac{1-\gamma_5}{2}\gamma_\beta C\ \bar{u}_L^T(p_2)e^{ip_2x_2}\langle N_f|T(J^\alpha(x_1)J^\beta(x_2))|N_i\rangle\ d^4x_1d^4x_2\end{aligned} \quad (8.33)$$

Initial nuclei in $0\nu\beta\beta$-decay are heavy nuclei like ^{76}Ge, ^{136}Xe, ^{130}Te, ^{100}Mo and others. The calculation of the nuclear part of the matrix element of the $0\nu\beta\beta$-decay is a complicated nuclear problem. In such a calculation different approximations are used. We will present now the matrix element of the $0\nu\beta\beta$-decay in a form which is appropriate for such approximate calculations.

Let us perform in (8.33) the integration over the time variables x_2^0 and x_1^0. The integral over x_2^0 can be presented in the form

$$\int_{-\infty}^{\infty}\ldots dx_2^0 = \int_{-\infty}^{x_1^0}\ldots dx_2^0 + \int_{x_1^0}^{\infty}\ldots dx_2^0\,. \quad (8.34)$$

After the integration over q^0 in the neutrino propagator, in the region $x_1^0 > x_2^0$ we find[3]

$$\frac{i}{(2\pi)^4}\int\frac{e^{-iq\ (x_1-x_2)}}{q^2-m_i^2}d^4q = \frac{1}{(2\pi)^3}\int\frac{e^{-iq_i^0\ (x_1^0-x_2^0)+i\mathbf{q}\ (\mathbf{x}_1-\mathbf{x}_2)}}{2\ q_i^0}d^3q\,, \quad (8.35)$$

where

[3] It is assumed that in the propagator $m_i^2 = m_i^2 - i\varepsilon$.

$$q_i^0 = \sqrt{\mathbf{q}^2 + m_i^2}\,. \tag{8.36}$$

In the region $x_1^0 < x_2^0$ we have

$$\frac{i}{(2\pi)^4} \int \frac{e^{-iq\,(x_1-x_2)}}{q^2 - m_i^2} d^4q = \frac{1}{(2\pi)^3} \int \frac{e^{-iq_i^0\,(x_2^0-x_1^0)+i\mathbf{q}\,(\mathbf{x}_2-\mathbf{x}_1)}}{2\,q_i^0} d^3q. \tag{8.37}$$

For the operators $J^\alpha(x)$ from the invariance under the translations we have

$$J^\alpha(x) = e^{iHx^0} J^\alpha(\mathbf{x}) e^{-iHx^0}, \tag{8.38}$$

where H is the total Hamiltonian. From this relation we find

$$\begin{aligned}&\langle N_f|J^\alpha(x_1)J^\beta(x_2)|N_i\rangle = \\ &\sum_n e^{i(E_f-E_n)x_1^0} e^{i(E_n-E_i)x_2^0}\,\langle N_f|J^\alpha(\mathbf{x_1})|N_n\rangle\langle N_n|J^\beta(\mathbf{x_2}))|N_i\rangle,\end{aligned} \tag{8.39}$$

where $|N_n\rangle$ is the vector of the state of the intermediate nucleus with 4-momentum $P_n = (E_n, \mathbf{p_n})$. In (8.39) the sum over the total system of the states $|N_n\rangle$ is assumed.

Taking into account that at $\pm\infty$ the interaction is turned off we have

$$\int_{-\infty}^{0} e^{iax_2^0}\,dx_2^0 \to \int_{-\infty}^{0} e^{i(a-i\varepsilon)x_2^0}\,dx_2^0 = \lim_{\varepsilon\to 0} \frac{-i}{a - i\varepsilon} \tag{8.40}$$

and

$$\int_{0}^{-\infty} e^{iax_2^0}\,dx_2^0 \to \int_{0}^{\infty} e^{i(a+i\varepsilon)x_2^0}\,dx_2^0 = \lim_{\varepsilon\to 0} \frac{i}{a + i\varepsilon}. \tag{8.41}$$

From (8.40) and (8.41) we find

$$\begin{aligned}&\int_{-\infty}^{\infty} dx_1^0 \int_{-\infty}^{x_1^0} dx_2^0 \sum_n \langle N_f|J^\alpha(\mathbf{x_1})|N_n\rangle\langle N_n|J^\beta(\mathbf{x_2})|N_i\rangle e^{i(E_f-E_n)x_1^0+i(E_n-E_i)x_2^0} \\ &e^{i(p_1^0x_1^0+p_2^0x_2^0)} \times e^{iq_i^0(x_2^0-x_1^0)} = -i\sum_n \frac{\langle N_f|J^\alpha(\mathbf{x_1})|N_n\rangle\langle N_n|J^\beta(\mathbf{x_2}))|N_i}{E_n + p_2^0 + q_i^0 - E_i - i\varepsilon}\,2\pi\delta \\ &\qquad (E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{8.42}$$

Taking into account all these relations, for the matrix element of the neutrinoless double β-decay we find the following expression

$$
\begin{aligned}
\langle f|S^2|i\rangle = & 2i\left(\frac{G_F}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}\bar{u}(p_1)\gamma_\alpha\gamma_\beta(1+\gamma_5)C\bar{u}^T(p_2)\int d^3x_1 d^3x_1 e^{-i\mathbf{p_1}\mathbf{x_1}-i\mathbf{p_2}\mathbf{x_2}}\times \\
& \sum_i U_{ei}^2 m_i \frac{1}{(2\pi)^3}\int\frac{e^{i\mathbf{q}\,(\mathbf{x_1}-\mathbf{x_2})}}{q_i^0}d^3q\left[\sum_n\frac{\langle N_f|J^\alpha(\mathbf{x_1})|N_n\rangle\langle N_n|J^\beta(\mathbf{x_2}))|N_i\rangle}{E_n+p_2^0+q_i^0-E_i-i\varepsilon}\right. \\
& \left.+\sum\nolimits_n\frac{\langle N_f|J^\beta(\mathbf{x_2})|N_n\rangle\langle N_n|J^\alpha(\mathbf{x_1}))|N_i\rangle}{E_n+p_1^0+q_i^0-E_i-i\varepsilon}\right]2\pi\delta(E_f+p_1^0+p_2^0-E_i)
\end{aligned}
\tag{8.43}
$$

Equation (8.43) is the exact expression for the matrix element of $0\nu\beta\beta$-decay in the second order of the perturbation theory. We will consider major $0^+ \to 0^+$ transitions of even-even nuclei. For such transitions the following approximations are standard.

1. Small neutrino masses can be safely neglected in q_i^0.
   The averaged momentum of the virtual neutrino is given by the relation $q \simeq \frac{1}{r}$, where r is the average distance between two nucleons in nucleus. Taking into account that $r \simeq 10^{-13}$ cm, we have $q \simeq 100$ MeV. Neutrino masses are smaller than 2.2 eV. Thus, we have $q_i^0 = \sqrt{\mathbf{q}^2 + m_i^2} \simeq q$
2. Long-wave approximation.
   We have $p_k x_k \leq p_k R$, where $R \simeq 1.2\, A^{1/3} \cdot 10^{-13}$ cm is the radius of nucleus ($k = 1, 2$). Taking into account that $p_k \lesssim 1$ Mev, we have $p_k x_k \ll 1$. Thus, we put $e^{-i\mathbf{p_1}\mathbf{x_1}-i\mathbf{p_2}\mathbf{x_2}} \simeq 1$ i.e. we assume that two electrons are emitted in the S-states.
3. Closure approximation.
   Energy of the virtual neutrino $q \simeq 100$ MeV is much larger than the excitation energy $(E_n - E_i)$ of states $|N_n\rangle$. Thus, we can change the energy of the intermediate states E_n by average energy $\overline{E}$. In this (closure) approximation we have

$$
\frac{\langle N_f|J^\alpha(\mathbf{x_1})|N_n\rangle\langle N_n|J^\beta(\mathbf{x_2}))|N_i\rangle}{E_n+p_2^0+q_i^0-E_i-i\varepsilon} \simeq \frac{\langle N_f|J^\alpha(\mathbf{x_1})J^\beta(\mathbf{x_2}))|N_i\rangle}{\overline{E}+p_2^0+q-E_i-i\varepsilon},
\tag{8.44}
$$

   where $\overline{E}$ is the average energy of the exited states.
4. The impulse approximation for the hadronic charged current $J^\alpha(\mathbf{x})$.
   Taking into account the major terms, the hadronic charged current takes the form[4]

$$
J^\alpha(\mathbf{x}) \simeq \sum_n \delta(\mathbf{x}-\mathbf{r}_n)\,\tau_+^n[\,g_V(q^2)g^{\alpha 0} + g_A(q^2)\sigma_i^n g^{\alpha i}\,]\,.
\tag{8.45}
$$

[4] The pseudoscalar term in the one-nucleon matrix element of the hadronic charged current induces a tensor term. From numerical calculations follow that its contribution to the matrix element can be significant.

Here $g_V(q^2)$ and $g_A(q^2)$ are vector and axial form factors , σ_i and τ_i are Pauli matrices, $\tau_+ = \frac{1}{2}\,(\tau_1 + i\tau_2)$ and index n runs over all nucleons in a nucleus. We have $g_V(0) = 1, \;\; g_A(0) = g_A \simeq 1.27$.

It is obvious that $\tau_+^n\;\tau_+^n\; = 0$ Thus, in the impulse approximation the hadronic currents satisfy the relation

$$J^{\alpha}(\mathbf{x}_1)\;J^{\beta}(\mathbf{x}_2) = J^{\beta}(\mathbf{x}_2)\;J^{\alpha}(\mathbf{x}_1). \tag{8.46}$$

Further, the matrix $\gamma_\alpha\gamma_\beta$ in the leptonic part of the matrix element (8.43) can be presented in the form

$$\gamma_\alpha\gamma_\beta = g_{\alpha\beta} + \frac{1}{2}\,(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha). \tag{8.47}$$

It follows from (8.46) that the second term of (8.47) does not give contribution to the matrix element. From (8.45) we have

$$J^{\alpha}(\mathbf{x_1})J_{\alpha}(\mathbf{x_2}) = \sum_{n,m}\tau_+^n\tau_+^m\delta(\mathbf{x}_1-\mathbf{r}_n)\;\delta(\mathbf{x}_2-\mathbf{r}_m)(g_V^2(q^2)-g_A^2(q^2)\,\sigma^n\cdot\sigma^m)\;, \tag{8.48}$$

Neglecting nuclei recoil, we obtain in the laboratory frame

$$M_i = M_f + p_2^0 + p_1^0,$$

where M_i and M_f are masses of the initial and final nuclei. From this relation we find

$$q + p_{1,2}^0 + \overline{E} - M_i = q \pm \left(\frac{p_1^0 - p_2^0}{2}\right) + \overline{E} - \frac{M_i + M_f}{2} \tag{8.49}$$

The term $(\frac{p_1^0 - p_2^0}{2})$, is much smaller than all other terms in the right-hand side of this relation. Neglecting this term, we have

$$q + p_{1,2}^0 + \overline{E} - M_i \simeq q + \overline{E} - \frac{M_i + M_f}{2} \tag{8.50}$$

Further, taking into account that $g_V(q^2) \simeq \frac{1}{1+\frac{q^2}{0.71\ \mathrm{GeV}^2}}$ and $g_A(q^2) \simeq \frac{1}{1+\frac{q^2}{M_A^2}}$, where $M_A \simeq 1\ \mathrm{GeV}^2$, we can neglect q^2-dependence of the formfactors. After the integration in the matrix element (8.43) over $\mathbf{x_1}$ and $\mathbf{x_2}$, for the neutrino propagator we find the following expression

$$\frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{q}\;\mathbf{r}_{nm}}d^3q}{q(q + \overline{E} - \frac{1}{2}(M_i + M_f))} = \frac{1}{4\pi R}H(r_{nm}, \overline{E}), \tag{8.51}$$

where

$$H(r,\overline{E})=\frac{2R}{\pi r}\int_0^\infty\frac{\sin qr\;dq}{q+\overline{E}-\frac{1}{2}(M_i+M_f)}. \tag{8.52}$$

Here R is the nuclei radius and $\mathbf{r}_{nm}=\mathbf{r}_n-\mathbf{r}_m$.

Taking into account all these relations, from (8.43) for the matrix element of $0\nu\beta\beta$-decay we obtain the following expression

$$\langle f|S^2|i\rangle=-i\left(\frac{G_F}{\sqrt{2}}\right)^2\frac{1}{(2\pi)^3}\frac{1}{\sqrt{p_1^0p_2^0}}\,m_{\beta\beta}\;g_A^2\;\frac{1}{R}\;\bar{u}(p_1)(1+\gamma_5)C\bar{u}^T(p_2)$$
$$\times M^{0\nu}\;\;\delta(p_1^0+p_2^0+M_f-M_i), \tag{8.53}$$

where

$$m_{\beta\beta}=\sum_i U_{ei}^2 m_i \tag{8.54}$$

is the effective Majorana mass and

$$M^{0\nu}=M_{GT}^{0\nu}-\frac{1}{g_A^2}\;M_F^{0\nu} \tag{8.55}$$

is the nuclear matrix element. Here

$$M_F^{0\nu}=\langle\Psi_f|\sum_{n,m}H(r_{n,m},\overline{E})\;\tau_+^n\tau_+^m|\Psi_i\rangle \tag{8.56}$$

is the Fermi matrix element and

$$M_{GT}^{0\nu}=\langle\Psi_f|\sum_{n,m}H(r_{n,m},\overline{E})\;\tau_+^n\tau_+^m\;\sigma^n\cdot\sigma^m)|\Psi_i\rangle \tag{8.57}$$

is the Gamov-Teller matrix element. In (8.56) and (8.57) $|\Psi_{i,f}\rangle$ are wave function of the initial and final nuclei.

From (8.53) we conclude that matrix element of $0\nu\beta\beta$-decay is a product of the effective Majorana mass $m_{\beta\beta}$, the electron matrix element and the nuclear matrix element which includes neutrino propagator (neutrino potential). Taking into account that $\overline{E}-\frac{1}{2}(M_i+M_f)$ is much smaller than $\bar{q}$ for the neutrino propagator we obtain the following approximate relation

$$H(r)\simeq\frac{2R}{\pi}\int_0^\infty\frac{\sin qr}{qr}dq=\frac{R}{r}\;. \tag{8.58}$$

Using the standard rules, from (8.53) we can easily obtain the decay rate of the $0\nu\beta\beta$-decay. The electron part of the decay probability is given by the trace

$$\mathrm{Tr}(1+\gamma_5)(\gamma \cdot p_2 - m_e)(1-\gamma_5)(\gamma \cdot p_1 + m_e) = 8 p_1 p_2 \,. \tag{8.59}$$

Taking into account the final state electromagnetic interaction of the electrons and nucleus for the decay rate of the $0\nu\beta\beta$-decay we find the following expression

$$\begin{aligned} d\Gamma^{0\nu} = |m_{\beta\beta}|^2\, |M^{0\nu}|^2\, \frac{1}{(2\pi)^5}\, G_F^4\, \frac{1}{R^2}\, g_A^4 (E_1 E_2 - p_1 p_2 \cos\theta) \times \\ F(E_1, (Z+2))\; F(E_2, (Z+2))\; p_1 p_2\; \sin\theta d\theta\, dE_2, \end{aligned} \tag{8.60}$$

where $E_{1,2} \equiv p^0_{1,2}$ is electron total energy ($E_2 = M_i - M_f - E_1$), θ is the angle between electron momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and

$$F(Z) \simeq \frac{2\pi\eta}{1 - e^{-2\pi\eta}}, \tag{8.61}$$

is the Fermi function ($\eta = Z\alpha\, \frac{m_e}{p}$).

From (8.60) follows that for the ultra relativistic electrons θ-dependence of the decay rate is given by the factor $(1 - \cos\theta)$. Thus, ultra relativistic electrons cannot be emitted in the same direction. This is connected with the fact that the helicity of the high energy electrons, produced in the weak interaction, is equal to -1. If electrons are emitted in the same direction, the projection of their total angular momentum on the direction of the momentum is equal to -1. It is obvious that such electrons cannot be produced in $O^+ \to O^+$ transition.

From expression (8.60) for the total decay rate we obtain the following expression

$$\Gamma^{0\nu} = \frac{1}{T^{0\nu}_{1/2}} = |m_{\beta\beta}|^2\, |M^{0\nu}|^2\, G^{0\nu}(Q, Z), \tag{8.62}$$

where[5]

$$\begin{aligned} G^{0\nu}(Q,Z) = \frac{1}{2(2\pi)^5}\, G_F^4\, \frac{1}{R^2}\, g_A^4 \int_0^Q dT_1 \int_0^\pi \sin\theta d\theta\, (E_1 E_2 - p_1 p_2 \cos\theta) p_1 p_2 \times \\ F(E_1, (Z+2))\; F(E_2, (Z+2)). \end{aligned} \tag{8.63}$$

Here $T_1 = E_1 - m_e$, $Q = M_i - M_f - 2m_e$ is the total released kinetic energy and $T^{0\nu}_{1/2}$ is the half-life of the $0\nu\beta\beta$-decay. In Table 8.2 we present numerical values of $G^{0\nu}(Q, Z)$ for some nuclei . The total rate of the $0\nu\beta\beta$-decay is the product of three factors:

[5] An additional factor $\frac{1}{2}$ is due to the fact that in the final state we have two identical electrons.

Table 8.2 The values of the factor $G^{0\nu}(Q, Z)$ for some nuclei

Nucleus	$G^{0\nu}(Q, Z)$ in units $10^{-25}\mathrm{y}^{-1}\mathrm{eV}^{-2}$
^{76}Ge	0.30
^{100}Mo	2.19
^{130}Te	2.12
^{136}Xe	2.26

1. The modulus squared of the effective Majorana mass.
2. Square of nuclear matrix element.
3. The known factor $G^{0\nu}(Q, Z)$.

The effective Majorana mass is given by the relation $m_{\beta\beta} = \sum_i U_{ei}^2\, m_i$. An information about the neutrino mixing angles θ_{ik} and neutrino mass-squared differences Δm_{ik}^2 was obtained from the data of the neutrino oscillation experiments. Taking into account these data, we will consider now possible values of the effective Majorana mass.

8.3 Effective Majorana Mass

From neutrino oscillation data follows that one mass-squared difference (solar) is much smaller than the other one (atmospheric). For three massive neutrinos two types of neutrino mass spectra are possible in this case.

1. Normal spectrum

$$m_1 < m_2 < m_3; \quad \Delta m_{12}^2 \ll \Delta m_{23}^2 \tag{8.64}$$

2. Inverted spectrum[6]

$$m_3 < m_1 < m_2; \quad \Delta m_{12}^2 \ll |\Delta m_{13}^2| \tag{8.65}$$

In the case of the normal spectrum the neutrino masses $m_{2,3}$ are connected with the lightest mass m_1 and two neutrino mass-squared differences Δm_{12}^2 and Δm_{23}^2 by the following relations

[6] In order to have the same notation Δm_{12}^2 for the solar-KamLAND neutrino mass-squared difference and to determine this quantity as a positive one the neutrino masses are usually labeled differently in the cases of the normal and inverted neutrino mass spectra. In the case of the normal spectrum $\Delta m_{23}^2 > 0$ and in the case of the inverted spectrum $\Delta m_{13}^2 < 0$. Thus, with such a notation the character of the neutrino mass spectrum is determined by the sign of the larger (atmospheric) neutrino mass-squared difference. It clear, however, that the sign of the atmospheric mass-squared difference has no physical meaning: it is a convention based on the labeling of the neutrino masses and the way how the neutrino mass-squared difference is determined ($\Delta m_{ik}^2 = m_k^2 - m_i^2$). In both cases of the neutrino mass spectrum for the mixing angles the same notations can be used.

$$m_2 = \sqrt{m_1^2 + \Delta m_{12}^2}, \quad m_3 = \sqrt{m_1^2 + \Delta m_{12}^2 + \Delta m_{23}^2} \tag{8.66}$$

In the case of the inverted spectrum we have

$$m_1 = \sqrt{m_3^2 + |\Delta m_{13}^2|}, \quad m_2 = \sqrt{m_3^2 + |\Delta m_{13}^2| + \Delta m_{12}^2} \tag{8.67}$$

It is obvious that effective Majorana mass is determined not only by the lightest neutrino mass and neutrino mass-squared differences but also by the character of the neutrino mass spectrum.

Usually the following three typical neutrino mass spectra are considered[7]

1. Hierarchy of the neutrino masses

$$m_1 \ll m_2 \ll m_3. \tag{8.68}$$

2. Inverted hierarchy of the neutrino masses

$$m_3 \ll m_1 < m_2 \tag{8.69}$$

3. Quasi-degenerate neutrino mass spectrum

$$m_1 \simeq m_2 \simeq m_3, \quad m_1(m_3) \gg \sqrt{\Delta m_{23}^2}\ (\sqrt{|\Delta m_{13}^2|}). \tag{8.70}$$

We will discuss now the possible values of the effective Majorana mass in the case of these three neutrino mass spectra.

I. Hierarchy of the neutrino masses

In this case we have

$$m_1 \ll \sqrt{\Delta m_{12}^2}, \quad m_2 \simeq \sqrt{\Delta m_{12}^2}, \quad m_3 \simeq \sqrt{\Delta m_{23}^2}. \tag{8.71}$$

Thus, in the case of neutrino mass hierarchy the neutrino masses m_2 and m_3 are determined by the neutrino mass-squared differences Δm_{12}^2 and Δm_{23}^2, correspondingly, and the lightest mass is very small. Neglecting the contribution of m_1 to the effective Majorana mass and using the standard parametrization of the neutrino mixing matrix we find

---

[7] Let us notice that these three neutrino mass spectra correspond to different mechanisms of neutrino mass generation. Masses of quarks and charged leptons satisfy hierarchy of the type (8.68). Hierarchy of neutrino masses is a typical feature of GUT models (like $SO(10)$) in which quarks and leptons are unified. Inverted spectrum and quasi-degenerate spectrum require specific symmetries of the neutrino mass matrix.

$$|m_{\beta\beta}| \simeq \left| \sin^2\theta_{12}\sqrt{\Delta m^2_{12}} + e^{2i\,\alpha}\sin^2\theta_{13}\sqrt{\Delta m^2_{23}} \right| . \tag{8.72}$$

Here α is (unknown) Majorana phase difference.

The first term in Eq.(8.72) is small because of the smallness of Δm^2_{12}. The contribution of the "large" Δm^2_{23} to $|m_{\beta\beta}|$ is suppressed by the small factor $\sin^2\theta_{13}$. Using the values (6.123) and the CHOOZ bound (6.136), we have

$$\sin^2\theta_{12}\sqrt{\Delta m^2_{12}} \simeq 2.8\cdot 10^{-3}\ \mathrm{eV}, \quad \sin^2\theta_{13}\sqrt{\Delta m^2_{23}} \lesssim 2.5\cdot 10^{-3}\mathrm{eV}. \tag{8.73}$$

Thus, if the value of the parameter $\sin^2\theta_{13}$ is close to the CHOOZ bound, the first term and the modulus of the second term of (8.72) are approximately equal and at $\alpha \simeq \pi/2$ the terms in the expression (8.72) practically cancel each other. In this case the Majorana mass $|m_{\beta\beta}|$ will be close to zero.

Even without this possible cancelation the effective Majorana mass in the case of the neutrino mass hierarchy is very small. In fact, from (8.72) and (8.73) we have the following upper bound

$$|m_{\beta\beta}| \le \left(\sin^2\theta_{12}\sqrt{\Delta m^2_{12}} + \sin^2\theta_{13}\sqrt{\Delta m^2_{23}}\right) \lesssim 5.3\cdot 10^{-3}\ \mathrm{eV}. \tag{8.74}$$

This bound is significantly smaller that the expected sensitivity of the future experiments on the search for $0\nu\beta\beta$-decay (see later).

II. Inverted hierarchy of the neutrino masses

For the neutrino masses we have in this case

$$m_3 \ll \sqrt{|\Delta m^2_{13}|},\ \ m_1 \simeq \sqrt{|\Delta m^2_{13}|},\ \ m_2 \simeq \sqrt{|\Delta m^2_{13}|}\left(1+\frac{\Delta m^2_{12}}{2\,|\Delta m^2_{13}|}\right). \tag{8.75}$$

In the expression for the effective Majorana mass $|m_{\beta\beta}|$ the lightest mass m_3 is multiplied by the small parameter $\sin^2\theta_{13}$. Neglecting the contribution of this term and also neglecting the small term $\frac{\Delta m^2_{12}}{2\,|\Delta m^2_{13}|}$ in (8.75) we find

$$|m_{\beta\beta}| \simeq \sqrt{|\Delta m^2_{13}|}\ (1-\sin^2 2\,\theta_{12}\ \sin^2\alpha)^{\frac{1}{2}}, \tag{8.76}$$

where α is the difference of the Majorana phases of the elements U_{e2} and U_{e1}. The phase difference α is the only unknown parameter in the expression for $|m_{\beta\beta}|$ in the case of the inverted hierarchy. From (8.76) we find

$$\cos 2\,\theta_{12}\sqrt{|\Delta m^2_{13}|} \le |m_{\beta\beta}| \le \sqrt{|\Delta m^2_{13}|}, \tag{8.77}$$

where the upper bound corresponds to $\alpha = 0, \pi$ and the lower bound corresponds to $\alpha = \pm\frac{\pi}{2}$. From (6.123) we find the following range of the possible values of the effective Majorana mass

$$1.8 \cdot 10^{-2} \leq |m_{\beta\beta}| \leq 4.9 \cdot 10^{-2} \text{ eV} \tag{8.78}$$

It is important that in the case of the inverted hierarchy of the neutrino masses the lower bound of the effective Majorana mass is different from zero.

The anticipated sensitivities to the effective Majorana mass of the next generation of the experiments on the search for the $0\nu\beta\beta$-decay are in the range (8.78) (see below). Thus, the future $0\nu\beta\beta$-decay experiments will probe the inverted hierarchy of the neutrino masses.

III. Quasi-degenerate neutrino mass spectrum

Neglecting the small contribution of $\sin^2\theta_{13}$, for the effective Majorana mass we obtain in the case of the quasi-degenerate neutrino mass spectrum the following expression

$$|m_{\beta\beta}| \simeq m_{\min}\,(1 - \sin^2 2\,\theta_{12}\,\sin^2\alpha)^{\frac{1}{2}}, \tag{8.79}$$

where $m_{\min}$ is the lightest neutrino mass and α is the Majorana phase difference. Thus, $|m_{\beta\beta}|$ depends in this case on two unknown parameters: $m_{\min}$ and α. From (8.79) we obtain the following range for the effective Majorana mass:

$$\cos 2\,\theta_{12}\,m_{\min} \leq |m_{\beta\beta}| \leq m_{\min}. \tag{8.80}$$

If $0\nu\beta\beta$-decay will be observed and the effective Majorana turn out to be relatively large ($|m_{\beta\beta}| \gg \sqrt{\Delta m^2_{23}|}$) it would be an evidence that neutrinos are Majorana particles and the spectrum of their mass is quasi-degenerate. In this case we could conclude that

$$|m_{\beta\beta}| \leq m_{\min} \leq 2.8\,|m_{\beta\beta}| \tag{8.81}$$

An information about the lightest neutrino mass can be obtained from experiments on the measurement of the end-point part of the β-spectrum of tritium. From existing data of the Mainz and Troitsk tritium experiments it was found the upper bound

$$m_{\min} < 2.2 \text{ eV}. \tag{8.82}$$

The expected sensitivity of the future KATRIN experiment is equal to

$$m_{\min} \simeq 0.2 \text{ eV} \tag{8.83}$$

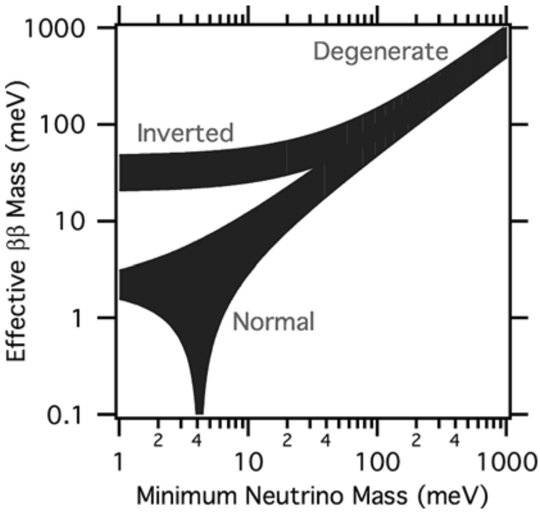

Fig. 8.2 Effective Majorana mass for the normal and inverted neutrino mass spectra as a function of $m_{\min}$

We have considered three neutrino mass spectra with special values of the lightest neutrino mass $m_{\min}$. In Fig. 8.2 the effective Majorana mass for the normal and inverted neutrino mass spectra as a function of $m_{\min}$ is presented. Uncertainties of the parameters Δm^2_{12}, Δm^2_{23} and $\tan^2\theta_{12}$ and possible values of the Majorana phase difference α are taken into account in Fig. 8.2. In conclusion let us notice that if in the KATRIN (or other) experiments the neutrino mass will be measured and in the $0\nu\beta\beta$-decay experiments, sensitive to the effective Majorana mass in the range (8.80), a positive signal will not be observed it would be an evidence that neutrinos with definite masses are Dirac particles.

8.4 On the Nuclear Matrix Elements of the $0\nu\beta\beta$-Decay

Effective Majorana mass $|m_{\beta\beta}|$ is not a directly measurable quantity. From the measurement of the half-life of the $0\nu\beta\beta$-decay only *the product of the effective Majorana mass and nuclear matrix element* can be obtained (see relation (8.62). In order to determine $m_{\beta\beta}$ we must know nuclear matrix elements (NME).

The calculation of NME is a complicated nuclear problem. Two different approaches are used for the calculation of NME: the Nuclear Shell Model (NSM) and the Quasiparticle Random Phase Approximation (QRPA). The latest QRPA calculations of NME of different nuclei are in an agreement with each other. However, NME calculated in the NSM are approximately two times smaller than the QRPA nuclear matrix elements. Further improvements of the NSM and QRPA approaches and new methods of the calculations are definitely needed.

Notice that if the $0\nu\beta\beta$-decays of *different nuclei* will be observed, from the ratios of the half-lifes the ratios of the corresponding NME can be obtained. The comparison of such ratios with the predictions of the models would allow to test models.

8.5 Data of Experiments on the Search for $0\nu\beta\beta$-Decay. Future Experiments

At present there exist data of many experiments on the search for $0\nu\beta\beta$-decay. The most stringent lower bounds on the half-life of $0\,\nu\beta\beta$-decay were obtained in the Heidelberg-Moscow and IGEX ^{76}Ge experiments and in the CUORICINO ^{130}Te experiment.

In the Heidelberg-Moscow experiment five crystals of 86 % enriched ^{76}Ge with a total mass 10.96 kg were used as a source (and detector). The released kinetic energy in the $0^+ \to 0^+$ transition $^{76}\mathrm{Ge} \to ^{76}\mathrm{Se} + e^- + e^-$ is equal to $Q_{\beta\beta} = 2,039.06 \pm 0.05$ KeV. For the half-life of ^{76}Ge the following lower bound was obtained in the experiment[8]

$$T^{0\,\nu}_{1/2}(^{76}\mathrm{Ge}) \geq 1.9 \cdot 10^{25}\mathrm{y} \;\; (90\%\ \mathrm{CL}) \tag{8.84}$$

Taking into account uncertainties in NME calculations, the following bounds were obtained from this result for the effective Majorana mass

$$|m_{\beta\beta}| \leq (0.3 - 1.2)\;\; \mathrm{eV}. \tag{8.85}$$

In the IGEX experiment the following lower bound was found for the half-life of the $0\nu\beta\beta$-decay of ^{76}Ge

$$T^{0\,\nu}_{1/2}(^{76}\mathrm{Ge}) \geq 1.6 \cdot 10^{25}\;\mathrm{y} \;\; (90\%\ \mathrm{CL}) \tag{8.86}$$

From this data was obtained

$$|m_{\beta\beta}| \leq (0.3 - 1.3)\;\; \mathrm{eV}. \tag{8.87}$$

In the cryogenic experiment CUORICINO the search for $0\nu\beta\beta$ decay of natural ^{130}Te was performed. An array of 62 TeO_2 crystals with a total mass of 40.7 kg was placed in a cryostat at the temperature $T = 8$ mK. Since the heat capacity is proportional to T^3, an increase of the temperature due to a tiny energy release can be recorded in the experiment. ^{130}Te has a large natural abundance (33.8%) and a relatively large $Q_{\beta\beta}$-value ($Q_{\beta\beta} = 2,528.8 \pm 1.3$ keV). No signal in the region of the $0\nu\beta\beta$ decay of ^{130}Te was found in the experiment. For the half-life the following lower bound was obtained

$$T^{0\,\nu}_{1/2}(^{130}\mathrm{Te}) \geq 3.0 \cdot 10^{24}\;\mathrm{y} \;\; (90\%\ \mathrm{CL}) \tag{8.88}$$

---

[8] Some participants of the Heidelberg-Moscow collaboration claim that they obtained an evidence for the $0\nu\beta\beta$-decay of ^{76}Ge with the following $3\,\sigma$ range for the half-life of ^{76}Ge: $0.69 \cdot 10^{25} \leq T^{0\,\nu}_{1/2}(^{76}\mathrm{Ge}) \leq 4.18 \cdot 10^{25}$ y. This claim is going to be checked by the GERDA collaboration.

For the upper bound of the effective Majorana mass the following range of values was inferred

$$|m_{\beta\beta}| \leq (0.19 - 0.68)\ \text{eV}. \tag{8.89}$$

Several future experiments on the search for $0\,\nu\beta\,\beta$-decay are in preparation at present. Detectors in these experiments will be much larger than in today's experiments (about hundreds of kg and even $\sim$1 ton). All groups plan to significantly decrease the background and to improve the energy resolution. The aim of future experiments is to reach a sensitivity of

$$|m_{\beta\beta}| \simeq \text{a few } 10^{-2}\ \text{eV}, \tag{8.90}$$

which will allow to test the Majorana nature of neutrinos in the case of the inverse hierarchy of the neutrino masses. The future CUORE experiment will use the same cryogenic technique as the CUORICINO experiment. An array of 988 TeO_2 crystals with a total mass of 741 kg of natural Te will be used. Significant reduction of the background and improvement of the energy resolution are planned to be reached.

The new ^{76}Ge experiment GERDA started at the Gran Sasso Laboratory with 17.9 kg enriched Ge source (detector). At the next step an additional 22 kg of enriched Ge will be added. At this stage the existing claim of the observation of the $0\nu\beta\beta$-decay will be checked. Later, a large (about 1 ton) GERDA-Majorana experiment is envisaged which will allow to test the Majorana nature of massive neutrinos in the case of the inverted mass hierarchy.

In the future EXO experiment, the $0\nu\beta\beta$-decay $^{136}\text{Xe} \to ^{136}\text{Ba}^{++} + e^- + e^-$ will be searched for. A new feature of the EXO experiment is a possibility to detect the daughter nuclei Ba^{++}. This ion will be neutralized to $^{136}\text{Ba}^+$ and localized. Then $^{136}\text{Ba}^+$ ion will be optically detected through the irradiation by photons from two lasers. It is expected that about 10^7 photons/s will be emitted by one ion. If this method of detection of $^{136}\text{Ba}^+$ will be realized it will drastically reduce the background. At the first stage of the experiment (without Ba tagging) $\sim$ 200 kg of 80% enriched ^{136}Xe will be used. After two years of running it is planned to reach $T^{0\,\nu}_{1/2}(^{136}\text{Xe}) > 6.4 \cdot 10^{25}$ y and $|m_{\beta\beta}| \lesssim 1.5 \cdot 10^{-1}$ eV.

The tracking detector NEMO is able to identify e^-, e^+, γ, α. Neutrinoless double β-decays of different nuclei can be searched for in the experiment. With 13.3 kg.y exposure for ^{100}Mo, $T^{0\,\nu}_{1/2}(^{100}\text{Mo}) > 5.8 \cdot 10^{23}$ y (90% CL) was found. From this result the bound $|m_{\beta\beta}| < (0.6 - 1.3)$ eV was obtained.

In the future Super-NEMO experiment, the search for the $0\nu\beta\beta$-decays of ^{82}Se or ^{150}Nd with $\sim$ 100 kg source is planned. It is expected to reach the sensitivity $T^{0\,\nu}_{1/2} \sim 2.0 \cdot 10^{26}$ y which corresponds to $|m_{\beta\beta}| \simeq (5 - 9)\ 10^{-2}$ eV.

Chapter 9
On absolute Values of Neutrino Masses

9.1 Masses of Muon and Tau Neutrinos

We have seen in the previous sections that neutrino oscillation experiments allow us to obtain the values only of the neutrino mass-squared differences. Information about the absolute value of the neutrino mass can be inferred from experiments which are based on the measurement of the neutrino mass via the kinematics of a decay.

The most precise upper bound on the "mass" of the muon neutrino was obtained from the measurement of the muon momentum in the decay

$$\pi^+ \to \mu^+ + \nu_\mu. \tag{9.1}$$

The state of the muon neutrino is the superposition of states of neutrinos with definite masses

$$|\nu_\mu\rangle = \sum_i U^*_{\mu i}|\nu_i\rangle. \tag{9.2}$$

From the energy-momentum conservation for the mass of ν_i, produced in the decay $\pi^+ \to \mu^+\nu_i$, we find the following expression

$$m_i^2 = m_\pi^2 + m_\mu^2 - 2m_\pi\sqrt{m_\mu^2 + (p^i_\mu)^2}. \tag{9.3}$$

Here m_π and m_μ are masses of the pion and muon and p^i_μ is the momentum of the muon (in the pion rest frame).

In the most precise PSI experiment for the muon momentum the value

$$p_\mu = (29.79200 \pm 0.00011)\ \text{MeV} \tag{9.4}$$

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was found. From the PSI data for the square of the neutrino mass was obtained[1]

$$m^2_{\nu_\mu} = (-0.016 \pm 0.023)\ \text{MeV}. \tag{9.5}$$

From this result for the upper bound of the muon mass was found

$$m_{\nu_\mu} < 190\ \text{keV}. \tag{9.6}$$

The upper bound for the mass of the tau neutrino was obtained from a study of the decays

$$\tau^- \to 2\pi^- + \pi^+ + \nu_\tau, \quad \tau^- \to 3\pi^- + 2\pi^+ + (\pi^0) + \nu_\tau \tag{9.7}$$

in the ALEPH experiment. From this experiment was found

$$m_{\nu_\tau} < 18.2\ \text{MeV}. \tag{9.8}$$

9.2 Neutrino Masses from the Measurement of the High-Energy Part of the β-Spectrum of Tritium

The most stringent upper bound on the absolute value of the neutrino mass was obtained from the detailed investigation of the high-energy part of the electron spectrum in the decay

$$^3\text{H} \to {}^3\text{He} + e^- + \bar{\nu}_e. \tag{9.9}$$

The effective Hamiltonian of the decay is given by the expression

$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}} 2\bar{e}_L \gamma_\alpha \nu_{eL}\ j^\alpha, \tag{9.10}$$

where j^α is the hadronic charged current and

$$\nu_{eL} = \sum_i U_{ei} \nu_{iL}. \tag{9.11}$$

For the electron spectrum in the decay (9.9) we obtain the following expression

---

[1] We have $p_\mu = \sum_i |U_{\mu i}|^2 p_i \simeq \frac{1}{2} m_\pi (1-r) - \frac{1+r}{1-r} \frac{m^2_{\nu_\mu}}{2m_\pi}$, where $r = \frac{m^2_\mu}{m^2_\pi}$. The effective mass-squared of ν_μ is given by the relation $m^2_{\nu_\mu} = \sum_i |U_{\mu i}|^2 m_i^2$.

$$\frac{d\Gamma}{dE} = \sum_i |U_{ei}|^2 \frac{d\Gamma_i}{dE}, \tag{9.12}$$

where

$$\frac{d\Gamma_i}{dE} = Cp(E + m_e)(E_0 - E)\sqrt{(E_0 - E)^2 - m_i^2}\, F(E)\, \theta(E_0 - E - m_i). \tag{9.13}$$

Here E is the kinetic energy of the electron, $E_0 \simeq 18.6$ keV is the energy released in the decay, and $F(E)$ is the Fermi function, which describes the Coulomb interaction of the final particles. The constant C is given by the expression

$$C = \frac{G_F^2 m_e^5}{2\pi^3} |M|^2, \tag{9.14}$$

where M is the (constant) nuclear matrix element.

Let us notice that the neutrino mass m_i enters in (9.13) through the neutrino momentum $p_i = \sqrt{(E_0 - E)^2 - m_i^2}$ and the step function $\theta(E_0 - E - m_i)$ provides the condition $E \leq E_0 - m_i$.

As is seen from (9.13), the largest distortion of the electron spectrum has to be observed in the region $(E^{\max} - m_i) \lesssim E \lesssim E^{\max}$, where $E^{\max} = E_0 - m_i$. However, if $m_i \simeq 1$ eV only about 10^{-11} decays of tritium give a contribution to this region. In order to increase the luminosity of the experiments a much larger part of the β-spectrum has to be used for the analysis of the effect of the neutrino mass. The best upper bounds on neutrino masses were obtained in the recent Mainz and Troitsk tritium experiments. In the Mainz experiment $\sim$70 eV of the end-point part of the spectrum was used. Taking into account that in the large part of the measured spectrum $(E_0 - E)^2 \gg m_i^2$ the electron spectrum can be presented in the form

$$\frac{d\Gamma}{dE} \simeq Cp(E + m_e)(E_0 - E)\sqrt{(E_0 - E)^2 - m_\beta^2}\, F(E), \tag{9.15}$$

where the effective mass m_β is given by the expression

$$m_\beta = \sqrt{\sum_i |U_{ei}|^2 m_i^2}. \tag{9.16}$$

In the Mainz experiment, frozen molecular tritium condensed on the graphite substrate was used as a tritium source. The electron spectrum was measured by an integral spectrometer with a retarding electrostatic filter. The resolution of the spectrometer was 4.8 eV. The data of the experiment were fitted with four free parameters: normalization C, background B, released energy E_0 and the effective neutrino mass-squared m_β^2. From the fit of the data $E_0 = 18.575$ eV was found.

From the combined analysis of all Mainz data, for the parameter m_β^2 was inferred

$$m_\beta^2 = (-1.2 \pm 2.2 \pm 2.1)\ \mathrm{eV}^2. \tag{9.17}$$

From this value the following upper bound for the neutrino mass was obtained

$$m_\beta < 2.2\ \mathrm{eV} \quad (95\%\ \mathrm{CL}). \tag{9.18}$$

In the Troitsk tritium experiment a gaseous molecular source was used. The electron spectrum was measured by an integral electrostatic spectrometer of the same type as in the Mainz experiment. The resolution of the spectrometer was (3.5–4) eV. A step function superimposed on the integral continuous spectrum was observed in the Troitsk experiment. This step function corresponds to a peak in the differential spectrum at a distance of a few eV from the end of the spectrum. The origin of this peak is unknown. Such an anomaly was not observed in the Mainz experiment.

In analysis of the Troitsk data six free parameters were used: C, B, E_0, m_β^2, the position of the step and its height.

From the analysis of the data, for the parameter m_β^2 was found

$$m_\beta^2 = (-2.3 \pm 2.5 \pm 2.0)\ \mathrm{eV}^2. \tag{9.19}$$

From (9.19) the following upper bound was obtained

$$m_\beta < 2.2\ \mathrm{eV} \quad (95\%\ \mathrm{CL}). \tag{9.20}$$

From the data of the Mainz and Troitsk experiments a bound on the lightest neutrino mass can be obtained. For example, in the case of the normal mass spectrum ($m_1 < m_2 < m_3$) we have

$$m_\beta^2 = m_1^2 + (1 - |U_{e1}|^2)\ \Delta m_{12}^2 + |U_{e3}|^2\ \Delta m_{23}^2. \tag{9.21}$$

From neutrino oscillation data follows that the sum of the last two terms in the right-hand side of (9.21) is smaller than $2 \cdot 10^{-4}\ \mathrm{eV}^2$. Thus, from the data of the Mainz and Troitsk experiments we have

$$m_1 < 2.2\ \mathrm{eV}.$$

Similarly, for the inverted neutrino mass spectrum ($m_3 < m_1 < m_2$) we obtain $m_3 < 2.2$ eV.

The experiment of the next generation on the measurement of the neutrino mass will be the Karlsruhe Tritium Neutrino Experiment (KATRIN). In this experiment two tritium sources will be used: a gaseous T_2 source and a frozen tritium source. The integral MAC-E-Filter spectrometer (Magnetic Adiabatic Collimator combined with an Electrostatic Filter) will have two parts: the pre-spectrometer, which will select electrons in the last $\sim$ 100 eV part of the spectrum, and the large main spectrometer. It has high luminosity, low background and high energy resolution (1 eV).

It is planned that after five years of running, a sensitivity to the effective neutrino mass of 0.2 eV at 95% CL will be reached (upper bound in the case if no positive signal will be found). The mass $m_\beta = 0.35$ eV can be measured with 5 σ significance and the $m_\beta = 0.30$ eV can be measured with 3 σ significance.

In conclusion we would like to mention that in addition to the measurement of the tritium β-spectrum there exist an alternative approach to the determination of the neutrino mass through the investigation of the β-decay. The groups in Genova and Milano developed the low-temperature cryogenic technique for the measurement of the β-decay of ^{187}Re. The energy release in the β-decay of this element is the lowest-known ($E_0 = 2.5$ keV). This means that the study of the β-decay of ^{187}Re is a very sensitive way for the investigation of the effects of the neutrino mass. The best upper bound that was reached in the MIBETA Rhenium experiment is $m_\beta < 15$ eV. A new cryogenic experiment MARE with a sensitivity to the neutrino mass of 2 eV (and in the second phase of 0.2 eV) is now under preparation.

Chapter 10
Neutrino Oscillation Experiments

10.1 Introduction

The long period of the discovery of neutrino oscillations started in 1970 with the pioneer Homestake solar neutrino radiochemical experiment by Davis et al. In this experiment, the observed rate of solar ν_e was found to be significantly smaller that the rate, predicted by the Standard Solar Model (SSM). This discrepancy was called *the solar neutrino problem (puzzle)*.

Even before the Homestake experiment started, B. Pontecorvo suggested that because of neutrino oscillations the observed flux of the solar neutrinos might be two times smaller than the predicted flux.[1] After the Davis results were obtained the idea of neutrino oscillations as a possible reason for the solar neutrino puzzle became more and more popular.

In the eighties, the second solar neutrino experiment Kamiokande was performed. In this direct-counting experiment a large water-Cherenkov detector was used. The solar neutrino rate measured by the Kamiokande experiment was also smaller than the rate predicted by the SSM.

In the Homestake and Kamiokande experiments high-energy solar neutrinos, produced mainly in the decay of ^{8}B, were detected. The flux of these neutrinos is much smaller than the total solar neutrino flux (about 10^{-4} of the total flux) and its predicted value strongly depends on the model.

In the nineties the new radiochemical solar neutrino experiments SAGE and GALLEX started. In these experiments neutrinos from all reactions of the pp and CNO cycles, including low-energy neutrinos from the reaction $pp \to de^{+}\nu_e$, were detected. This reaction gives the largest contribution to the flux of the solar neutrinos. The flux of the pp neutrinos can be predicted in a practically model independent way. The event rates measured in the SAGE and GALLEX experiments were approximately two times smaller than the predicted rates. Thus, in these experiments important evidence was obtained in favor of the disappearance of solar ν_e on the way from the central region of the sun, where solar neutrinos are produced, to the earth.

[1] Only two types of neutrinos were known at that time.

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Another indications in favor of neutrino oscillations were obtained in the nineties in the Kamiokande and IMB neutrino experiments in which atmospheric muon and electron neutrinos were detected. These neutrinos are produced in decays of pions and kaons, created in interactions of cosmic rays with nuclei in the atmosphere, and also in decays of muons, which are produced in the decays of pions and kaons. It was found in these experiments that the ratio of the numbers of ν_μ and ν_e events is significantly smaller than the (practically model independent) predicted ratio.

On the other side, no indications in favor of neutrino oscillations were found in the eighties and nineties in numerous reactor and accelerator short baseline experiments.

The situation with the search for neutrino oscillations drastically changed in 1998 when in the water-Cherenkov Super-Kamiokande experiment a significant up-down asymmetry of the high-energy atmospheric neutrino muon events was observed. It was discovered in this experiment that the number of up-going high-energy muon neutrinos, passing through the earth, is about two times smaller than the number of the down-going muon neutrinos coming directly from the atmosphere.

Indications in favor of a disappearance of solar ν_e became evidence in 2002 with the solar neutrino SNO experiment in which solar neutrinos were detected through the observation of CC and NC reactions. *A Model independent evidence of the disappearance of solar ν_e* was obtained in this experiment. It was shown that the flux of the solar ν_e is approximately three times smaller than the flux of ν_e, ν_μ and ν_τ.

In 2002 in the KamLAND reactor neutrino experiment was found that the number of reactor $\bar{\nu}_e$ events at the average distance of ~ 170 km from the reactors is about 0.6 of the number of the expected events. A significant distortion of the $\bar{\nu}_e$ spectrum was observed in the experiment in 2004.

Neutrino oscillations were observed also in the accelerator long-baseline K2K and MINOS experiments. These experiments perfectly confirm the results obtained in the atmospheric Super-Kamiokande experiment.

Two important reactor neutrino experiments CHOOZ and Palo Verde were performed in which neutrino oscillations were not found. In the CHOOZ experiment was shown that the element $|U_{e3}|^2$, which determines the size of such effects of the three-neutrino mixing as CP violation, is less than $\sim 5 \cdot 10^{-2}$.

For many years existed an additional indication in favor of neutrino oscillations which was obtained in the accelerator short-baseline LSND experiment. The explanation of the LSND result would require an existence of sterile neutrinos. The recent MiniBooNE experiment does not confirm LSND indication.

All these experiments complete the first period of the brilliant discovery of neutrino oscillations. It was proven that neutrinos have small masses and that the flavor neutrinos ν_e, ν_μ, ν_τ are "mixed particles". All observed data can perfectly be described if we assume three-neutrino mixing. The values of four neutrino oscillation parameters (two-mass squared differences and two mixing angles) were determined.

In this chapter we will briefly discuss the major neutrino oscillation experiments.

10.2 Solar Neutrino Experiments

10.2.1 Introduction

Solar electron neutrinos are produced in reactions of the thermonuclear *pp* and *CNO* cycles in which the energy of the sun is generated. The thermonuclear reactions are going on in the central, most hot region of the sun. In this region the temperature is about $15 \cdot 10^6$ K. At such a temperature the major contribution to the energy production is given by the *pp* cycle. The estimated contribution of the *CNO* cycle to the sun's energy production is about 1%.[2] In the beginning of the *pp* cycle, deuterium is produced in the reactions

$$p + p \to d + e^+ + \nu_e \quad \text{and} \quad p + e^- + p \to d + \nu_e. \tag{10.1}$$

The first *pp* reaction gives the dominant contribution to the deuterium production (99.71%). The contribution of the second *pep* reaction is 0.23%.

Deuterium and proton produce ^{3}He in the reaction

$$p + d \to {}^3\text{He} + \gamma. \tag{10.2}$$

The produced ^{3}He disappears due to three reactions

$$^3\text{He} + {}^3\text{He} \to {}^4\text{He} + p + p \quad (84.82\%). \tag{10.3}$$

$$^3\text{He} + p \to {}^4\text{He} + +e^+ + \nu_e \quad (\text{about } 10^{-5}\%) \tag{10.4}$$

$$^3\text{He} + {}^4\text{He} \to {}^7\text{Be} + \gamma \quad (15.08\%) \tag{10.5}$$

In the first two reactions ^{4}He is produced. Nuclei ^{7}Be take part in two chains of reactions terminated with the production of ^{4}He nuclei

$$^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e \quad {}^7\text{Li} + p \to {}^4\text{He} + {}^4\text{He}. \tag{10.6}$$

$$p + {}^7\text{Be} \to {}^8\text{B} + \gamma \quad {}^8\text{B} \to {}^8\text{Be}^* + e^+ + \nu_e \quad {}^8\text{Be}^* \to {}^4\text{He} + {}^4\text{He}. \tag{10.7}$$

Positrons, produced in different reactions, annihilate with electrons. Thus, the energy of the sun is produced in the transition[3]

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[2] In stars significantly heavier than the sun the central temperatures are higher and the *CNO* cycle gives the dominant contribution to the energy production.

[3] The *CNO* cycle is the following chain of reactions: $p + {}^{12}\text{C} \to {}^{13}\text{N} + \gamma$, ${}^{13}\text{N} \to {}^{13}\text{C} + e^+ + \nu_e$, $p + {}^{13}\text{C} \to {}^{14}\text{N} + \gamma$, $p + {}^{14}\text{N} \to {}^{15}\text{O} + \gamma$, ${}^{15}\text{O} \to {}^{15}\text{N} + e^+ + \nu_e$. There are two branches of reactions with nuclei ^{15}N, terminated with the production of ^{4}He: $p + {}^{15}\text{N} \to {}^{12}\text{C} + {}^4\text{He}$ or $p + {}^{15}\text{N} \to {}^{16}\text{O} + \gamma$, $p + {}^{16}\text{O} \to {}^{17}\text{F} + \gamma$, ${}^{17}\text{F} \to {}^{17}\text{O} + e^+ + \nu_e$, $p + {}^{17}\text{O} \to {}^{14}\text{N} + {}^4\text{He}$.

$$4p + 2e^- \to {}^4\mathrm{He} + 2\nu_e + Q, \tag{10.8}$$

where

$$Q = 4m_p + 2m_e - m_{^4\mathrm{He}} \simeq 26.73 \text{ MeV} \tag{10.9}$$

is the energy per production of a ^{4}He nucleus.

From (10.8) follows that the production of solar energy equal to $\frac{1}{2}Q \simeq 13.36$ MeV is accompanied by the emission of one neutrino. The energy, produced by the sun, is emitted in the form of photons (about 98%) and neutrinos (about 2%). Let us consider neutrinos with an energy E. The emission of such neutrinos is accompanied by the production of luminous energy equal to $\frac{1}{2}Q - E$. If $\phi_r(E)$ is the flux on the earth of neutrinos from the source r ($r = pp, {}^7\mathrm{Be}, {}^8\mathrm{B}, \ldots$) we have the following relation

$$\int \sum_r \left(\frac{1}{2}Q - E\right) \phi_r(E)\, dE = \frac{\mathcal{L}_\odot}{4\pi R^2}, \tag{10.10}$$

where $\mathcal{L}_\odot$ is the luminosity of the sun and R is the sun-earth distance. The relation (10.10) is called *luminosity relation*. It is a general constraint on the fluxes of solar neutrinos. The luminosity relation is based on the following assumptions

1. The solar energy is of thermonuclear origin.
2. The sun is in a stationary state.

The last assumption is connected with the fact that neutrinos observed in a detector were produced about 8 min before the detection. On the other side it takes about 10^5 years for photons produced in the central region of the sun to reach the surface of the sun.

We can rewrite the luminosity relation in the form

$$\sum_r \left(\frac{Q}{2} - \overline{E}_r\right) \Phi_r = \frac{\mathcal{L}_\odot}{4\pi R^2}. \tag{10.11}$$

Here

$$\overline{E}_r = \frac{1}{\Phi_r} \int E\, \phi_r(E)\, dE \tag{10.12}$$

is the average neutrino energy from the source r and $\Phi_r = \int \phi_r(E)\, dE$ is the total flux of neutrinos from the source r. For the calculation of neutrino fluxes it is necessary to use a solar model. Usually the results of the calculations in the framework

Table 10.1 Solar reactions, in which neutrinos are produced, and neutrino fluxes predicted by the SSM

Abbreviation	Reaction	SSM flux ($cm^{-2}\ s^{-1}$)	Neutrino energy (MeV)
pp	$p + p \to d + e^+ + \nu_e$	$6.05 \cdot 10^{10}$	≤ 0.42
pep	$p + e^- + p \to d + \nu_e$	$1.45 \cdot 10^{8}$	1.44
^{7}Be	$e^- + {}^7\mathrm{Be} \to {}^7\mathrm{Li} + \nu_e$	$4.38 \cdot 10^{9}$	0.86
^{8}B	${}^8\mathrm{B} \to {}^8\mathrm{Be}^* + e^+ + \nu_e$	$4.59 \cdot 10^{6}$	$\lesssim 15$
hep	${}^3\mathrm{He} + p \to {}^4\mathrm{He} + e^+ + \nu_e$	$8.23 \cdot 10^{3}$	≤ 18.8
^{13}N	${}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \nu_e$	$2.03 \cdot 10^{8}$	≤ 1.20
^{15}O	${}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \nu_e$	$1.47 \cdot 10^{8}$	≤ 1.73
^{17}F	${}^{17}\mathrm{F} \to {}^{17}\mathrm{O} + e^+ + \nu_e$	$3.31 \cdot 10^{6}$	≤ 1.74

of the so-called Standard Solar Model (SSM) are used.[4] In Table 10.1 we present the predicted by SSM fluxes of ν_e from different reactions. In this table we included also predicted fluxes from three reactions of the CNO cycle: ${}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \nu_e$, ${}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \nu_e$ and ${}^{17}\mathrm{F} \to {}^{17}\mathrm{O} + e^+ + \nu_e$. In the last column of Table 10.1 neutrino energies from different reactions are given.

As we will see later, the main contributions to the event rates measured in modern solar neutrino experiments give

1. low energy pp neutrinos,
2. medium energy monochromatic ^{7}Be neutrino,
3. high energy ^{8}B neutrinos.

It is evident from Table 10.1 that the second term of the luminosity relation (10.11) is much smaller than the first one. If we neglect this term, we find the following expression for the total flux of neutrinos

$$\Phi = \sum_r \Phi_r \simeq \frac{\mathcal{L}_\odot}{2\pi R^2 Q}. \tag{10.13}$$

[4] The Standard Solar Model is based on the assumption that the sun is a spherically symmetric plasma sphere in hydrostatic equilibrium. The effects of rotation and of the magnetic field are neglected.

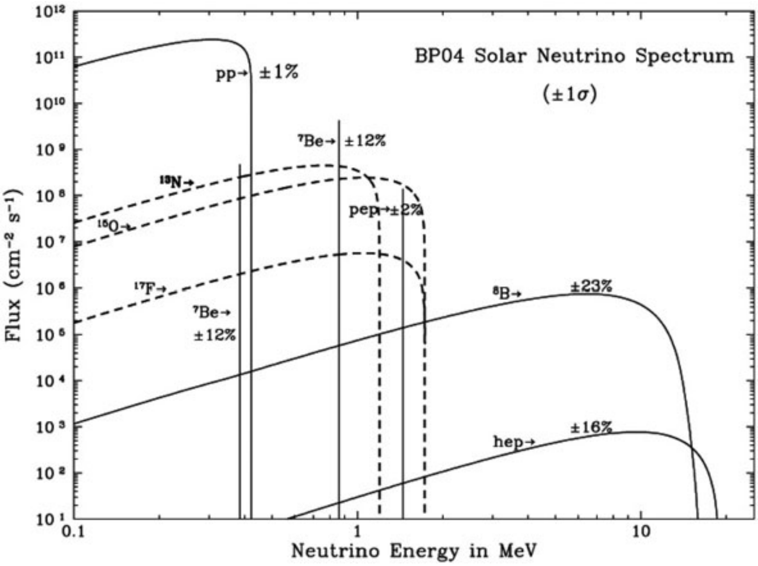

Fig. 10.1 Spectra of solar neutrinos from different reactions predicted by the Standard Solar Model

Taking into account that $\mathcal{L}_{\odot} = 2.40 \cdot 10^{39}\,\text{MeV}\,\text{s}^{-1}$ and $R = 1.496 \cdot 10^{13}\,\text{cm}$ we find

$$\Phi \simeq 6.4 \cdot 10^{10}\,\text{cm}^{-2}\,\text{s}^{-1}\,. \tag{10.14}$$

In Fig. 10.1 spectra of neutrinos from different reactions, predicted by the SSM, are presented.

10.2.2 Homestake Chlorine Solar Neutrino Experiment

The first experiment, in which solar electron neutrinos were detected, was the Homestake experiment by R. Davis et al.[5] The experiment continued from 1968 till 1994. In the Davis experiment radiochemical chlorine-argon method, proposed by B. Pontecorvo in 1946, was used. Solar electron neutrinos were detected through the observation of the reaction

$$\nu_e + {}^{37}\text{Cl} \to e^- + {}^{37}\text{Ar}\,. \tag{10.15}$$

The threshold of this process is equal to 0.814 Mev. The ^{37}Ar atoms are radioactive. They decay via electron-capture with emission of Auger electrons. The half-life of the decay is 34.8 days.

[5] For this experiment R. Davis was awarded with the Nobel Prize in 2002.

A tank filled with 615 tons of liquid perchloroethylene (C_2Cl_4) was used as an detector in the Davis experiment. In order to decrease the cosmic ray background, the experiment was performed in the Homestake mine (USA) at depth of about 1,480 m (4,100 m of water equivalent).

The radioactive ^{37}Ar atoms, produced by solar ν_e via the reaction (10.15) during the exposure time (about two months), were extracted from the tank by purging with ^{4}He gas. The gas with radioactive ^{37}Ar atoms was put in a low-background proportional counter in which the signal from Auger electrons was detected. An important feature of the experiment was the measurement of the rise time of the signal. This allowed to suppress the background. About 16 atoms of ^{37}Ar were extracted during one exposure run.

The energy threshold of the Cl-Ar reaction is larger than the upper bound for energies of pp neutrinos, constituting the major part of the solar neutrinos (see Table 10.1). At high ^{8}B energies, the transition to an excited state of ^{37}Ar significantly increase the cross section of the process (10.15). As a result, the main contribution to the counting rate give the high energy ^{8}B neutrinos. According to the SSM the contribution of the ^{8}B neutrinos to the event rate is equal to 5.76 SNU.[6] The predicted contribution of ^{7}Be neutrinos is equal to 1.15 SNU. Other much smaller contributions to the event rate come from pep and CNO neutrinos.

The event rate measured in the Homestake experiment, averaged over 108 runs between 1970 and 1994, is equal to

$$R_{\rm Cl} = (2.56 \pm 0.16 \pm 0.16)\ \ \text{SNU} \tag{10.16}$$

The measured event rate is significantly smaller than the rate predicted by the SSM (assuming that there are no neutrino oscillations):

$$R_{\rm SSM} = 8.1 \pm 1.3\ \ \text{SNU} \tag{10.17}$$

10.2.3 Radiochemical GALLEX-GNO and SAGE Experiments

Neutrinos from all solar neutrino reactions including low-energy neutrinos from pp reaction were detected in the radiochemical gallium GALLEX-GNO and SAGE experiments. In these experiments neutrinos were detected by the radiochemical method through the observation of the reaction

$$\nu_e + {}^{71}\text{Ga} \to e^- + {}^{71}\text{Ge}, \tag{10.18}$$

in which radioactive ^{71}Ge was produced. The threshold of this reaction is equal to 0.233 MeV. The half-life of ^{71}Ge is equal to 11.43 days.

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[6] One solar neutrino unit (SNU) is determined as follows: 1 SNU $= 10^{-36}$events atom^{-1} s^{-1}.

The detector in the GNO-GALLEX experiment was a tank containing 100 tons of a water solution of gallium chloride (30.3 tons of ^{71}Ga). The experiment was done in the underground Gran Sasso Laboratory (Italy). During 1991–2003 there were 123 GALLEX and GNO exposure runs. The duration of one run was about 4 weeks. About 10 atoms of ^{71}Ge were produced during one run. Radioactive ^{71}Ge atoms were extracted from the detector by a chemical procedure and introduced to a small proportional counter in which Auger electrons, produced in the capture $e^- + {}^{71}\mathrm{Ge} \to {}^{71}\mathrm{Ga} + \nu_e$, were detected.

The measured event rate averaged over 123 runs is equal to

$$R_{\mathrm{Ga}} = (67.5 \pm 5.1)\ \mathrm{SNU}. \quad (\mathrm{GALLEX} - \mathrm{GNO}) \tag{10.19}$$

The event rate predicted by the SSM

$$(R_{\mathrm{Ga}})_{\mathrm{SSM}} = (128\ {}^{+9}_{-7})\ \mathrm{SNU}. \tag{10.20}$$

is about two times larger than the measured rate.

The major contribution to the predicted event rate comes from the pp neutrinos (69.7 SNU). Contributions of ^{7}Be and ^{8}B neutrinos to the predicted event rate are equal to 34.2 SNU and 12.1 SNU, respectively. Combining the GALLEX-GNO results with the results of SNO and BOREXINO experiments from which the fluxes of the ^{8}B and ^{7}Be neutrinos can be obtained, it is possible to determine the average probability of the low-energy pp neutrinos to survive. It was found that $P_{pp}(\nu_e \to \nu_e) = 0.52 \pm 0.12$.

In another gallium experiment SAGE about 50 tons of ^{71}Ga in the form of liquid metal are used. The experiment is doing in the Baksan Neutrino Observatory (Caucasus mountains, Russia) in a hall with an overburden of 4,700 m water equivalent. Neutrinos are detected through the observation of the reaction (10.18). An exposure time in this experiment is about 4 weeks. The ^{71}Ge atoms, produced by the solar neutrinos, are chemically extracted from the target and are converted to GeH_4. Auger electrons, produced in decay of germanium, are detected in a tiny proportional counter.

The germanium production rate, measured in the SAGE experiment, averaged over 92 runs during 1990–2001 is equal to

$$R_{\mathrm{Ga}} = (70.8^{+5.3}_{-5.2}(\mathrm{stat})^{+3.7}_{-3.2}(\mathrm{syst}))\ \mathrm{SNU}. \quad (\mathrm{SAGE}) \tag{10.21}$$

As it is seen from (10.19) and (10.21), the rates measured in the SAGE and in the GALLEX-GNO experiments are in a good agreement.

Both GALLEX and SAGE Collaborations performed tests (source) experiments in which intense ^{51}Cr neutrino sources (in two GALLEX and one SAGE experiments), a ^{37}Ar neutrino source (in the SAGE experiment) and ^{71}As source (in the GALLEX experiment) were used. They found agreement between observed and predicted ^{71}Ge production rates. This agreement is an additional confirmation of the reliability of the experiments.

10.2.4 Kamiokande and Super-Kamiokande Solar Neutrino Experiments

In radiochemical experiments the neutrino direction cannot be determined. The first experiment in which the neutrino direction was measured was Kamiokande. It was proved that the detected neutrinos were coming from the sun.

In the Kamiokande experiment a 2,140 ton water-Cherenkov detector was used. The experiment was done in the Kamioka mine (Japan) at a depth of about 1,000 m (2,700 m water equivalent).

In the Kamiokande experiment the solar neutrinos were detected through the observation of recoil electrons from the elastic neutrino-electron scattering

$$\nu_x + e \to \nu_x + e. \quad (x = e, \mu, \tau) \tag{10.22}$$

All types of flavor neutrinos are observed in the experiment. However, because $\sigma(\nu_{\mu,\tau} e \to \nu_{\mu,\tau} e) \simeq 0.16\, \sigma(\nu_e e \to \nu_e e)$ the Kamiokande experiment allowed to determine mainly the flux of the solar ν_e's on the earth.

Electron neutrinos from the process (10.22) were detected via the observation of the Cherenkov radiation in water. 1,000 large (50 cm in diameter) photomultipliers. which covered about 20% of the surface of the detector, were utilized in the experiment. Because of the contamination of Rn in the water it was necessary to apply a 7.5 MeV energy threshold for the recoil electrons.

At high energies recoil electrons are emitted in a narrow (about 15°) cone around the initial neutrino direction. In the experiment a strong correlation between the direction of recoil electrons and the direction to the sun was observed This correlation was an important signature which allowed to suppress background. It was proved that the observed events were due to neutrinos from the sun.

Because of the high threshold only ^{8}B neutrinos could be detected in the Kamiokande experiment.[7] The total flux of high energy ^{8}B neutrinos obtained from the results of the Kamiokande experiment is equal to

$$\Phi_\nu^{\mathrm{K}} = (2.80 \pm 0.19 \pm 0.33) \cdot 10^6\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}. \tag{10.23}$$

The ratio of the measured solar neutrino flux to the flux predicted by the SSM under the assumption that there are no neutrino oscillations is equal to $R^{\mathrm{K}} = 0.51 \pm 0.04 \pm 0.06$.

The result of the Kamiokande experiment was an important confirmation of the existence of the solar neutrino problem, discovered in Davis et al. in the Homestake experiment.[8]

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[7] According to the SSM the contribution of *hep* neutrinos is negligible.

[8] In 1987 the Kamiokande Collaboration (and also the IMB and Baksan Collaborations) observed neutrinos from the explosion of the supernova SN1987A. This was the first observation of

The Kamiokande experiment was running during 9 years from 1987 till 1995. In 1996 the experiment of the next generation (Super-Kamiokande) was started. In this experiment a huge 50 kiloton water-Cherenkov detector (fiducial volume 22.5 kilotons) was used. There were three phases of the Super-Kamiokande experiment. The SK-I phase started in 1996 and finished in 2001. In this phase 11,000 photomultipliers were used. In 2001 an accident happened in which many of the photomultipliers were destroyed. After about a year, half of the photo-tubes were restored and SK-II started. In July 2006 after restoring the total number of photo-tubes SK-III started.

There were many improvements in the Super-Kamiokande experiment with respect to Kamiokande. An important one was the energy calibration at a 2% level in the whole energy range. During 1,496 days of SK-I the large number of $22,404 \pm 226^{+784}_{-717}$ neutrino-electron scattering events was observed. During the first 280 days the threshold was 6.5 MeV. For the remaining days the threshold was 5 MeV. The total flux of the solar ^{8}B neutrinos was determined in SK-I experiment with a $\sim 1\%$ statistical accuracy:

$$\Phi_{\nu}^{\text{SK-I}} = (2.35 \pm 0.02 \pm 0.08) \cdot 10^6 \, \text{cm}^{-2}\,\text{s}^{-1}. \tag{10.24}$$

No distortion of the spectrum of recoil electrons with respect to the expected spectrum was observed.[9] The high statistics of the events allowed the Super-Kamiokande Collaboration to determine the dependence of the number of events on the zenith angle and to measure a day-night asymmetry which could give an information on the earth matter effects. No significant zenith angle dependence was found. For the day-night asymmetry a value

$$A_{\text{D-N}} = -1.7\% \pm 1.6\% \pm 1.3\% \tag{10.25}$$

was obtained.

The SK-II phase of the Super-Kamiokande experiment started in December 2002 and finished in October 2005. There was a reduced number of photo-tubes (protected against possible blast) in this run. The energy threshold was 7 MeV. $7,212.8 \;{}^{+152.9}_{-150.9}(\text{stat}) \;{}^{+483.3}_{-461.6}(\text{syst})$ neutrino-electron events were observed. The total flux of the solar ^{8}B neutrinos determined in the SK-II experiment

$$\Phi_{\nu}^{\text{SK-II}} = (2.38 \pm 0.05 \pm 0.16) \cdot 10^6 \, \text{cm}^{-2}\,\text{s}^{-1} \tag{10.26}$$

is in a perfect agreement with the flux (10.24) measured in the SK-I experiment.

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supernova neutrinos. The experiment confirmed the general theory of the gravitational collapse (see Sect. 11.8).

[9] The initial ^{8}B solar neutrino spectrum is determined by the weak decay $^8\text{B} \to e^+ + \nu_e + 2\alpha$. This spectrum can be obtained from the measurement of the α-spectrum in a laboratory. The fact that the electron spectrum, measured in the Super-Kamiokande experiment, is in an agreement with the expected spectrum means that in the high-energy ^{8}B region the probability of the solar ν_e to survive is constant.

10.2.5 SNO Solar Neutrino Experiment

The fluxes of solar neutrinos, measured in the Homestake, GALLEX-GNO, SAGE, Kamiokande and Super-Kamiokande experiments, were significantly smaller than the fluxes, predicted by the Standard Solar model. From the analysis of the data of these experiments, strong indications in favor of neutrino transitions driven by neutrino masses and mixing were obtained.

The first model-independent evidence for transitions of solar ν_e into $\nu_{\mu,\tau}$ was obtained in the SNO solar neutrino experiment. The experiment was done in the Creighton mine (Sudbury, Canada) at a depth of 2,092 m (6,010 m water equivalent). The target in the SNO experiment was 1,000 tons of pure heavy water D_2O contained in a 12 m diameter acrylic vessel. Cherenkov light was detected by 9,456 photo-multipliers of 20 cm in diameter.

A new, crucial feature of the SNO experiment was the observation of solar neutrinos *via three different processes*.

1. The CC process

$$\nu_e + d \to e^- + p + p \,. \tag{10.27}$$

2. The NC process

$$\nu_x + d \to \nu_x + p + n \quad (x = e, \mu, \tau) \tag{10.28}$$

3. Elastic neutrino-electron scattering (ES)

$$\nu_x + e \to \nu_x + e \,. \tag{10.29}$$

The CC and ES processes were observed through the detection of the Cherenkov light produced by electrons in the heavy water. The NC process was observed via the detection of neutrons. There were three phases of the SNO experiment in which different methods of the detection of neutrons were used.

- D_2O phase (306 days). During this phase neutrons from the NC process (10.28) were detected through the observation of γ-quanta produced in the process $n + d \to t + \gamma$.
- Salt phase (391 days). In the heavy water about 2 tons of NaCl were dissolved. Neutrons were detected through the observation of γ-quanta from the capture $n + {}^{35}\mathrm{Cl} \to {}^{35}\mathrm{Cl} + \gamma$. For thermal neutrons the cross section of this process ($\simeq$44 b) is much higher than the cross section of the process $nd \to t\gamma$ ($\simeq$0.5 mb). Thus, the addition of the salt significantly enhanced the NC signal.
- ^{3}He proportional counter phase (385 days). During this phase neutrons were detected through the observation of the process $n + {}^3\mathrm{He} \to p + t$ in counters deployed in the heavy water detector.

The SNO Collaboration started to collect data in 1999. The last phase was finished in 2006. We will present results obtained during the salt and the ^{3}He proportional counter phases.

During the salt phase $(2,176 \pm 78)$ CC events, $(2,010 \pm 85)$ NC events and (279 ± 26) ES events were observed. The threshold for the detection of the electrons from the CC and the ES processes was equal to $T_{\rm thr} = 5.5$ MeV. The neutrino energy threshold for NC process is 2.2 MeV (the deuterium bounding energy). Thus, in the SNO experiments mostly high energy solar ^{8}B neutrinos can be detected.

The initial spectrum of ν_e from the ^{8}B decay is known. It was obtained from the measurement of α-spectrum from the ^{8}B decay. From the analysis of all existing solar neutrino data follows that in the high-energy region $\nu_e \to \nu_e$ survival probability is a constant. Thus, the CC and ES electron spectra can be predicted. The electron CC and ES spectra measured in the SNO experiment are in good agreement with this prediction.

Assuming the undisturbed ^{8}B neutrino spectrum from the observation of the CC events for the flux ν_e the following value was obtained in the SNO experiment

$$\Phi^{\rm CC}_{\nu_e} = (1.68 \pm 0.06^{+0.08}_{-0.09}) \cdot 10^6 \,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}. \tag{10.30}$$

Because of the $\nu_e - \nu_\mu - \nu_\tau$ universality of the NC neutrino-hadron interaction the observation of NC events allows to determine the total flux of all flavor neutrinos. In the SNO experiment was found that the total flux of all flavor neutrinos is equal to

$$\Phi^{\rm NC}_{\nu_{e,\mu,\tau}} = (4.94 \pm 0.21^{+0.38}_{-0.34}) \cdot 10^6 \,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,. \tag{10.31}$$

Finally, for the neutrino flux determined from the observation of the ES events it was found

$$\Phi^{\rm ES}_{\nu} = \Phi_{\nu_e} + 0.16\,\Phi_{\nu_{\mu,\tau}} = (2.35 \pm 0.22 \pm 0.15) \cdot 10^6 \,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}. \tag{10.32}$$

The SNO experiment provides the solution of the solar neutrino problem. If we compare the flux of ν_e with the total flux of ν_e, ν_μ and ν_τ, we come to the model independent conclusion that solar ν_e on the way from the sun to the earth are transformed into ν_μ and ν_τ. From an analysis of the data of the SNO and other solar neutrino experiments follows that these transitions are due to neutrino masses and mixing. For the ratio of the fluxes of ν_e and ν_e, ν_μ and ν_τ we have

$$\frac{\Phi^{\rm CC}_{\nu_e}}{\Phi^{\rm NC}_{\nu_{e,\mu,\tau}}} = 0.340 \pm 0.023^{+0.029}_{-0.031}. \tag{10.33}$$

The ratio of the ν_e flux and the ES flux is equal to

$$\frac{\Phi^{\rm CC}_{\nu_e}}{\Phi^{\rm ES}} = 0.712 \pm 0.075^{+0.045}_{-0.044}. \tag{10.34}$$

The value (10.31) of the total flux of all flavor neutrinos is in agreement with the flux of the ^{8}B neutrinos predicted by the Standard Solar Model (see Table10.1). Thus, no indications in favor of a transition of the solar neutrinos into sterile states were obtained in the SNO experiment.

In the SNO experiment the day-night asymmetries were also measured. Values compatible with zero were found. For the CC events was found

$$A^{\mathrm{CC}} = -0.015 \pm 0.058 \pm 0.027. \tag{10.35}$$

From the two-neutrino analysis of the SNO data the following values were obtained for the neutrino oscillation parameters

$$\Delta m^2_{12} = (5.0^{+6.2}_{-1.8}) \cdot 10^{-5}\ \mathrm{eV}^2, \quad tan^2\theta_{12} = (0.45^{+0.11}_{-0.10}). \tag{10.36}$$

From an analysis of the SNO data and the data of all other solar neutrino experiments it was found

$$\Delta m^2_{12} = (6.5^{+4.4}_{-2.3}) \cdot 10^{-5}\ \mathrm{eV}^2, tan^2\theta_{12} = (0.45^{+0.09}_{-0.08}). \tag{10.37}$$

Finally, from an analysis of the data of the solar neutrino experiments and the data of the KamLAND reactor experiment the following values were inferred for the neutrino oscillation parameters

$$\Delta m^2_{12} = (8.0^{+0.6}_{-0.4}) \cdot 10^{-5}\ \mathrm{eV}^2, \quad \tan^2\theta_{12} = (0.45^{+0.09}_{-0.07}). \tag{10.38}$$

In 2004, in the SNO heavy water detector, 36 strings of ^{3}He proportional counters were deployed. Neutrons from NC events are detected in the ^{3}He counter through the observation of the process $n + {}^3\mathrm{He} \to {}^3\mathrm{H} + p$ in which a kinetic energy of 764 keV is released. During the ^{3}He phase of the experiment (983^{+77}_{-76}) NC events, $(1,867^{+91}_{-101})$ CC events and (171^{+24}_{-22}) ES events were observed.

For the fluxes of ν_e, of $\nu_{e,\mu,\tau}$ and of ES neutrinos the following values were obtained

$$\Phi^{\mathrm{CC}}_{\nu_e} = (1.67^{+0.05+0.07}_{-0.04-0.08}) \cdot 10^6\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}, \tag{10.39}$$

$$\Phi^{\mathrm{NC}}_{\nu_{e,\mu,\tau}} = (5.54^{+0.33+0.36}_{-0.31-0.34}) \cdot 10^6\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}, \tag{10.40}$$

$$\Phi^{\mathrm{ES}} = (1.77^{+0.24+0.09}_{-0.21-0.10}) \cdot 10^6\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}. \tag{10.41}$$

The value (10.40) for the total flux of all active neutrinos, which was obtained via the detection of neutrons by the ^{3}He proportional counters, is in agreement with the values which were obtained by the other methods for the detection of neutrons.

From the two-neutrino analysis of all solar neutrino data, including the ^{3}He SNO and the Borexino data, and the data of the KamLAND experiment, the following values were obtained for the neutrino oscillation parameters

$$\Delta m^2_{12} = (7.94^{+0.42}_{-0.26}) \cdot 10^{-5}\ \text{eV}^2, \quad \theta_{12} = (33.8^{+1.4}_{-1.3})\ \text{degrees}. \tag{10.42}$$

10.2.6 Borexino Solar Neutrino Experiment

After many years of preparation in 2007 the new solar neutrino experiment Borexino started. In this experiment, monochromatic ^{7}Be neutrinos with energy 0.862 MeV are observed in real time.

The Borexino detector is a scintillator with a mass of 278 tons (fiducial mass 78.5 tons) contained in a thin nylon vessel. The solar neutrinos are observed through the detection of recoil electrons from the elastic neutrino-electron scattering

$$\nu_x + e \to \nu_x + e. \tag{10.43}$$

The scintillation light is detected by 2,212 photomultipliers uniformly distributed on the inner surface of the detector. The measurement of the scintillation light allows to determine the energy of the electrons. There is no information about the direction of the electrons. Because the energy threshold in the Borexino experiment must be low, the major requirement is an extremely low radioactive contamination of the scintillator.

The recoil electrons produced by monochromatic neutrinos have a characteristic spectrum with a Compton-like edge at 665 keV. This important feature allow the Borexino Collaboration to separate the signal from background from the decay of ^{85}Cr , ^{210}Bi , ^{11}C, etc. From results of the running of the experiment during 192 days for the interaction rate of the ^{7}Be neutrinos is found the value

$$R_{\text{Borexino}} = (49 \pm 3 \pm 4)\ \text{counts}/(\text{day } 100\ \text{ton}). \tag{10.44}$$

Assuming that the flux of the ^{7}Be neutrinos is given by the SSM ($\Phi(^7\text{Be}) = (5.08 \pm 0.25) \cdot 10^9\ \text{cm}^{-2}\ \text{s}^{-1}$) and that there are no neutrino transitions we find for the rate

$$R_{\text{SSM}} = (74 \pm 4)\ \text{counts}/(\text{day } 100\ \text{ton}). \tag{10.45}$$

Assuming the SSM ^{7}Be flux and using the values of the neutrino oscillation parameters Δm^2_{12} and $\tan^2\theta_{12}$ obtained from the global fit of the data of all solar neutrino experiments (except Borexino), the interaction rate was found to be in good agreement with (10.44):

$$R_{\text{glob.fit}} = (48 \pm 4)\ \text{counts}/(\text{day } 100\ \text{ton}). \tag{10.46}$$

Finally, for the ν_e survival probability at the ^{7}Be energy of 0.862 MeV the value $\text{P}(\nu_e \to \nu_e) = 0.56 \pm 0.10$ was found which is in good agreement with the ν_e survival probability $\text{P}(\nu_e \to \nu_e) = 0.541 \pm 0.17$ obtained from the global fit of the solar and KamLAND data.

10.3 Super-Kamiokande Atmospheric Neutrino Experiment

The water-Cerenkov Super-Kamiokande detector is a multi-purpose detector. In the previous subsection we discussed the Super-Kamiokande solar neutrino experiment. In this subsection we will consider the Super-Kamiokande experiment on the detection of atmospheric neutrinos.

In the Super-Kamiokande atmospheric neutrino experiment the first model independent evidence in favor of neutrino oscillations was obtained (1998). This finding opened a new era in the study of neutrino oscillations.

There were three stages in the Super-Kamiokande atmospheric neutrino experiment (as in the solar neutrino experiment). The SK-I run started in April 1996 and finished in November 2001 after the accident with photo-tubes happened. The SK-II run with only half of the photo-tubes operating continued for 800 days. In June 2006 after the total number of photo-tubes was restored, the SK-III run started.

The Super-Kamiokande detector consists of two optically separated water-Cherenkov cylindrical detectors with a total mass of 50 kilotons of water. The inner detector with 11,146 photo-tubes has a radius of 16.9 m and a height of 36.2 m. The outer detector is a veto detector. It allows to reject cosmic ray muons. The fiducial mass of the detector is 22.5 kilotons.

In the Super-Kamiokande atmospheric neutrino experiment neutrinos are detected in a wide range of energies from about 100 MeV to about 10 TeV. At smaller energies from 100 MeV to 10 GeV, neutrinos are detected through the observation of electrons and muons produced in the following process of the interaction of neutrinos and antineutrinos with the nuclei in the detector:

$$\nu_l(\bar{\nu}_l) + N \to l^-(l^+) + X. \quad (l = e, \mu) \tag{10.47}$$

There are two types of events of this category. If all energy is deposited in the inner detector such an event is called fully content event (FC). If a high energy muon escapes the inner detector and deposits its energy in the outer veto detector such an event is called a partially contained event (PC).

In the Super-Kamiokande detector high energy muons are observed which are produced in the processes of the interaction of muon neutrinos with the nuclei of the rock surrounding the detector. Down-going muons produced in such processes cannot be distinguished from cosmic ray muons. Up-going muons are of neutrino origin. There are two categories of such events. Upward stopping muons are those muons of neutrino origin which come to rest in the detector. Upward through-going muons are those muons which pass the whole detector.

FC events are produced by neutrinos with energies of a few GeV. PC events are produced by neutrinos with energies about an order of magnitude higher. The energies of neutrinos which produce upward stopping muons is about 10 GeV. Upward through-going muons are produced by neutrinos with an average energy of about 100 GeV.

In the processes of interaction of cosmic rays with nuclei of the atmosphere, pions and kaons are produced. Atmospheric neutrinos originate from the decays of such pions and kaons and consequent decays of muons. Neutrinos with energies $\lesssim 5$ GeV are produced mainly in the decays of pions and muons

$$\pi^+ \to \mu^+ + \nu_\mu, \quad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \tag{10.48}$$

and

$$\pi^- \to \mu^- + \bar{\nu}_\mu, \quad \mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{10.49}$$

In the region $E \lesssim 1$ GeV both pions and muons decay in the atmosphere. It is evident that in this region the ratio r of the fluxes of $\nu_\mu + \bar{\nu}_\mu$ and $\nu_e + \bar{\nu}_e$ is equal to two.[10] However, muons which could produce neutrinos with energies larger than about 1 GeV can reach the earth surface before decay. Thus, at $E \gtrsim 1$ GeV neutrino fluxes from muon decay decrease and the ratio r increases. At neutrino energies ≤ 5 GeV, the ratio r is known with an accuracy of 3%. At higher energies the contribution of kaons becomes more important. In this region, the uncertainty of the ratio r becomes larger (because of the rather poor knowledge of the K/π production ratio) and at $E \simeq 10^2$ GeV it is about 10%.

A first model-independent evidence in favor of neutrino oscillations was obtained by the Super-Kamiokande Collaboration through the investigation of the zenith-angle dependence of the atmospheric electron and muon events. The zenith angle θ is determined in such a way that neutrinos going vertically downward have $\theta = 0$ and neutrinos coming vertically upward through the earth have $\theta = \pi$. Because of the geomagnetic cutoff at small energies (0.3–0.5 GeV) the flux of downward going neutrinos is lower than the flux of upward going neutrinos. At neutrino energies $E \geq 0.9$ GeV the fluxes of muon and electron neutrinos are symmetric under the change $\theta \to \pi - \theta$. Thus, if there are no neutrino oscillations at high energies the numbers of electron and muon events must satisfy the relation

$$N_l(\cos\theta) = N_l(-\cos\theta) \quad l = e, \mu. \tag{10.50}$$

We will see later that a significant violation of this relation was discovered in the Super-Kamikande experiment.

The characteristic feature of atmospheric neutrinos is an enhancement of the horizontal neutrino flux with respect to the vertical flux. This enhancement cannot be observed at neutrino energies below 1 GeV because at such energies the angular correlation between the neutrino and lepton directions is rather poor. At higher energies a $\cos\theta$-dependence of the numbers of the events can be measured.

For the study of flavor neutrino oscillations it is crucial to distinguish electrons and muons produced in the processes (10.47). In the Super-Kamiokande experiment leptons are observed through the detection of the Cherenkov radiation. The shapes of the Cherenkov rings of electrons and muons are completely different. In the case

[10] The Super-Kamiokande detector does not allow to determine the charges of the leptons.

of electrons the Cherenkov rings exhibit a more diffuse light than in the muon case. The probability of a misidentification of electrons and muons is below 2%. Fully contained events are divided into two samples: sub-GeV events are events with visible energy less than 1.3 GeV and multi-GeV events are events with visible energy more than 1.3 GeV.

An indication in favor of neutrino oscillations came from the measurement of the ratio r of the $(\nu_\mu + \bar{\nu}_\mu)$ and $(\nu_e + \bar{\nu}_e)$ fluxes. This ratio can be predicted with an accuracy of about 3%. In the SK-I run, for the double ratio $R = \frac{r_{\mathrm{meas}}}{r_{\mathrm{MC}}}$ (r_{meas} is the measured and r_{MC} is the predicted $\frac{\mu}{e}$ ratios) following value was obtained in the sub-GeV region

$$R_{\text{sub-Gev}} = 0.658 \pm 0.016 \pm 0.035. \tag{10.51}$$

In the multi-GeV region was found

$$R_{\text{multi-Gev}} = 0.702 \pm 0.032 \pm 0.101. \tag{10.52}$$

It is obvious that if there are no neutrino oscillations the double ratio R must be equal to one.[11]

We will discuss now the most important Super-Kamiokande result: the zenith-angle distribution of the electron and muon events. The results of the measurements of these distributions are presented in Fig. 10.2. As is seen from Fig. 10.2, the distributions of sub-GeV and multi-GeV electron events are in agreement with the expected distributions. In the distributions of the muon events at energies larger than 400 MeV a significant deficit of upward-going muons is observed.

This result can naturally be explained by the disappearance of muon neutrinos due to neutrino oscillations. As we have seen before, in the case of neutrino oscillations the probability of ν_μ to survive depends on the distance between neutrino source and neutrino detector. Downward going neutrinos ($\theta \simeq 0$) pass a distance of about 15–20 km. On the other side upward going neutrinos ($\theta \simeq \pi$) pass a distance of about 13,000 km (earth diameter). The measurement of the dependence of the numbers of the electron and muon events on the zenith angle θ allows to span the whole region of distances from about 15 km to about 13,000 km.

From the data of the Super-Kamiokande experiment for multi-GeV electron events was found

$$\left(\frac{U}{D}\right)_e = 0.961^{+0.086}_{-0.079} \pm 0.016. \tag{10.53}$$

For multi-GeV and PC muon events was obtained the value

[11] In the Kamiokande atmospheric neutrino experiment was found that the double ratio R is about 06. This was a first indication in favor of neutrino oscillations of atmospheric neutrinos.

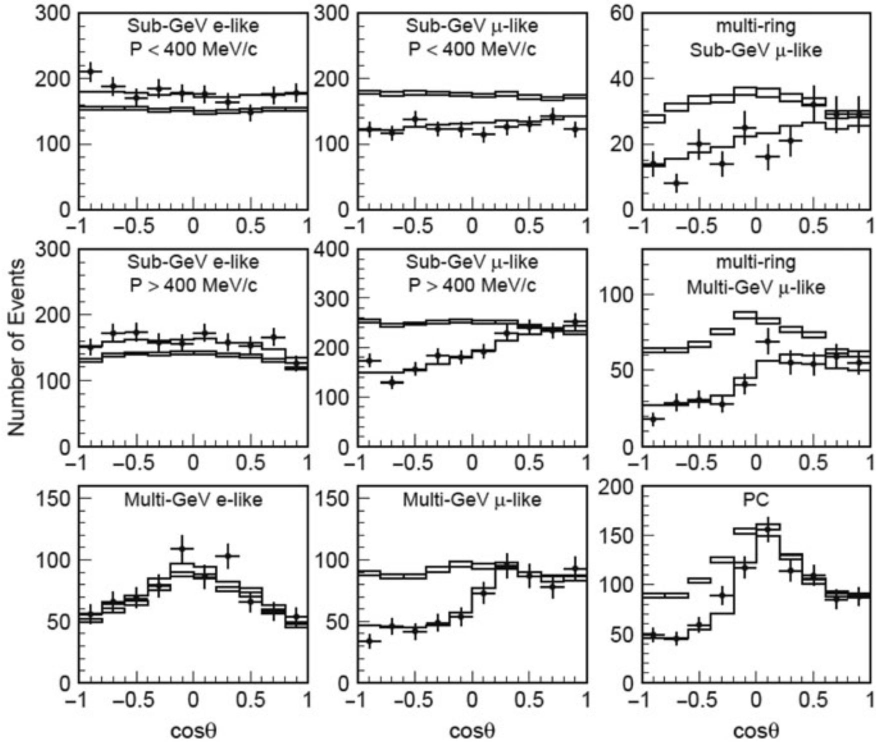

Fig. 10.2 Zenith angle dependence of the numbers of electron and muon events measured in the Super-Kamiokande atmospheric neutrino experiment. *Box histograms* show expected numbers of events in the case of no oscillations. The best-fit two-neutrino oscillation curve is also plotted (arXiv:hep-ex/0501064)

$$\left(\frac{U}{D}\right)_{\mu} = 0.551^{+0.035}_{-0.033} \pm 0.004. \tag{10.54}$$

Here U is the total number of upward going leptons ($-1 < \cos\theta < -0.2$) and D is the total number of downward going leptons ($0.2 < \cos\theta < 1$).

In the SK-I run (1996–2001) about 15,000 atmospheric neutrino events in the energy range 100 MeV–10 TeV and the distance range (15–13,000) km were collected. These wide ranges of energies and distances allow the Super-Kamikande Collaboration to study neutrino oscillations in details. From the two-neutrino analysis of all data, obtained in the SK-I run, the following ranges for neutrino oscillation parameters were obtained (90% CL)

$$1.5 \cdot 10^{-3} < \Delta m^2_{23} < 3.4 \cdot 10^{-3}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{23} > 0.92. \tag{10.55}$$

The best-fit values of the parameters are

$$\Delta m_{23}^2 = 2.1 \cdot 10^{-3}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{23} = 1 \quad (\chi^2_{\min}/\mathrm{dof} = 174.8/177). \tag{10.56}$$

The Super-Kamiokande data can be explained by $\nu_\mu \leftrightarrows \nu_\tau$ oscillations. Let us notice that because the threshold of τ production in $\nu_\tau - N$ processes is about 3.5 GeV and the majority of the atmospheric neutrinos have energies which are below this threshold, $\nu_\mu \leftrightarrows \nu_\tau$ oscillations manifest themselves in the Super-Kamiokande experiment mainly in the disappearance of muon neutrinos.[12]

The two-neutrino ν_μ survival probability depends on $\frac{L}{E}$ and is given by the well known expression

$$\mathrm{P}(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta_{23}\ \sin^2 \Delta m_{23}^2 \frac{L}{4E}. \tag{10.57}$$

This probability reaches the first minimum at

$$\left(\frac{L}{E}\right)_{\min} = \frac{2\pi}{\Delta m_{23}^2}\ . \tag{10.58}$$

In the standard Super-Kamiokande analysis of the data the dependence of the probability on $\frac{L}{E}$ is practically washed out because of the poor resolution. In order to reveal the oscillatory behavior of the probability the Super-Kamiokande Collaboration made a special analysis. A subset of events with high resolution in the variables L and E was chosen for the analysis. This allowed to determine the ν_μ survival probability as a function of $\frac{L}{E}$ and to reveal the first minimum of the survival probability (see Fig. 10.3). From analysis of this selected data the following ranges were obtained for the neutrino oscillation parameters (90% CL)

$$1.9 \cdot 10^{-3} < \Delta m_{23}^2 < 3.0 \cdot 10^{-3}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{23} > 0.90. \tag{10.59}$$

It is seen from Fig. 10.3 that the minimum of the survival probability corresponds to $\left(\frac{L}{E}\right)_{\min} \simeq 500\ \frac{\mathrm{km}}{\mathrm{GeV}}$. It is easy to estimate from this number the value of the neutrino mass-squared difference Δm_{23}^2. In fact, taking into account that $\sin^2 2\theta_{23} \simeq 1$ we obtain the value

$$\Delta m_{23}^2 = \frac{2\pi \hbar\, c}{\left(\frac{L}{E}\right)_{\min}} \simeq 2.5 \cdot 10^{-3}\ \mathrm{eV}^2, \tag{10.60}$$

which is in agreement with (10.59).

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[12] The long baseline experiment OPERA is aimed at the study of $\nu_\mu \to \nu_\tau$ transitions. In this experiment the production of τ in ν_τ-nuclei processes will be observed in an emulsion. The distance between the source of ν_μ (CERN, Switzerland) and the detector (Gran Sasso Laboratory, Italy) is about 730 km.

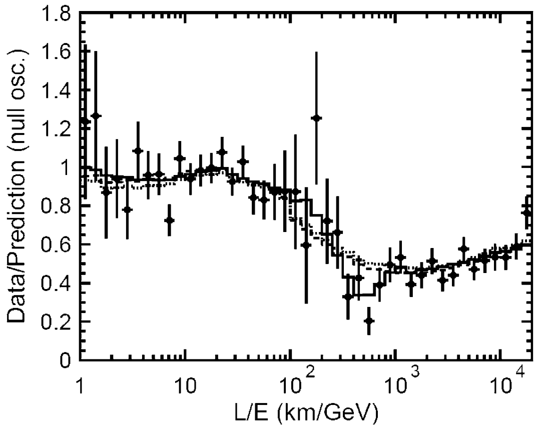

Fig. 10.3 Values of the probability $P(\nu_\mu \to \nu_\mu)$ as a function of the parameter $\frac{L}{E}$, determined from the data of the Super-Kamiokande atmospheric neutrino experiment. The best-fit two-neutrino oscillation curve is also plotted (arXiv:hep-ex/0404034)

10.4 KamLAND Reactor Neutrino Experiment

Compelling evidence in favor of oscillations of reactor $\bar{\nu}_e$ was obtained in the KamLAND experiment. The experiment was done in the Kamioka mine (Japan) at a depth of about 1 km. In the KamLAND detector, 1 kiloton of high-purity liquid scintillator is used. The scintillator is contained in a 13 m-diameter transparent nylon balloon, suspended in 1,800 m^3 non-scintillating buffer oil. Balloon and buffer oil are contained in an 18 m-diameter stainless-steel vessel. On the inner surface of the vessel 1,879 photomultipliers are mounted. Outside the steel container is a tank with 3.2 kiloton of water and 225 phototubes. This water-Cherenkov detector allows to eliminate muons of cosmic-ray origin.

In the KamLAND experiment, electron antineutrinos from 55 Japanese reactors situated at an average distance of 180 km (175 ± 35 km) from the Kamioka mine are detected.

Reactor $\bar{\nu}_e$'s are produced in decays of nuclei, which are products of fission of ^{235}U (57%), ^{238}U (7.8%), ^{239}Pu (29.5%) and ^{241}Pu (5.7%). Each fission, in which about 200 MeV is produced, is accompanied by the emission of 6 $\bar{\nu}_e$. A reactor with power about 3 GW$_{th}$ emits $\simeq 6 \cdot 10^{20}$ $\bar{\nu}_e/\mathrm{s}$.

The $\bar{\nu}_e$ spectrum of a reactor is well known. It was determined from the measurements of β-spectra, resulting from fission of uranium and plutonium.

Antineutrinos are detected in the KamLAND experiment through the observation of the inverse β-decay

$$\bar{\nu}_e + p \to e^+ + n. \tag{10.61}$$

Two γ-quanta from the annihilation of e^+ (prompt signal) and a γ-quantum from the process $n + p \to d + \gamma$ (delayed signal) are detected in the experiment. The signature of the event in the KamLAND experiment (and in other reactor neutrino experiments) is a coincidence between the prompt signal and the delayed signal.

The energy of the delayed γ-quantum is $\simeq$ 2.2 MeV and the neutron capture time is (207.5 $\pm$ 2.8) μs. The threshold neutrino energy is equal to 1.8 MeV. The prompt energy E_p is connected with the neutrino energy E by the relation

$$E \simeq E_p + \bar{E}_n + 0.8\ \text{MeV}. \tag{10.62}$$

where $\bar{E}_n$ is average neutron recoil energy ($\simeq$10 keV. The prompt energy includes the kinetic energy of the positron and the annihilation energy (2 m_e).

In the KamLAND experiment not only $\bar{\nu}_e$ from reactors but also electron antineutrinos which are produced in decay chains of ^{238}U and ^{232}Th in the interior of the earth (geo-neutrinos) are detected. The prompt energy released in the interaction of geo-neutrinos with protons is less than 2.6 MeV. The expected flux of geo-neutrinos is calculated in a reference geological model.

The average energy of the reactor antineutrinos is 3.6 MeV. It is easy to see that for such energies, distances of about $\simeq$100 km are appropriate to study neutrino oscillations driven by the solar neutrino mass-squared difference Δm^2_{12}. In fact, for the oscillation length we have

$$L_{12} \simeq 2.5\, \frac{E}{\Delta m^2_{12}}\ \text{m}, \tag{10.63}$$

where E is the neutrino energy in MeV and Δm^2_{12} is the neutrino mass-squared difference in eV^2. For $E = 3.6$ MeV and $\Delta m^2_{12} \simeq 8 \cdot 10^{-5}$ eV^2 we have $L_{12} \simeq 120$ km.

In the case of the oscillations driven by small Δm^2_{12} the averaged three-neutrino probability of $\bar{\nu}_e$ to survive in vacuum is given by the expression

$$P(\bar{\nu}_e \to \bar{\nu}_e) = |U_{e3}|^4 + (1 - |U_{e3}|^2)^2\ P^{12}(\bar{\nu}_e \to \bar{\nu}_e), \tag{10.64}$$

where

$$P^{12}(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \sin^2 2\theta_{12}\ \sin^2 \Delta m^2_{12} \frac{L}{4E} \tag{10.65}$$

is the two-neutrino survival probability. The small quantity $|U_{e3}|^2 < 5 \cdot 10^{-2}$ can be neglected. Thus, for the analysis of the KamLAND data the two-neutrino survival probability (10.65) can be used.

We will discuss here the KamLAND data collected from March 2002 till May 2007. If there are no $\bar{\nu}_e$ oscillations, 2, 179$\pm$89 events from reactor antineutrinos are expected. In the reactor antineutrino energy region the background is 276.1 $\pm$ 23.5. The observed number of events is equal to 1,609 events.

The prompt energy spectrum is presented in Fig. 10.4.

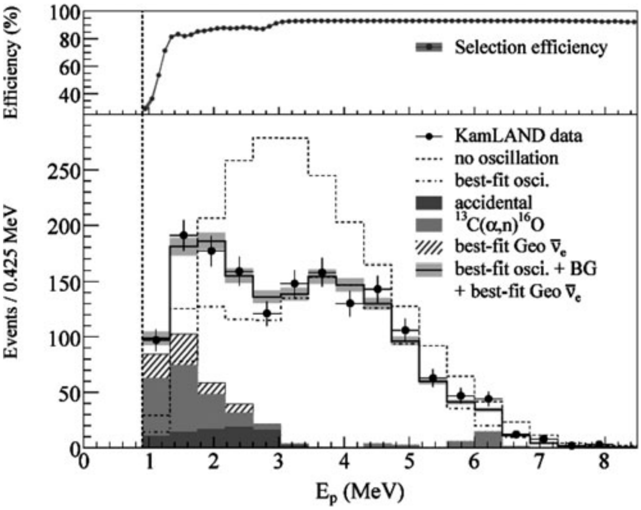

Fig. 10.4 Prompt event energy spectrum measured in the KamLAND experiment. The *dashed line* shows the predicted spectrum in the case of no oscillations. Best-fit oscillation curve is presented. In the *shaded areas* different backgrounds are shown (arXiv:0801.4589)

From the two-neutrino analysis of the KamLAND data, which include geo-neutrinos and background, the following values were obtained for neutrino oscillation parameters

$$\Delta m^2_{12} = (7.58^{+0.14}_{-0.13}(\mathrm{stat})^{+0.15}_{-0.15}(\mathrm{syst}))\cdot 10^{-5}\ \mathrm{eV}^2,\ \tan^2\theta_{12} = 0.56^{+0.10}_{-0.07}(\mathrm{stat})^{+0.10}_{-0.06}(\mathrm{syst}). \tag{10.66}$$

From the analysis of reactor neutrino events (with the cut $E_p > 2.6$ Mev applied) for the neutrino oscillation parameters was found the values

$$\Delta m^2_{12} = (7.66^{+0.22}_{-0.20}) \cdot 10^{-5}\ \mathrm{eV}^2, \quad \tan^2\theta_{12} = 0.52^{+0.16}_{-0.10}, \tag{10.67}$$

which are in agreement with (10.66).

From a joint analysis of the data of the KamLAND experiment and the data of the solar neutrino experiments a much better accuracy for the parameter $\tan^2\theta_{12}$ can be inferred[13]:

$$\Delta m^2_{12} = (7.59^{+0.21}_{-0.21}) \cdot 10^{-5}\ \mathrm{eV}^2, \quad \tan^2\theta_{12} = 0.47^{+0.06}_{-0.05}. \tag{10.68}$$

The KamLAND Collaboration obtained from their data values for the $\bar{\nu}_e$ survival probability at different values of the parameter $\frac{L_0}{E}$, where $L_0 = 180$ km is a flux-weighted average distance between the reactors and the detector. In Fig. 10.5 the probability $P(\bar{\nu}_e \to \bar{\nu}_e)$ as a function of $\frac{L_0}{E}$ is presented (geo-neutrino and

[13] This analysis is based on the assumption of the CPT invariance.

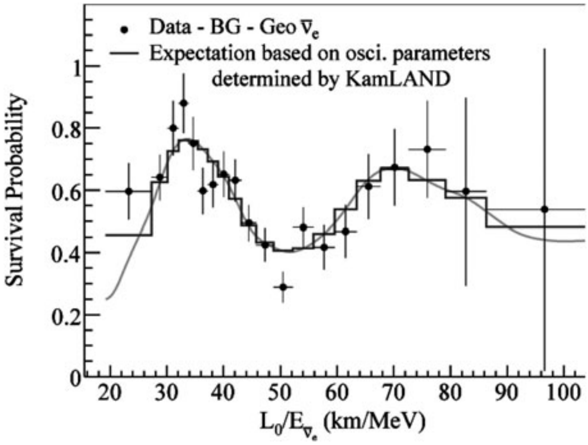

Fig. 10.5 Ratio of $\bar{\nu}_e$ spectrum measured in the KamLAND experiment to the spectrum, expected in the case of no oscillations as a function of $\frac{L_0}{E}$ ($L_0 = 180$ km is flux-weighted average distance from reactors to the detector). The expected ratio calculated with the values of the oscillation parameters obtained by the KamLAND collaboration is also shown (arXiv:0801.4589)

background events were subtracted). The oscillatory behavior of the probability is evident from Fig. 10.5.

10.5 Long-Baseline Accelerator Neutrino Experiments

10.5.1 K2K Accelerator Neutrino Experiment

Neutrino oscillations driven by the atmospheric neutrino mass-squared difference Δm^2_{23} were studied in the long baseline accelerator experiments $K2K$ and MINOS. For a neutrino energy of $\simeq 1$ GeV and $\Delta m^2_{23} \simeq 2.5 \cdot 10^{-3}$ eV^2 the oscillation length L_{23} is given by

$$L_{23} \simeq 2.5 \, \frac{E}{\Delta m^2_{23}} \text{ m} \simeq 10^3 \text{ km}. \tag{10.69}$$

In the first long baseline K2K experiment the distance between the neutrino source (KEK accelerator, Japan) and the neutrino detector (Super-Kamiokande) is 250 km.

Protons with an energy of 12 GeV from the KEK-PS accelerator bombard an aluminum target in which secondary particles were produced. Positively charged particles (mainly π^+) were focused in horns and decay in a 200 m-long decay pipe. After a beam dump in which all hadrons and muons were absorbed a neutrino beam was produced (there were 97.3% of ν_μ , 1.3% of ν_e and 1.4% of $\bar{\nu}_\mu$ in the beam). The neutrinos had energies in the range (0.5–1.5) GeV.

At a distance of about 300 m from the pion production target there were near detectors: a one kiloton water-Cherenkov detector (similar to the Super-Kamiokande detector) and a fine-grained detector system. The measurement of the neutrino energy spectrum in the near detectors allowed to predict the neutrino spectrum in the Super-Kamiokande detector for the case if there are no neutrino oscillations.

The disappearance of muon neutrinos was searched for in the K2K experiment. The two-neutrino survival probability has the familiar form

$$\mathrm{P}(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta_{23}\ \sin^2\left(1.27\ \Delta m^2_{23}\frac{L}{E}\right), \tag{10.70}$$

where E is the neutrino energy in GeV, L is the source-detector distance in km and Δm^2_{23} is the neutrino mass-squared difference in eV^2.

The disappearance of muon neutrinos manifests itself in the suppression of the total flux and in the distortion of the neutrino spectrum in the Super-Kamiokande detector. For the measurement of the total flux all observed events were used. For the measurement of the distortion of the neutrino spectrum a subset of one-ring contained muon events which are due to the quasi-elastic process

$$\nu_\mu + n \to \mu^- + p \tag{10.71}$$

was utilized. The measurement of the energy of μ^- and the angle between the neutrino and the μ^- directions allow to determine the neutrino energy.

From 1999 till 2004 in the K2K experiment 112 neutrino events were detected. The number of expected events in the case if there are no neutrino oscillations is equal to $158^{+9.2}_{-8.6}$. The distortion of the neutrino spectrum in the low energy region was observed in the K2K experiment.

From the two-neutrino analysis of the K2K data for the parameter Δm^2_{23} at $\sin^2 2\theta_{23} = 1$, the following 90% CL range

$$1.9 < \Delta m^2_{23} < 3.5 \cdot 10^{-3}\ \mathrm{eV}^2. \tag{10.72}$$

was obtained. The best-fit value of the parameter is $\Delta m^2_{23} = 2.8\ \mathrm{eV}^2$.

The K2K experiment was the first experiment with artificially produced neutrinos which confirmed the existence of neutrino oscillations discovered in the atmospheric Super-Kamiokande neutrino experiment.

10.5.2 MINOS Accelerator Neutrino Experiment

In the long baseline MINOS experiment muon neutrinos produced at the Fermilab Main Injector facility are detected in the Sudan mine at a distance of 735 km. Protons with an energy of 120 GeV from the Main Injector bombard a graphite target and produce pion and kaons. Positively charged particles are focused in magnetic horns and decay in a 675 m long decay pipe. After the pipe there is an absorber for hadrons and 300 m of rock in which muons are stopped. Neutrinos pass through the absorber and the rock to neutrino detectors. A change of the distance between the graphite target and the horn system allows to change the neutrino spectrum. The majority of the MINOS data was obtained with the low-energy neutrino beam configuration ($1 \lesssim E \lesssim 5$ GeV). Some data was obtained with the medium-energy beam configuration ($5 \lesssim E \lesssim 10$ GeV). The initial neutrino beam consists of ν_μ (92%), $\bar{\nu}_\mu$ (5.8%), ν_e (1.2%) and $\bar{\nu}_e$ (0.1%).

There are two identical neutrino detectors in the MINOS experiment. The near detector (ND) with a mass of 1 kiloton is at a distance about 1 km from the target and about 100 m underground. The far detector (FD) with a mass of 5.4 kilotons is at a distance of 735 m from the target and about 700 m underground. The detectors are steel (2.54 cm thick)-scintillator (1 cm thick) calorimeters magnetized to 1.3 T. The measurement of the curvature of the muon tracks allows to distinguish ν_μ from 6% admixture of $\bar{\nu}_\mu$ and to measure energy of muons which leave the detector. The energies of the muons which are stopped in the detector are determined by their ranges.

Muon neutrinos are detected in the MINOS experiment via the observation of the process

$$\nu_\mu + \mathrm{Fe} \to \mu^- + X \tag{10.73}$$

The neutrino energy is given by the sum of the muon energy and the energy of hadronic shower.

The ν_μ-survival probability as a function of the neutrino energy is measured in the MINOS experiment. The two-neutrino ν_μ-survival probability is given by Eq. (10.70).

In the near detector the initial neutrino spectrum (before neutrino oscillations could take place) is measured. This measurement allows to predict the expected spectrum of the muon neutrinos in the far detector in the case if there were no neutrino oscillations.

We will discuss here the results of the analysis of the data collected from May 2005 till July 2007. During this time, in the far detector 848 muon events were observed. The number of the events expected in the case if there were no neutrino oscillations is $1,065 \pm 60$ (syst). The spectrum of the muon neutrinos measured in the FD is presented in Fig. 10.6. A strong distortion of the spectrum is observed in the low-energy region. From the two-neutrino analysis of the MINOS data the following values were found for the neutrino oscillation parameters

$$\Delta m^2_{23} = (2.43 \pm 0.13) \cdot 10^{-3}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{12} > 0.90\ (90\%\ \mathrm{CL}). \tag{10.74}$$

Thus, in the long-baseline MINOS experiment convincing evidence in favor of neutrino oscillations in the atmospheric neutrino mass-squared difference range was obtained. The value of the parameter Δm^2_{23} was measured in the MINOS experiment with an accuracy which at the moment is better than in all other measurements.

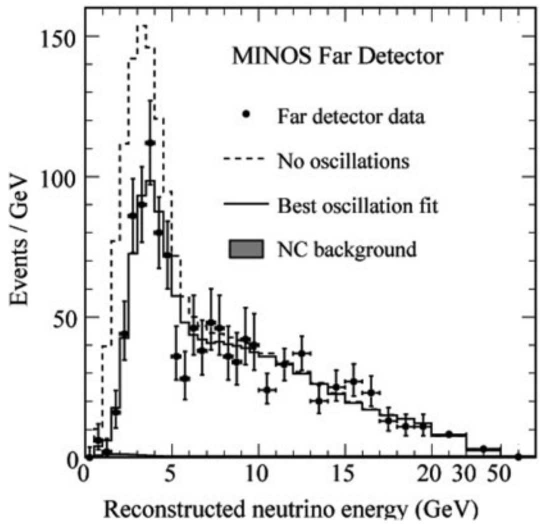

Fig. 10.6 Muon neutrino energy spectrum measured in the MINOS experiment. The *dashed curve* shows the expected spectrum in the case of no oscillations. The neutrino oscillation best-fit spectrum is also shown (arXiv:0806.2237)

10.6 MiniBooNE Accelerator Neutrino Experiment

In the MiniBooNE experiment at Fermilab LSND indications in favor of neutrino oscillations with $\Delta m^2 \simeq 1$ eV^2 were checked.

The LSND experiment was performed at the Los Alamos Meson Physics Facility (LAMPF). This was a short-baseline neutrino oscillation experiment with a distance between neutrino source and detector about 30 m. The LAMPF proton beam with a kinetic energy of 800 MeV produced pions by hitting a 30 cm long water target. Most of the π^+'s were stopped in the target and decay dominantly via $\pi^+ \to \mu^+ + \nu_\mu$. Muons came to rest in the target as well and decay via $\mu^+ \to e^+ + \bar{\nu}_\mu + + \nu_e$. In the detector (cylindrical tank 8.3 m long by 5.7 m in diameter, filled with a scintillator) $\bar{\nu}_e$'s were searched for through the observation of the reaction

$$\bar{\nu}_e + p \to e^+ + n \tag{10.75}$$

The $\bar{\nu}_e$ signature was a coincidence between e^+ and a delayed 2.2 MeV γ from the capture $n + p \to d + \gamma$. In the LSND experiment an excess of $87 \pm 22.4 \pm 6.0$ events with e^+ energy in the range 20–60 MeV above the expected neutrino-induced background was observed. This excess could be explained by $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions and corresponds to the transition probability $P(\bar{\nu}_\mu \to \bar{\nu}_e) = (0.264 \pm 0.067 \pm 0.045)\%$. From the analysis of the LSND data for the parameter Δm^2 the following range

$$0.2 < \Delta m^2 < 10 \cdot \mathrm{eV}^2 \tag{10.76}$$

was found.

As we have discussed before, from the analysis of the data of the solar, KamLAND, atmospheric and LBL accelerator experiments follows that two neutrino mass-squared differences ($\Delta m^2_{12} \simeq 8 \cdot 10^{-5}\mathrm{eV}^2$ and $\Delta m^2_{23} \simeq 2.4 \cdot 10^{-3}\mathrm{eV}^2$) are responsible for neutrino oscillations. In order to have two independent neutrino mass-squared differences we must have three neutrinos with definite masses. These three massive neutrinos correspond to the three flavor neutrinos ν_e, ν_μ, ν_τ. If the LSND result is correct we have to assume that exist at least four massive neutrinos and in addition to the three flavor neutrinos at least one sterile neutrino. The existence of sterile neutrinos would be an important ingredient for our understanding of the origin of neutrino masses and mixing. The check of the LSND claim was a very important problem.

In the MiniBooNE experiment the neutrino beam, produced at the Fermilab Booster facility, was used. 8 GeV protons bombarded a 71 cm-long beryllium target. Positively charged pions and kaons, produced in the target, were focused by a horn and decayed in a 50 m-long pipe. The neutrino energy spectrum had a maximum at the energy about 700 MeV. The distance between the target and a neutrino detector in the MiniBooNE experiment was about 540 m. This distance is comparable with the oscillation length which corresponds to $\Delta m^2 \simeq 1\ \mathrm{eV}^2$.

The MiniBooNE detector is a spherical tank with an inner radius of 610 cm filled with 800 tons of pure mineral oil. Charged particles were detected through the observation of directional Cherenkov radiation and isotropic scintillation light. 380 candidates for CC quasielastic events

$$\nu_e + n \to e^- + p, \tag{10.77}$$

were observed.

No significant excess over the expected background was observed in the neutrino energy range $475 < E < 1,250$ MeV ($22 \pm 19 \pm 35$ excess events). However, in the small-energy region $E < 475$ MeV an excess of $96 \pm 17 \pm 20$ events was found. These events cannot be explained by neutrino oscillations. The exclusion region in the plane of the parameters Δm^2 and $\sin^2 2\theta$ which was obtained from the two-neutrino analysis of the MiniBooNE data fully covers the LSND-allowed region. The authors came to the conclusion that their data excludes, at 98% CL, an existence of neutrino oscillations in the LSND region of parameters.

10.7 CHOOZ Reactor Neutrino Experiment

In the CHOOZ experiment the disappearance of reactor $\bar{\nu}_e$'s due to neutrino oscillations, driven by the atmospheric neutrino mass-squared difference Δm^2_{23}, was searched for. Though in this experiment (and also in the similar experiment Palo Verde) no indications in favor of neutrino oscillations were found, its results are extremely important for the neutrino mixing.

The two-neutrino $\bar{\nu}_e$-survival probability is given by the expression

$$P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \sin^2 2\theta_{13}\ \sin^2\left(1.27\ \Delta m^2_{23}\frac{L}{E}\right). \tag{10.78}$$

Here Δm^2_{23} is the neutrino mass-squared difference in eV^2, L is the source-detector distance in m and E is the antineutrino energy in MeV. The average energy of the reactor electron antineutrinos is equal to 3.6 MeV. The corresponding oscillation length is equal to 3.6 km.

In the CHOOZ experiment the detector was at a distance of about 1 km from each of the two reactors of the CHOOZ power station (8.5 GW$_{\mathrm{th}}$). The detector had 300 m water equivalent of rock overburden which reduced the cosmic muon flux.

The CHOOZ detector comprised 5 tons of Gd-loaded liquid scintillator contained in an acrylic vessel (neutrino target), the intermediate region with 17 tons of unloaded liquid scintillator which was viewed by 192 photomultipliers and the optically separated veto region with 90 tons of unloaded liquid scintillator and 48 photomultipliers. The antineutrinos were detected through the observation of the classical reaction

$$\bar{\nu}_e + p \to e^+ + n. \tag{10.79}$$

A prompt signal from the annihilation of the positron and a delayed signal from the capture of the neutron by Gd (in this process γ's with a total energy of $\simeq$8 MeV are produced) were recorded in coincidence.

From April 1997 till July 1998, in the CHOOZ experiment 3,600 antineutrino events were recorded. The observed dependence of the number of events on the prompt energy is in good agreement with prediction. For the ratio R of the total number of detected $\bar{\nu}_e$ events to the expected events was found

$$R = 1.01 \pm 2.8\%\ (\mathrm{stat}) \pm \pm 2.7\%\ (\mathrm{syst}). \tag{10.80}$$

The data of the experiments were analyzed in the framework of two-neutrino oscillations with the $\bar{\nu}_e$-survival probability given in Sect. 6.4. From the exclusion plot in the plane of the oscillation parameters Δm^2_{23} and $\sin^2 2\,\theta_{13}$ the following upper bound

$$\sin^2 2\,\theta_{13} \lesssim 2 \cdot 10^{-1}. \tag{10.81}$$

can be inferred.

As we have seen before, in neutrino oscillation experiments two mixing angles θ_{23} and θ_{12} were measured. It occurred that these angles are large: $\theta_{23} \simeq 45°$, $\theta_{12} \simeq 30°$. The angle θ_{13} is unknown. We have only the CHOOZ upper bound (10.81) from which follows that the angle θ_{13} is small.

The parameter $\sin\theta_{13}$ characterizes the element U_{e3} of the PMNS mixing matrix. It enters into the mixing matrix together with the CP phase δ ($U_{e3} = \sin\theta_{13}\ e^{-i\delta}$). The value of the parameter $\sin^2\theta_{13}$ determine the possibilities to study CP-violation in the lepton sector and to reveal the type of the neutrino mass spectrum (normal or inverted). The measurement of the parameter $\sin\theta_{13}$ is one of the major problems of future neutrino oscillation experiments.

10.8 Future Neutrino Oscillation Experiments

With the neutrino oscillation experiments we have discussed before, the first discovery stage of neutrino masses and mixing was finished. Now the second stage of the investigation of this new beyond the Standard Model phenomenon started. It includes the preparation of neutrino oscillation experiments of the next generation, the study of the possibilities of the construction of such new neutrino facilities as the Super-beam, β-beam, Neutrino factory, etc.

The main goal of future neutrino oscillation experiments is:

1. to determine the value of the parameter $\sin^2 2\theta_{13}$ (or significantly improve the CHOOZ upper bound);
2. if the parameter $\sin^2\theta_{13}$ is not too small,
   - to investigate effects of CP violation in the lepton sector and to determine the phase δ,
   - to establish the character of the neutrino mass spectrum;
3. to measure the neutrino oscillation parameters Δm^2_{23}, Δm^2_{12}, $\sin^2 2\theta_{23}$ and $\tan^2\theta_{12}$ with accuracies which are much better than those achieved at present;
4. to search for sterile neutrinos, etc.

We will briefly discuss future reactor and accelerator experiments in which the fundamental parameter $\sin^2 2\theta_{13}$ will be measured.

Three new reactor experiments (Double CHOOZ (France), Daya Bay and RENO (South Korea) will start to run in 2010–2011.

In the CHOOZ experiment there was only one detector. In all future reactor neutrino experiments there will be a far and an identical near detector. This will allow to make relative measurements and to minimize systematic errors related to reactors. In all future reactor experiments detectors, larger than in the CHOOZ experiment, will be used. In the Double CHOOZ experiment two 10 ton liquid scintillator neutrino detectors will be used. In the Daya Bay experiment there will be eight detectors. Each detector will comprise a 20 ton target of Gd-loaded liquid scintillator. Two 16.5 ton Gd-loaded liquid scintillators will be used in the RENO experiment. In the Double CHOOZ, Daya Bay and RENO experiments electron antineutrinos will be detected, correspondingly, from two reactors with a thermal power of 8.7 $\mathrm{GW_{th}}$, four reactors with thermal power 11.6 $\mathrm{GW_{th}}$ (after 2010 six reactors with a thermal power of 17.4 $\mathrm{GW_{th}}$), six reactors with a thermal power of 17.4 $\mathrm{GW_{th}}$.

In Table 10.2 we present some features of these three experiments. In the second column distances in meters from reactors to the near and far detectors are shown.. In the third column overburdens of detectors in meters of the water equivalent are

Table 10.2 Future reactor experiments aiming at the measurement of the parameter $\sin^2 2\theta_{13}$

Experiment	Distances near/far (m)	Depth near/far (mwe)	Sensitivity to $\sin^2 2\theta_{13}$
Double CHOOZ	280/1,050	60/300	$3 \cdot 10^{-2}$
Daya Bay	360(500)/1,985(1,613)	260/910	$1 \cdot 10^{-2}$
RENO	290/1,380	120/450	$2 \cdot 10^{-2}$

indicated. In the last column the projected sensitivities to the parameter $\sin^2 2\theta_{13}$ to be achieved after three years of running time are presented.

In the long-baseline accelerator T2K experiment neutrinos produced at the new JPARC facility in Japan will be detected at a distance of 295 km by the water-Cherenkov Super-Kamiokande detector. Protons with an energy of 30 GeV from the synchrotron at JPARC will bombard a 90 cm-long graphite target producing pions and kaons. After the decay pipe (110 m-long) and the shielding, the beam of ν_μ's with about 0.4% admixture of ν_e's will be produced.

T2K will be an off-axis neutrino experiment: the angle between the direction to the detector and the direction of the parent π^+'s will be 2.5°. This allows to obtain a narrow-band neutrino beam with a maximal intensity at $E \simeq 0.7$ GeV. This energy is close to the energy E_0($E_0 = \frac{2.54}{\pi} \Delta m_{23}^2 L$) which at the distance of $L = 295$ km corresponds to the maximum of oscillations, driven by $\Delta m_{23}^2 \simeq 2.5 \cdot 10^{-3}\mathrm{eV}^2$.

At a distance of about 280 m from the target it will be several detectors in a 0.2 T uniform magnetic field. Near detectors will be used for the measurement of the neutrino spectrum and flux and also for the measurement of cross sections of different CC and NC processes.

The major aim of the T2K experiment is a high-precision measurement of neutrino oscillation parameters Δm_{23}^2 and $\sin^2 2\theta_{23}$ by the investigation of $\nu_\mu \to \nu_\mu$ transitions (through the detection of ν_μ's in Super-Kamiokande) and the determination of the parameter $\sin^2 2\theta_{13}$ by the search for $\nu_\mu \to \nu_e$ appearance (through the detection of ν_e's in Super-Kamiokande). It is anticipated that after 5 years of running time the parameters Δm_{23}^2 and $\sin^2 2\theta_{23}$ will be measured with the accuracies

$$\delta(\Delta m_{23}^2) < 1 \cdot 10^{-4}\ \mathrm{eV}^2, \quad \delta(\sin^2 2\theta_{23}) \simeq 10^{-2}. \tag{10.82}$$

The T2K sensitivity to the parameter $\sin^2 2\theta_{13}$ is equal to[14]

$$\sin^2 2\theta_{13} \simeq 0.008. \tag{10.83}$$

Thus, by about a factor of 20 an improvement with respect to the CHOOZ sensitivity is planned to be reached.

The neutrino oscillation experiments, we discussed, will be performed in the next years. There are many new, ambitious projects of neutrino oscillation experiments which are in a research and development stage. Information about these projects can be found in the arXiv, the proceedings of neutrino conferences, etc.

[14] Assuming that the CP phase δ is equal to 0 or π.

Chapter 11
Neutrino and Cosmology

11.1 Introduction

Photons and neutrinos are the most abundant particles in the Universe. Neutrinos played a very important role in the evolution of the Universe. Modern high precision cosmological data allow to obtain strong bounds on neutrino properties. In this section we will discuss neutrino decoupling in the Early Universe, the Big Bang nucleosynthesis and the number of the light neutrinos, the limit on the sum of the neutrino masses which can be inferred from the large-scale structure of the Universe and Cosmic Microwave Background radiation data. We will also consider supernova neutrinos.

We will start with a brief discussion of the standard cosmology. The standard cosmology is based on the Cosmological Principle and Einstein equations of the General Theory of Relativity. According to the Cosmological Principle the Universe observed from any spacial position is isotropic and homogeneous at large scales. One of the major evidence in favor of the isotropy of the Universe was obtained from the measurement of the temperatures of the cosmic microwave background radiation (CMB): the relative difference of the temperatures measured by antennas directed in different directions is not more than $\sim 10^{-5}$. It is a general consequence of the cosmological principle that the space-time metric of the Universe is the Robertson-Walker metric.

11.2 Standard Cosmology

Let $x^{\alpha} = (x^0, \mathbf{x})$ be the time-space coordinate of a point in some coordinate system. The square of the element of length (interval) has the following general form

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}. \tag{11.1}$$

In Euclidian space we have

$$ds^2 = \eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta}, \tag{11.2}$$

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where $\eta_{\alpha\beta} = (1, -1, -1, -1)$. As well known, the interval ds^2 in the Euclidian space is invariant under the Lorentz transformation.

In the general case of a non-Euclidean space the metric tensor $g_{\alpha\beta}$ depends on x. It determines the geometry of the space. The metric of the isotropic and homogeneous space is the Robertson-Walker metric which we will discuss now. Let us consider the space part of the metric tensor. In the Euclidean space we have

$$dl^2 = \sum_{i=1}^{3} dx^{i2}. \tag{11.3}$$

The Cartesian coordinates x^1, x^2, x^3 are connected with the spherical coordinates ρ, θ and ϕ by the relations

$$x^1 = \rho \sin\theta \cos\phi, \quad x^2 = \rho \sin\theta \sin\phi, \quad x^3 = \rho \cos\theta. \tag{11.4}$$

In spherical coordinates we obviously have

$$dl^2 = d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{11.5}$$

In the general case of the isotropic Universe we have

$$dl^2 = d\rho^2 + f^2(\rho)\, (d\theta^2 + \sin^2\theta d\phi^2). \tag{11.6}$$

The condition of isotropy allows to determine possible functions $f(\rho)$. Let us consider Fig. 11.1. Assuming that angles α and β are infinitesimally small, we have

$$CB = f(2\rho)\alpha = f(\rho)\beta, \quad DE = f(\rho - x)\alpha + f(x)\beta = f(\rho + x)\alpha. \tag{11.7}$$

From these relations we find

$$f(\rho - x) + f(x)\frac{f(2\rho)}{f(\rho)} = f(\rho + x) \tag{11.8}$$

Now if we take the derivative over x and put $x = 0$ we find the following relation from (11.8)

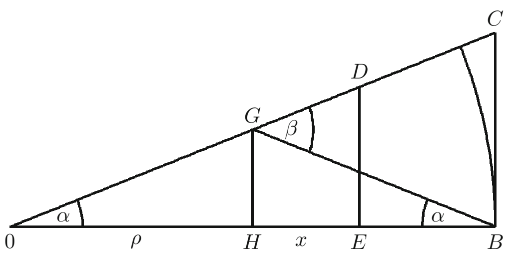

Fig. 11.1 Robertson-Walker geometry of a homogeneous and isotropic space

$$\frac{d\ f(\rho)}{d\rho} = \frac{f(2\rho)}{2f(\rho)}. \tag{11.9}$$

In deriving (11.9) we took into account that $f(\rho) \simeq \rho$ at $\rho \to 0$.

It is obvious that

$$f_1(\rho) = \sin\rho, \quad f_0(\rho) = \rho, \quad f_{-1}(\rho) = \sinh\rho \tag{11.10}$$

are solutions of Eq. (11.9). It is possible to show that there are no other solutions of this equation.

It follows from the cosmological principle that the most natural reference system in cosmology is the comoving system. The comoving system is the system in which the matter is at rest. Because comoving observers see the same sequence of events they have the same time which is called cosmic time. In the comoving system the cosmic time is the proper time. In the comoving system the Robertson-Walker metric has the form[1]

$$ds^2 = dt^2 - a^2(t)(d\rho^2 + f_k^2(\rho)\ (d\theta^2 + \sin^2\theta\ d\phi^2)), \tag{11.11}$$

where the functions $f_k(\rho)$ are given by (11.10). The function $a(t)$ cannot be determined from the requirements of isotropy. It has the dimension of time and is called the scale factor.

The Robertson-Walker metric can be presented in another form. Let us introduce the variable $r = f(\rho)$. We have

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\ (d\theta^2 + \sin^2\theta d\phi^2)\right), \quad k = 1, 0, -1. \tag{11.12}$$

At k=1 the space is a sphere with radius a. In fact, let us consider a sphere of radius a in a four-dimensional Euclidean space. We have

$$\sum_{i=1}^{4}(x^i)^2 = a^2. \tag{11.13}$$

The metric on the sphere is given by

$$dl^2 = \sum_{i=1}^{4}(dx^i)^2. \tag{11.14}$$

On the surface of the sphere one of the coordinates (say x^4) is determined by the other three. From (11.13) follows that

[1] From isotropy it is obvious that $g_{0i} = 0, \quad i = 1, 2, 3$.

$$dx^4 = -\frac{\sum_{i=1}^3 x^i dx^i}{\sqrt{a^2 - \sum_{i=1}^3 (x^i)^2}}. \tag{11.15}$$

Let us introduce the spherical coordinates

$$x^1 = r' \sin\theta \cos\phi, \quad x^2 = r' \sin\theta \sin\phi, \quad x^3 = r' \cos\theta. \tag{11.16}$$

For the metric we have

$$dl^2 = dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r'^2 dr'^2}{a^2 - r'^2} = \frac{a^2}{a^2 - r'^2} dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{11.17}$$

Finally, introducing the dimensionless variable $r = \frac{r'}{a}$, for the metric of the three-dimensional sphere we find the following expression

$$dl^2 = a^2(t)\left(\frac{dr^2}{1 - r^2} + r^2\,(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{11.18}$$

The case $k = 0$ in (11.12) corresponds to the flat space. The case $k = -1$ can be obtained from the expression (11.17) if we make the change $a \to ia$. It corresponds to the space with negative curvature (hyperboloid in the four-dimensional space). Let us consider the transformation of the coordinates

$$x^\alpha = x^\alpha(x'). \tag{11.19}$$

We have

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x'^\rho} dx'^\rho. \tag{11.20}$$

The contravariant vector A^α is transformed as dx^α:

$$A^\alpha = \frac{\partial x^\alpha}{\partial x'^\rho} A'^\rho. \tag{11.21}$$

For the interval we have

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = dx_\beta dx^\beta, \tag{11.22}$$

where $dx_\beta = g_{\beta\alpha}\, dx^\alpha$. From (11.20) we find

$$ds^2 = dx_\beta \frac{\partial x^\beta}{\partial x'^\rho} dx'^\rho. \tag{11.23}$$

Now taking into account that ds^2 is invariant, we have

$$ds^2 = dx'_\rho dx'^\rho, \tag{11.24}$$

where

$$dx'_\rho = \frac{\partial x^\beta}{\partial x'^\rho}\, dx_\beta. \tag{11.25}$$

If we multiply this relation by $\frac{\partial x'^\rho}{\partial x^\alpha}$, sum over ρ and take into account the relation

$$\frac{\partial x^\beta}{\partial x'^\rho}\frac{\partial x'^\rho}{\partial x^\alpha} = \frac{\partial x^\beta}{\partial x^\alpha} = \delta^\beta_\alpha \tag{11.26}$$

we find

$$dx_\alpha = \frac{\partial x'^\rho}{\partial x^\alpha} dx'_\rho. \tag{11.27}$$

The covariant vector B_α is transformed as dx_α:

$$B_\alpha = \frac{\partial x'^\rho}{\partial x^\alpha} B'_\rho. \tag{11.28}$$

It is easy to show that $B_\alpha A^\alpha$ is invariant. In fact, we have

$$B_\alpha A^\alpha = B'_\rho \frac{\partial x'^\rho}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\sigma} A'^\sigma = B'_\rho \frac{\partial x'^\rho}{\partial x'^\sigma} A'^\sigma = B'_\rho A'^\rho. \tag{11.29}$$

We have

$$B_\alpha = g_{\alpha\beta}\; B^\beta. \tag{11.30}$$

Let us introduce the tensor $g^{\alpha\beta}$ by the relation

$$B^\alpha = g^{\alpha\beta}\; B_\beta. \tag{11.31}$$

If we multiply this relation by $g_{\alpha\rho}$ we find

$$B^\alpha g_{\alpha\rho} = B_\rho = g^{\alpha\beta} g_{\alpha\rho}\; B_\beta. \tag{11.32}$$

Thus, the tensor $g_{\alpha\rho}$ must satisfy the relation

$$g^{\alpha\beta} g_{\alpha\rho} = \delta^\beta_\rho. \tag{11.33}$$

Let us also remark that tensors are transformed as products of vectors. For example, for the tensor A^α_β we have

$$A^{\alpha}_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\rho}} \frac{\partial x'^{\sigma}}{\partial x^{\beta}} A'^{\rho}_{\sigma}. \tag{11.34}$$

We have seen that the metric of an isotropic and homogeneous Universe is characterized by the scale function $a(t)$. The function $a(t)$ determines the important cosmological Hubble parameter $H(t)$. In fact, at relatively small distances two Galaxies are separated by the radial distance $\Delta l(t) = a(t)\, \Delta r$. Thus, for the relative velocity we obtain the following expression

$$\Delta v = \frac{d\Delta l}{dt} = \frac{da}{dt}\, \Delta r = H(t)\, \Delta l, \tag{11.35}$$

where

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{11.36}$$

is the Hubble parameter. The relation (11.35) is the famous Hubble law which connects the relative velocity of any two Galaxies with the distance between them. This law is in perfect agreement with experiment. From experimental data follows that the Universe is expanding.

Let us determine the parameter

$$z = \frac{\lambda_{\mathrm{o}} - \lambda_{\mathrm{e}}}{\lambda_{\mathrm{e}}}, \tag{11.37}$$

where λ_{o} is the wavelength of the light observed at the time t_{o} and λ_{e} is the wavelength of the light emitted at the time t_{e} ($t_{\mathrm{e}} < t_{\mathrm{o}}$). The cosmological change in time of all lengths is determined by the scale factor $a(t)$. We have

$$\frac{\lambda_{\mathrm{o}}}{\lambda_{\mathrm{e}}} = \frac{a(t_{\mathrm{o}})}{a(t_{\mathrm{e}})} = z + 1. \tag{11.38}$$

All light spectra observed from different Galaxies are red-shifted: $\lambda_{\mathrm{o}} > \lambda_{\mathrm{e}}$. Thus, we have $a(t_{\mathrm{o}}) > a(t_{\mathrm{e}})$. The observation of red shifts of the light emitted by Galaxies is the direct evidence in favor of the expansion of the Universe. The evolution of the scale factor is determined by the Einstein equation of the General Theory of Relativity. The Einstein equation is based on the assumption that the curvature of the space is determined by the energy-momentum tensor. It has the following tensor form

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{11.39}$$

Here

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \tag{11.40}$$

is the conserved Einstein tensor, $T^{\mu\nu}$ is the conserved energy-momentum tensor, G is the gravitational constant and Λ is the cosmological constant. In the units $\hbar = c = 1$ the constant G has the dimension M^{-2} and the constant Λ has the dimension M^2. We have $\frac{1}{\sqrt{G}} = M_P$, where $M_P \simeq 1.2 \cdot 10^{19}$ GeV is the Planck mass. In (11.40) $R_{\mu\nu}$ is the Ricci curvature tensor which is determined by the metric tensor and its first and second derivatives and $R = R_{\mu\nu} g^{\mu\nu}$ is the scalar curvature.

The standard cosmology is based on the assumption that the Universe can be considered as a perfect fluid. In this case the energy-momentum tensor $T^{\mu\nu}$ is given by the expression

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p\, g_{\mu\nu}. \tag{11.41}$$

Here ρ, p and u_μ are energy density, pressure and velocity. In the comoving system $u = (1, 0, 0, 0)$ and ρ, p can depend only on t. From the Einstein equation (11.39) and (11.41) we obtain the Friedman equations

$$\left(\frac{\hat{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{11.42}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{11.43}$$

We can present the Einstein equation in the form

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu}\right) \tag{11.44}$$

and interpret the Λ-term as an additional contribution to density and pressure. From (11.41) and (11.44) we have

$$p_\Lambda = -\frac{\Lambda}{8\pi G}, \quad \rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}. \tag{11.45}$$

If we calculate the derivative of (11.42) and with the help of (11.43) exclude $\ddot{a}$ we come to the equation

$$\dot{\rho} = -3H(\rho + p) \tag{11.46}$$

in which the curvature k and the cosmological constant Λ do not enter.

It is easy to see that Eq. (11.46) is the consequence of the first law of thermodynamics

$$d\mathcal{E} = T dS - p\, dV, \tag{11.47}$$

where $\mathcal{E}$, T, S and V are the total energy, temperature, entropy and volume of the system.

In the case of adiabatic expansion, $S = 0$. Thus, we have

$$\dot{\rho} = -(\rho + p)\frac{dV}{dt}\frac{1}{V}, \tag{11.48}$$

where we used the relation $\mathcal{E} = \rho V$. Now, taking into account that $V \sim a^3$ we find

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}. \tag{11.49}$$

Because $\frac{\dot{a}}{a} = H$, Eq. (11.49) coincides with (11.46).

Let us consider the first Friedman equation (11.42). Taking into account (11.45), we can write this equation in the form

$$H^2 = \frac{8\pi G}{3}\rho_{\rm tot} - \frac{k}{a^2}, \tag{11.50}$$

where

$$\rho_{\rm tot} = \rho + \rho_\Lambda. \tag{11.51}$$

Further, we have

$$\frac{k}{a^2} = H^2\left(\frac{\rho_{\rm tot}}{\rho_{\rm c}} - 1\right), \tag{11.52}$$

where

$$\rho_{\rm c} = \frac{3H^2}{8\pi G}. \tag{11.53}$$

Let us introduce the quantity

$$\Omega_{\rm tot} = \frac{\rho_{\rm tot}}{\rho_{\rm c}}. \tag{11.54}$$

The relation (11.52) takes the form

$$\frac{k}{a^2} = H^2(\Omega_{\rm tot} - 1). \tag{11.55}$$

The case $\Omega_{\rm tot} = 1$ ($\rho_{\rm tot} = \rho_{\rm c}$) corresponds to $k = 0$ (the flat Universe). If $\Omega_{\rm tot} > 1$, $k = 1$ (closed Universe). If $\Omega_{\rm tot} < 1$, $k = -1$ (open Universe).

The density ρ_c is called the critical density. Let us determine the dimensionless quantity[2]

$$h = \frac{H}{100\ \text{km s}^{-1}\ \text{Mpc}^{-1}}. \tag{11.56}$$

The critical density is given by

$$\rho_c = 1.88\, h^2\, 10^{-29} \text{g cm}^{-3}. \tag{11.57}$$

Let us introduce the dimensionless deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}}^2, \tag{11.58}$$

which characterizes the slowing down (or acceleration) of the expansion of the Universe. From the second Friedman equation (11.43) we find

$$q = \frac{1}{2\rho_c}\left(\rho + 3p - \frac{\Lambda}{4\pi G}\right). \tag{11.59}$$

In the case of $\Lambda = 0$ the deceleration parameter is positive and the expansion of the Universe is slowing down. It is obvious that this effect is due to gravitational attraction. From modern data follows, however, that $q < 0$. Thus, the expansion of the Universe is accelerating. This means that the negative term in the right-handed part of (11.59) (cosmological constant or more generally, dark energy (see below)) gives dominant contribution. We will now consider solutions of the Friedman equation. Let us write down the equation of state in the form

$$p = w\,\rho. \tag{11.60}$$

In the case of non-relativistic matter or dust $w_{\text{mat}} = 0$. For the radiation $w_r = \frac{1}{3}$. From (11.45) we conclude that in the case of the cosmological constant $w_\Lambda = -1$.

From (11.49) and (11.60) we have

$$\dot{\rho} = -3(1 + w)\rho\,\frac{\dot{a}}{a}. \tag{11.61}$$

This equation can easily be solved. We find

$$\rho \propto a^{-3(1+w)}. \tag{11.62}$$

Thus, in the case of matter we have

$$\rho_{\text{mat}} \propto a^{-3}. \tag{11.63}$$

[2] 1 pc= 3.26 light years=3.26 c · year.

This result is obvious: for matter $\rho_{\mathrm{mat}} \propto \frac{1}{V}$ and $V \propto a^3$.

In the case of radiation

$$\rho_{\mathrm{r}} \propto a^{-4}. \tag{11.64}$$

This behavior is due to the fact that the density of radiation is proportional to $\frac{\omega}{V} \propto \frac{1}{aV}$.

For the case of the cosmological constant $w = -1$ and we find from (11.61)

$$\rho_{\Lambda} = \mathrm{const.} \tag{11.65}$$

From (11.63), (11.64) and (11.65) we conclude that radiation dominates at the beginning of the Universe and the cosmological constant dominates at late times. Let us consider the solutions of the Friedman equation (11.42) for the early Universe at the time when the first term of the right-hand side of the equation dominates (we assume that $w > -\frac{1}{3}$). In this case we have

$$\dot{a} \propto a^{-\frac{3}{2}(1+w)+1}. \tag{11.66}$$

Taking into account the initial condition $a(0) = 0$ we easily find that the solution of this equation has the form

$$a^{\frac{3}{2}(1+w)} \propto t. \tag{11.67}$$

Finally, we have

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{11.68}$$

In the case of matter

$$a(t) \propto t^{\frac{2}{3}} \quad \text{and} \quad H(t) = \frac{\dot{a}}{a} = \frac{2}{3t}. \tag{11.69}$$

For radiation (relativistic particles) we have

$$a(t) \propto t^{\frac{1}{2}} \quad \text{and} \quad H(t) = \frac{\dot{a}}{a} = \frac{1}{2t}. \tag{11.70}$$

If the cosmological constant dominates, the Friedman equation has the form

$$\left(\frac{\hat{a}}{a}\right)^2 = H^2 \simeq \frac{\Lambda}{3} = \mathrm{const.} \tag{11.71}$$

The solution of this equation is given by

$$a(t) \propto e^{Ht}. \tag{11.72}$$

11.3 Early Universe; Neutrino Decoupling

In this section we briefly consider the early Big Bang Universe. In the early Universe the particles were in thermal equilibrium. The equilibrium number density of Fermi (Bose) particles of the type i is given by the expression

$$n_i = \frac{g}{2\pi^2} \int_0^\infty \frac{\sqrt{E^2 - m_i^2}\; E\, dE}{e^{\frac{E-\mu_i}{kT_i}} \pm 1}. \tag{11.73}$$

Here g is the number of the internal degrees of freedom, m_i is the mass of the particle i, μ_i is the chemical potential, T_i is the temperature and $k \simeq 1.38 \cdot 10^{-16}$ erg/grad is the Boltzmann constant. For the equilibrium energy density we have

$$\rho_i = \frac{g}{2\pi^2} \int_0^\infty \frac{\sqrt{E^2 - m_i^2}\; E^2\, dE}{e^{\frac{E-\mu_i}{kT_i}} \pm 1}. \tag{11.74}$$

In the ultra-relativistic case $kT_i \gg m_i,\ kT_i \gg \mu_i$ and from (11.73) we find

$$n_i = \frac{\zeta(3)}{\pi^2} g_i\ (kT_i)^3\ \text{(Bose)}, \quad n_i = \left(\frac{3}{4}\right) \frac{\zeta(3)}{\pi^2} g_i\ (kT_i)^3\ \text{(Fermi)}, \tag{11.75}$$

where $\zeta(3) = 1.202$ ($\zeta(n)$ is the Riemann zeta function). The energy densities are given by the expressions

$$\rho_i = \frac{\pi^2}{30} g_i\ (kT_i)^4\ \text{(Bose)}, \quad \rho_i = \left(\frac{7}{8}\right) \frac{\pi^2}{30} g_i\ (kT_i)^4\ \text{(Fermi)}. \tag{11.76}$$

The total energy density can be presented in the form

$$\rho = \sum_i \rho_i = \frac{\pi^2}{30} g_* (kT)^4. \tag{11.77}$$

Here T is the photon temperature and

$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \left(\frac{7}{8}\right) \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4. \tag{11.78}$$

is the effective number of degrees of freedom of ultra-relativistic particles.

At 1 eV $\ll T \ll$ 1 MeV the only relativistic particles are the photon ($g = 2$), three neutrinos and three antineutrinos ($g_\nu = 6$). Taking into account that $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ (see later) we have, at such temperatures, $g_* = 3.36$. At 1 MeV $\leq$

$T \leq 100$ MeV electrons and positrons are also relativistic particles (four additional degrees of freedom). Taking into account that in this range $T_\nu = T_e = T_\gamma$ we have $g_* = 2 + \frac{7}{8}(10) = 10.75$.

In the case of non-relativistic particles $T \ll m$ the distribution functions both for bosons and fermions are given by the Boltzmann factor

$$f(p) = \frac{1}{(2\pi)^3}\, e^{\frac{-(E-\mu)}{kT}}. \tag{11.79}$$

Taking into account that $E \simeq m + \frac{p^2}{2m}$ we find for the number density

$$n = g \int_0^\infty f(p) 4\pi p^2 dp = \frac{g}{(2\pi)^{3/2}}\,(m\, kT)^{3/2}\, e^{\frac{-(m-\mu)}{kT}}. \tag{11.80}$$

For the energy density in the non-relativistic case we have

$$\rho = mn \tag{11.81}$$

Let us consider now the entropy of the Universe. The second law of thermodynamics has the form

$$T dS = d(\rho V) + p\, dV = d((\rho + p)V) - V\, dp \tag{11.82}$$

From (11.82) we find the relation

$$dp = \frac{(\rho + p)}{T} dT, \tag{11.83}$$

which follows from the condition $\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}$. We have

$$dS = \frac{1}{T} d((\rho + p)V) - \frac{\rho + p}{T^2} V dT = d\left(\frac{\rho + p}{T} V\right) \tag{11.84}$$

Thus, for the total entropy of a system the following expression can be chosen

$$S = \frac{(\rho + p)}{kT} V. \tag{11.85}$$

Further, for the system in thermal equilibrium the total entropy is conserved:

$$S = \text{const.} \tag{11.86}$$

In the case of ultra-relativistic particles we have $p_i = \frac{1}{3}\rho_i$. From (11.76) we find the following expression for the entropy

$$S = \frac{2\pi^2}{45} g_{*s} (kT)^3 V, \tag{11.87}$$

where

$$g_{*s} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \left(\frac{7}{8}\right) \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \tag{11.88}$$

It is obvious that at $T_i = T$ we have $g_{*s} = g_*$.

Because $V \propto a^3$, from (11.86) and (11.87) we find the following relation between the temperature and the scale factor

$$T \propto g_{*s}^{-1/3}\, a^{-1}(t). \tag{11.89}$$

Thus, with expansion of the Universe the temperature drops as $a^{-1}(t)$.

From (11.63) and (11.64) we have for the ratio of energy densities of matter and relativistic particles

$$\frac{\rho_{\text{mat}}}{\rho_{\text{r}}} \propto a(t). \tag{11.90}$$

Taking into account that $a(t) \to 0$ at $t \to 0$ we conclude from (11.90) that in the early Universe ultra-relativistic particles dominate. At this stage the contribution of the curvature term and the cosmological constant term in the Friedman equation can be neglected. For the Hubble parameter we have in this case

$$H = \sqrt{\frac{8\pi G}{3}\rho} = \sqrt{\frac{4\pi^3}{45}}\, g_*^{1/2} \frac{(kT)^2}{M_{\text{P}}}, \tag{11.91}$$

where $M_{\text{P}} = \frac{1}{\sqrt{G}} \simeq 1.2 \cdot 10^{19}$ GeV is the Planck mass. From (11.91) we obtain

$$H = 0.21\, g_*^{1/2} \left(\frac{kT}{\text{MeV}}\right)^2 \text{ s}^{-1}. \tag{11.92}$$

From (11.91) and (11.70) we find that the time of expansion t is connected with the equilibrium temperature T by the following relation

$$t = \frac{1}{2H} = 2.38\, g_*^{-1/2} \left(\frac{\text{MeV}}{kT}\right)^2 \text{ s}. \tag{11.93}$$

The thermodynamic equilibrium takes place if the interaction rates of reactions which are responsible for the equilibrium are larger than the Hubble parameter H which characterizes the expansion rate. When the Universe expands the temperature drops and at some temperatures the interaction rates of reactions which provide thermodynamical equilibrium become comparable with the expansion rate. At such temperatures the equilibrium will not be maintained (the particles decouple with freeze-out abundance). In order to determine the interaction rate let us consider the reaction $a + b \to a' + b'$. The cross section of the reaction is given by the relation

$$\sigma_{fi} = \frac{w_{fi}}{j}, \tag{11.94}$$

where w_{fi} is the number of transitions in unit volume during unit time and $j = n_a n_b v$ ($n_a(n_b)$ being the number density of the particles a (b) and v is the relative velocity). The interaction rate is determined by the relation

$$\Gamma = \sigma_{fi} v n_b = \frac{w_{fi}}{n_a}. \tag{11.95}$$

It is obvious that Γ has the dimension $[M]$ (or s^{-1} in usual units). Let consider the decoupling of neutrinos. In early Universe neutrino equilibrium is kept by the reactions

$$e^+ + e^- \rightleftarrows \nu_l + \bar{\nu}_l, \quad \nu_l(\bar{\nu}_l) + e \rightleftarrows \nu_l(\bar{\nu}_l) + e \;\; (l = e, \mu, \tau) \tag{11.96}$$

We will now estimate the neutrino freeze-out temperature. Thermally averaged cross sections of the weak processes (11.96) are of the order

$$\sigma \simeq G_F^2 (kT)^2. \tag{11.97}$$

Taking into account that the number density of the ultra-relativistic particles is given by $n \simeq (kT)^3$ we have for the neutrino interaction rate

$$\Gamma \simeq G_F^2 (kT)^5. \tag{11.98}$$

The Hubble parameter is given by

$$H \simeq G^{1/2}\,(kT)^2 = \frac{(kT)^2}{M_{\mathrm{P}}}. \tag{11.99}$$

The freeze-out temperature can be estimated from the relation

$$\frac{\Gamma}{H} \simeq 1. \tag{11.100}$$

From (11.98) and (11.99) we have[3]

$$\frac{\Gamma}{H} \simeq G_F^2 M_{\mathrm{P}} (kT)^3 \simeq \left(\frac{kT}{1\,\mathrm{MeV}}\right)^3. \tag{11.101}$$

[3] We take into account that in the units $\hbar = c = 1$ the Fermi constant is equal to $G_F \simeq 1.026 \cdot 10^{-5}\,\frac{1}{m_p^2}$.

Thus, neutrinos decouple at temperatures of the order 1 MeV. At such temperatures the Universe is transparent for neutrinos and the neutrino temperature is proportional to a^{-1}.

At $kT \simeq 1$ MeV γ's and $e^{\pm}$ are in thermal equilibrium. When the temperature drops, $e^{\pm}$ begin to annihilate. The released energy heats up only γ's because neutrinos are decoupled. Thus, after the decoupling of photons their temperature will be higher than the neutrino temperature.

In fact, the effective number of degrees of freedom of γ's and $e^{\pm}$ is equal to $g_* = 2 + \frac{7}{8} \cdot 4 = \frac{11}{2}$. After the annihilation of electrons and positrons we have $g_* = 2$. Taking into account that the entropy is conserved we find

$$\frac{11}{2}(Ta)^3_{\text{before}} = 2(Ta)^3_{\text{after}}. \tag{11.102}$$

Thus, we have

$$\frac{(T)_{\text{after}}}{(T)_{\text{before}}} = \left(\frac{11}{4}\right)^{1/3} \frac{(a)_{\text{before}}}{(a)_{\text{after}}}. \tag{11.103}$$

From the conservation of the entropy of the decoupled neutrinos analogously we find analogously

$$\frac{(T_\nu)_{\text{after}}}{(T_\nu)_{\text{before}}} = \frac{(a)_{\text{before}}}{(a)_{\text{after}}}. \tag{11.104}$$

From (11.103) and (11.104) we find

$$\frac{(T)_{\text{after}}}{(T)_{\text{before}}} = \left(\frac{11}{4}\right)^{1/3} \frac{(T_\nu)_{\text{after}}}{(T_\nu)_{\text{before}}}. \tag{11.105}$$

Now taking into account that $(T_\nu)_{\text{before}} = (T)_{\text{before}}$ we come to the conclusion that after decoupling of the photons the temperatures of neutrinos and photons are connected by the relation

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T. \tag{11.106}$$

This relation between the neutrino and photon temperatures holds also at present. From the study of the cosmic microwave background radiation (CMB) it was found that $T = 2.725$ K. Thus, the neutrino temperature at present is equal to $T_\nu = 1.945$ K and $kT_\nu = 1.676 \cdot 10^{-4}$ eV.

The value of the lightest neutrino mass at present is unknown. Using the existing neutrino oscillation data we have

$$m_2 = \sqrt{m_1^2 + \Delta m_{12}^2} \gtrsim 8.7 \cdot 10^{-3}\text{eV}, \quad m_3 = \sqrt{m_1^2 + \Delta m_{12}^2 + \Delta m_{23}^2} \gtrsim 5 \cdot 10^{-2}\text{eV} \tag{11.107}$$

Thus, at least two neutrinos satisfy the condition $m_{2,3} \gg kT$ and are non-relativistic.[4]

11.4 Gerstein-Zeldovich Bound on Neutrino Masses

After $e^+ - e^-$ annihilation, the number density of γ's is given by the expression

$$n_\gamma = \frac{\zeta(3)}{\pi^2} g_\gamma (kT)^3, \quad g_\gamma = 2. \tag{11.108}$$

The number density of all types of neutrinos and antineutrinos at the temperatures when neutrinos are still ultra-relativistic is equal to

$$n_\nu = \frac{3}{4}\frac{\zeta(3)}{\pi^2} g_\nu (kT_\nu)^3, \quad g_\nu = 6. \tag{11.109}$$

Taking into account (11.106), we obtain the following relation

$$n_\nu = \frac{9}{11} n_\gamma. \tag{11.110}$$

Because neutrinos are decoupled this relation is also valid at present. Taking into account that the photon temperature at present is equal to $T = 2.725$ K we find from (11.108) for the present photon number density

$$n_\gamma = 410.5 \text{ cm}^{-3}. \tag{11.111}$$

From (11.110) follows that neutrino density at present is equal to

$$n_\nu = 336 \text{ cm}^{-3}. \tag{11.112}$$

These numbers can be compared with the number density of baryons in the Universe

$$n_B = 2.5 \cdot 10^{-7} \text{ cm}^{-3}. \tag{11.113}$$

Thus, *photons and neutrinos are the most abundant particles in the Universe*.

Because the neutrino number density is so large it is possible to obtain a limit on neutrino masses directly from the measurement of the cosmological parameters. (Gerstein and Zeldovich). Let us assume that all neutrinos at present are

[4] We assumed the normal neutrino mass spectrum. It is obvious that in the case of the inverted spectrum this conclusion is also valid.

non-relativistic and neutrino mass spectrum is quasi-degenerate ($m_1 \simeq m_2 \simeq m_3 \simeq \frac{1}{3}\sum_i m_i$). We have

$$\Omega_\nu = \frac{\sum_i m_i n_i}{\rho_c} \simeq \frac{\sum_i m_i \, n_\nu}{3 \, \rho_c}, \tag{11.114}$$

where $\rho_c = \frac{3H^2}{8\pi G}$ is the critical density. From (11.113) we find[5]

$$\Omega_\nu \simeq \frac{\sum_i m_i}{94 \, h^2 \text{ eV}}. \tag{11.115}$$

Neutrinos can be a part of the dark matter. It is obvious, however, that

$$\Omega_\nu \leq \Omega_{\text{DM}}. \tag{11.116}$$

From the analysis of the existing data follows that $\Omega_{\text{DM}} \simeq 0.20$ and $h \simeq 0.73$. Thus, from (11.115) and (11.116) we find

$$\sum_i m_i \leq 10 \text{ eV}. \tag{11.117}$$

From (11.117) we have for one neutrino mass $m_i \leq 3.3$ eV. Notice that from the measurement of the tritium β-spectrum (Troitsk, Meinz) the comparable bound ($m_i \leq 2.2$ eV) was found. We will see later that from CMB data and from data on the investigation of the large scale structure of the Universe bounds of about one order of magnitude better than (11.117) can be derived. Unlike the relic photons, the relic neutrinos have not been observed. Their observation is the extremely challenging problem. The cross section of the neutrino-nucleon scattering is so small ($\sim 10^{-62}$ cm^2) that the observation of this process does not look possible.

In principle there exist a possibility to reveal the existence of the relic neutrinos through the observation of the effect of the resonance production of Z^0-bosons in the interaction of ultra-high energy cosmic neutrinos with relic neutrinos

$$\nu(\bar{\nu}) + \bar{\nu}_i(\nu_i) \to Z^0. \tag{11.118}$$

The cross section of the process (11.118) averaged over the resonance is large and it is given by the expression

$$\sigma_R \simeq \frac{4\pi G_F}{\sqrt{2}} = 4.2 \cdot 10^{-32} \text{ cm}^2. \tag{11.119}$$

The effect of relic neutrinos can be observed as dips in the spectrum of ultra-high energy neutrinos at resonance energies

[5] In fact, we have $\frac{\rho_c}{\frac{1}{3}n_\nu} = \frac{1.054 \cdot 10^4 \text{eV cm}^{-3} \, h^2}{112 \text{ cm}^{-3}} \simeq 94 \, h^2$ eV.

$$E_i^R = \frac{m_Z^2}{2m_i} \simeq 4 \frac{1}{m_i/\text{eV}}\ 10^{21}\text{eV}. \tag{11.120}$$

This relation follows from the condition: $(k + k_i)^2 = m_Z^2$ where k and k_i are the momenta of cosmic and relic neutrinos, respectively.

The dips in the spectrum could be observed by the Auger, ANITA, EUSO, Ice-Cube and other detectors for ultra-high energy neutrinos. However, the existence of sources of neutrinos of such high energy is an open problem.

11.5 Big Bang Nucleosynthesis and the Number of Light Neutrinos

The measurement of the abundances of the light elements D, ^{3}He, ^{4}He and ^{7}Li produced in the end of the first three minutes of the evolution of the Universe provides one of the most important confirmations of the Big Bang theory. The detailed study of the primordial nucleosynthesis allows to obtain information about the number of light neutrinos. We will discuss here briefly the primordial nucleosynthesis.

The thermodynamic equilibrium between protons and neutrons in the early Universe is due to the neutrino processes

$$\nu_e + n \rightleftarrows e^- + p \tag{11.121}$$

and

$$\bar{\nu}_e + p \rightleftarrows e^+ + n. \tag{11.122}$$

Let us estimate the temperature at which neutrons and protons decouple. The $n \rightleftarrows p$ conversion rate due to the weak processes (11.121) and (11.122) is determined by the square of the Fermi constant and is given by

$$\Gamma_{n \rightleftarrows p} \simeq G_F^2 (k\,T)^5. \tag{11.123}$$

The decoupling temperature can be estimated from the condition $\Gamma_{n \rightleftarrows p} \simeq H$. We have

$$G_F^2 (k\,T_{\text{dec}})^5 \simeq \frac{(k\,T_{\text{dec}})^2}{M_P}, \tag{11.124}$$

where M_P is the Planck mass. From this relation we have

$$k\,T_{\text{dec}} \simeq 1\ \text{MeV}. \tag{11.125}$$

This temperature corresponds to $t \simeq 1$ s.

The equilibrium number densities of protons and neutrons are equal to (see (11.80))

$$n_{p,n} = \frac{2}{(2\pi)^{3/2}}\,(m_{p,n}kT)^{3/2}\,e^{\frac{-(m_{p,n}-\mu_{p,n})}{kT}}. \tag{11.126}$$

Taking into account that $\frac{\mu_{p,n}}{kT} \ll 1$ we obtain for the ratio $\frac{n_n}{n_p} = \frac{n}{p}$ at the temperature of decoupling

$$\frac{n}{p} \simeq e^{-\frac{\Delta m}{kT_{\rm dec}}}, \tag{11.127}$$

where $\Delta m = m_n - m_p = 1.293$ MeV is the neutron-proton mass difference. We have

$$\frac{n}{p} \simeq \frac{1}{6}. \tag{11.128}$$

Before discussing the nucleosynthesis let us determine the quantity

$$\eta = \frac{n_B}{n_\gamma}, \tag{11.129}$$

where n_γ is the number density of γ's and

$$n_B = n_p + n_n + An_A \tag{11.130}$$

is the baryon number density (n_A is the number density of nuclei with atomic number A). From experimental data follows that (see later)

$$\eta \simeq 6 \cdot 10^{-10}. \tag{11.131}$$

For the nucleosynthesis to start it is necessary that deuterium will be produced. Deuterium is produced in the process

$$n + p \to d + \gamma. \tag{11.132}$$

However, at $kT \simeq 1$ MeV because of the large number density of γ's (see 11.131) and the small bounding energy of D ($\varepsilon_D \simeq 2.23$ MeV) deuterium is dissociated due to the reaction

$$\gamma + d \to n + p. \tag{11.133}$$

Nucleosynthesis starts at the time when the temperature drops to $kT \simeq 0.1$ MeV.[6] When nucleosynthesis starts, practically all neutrons will be bound in ^{4}He, a light nucleus with the largest binding energy ($\varepsilon_{^4\mathrm{He}} \simeq 28.3$ MeV) through the chain of reactions

$$n + p \to d + \gamma, \quad p + d \to ^3\mathrm{He} + \gamma, \quad d + ^3\mathrm{He} \to ^4\mathrm{He} + p \text{ and others} \quad (11.134)$$

Due to the decay of neutrons ($\tau_n = 885.7 \pm 0.8$ s) during the time from the decoupling of neutrons till the beginning of the nucleosynthesis the ratio $\frac{n}{p}$ drops from $\sim \frac{1}{6}$ to

$$\frac{n}{p} \simeq \frac{1}{7}. \quad (11.135)$$

It is easy to estimate the mass fraction of ^{4}He produced during the nucleosynthesis. Taking into account that $n_{^4\mathrm{He}} \simeq \frac{n_n}{2}$ we find

$$Y_p = \frac{4n_{^4\mathrm{He}}}{n_N} \simeq \frac{2\frac{n}{p}}{1+\frac{n}{p}} = 0.25. \quad (11.136)$$

When the reaction rates of nucleosynthesis are smaller than the expansion rate (at the small number density of deuterium) the abundances of D and ^{3}He are frozen out. The predicted abundances of D and ^{3}He are decreased with the increase of η and are in the range $(10^{-5} - 10^{-4})$.

In the primordial nucleosynthesis a small amount of ^{7}Li is produced in the reactions

$$^3\mathrm{H} + ^4\mathrm{He} \to ^7\mathrm{Li} + \gamma, \quad ^3\mathrm{He} + ^4\mathrm{He} \to ^7\mathrm{Be} + \gamma, \quad ^7\mathrm{Be} + e^- \to ^7\mathrm{Li} + \nu_e. \quad (11.137)$$

The predicted abundance of ^{7}Li depends on η and lies in the range $(10^{-10} - 10^{-9})$.

Light elements were produced in the Big Bang nucleosynthesis during the time from 1 to 3 min after the Big Bang. Due to effects of the following stellar nucleosynthesis, the estimation of the systematic errors in the present-day measurements of primordial abundances of light elements is a complicated problem. For the abundances of D,^{4} He and ^{7}Li the values were found

$$\frac{\mathrm{D}}{\mathrm{H}} = (2.84 \pm 0.26)\ 10^{-5}, \quad Y_p = 0.249 \pm 0.009, \quad \frac{\mathrm{Li}}{\mathrm{H}} = (1.7^{+1.2}_{-0.02})\ 10^{-10}. \quad (11.138)$$

[6] The temperature of the nucleosynthesis can be estimated from the condition $\bar{n}_\gamma \lesssim n_N$, where $\bar{n}_\gamma \simeq e^{-\frac{\varepsilon_D}{kT}}\ n_\gamma$ is the number of γ's above the threshold of the reaction (11.133). From this condition we have $kT \simeq -\frac{\varepsilon_D}{\ln \eta} \simeq 0.1$ MeV.

Notice that the value for the primordial abundance of ^{4}He is in good agreement with the estimate (11.136). No reliable data for the primordial abundance of ^{3}He exists at the moment.

In the framework of the Standard Model with three types of neutrinos the primordial abundances of the light elements depend only on the parameter η. The values presented in (11.138) are compatible at the 95% level if the parameter η lies in the range

$$4.7 \cdot 10^{-10} \leq \eta \leq 6.5 \cdot 10^{-10}. \tag{11.139}$$

From (11.139) we have for relative baryon density $\Omega_B = \frac{m_N n_B}{\rho_c}$

$$0.017 \leq \Omega_B h^2 \leq 0.024. \tag{11.140}$$

The parameter η and, consequently $\Omega_B h^2$, was determined from the WMAP measurements of the temperature fluctuations of the cosmic microwave background radiation. It was found

$$\eta = (6.11 \pm 0.19) \cdot 10^{-10} \ \text{ and } \ \Omega_B h^2 = 0.0223 \pm 0.0007. \tag{11.141}$$

The agreement of (11.139), (11.140) and (11.141) manifests a success of the theory of the Big Bang nucleosynthesis. The agreement of the theory of the Big Bang nucleosynthesis with the measurements allows to limit the number of possible additional light neutrino types. At temperatures $kT \simeq 1$ Mev the energy density of the Universe is determined by photons, $e^{\pm}$, neutrinos and antineutrinos. The effective number of the relativistic degrees of freedom can be written as

$$g_* = 5.5 + 2\frac{7}{8}\, N_\nu, \tag{11.142}$$

where N_ν is the number of neutrino types. In the Standard Model $N_\nu = 3$. If $N_\nu > 3$ the expansion rate

$$H = \sqrt{\frac{8\pi}{3} G g_*}\, (k\,T)^2 \tag{11.143}$$

will be larger and, as a result, the decoupling temperature will also be larger than in the standard case. If the decoupling temperature will be larger the ratio $\frac{n}{p}$ will be larger and the primordial abundance of ^{4}He will be larger than in the case of $N_\nu = 3$. From the primordial abundance of ^{4}He together with the CMB value of the parameter η the value

$$N_\nu = 3.14^{+0.70}_{-0.65} \tag{11.144}$$

was found for the number of neutrino types. The value (11.144) is in an agreement with the number of neutrino types measured via investigation of the decay $Z^0 \to \nu + \bar{\nu}$ at LEP:

$$N_\nu = 2.9840 \pm 0.0082. \tag{11.145}$$

11.6 Large Scale Structure of the Universe and Neutrino Masses

Information about the sum of neutrino masses $\sum_i m_i$ can be inferred from the study of the Large Scale Structure (LSS) of the Universe. There are two recent galaxy surveys: the 2dF Galaxy Redshift Survey (220,000 galaxies) and the Sloan Digital Sky Survey (SDSS) (about one million galaxies). From these surveys it is possible to obtain the so-called power spectrum.

The density fluctuation $\delta(\mathbf{x})$ is given by the relation

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x}) - < \rho >}{< \rho >}, \tag{11.146}$$

where $< \rho >$ is the average density. The Fourier component of the function $\delta(\mathbf{x})$ is given by

$$\delta(\mathbf{k}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k}\mathbf{x}} \delta(\mathbf{x})\, d^3x. \tag{11.147}$$

The power spectrum $P(k)$ is determined as follows

$$P(\mathbf{k}) =< |\delta(\mathbf{k})|^2 >= P(k). \tag{11.148}$$

The primordial spectrum $P(k)$ is determined by the initial conditions at early times. It is usually assumed that the primordial spectrum has a power-law form

$$P(k) = A\, k^n, \tag{11.149}$$

where A is a constant. From experimental data follows that the primordial power spectrum is of the Harrison-Zeldovich form, i.e. $n \simeq 1$.

The power spectrum at present is given by the relation

$$P(k, t_0) = T^2(k)\; P(k), \tag{11.150}$$

where the transfer function $T(k)$ is determined by the evolution of initial perturbations. This function can be obtained from calculations which must take into account complicated physics and mathematics connected with the growth of the original density perturbations (such as nonlinear effects at a later stage of galaxy structure formation, etc.).

The measurement of the LSS of the Universe allows us to determine the power spectrum $P_g(k, t_0)$ of visible matter. It follows from calculations that the power spectrum of all matter (visible and dark) $P_m(k, t_0)$ is different from the power spectrum of visible galaxies. Thus, for the comparison of the measurements and theory we need to know the bias parameter

$$b^2(k) = \frac{P_g(k, t_0)}{P_m(k, t_0)}. \tag{11.151}$$

Notice that this parameter, in principle, can be determined from higher order correlations.

Although the contribution of neutrinos to the matter density of the Universe is small ($\frac{\Omega_\nu}{\Omega_m} < 7\%$) the analysis of the modern high precision cosmological data allow to obtain rather stringent limits on the sum of neutrino masses $\sum_i m_i$. We will present here a qualitative argument in favor of the high sensitivity of the LSS data to neutrino masses.

The growth of density fluctuations induced by the gravitational attraction is given by the relation

$$\delta\rho \backsim a^p. \tag{11.152}$$

If all matter is able to cluster, $p = 1$ and $\delta\rho$ is proportional to a. In general

$$p = \Omega_*^{3/5}, \tag{11.153}$$

where Ω_* is fraction of matter which can cluster. On the scale where neutrinos are not clustering we have

$$\Omega_* = \frac{\Omega_m - \Omega_\nu}{\Omega_m} = 1 - f_\nu, \tag{11.154}$$

where Ω_m is the density of all matter, Ω_ν is the neutrino density and

$$f_\nu = \frac{\Omega_\nu}{\Omega_m}. \tag{11.155}$$

Density fluctuations start growing at the beginning of the mater dominated era (scale factor a_M) and they stop growing at the time when the dark energy dominated era starts (scale factor a_Λ). The growth of the fluctuations during this time is given by the factor

$$\left(\frac{a_\Lambda}{a_M}\right)^{\Omega_*^{3/5}} \simeq \left(\frac{a_\Lambda}{a_M}\right) e^{-\frac{3}{5} f_\nu \ln \frac{a_\Lambda}{a_M}}, \tag{11.156}$$

where the exponent gives the suppression of the growth of the fluctuations due to neutrino non-clustering. For the suppression of the power spectrum at a scale where neutrinos do not cluster we have

$$\frac{P(k,\sum m_i)}{P(k,0)} \simeq e^{-\frac{6}{5}f_\nu \ln\frac{a_\Lambda}{a_M}}. \tag{11.157}$$

For the fraction of neutrinos f_ν we have (see (11.115))

$$f_\nu = \frac{\sum m_i}{94\ \mathrm{eV}\ \Omega_m h^2}. \tag{11.158}$$

From analysis of CMB data it was found that $\Omega_m h^2 \simeq 0.15$. Thus, we have

$$f_\nu \simeq \frac{\sum m_i}{14\ \mathrm{eV}}. \tag{11.159}$$

If $\sum m_i \simeq 1$ eV in this case $f_\nu \simeq 0.07$. High sensitivity of LSS data to the parameter $\sum m_i$ is connected with the fact that the ratio $\left(\frac{a_\Lambda}{a_M}\right)$ is large. It was found from calculations that

$$\frac{P(k,\sum m_i)}{P(k,0)} \simeq e^{-8f_\nu}. \tag{11.160}$$

At $\sum m_i \simeq 1$ eV, the suppression factor is about two.

The large velocities of neutrinos prevent their clustering with matter at scales smaller that the free-streaming wave length λ which is determined by the distance that neutrino can pass during the Hubble time $\frac{1}{H}$. The free-streaming wave length is determined as

$$\lambda = 2\pi\sqrt{\frac{2}{3}}\frac{v}{H}, \tag{11.161}$$

where v is the neutrino velocity. For non-relativistic neutrinos with mass m we have

$$v \simeq \frac{\bar{p}}{m} \simeq 3\frac{kT}{m}. \tag{11.162}$$

In order to estimate v we will take into account that $T \backsim a^{-1}$ and that the neutrino temperature T is connected with the photon temperature T_γ by the relation $T = (\frac{4}{11})^{1/3}T_\gamma$. We have $T = (1+z)(\frac{4}{11})^{1/3}T^0_\gamma$, where $T^0_\gamma = 2.73$ K is the temperature of the cosmic microwave background radiation and z is red-shift ($z = \frac{a^0}{a}$). We have

$$v \simeq 150\ (1+z)\left(\frac{1\ \mathrm{eV}}{m}\right). \tag{11.163}$$

For the wave number we have

$$k = 2\pi \frac{a}{\lambda} = \frac{\sqrt{4\pi G\rho a^2}}{v} \tag{11.164}$$

The minimal wave length, which corresponds to the time when neutrinos became non relativistic, is given by the relation

$$k_{\mathrm{nr}} \simeq 2 \cdot 10^{-2} \Omega_m^{1/2} \left(\frac{m}{1\ \mathrm{eV}}\right) h\ \mathrm{Mpc}^{-1}, \tag{11.165}$$

which can be obtained from the condition $\bar{p} \simeq m$.

At $k > k_{\mathrm{nr}}$ the power-spectrum suppression due to neutrino masses can be observed. In the region $k < k_{\mathrm{nr}}$, corresponding to scales larger than the horizon, there is no suppression of the power spectrum.

At Fig. 11.2 the power spectrum is presented. If $\sum m_i \simeq 1$ eV the spectrum is suppressed by a factor of two (with respect to the $\sum m_i = 0$ spectrum) at $k > k_{\mathrm{nr}}$.

From the analysis of cosmological data many different bounds on the sum of the neutrino masses in the interval $\sum m_i \leq (0.4\text{–}1.7)$ eV were inferred. From the analysis made by the SDSS collaboration a conservative bound

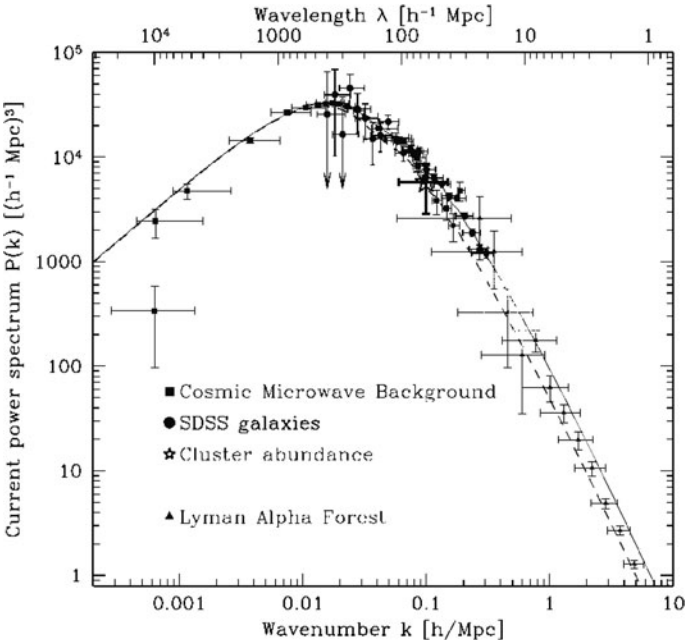

Fig. 11.2 Power spectrum of the distribution of Galaxies in the Universe. The *solid curve* was calculated for the values of matter density, Hubble parameter and barion fraction equal to $\Omega_m = 0.28$, $h = 0.72$, and $\frac{\Omega_b}{\Omega_m} = 0.16$, correspondingly. The *dashed curve* was calculated under the assumption that neutrinos constitute 7% of the dark matter. This assumption corresponds to the value of the sum of neutrino masses equal to $\sum m_i = 1$ eV (arXiv: hep-ph/0503257)

$$\sum m_i \leq 1.72 \text{ eV} \tag{11.166}$$

was obtained.

The bound on the sum of neutrino masses depends on the values of other cosmological parameters. For example, a change of the spectral index n in the primordial power spectrum (11.149) can partially mimic the effect of neutrino masses. The joint analysis of the SDSS data and the Cosmic Microwave Background radiation data, which strongly constrain the values of the cosmological parameters, allows to obtain more stringent and reliable bound on the parameter $\sum m_i$. We will briefly discuss these data in the next section.

We mentioned before that one of the problems of the determination of the parameter $\sum m_i$ from cosmological data is connected with the fact that the existing LSS surveys give information about the distribution of luminous mater which is only about 15% of all matter. Let us notice that future weak gravitational lensing measurements will allow us to obtain information about the power spectrum of the distribution of all matter including the dark matter.

11.7 Cosmic Microwave Background Radiation

The measurements of anisotropies of the Cosmic Microwave Background (CMB) radiation provide a profound confirmation of the Big Bang cosmology. These measurements allow one to obtain the most precise values of cosmological parameters.

The spectrum of the CMB radiation is an ideal Planck spectrum which is characterized by the temperature. The mean measured temperature is $\bar{T} = (2.275 \pm 0.001)$ K. Starting from the pioneering COBE satellite observation, in different experiments the measurement of the temperature variation of the CMB radiation at the level of 10^{-5} was performed. The most precise recent results were obtained by the Wilkinson Microwave Anisotropy Probe (WMAP).

The cosmic microwave background radiation (relic radiation) consists of photons which were emitted at the time when protons and electrons produced hydrogen atoms (recombination era). The temperature at that time (about 380,000 years after the Big Bang) dropped to 3,000 K. Produced during the recombination era photons decouple from matter and freely propagate in the Universe.

Anisotropies of the CMB are due to the Sachs-Wolfe effect (the fluctuation of temperature due to perturbations of the gravitational potential), due to fluctuations of the matter density and other effects.

Let us denote by $\delta T(\mathbf{n})$ the temperature fluctuation in the direction characterized by the vector $\mathbf{n}$ ($\delta T = T - \bar{T}$). The two-point correlation function is determined as follows

$$C(\theta) =< \delta T(\mathbf{n})\, \delta T(\mathbf{n}') >, \tag{11.167}$$

where $\mathbf{n}\cdot\mathbf{n}' = \cos\theta$ and averaging over all pairs separated by the angle θ is assumed. Notice that because of the isotropy of the space the correlation function depends only on the angle θ. The correlation function can be expanded over Legendre polynomials. We have

$$C(\theta) = \sum_l \frac{(2l+1)}{4\pi} C_l \, P_l(\cos\theta). \tag{11.168}$$

The power spectrum is determined as follows

$$\Delta_l = \frac{l(l+1)}{2\pi} C_l. \tag{11.169}$$

At large l a spherical harmonic analysis becomes a Fourier analysis in two dimensions; l becomes the Fourier wave number and

$$\theta \sim \frac{2\pi}{l}. \tag{11.170}$$

The measured power spectrum has a characteristic peak structure.[7] The position of the first, main peak is determined by the curvature of space:

$$l_1 \simeq \frac{200}{\sqrt{1-\Omega_k}}. \tag{11.171}$$

From existing data follows that $\Omega_k = 0$, i.e. that out Universe is flat.

From the analysis of the latest WMAP data, collected during 5 years, the following values were obtained for the basic cosmological parameters :

$$\text{Hubble parameter} : \quad H_0 = 71.9^{+2.6}_{-2.7}\,\text{kms}^{-1}\text{Mpc}^{-1}, \tag{11.172}$$

$$\text{Baryon density} : \quad \Omega_b = 0.0441 \pm 0.0030, \tag{11.173}$$

$$\text{Dark matter density} : \quad \Omega_c = 0.214 \pm 0.027, \tag{11.174}$$

$$\text{Dark energy density} : \quad \Omega_\Lambda = 0.742 \pm 0.030. \tag{11.175}$$

From the analysis of the WMAP data the following bound was obtained for the neutrino density

$$\Omega_\nu h^2 < 0.014. \tag{11.176}$$

Taking into account that $\sum_i m_i/94\,\text{eV} = \Omega_\nu h^2$ we have the following upper bound for the sum of the neutrino masses

[7] In the early Universe baryons and photons can be treated as a fluid. The combination of effects of gravity and pressure of radiation creates longitudinal acoustic oscillations in the photon-baryon fluid. The sound wave can be decomposed into a superposition of modes with different wave numbers $k \simeq \frac{1}{\lambda}$. The wave length λ corresponds to an observable angle θ. As follows from (11.170), the observed peaks are relevant to characteristic distances.

$$\sum_i m_i < 1.3 \text{ eV}. \tag{11.177}$$

Thus, the existing CMB data allow to obtain only a rather moderate restriction on the neutrino masses. This is connected with the fact that at the epoch of the recombination, neutrino energies are at 1 eV level. In fact, at this epoch the temperature of γ's was equal to $T_\gamma \simeq 3,000$ K and $kT_\gamma \simeq 0.26$ eV. For the average neutrino energy we have $< E >= 3.15\, kT_\nu = 3.15 \left(\frac{4}{11}\right)^{1/3} kT_\gamma \simeq 0.6$ eV. Therefore, if the neutrino masses are much less than ~ 0.6 eV, neutrinos are ultra-relativistic particles and the effect of neutrino masses cannot be revealed through the observation of the CMB radiation.

Bound on the sum of neutrino masses stronger than (11.177) can be obtained if WMAP data are analysed together with other cosmological data. From the analysis of the WMAP 5-year data together with the baryonic acoustic oscillation data and supernova data it was found

$$\sum_i m_i < 0.61 \text{ eV}. \tag{11.178}$$

Notice that from WMAP data some constraint on the effective number of neutrinos can be obtained. From the WMAP 5-year data was found

$$N_\nu = 4.4 \pm 1.5. \tag{11.179}$$

11.8 Supernova Neutrinos

Neutrinos play an extremely important role in gravitational collapse of some stars (supernovae explosion). The observation of solar neutrinos and observation of neutrinos from the explosion of the supernovae SN1987A opened a new field in astronomy, neutrino astronomy. In 2002 M. Koshiba and R. Davis were awarded with Nobel Prize for the discovery of this new field .

The supernova explosion is a very complicated phenomenon. Its understanding is in the process of permanent improvement. Here we will discuss briefly main stages of the gravitational collapse of a supernova with a mass larger than eight solar masses (type II supernova) in which a large number of neutrinos are produced. Such a star evolves through nuclear fusion of hydrogen, helium, carbon and other nuclei and takes an onion-like structure with an iron core surrounded by shells of hydrogen, helium, carbon, neon, oxygen and silicon. Since iron nuclei are the most tightly bound nuclei, production of iron is the final stage of the chains of reactions of nuclear fusion.

The iron core typically has a mass of about one solar mass, a radius of a few thousand kilometer, a central density of about 10^{10} g cm^{-3} and a central temperature of about 1 MeV.

The gravitational contraction of the iron core is equilibrated by the electron degeneracy pressure. When the iron core grows to about the Chandrasecar mass

(1.44 solar mass) an electron degeneracy pressure cannot prevent the gravitational contraction and the core collapses.

During the collapse a huge amount of energy is released. The released gravitational energy is given by the binding energy of the core

$$\Delta E \simeq \frac{3}{5}\frac{GM^2}{R}, \tag{11.180}$$

where M and R are mass and radius of the core. At $M \simeq M_{\odot}$ and $R \simeq 10$ km the energy $\simeq 3 \cdot 10^{53}$ erg is released. About 99% of this energy is carried away by neutrinos.

At an earlier stage of the collapse, photodissociation of iron nuclei

$$\gamma +^{56}\mathrm{Fe} \to 13\alpha + 4n \tag{11.181}$$

(due to increase of the temperature) and electron capture processes

$$e^- + (A.Z) \to \nu_e + (A, Z-1), \tag{11.182}$$

and

$$e^- + p \to \nu_e + n \tag{11.183}$$

reduce the core energy and the electron density and accelerate the collapse.

When the density of the core reaches a value of $\sim 10^{12}$ g cm^{-3} neutrinos become trapped because their mean free path at such a density is much smaller than the radius of the core. At this stage, the inner part of the core ($\sim 0.8 M_{\odot}$) collapses with subsonic velocities proportional to the radius (homologous collapse). The outer part of the core collapses with supersonic free-fall velocities.

When the density of the core reaches nuclear density of $\sim 10^{14}$ g cm^{-3} the pressure of the degenerate nucleons stops the collapse of the inner part of the core. The radius of the core at this stage is about 10 km.

The stop of the collapse of the inner core creates shock wave which propagates outward through the outer part of the core.

The shock wave propagating through infalling matter of the outer part of the core dissociates nuclei into protons and neutrons. The capture rate of electrons on protons ($e^- p \to \nu_e + n$) is larger than on nuclei. As a result protons will be transferred into neutrons with the emission of ν_e's. When the shock reaches the region with a density of $\sim 10^{11}$ g cm^{-3} neutrinos produced behind the shock leave the star (neutronization burst). Emitted ν_e's carry away about 10^{51} erg during a few milliseconds.

Because of the loss of energy through photodisintegration of nuclei and neutrino emission the shock at a 100–200 km radius is weakened, stalled and turned into an accretion disc. A remnant begins to form by accretion of infalling material. The remnant evolves to a neutron star or a black hole depending on the mass of the progenitor star.

It is a standard assumption, confirmed by supernova simulations, that the stalled shock is revived by neutrinos streaming off the hot core of the proto-neutron star. These neutrinos carry out most of the released energy. The energy input by neutrinos drives the shock outward leading to the supernova explosion in about 0.5 s after the bounce.

Neutrinos are produced in the hot core via the NC reactions

$$\begin{aligned} &e^- + e^+ \to \nu_l + \bar{\nu}_l, \quad e^\pm + N \to e^\pm + N + \nu_l + \bar{\nu}_l \\ &N + N \to N + N + \nu_l + \bar{\nu}_l, \quad \gamma + e^\pm \to e^\pm + \nu_l + \bar{\nu}_l, \ldots \end{aligned} \tag{11.184}$$

where $l = e, \mu.\tau$.

Neutrinos are trapped in the inner high-density part of the proto-neutron star. It takes a fraction of a second for the trapped neutrinos to diffuse out to the neutrino-sphere where the density is about $\sim 10^{11}$ g cm^{-3}. Neutrinos can leave the star from this region. Because ν_e and $\bar{\nu}_e$ have both CC and NC interactions and $\nu_{\mu,\tau}$ and $\bar{\nu}_{\mu,\tau}$ have only (universal) NC interaction there are three different energy-dependent neutrino-spheres with radii from about 50 to 100 km. Neutrinos are emitted from the neutrino-spheres with black-body spectrum and average energies in the range (10–20) MeV. The emission of thermal neutrinos of all flavors continues for a few second. These neutrinos carry out practically all energy produced in the supernova explosion ($\sim 3 \cdot 10^{53}$ erg).

On 23 February 1957 in the Large Magellanic Claud (a nearby galaxy) at a distance of about 51.4 kpc from the earth a supernova explosion was observed. This supernova was named SN1987A. In three big underground neutrino detectors Kamiokande II, IMB and Baksan at the same time (up to uncertainties in time calibrations) neutrino bursts with neutrino energies of about 15 MeV were observed. The neutrino events were detected about three hours before the first optical observation of SN1987A was done.[8] The observed neutrino bursts lasted about 10 s.

In all three underground detectors antineutrinos were observed via the reaction

$$\bar{\nu}_e + p \to e^+ + n. \tag{11.185}$$

The cross section of this reaction

$$\sigma \simeq 8.5 \cdot 10^{-44} \left(\frac{E}{\text{MeV}}\right)^2 \text{cm}^2 \tag{11.186}$$

is much larger than the cross sections of other possible reactions. Eleven antineutrino events were observed in the Kamiokande II detector, eight events in the IMB detector and five candidate-events in the Baksan detector.

[8] This corresponds to the general theory of the supernova explosion: neutrinos are produced in about 10 s after core collapse and visible light is produced later after the shock reaches the surface of the star.

The first observation of supernova neutrinos opened a new era in astronomy. Although only about twenty neutrino events were detected, these observations confirm general features of a core-collapse supernova explosion we discussed before.

From one to three core-collapse supernovae per century is expected in our galaxy. When this happens, in modern and future neutrino detectors thousands of supernova neutrino events will be detected. The detailed investigation of supernova neutrinos will be very important for the study of the mechanism of supernova explosions and for obtaining (because of the MSW matter effects) information about the mixing angle θ_{13} and the character of neutrino mass spectrum. Recently some new collective nonlinear matter effects due to neutrino–neutrino interaction were studied in details. A discussion of all these subjects is out of the scope of this book.

11.9 Baryogenesis Through Leptogenesis

The existing data represent evidence that our Universe predominantly consists of matter. Thus, the baryon-antibaryon asymmetry of the Universe determined by the ratio

$$\eta_B = \frac{n_B - n_{\bar{B}}}{n_\gamma} \tag{11.187}$$

($n_B(n_{\bar{B}})$ is the baryon (antibaryon) number density, n_γ is the photon number density) practically coincide with the ratio

$$\eta = \frac{n_B}{n_\gamma} = (6.11 \pm 0.19) \cdot 10^{-10}, \tag{11.188}$$

determined from measurements of the primordial abundances of D, ^{3}He, ^{4}He and ^{7}Li and from the measurement of the anisotropy of the Cosmic Microwave Background Radiation.

In the Big Bang Universe there was no initial baryon asymmetry. The explanation of the baryon asymmetry of the Universe is a big problem for modern cosmology. There are several approaches to the solution of this problem. During the last years, the idea of the generation of the baryon asymmetry of the Universe through the lepton asymmetry, produced by CP violating decays of heavy Majorana particles, has actively been developing. This approach was inspired by the discovery of nonzero neutrino masses and by the seesaw mechanism of the neutrino mass generation.

In this section we will briefly consider the main ideas of leptogenesis. In order that in the evolution of the Universe, the baryon asymmetry was created, the following three conditions, formulated by Saharov, must be satisfied:

- The baryon number has to be violated at some stage of the evolution.
- C and CP must be violated.
- Departure from thermal equilibrium must take place.

In fact, if the baryon number is conserved and invariance under C and CP transformation holds, the baryon asymmetry cannot be generated if the initial state is a state with $B = 0$ and definite C and CP values. Due to the CPT invariance, the masses of a baryon and an antibaryon are equal. This means that in the case of thermal equilibrium, the number densities of baryons and antibaryons are the same.

In the Standard Model, C and CP are not conserved. Because left-handed and right-handed fields enter differently in the Lagrangian of the interaction of the fermion and vector fields, baryon and lepton currents in the SM are also not conserved. It can be shown that the SM allows processes with

$$\Delta B = \Delta L = \pm 3. \tag{11.189}$$

From (11.189) follows that in the SM, the sum $(B+L)$ is violated and the difference $(B-L)$ is conserved.

The baryon and lepton numbers are not conserved in processes of transitions between different vacua which have different topological charges and are separated by a potential barrier. The heights of the barriers are given by the sphaleron energy (a saddle point of the energy of gauge and Higgs fields). At $T \ll 100$ GeV the rate of such tunnel transitions is determined by the instanton action and is negligibly small ($\Gamma \simeq e^{-\frac{4\pi}{\alpha}} \simeq 10^{-165}$). At temperatures higher than $\simeq$100 GeV, transitions over the barrier due to thermal fluctuations become important and the rate of $(B+L)$-violating processes can be significant. However, *the SM cannot explain the baryon asymmetry of the Universe*.

- In the Standard Model the CP asymmetry is different from zero only if all three families are involved. This means that masses of all quarks and all mixing angles must enter into the CP asymmetry. As a result, the CP asymmetry in the SM is suppressed by the smallness of masses of light quarks with respect to the scale of the electroweak breaking and by the smallness of the product $s_{12}s_{23}s_{13}$. The estimated value of the asymmetry ($\sim 10^{-18}$) is too small for the explanation of the baryon asymmetry of the Universe.
- The departure from equilibrium can be satisfied if the mass of the Higgs boson m_H is less than $\sim$ 70 GeV. From the data of the LEP experiments follows that $m_H > 115$ GeV.
- The conservation of $(B-L)$ prevents the generation of the baryon asymmetry (sphaleron processes wash out the baryon asymmetry).

Thus, the baryon asymmetry of the Universe can only be explained in the framework of physics beyond the SM. This new physics must insure: (1) a new source of CP violation; (2) $(B-L)$ violation; (3) out of equilibrium processes.

The leptogenesis is a scenario in which new physical processes generate a lepton asymmetry of the Universe which is partially converted into a baryon asymmetry through sphaleron processes.

Let us assume that there exist Majorana particles N_i with masses M_i much larger than the electroweak vacuum expectation value $v \simeq 246$ GeV and that the fields

$N_i(x)$ ($N_i(x) = N_i^c(x)$) are singlets of the electroweak $SU(2) \times U(1)$ group. The mass term of N_i and the Lagrangian of the Yukawa interaction have the form

$$\mathcal{L} = -\frac{1}{2}\sum_i \bar{N}_i M_i N_i - (\sum_{l,i} \bar{L}_{lL}\tilde{\phi}\, Y_{il} N + \text{h.c.}). \tag{11.190}$$

Here

$$L_L = \begin{pmatrix} \nu_{1L} \\ l_L \end{pmatrix} \quad \text{and} \quad \tilde{\phi} = i\tau_2\phi^* = \begin{pmatrix} \phi_0^* \\ -\phi_+^* \end{pmatrix}$$

are lepton and Higgs doublets and Y_{li} are Yukawa coupling constants.

It is obvious that the Lagrangian (11.190)

- does not conserve the lepton number L,
- conserves the baryon number (thus, it does not conserve $(B-L)$).
- violates CP in the case of a complex Y.

In the region of energies much smaller than M_i in second order of Y the Lagrangian (11.190) generates the following effective Lagrangian for processes with virtual N_i

$$\mathcal{L} = -\sum_{l',l,i} \bar{L}_{l'L}\tilde{\phi}\, Y_{il'}\frac{1}{M_i}\, Y_{il} C\tilde{\phi}^T \bar{L}_{lL}^T + \text{h.c.} \tag{11.191}$$

After the spontaneous symmetry breaking in the unitary gauge we have

$$\tilde{\phi} = \begin{pmatrix} \frac{v+H}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{11.192}$$

From (11.191) and (11.192) we obtain the Majorana neutrino mass term

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\sum_{l',l} \bar{\nu}_{l'L} M_{l'l}^{\mathrm{M}} C\bar{\nu}_{lL}^T + \text{h.c.} \tag{11.193}$$

with the seesaw mass matrix

$$M^{\mathrm{M}} = Y^T \frac{1}{M} Y\, v^2. \tag{11.194}$$

Thus, if we assume the existence of heavy Majorana particles with masses $M_i \gg v$ which have the Yukawa interaction (11.190), we come to the seesaw mechanism of neutrino mass generation.

The leptogenesis started in the Early Universe at $T \simeq M_i$. Majorana particles N_i can decay into both CP-conjugated final states LH and $\bar{L}\bar{H}$. The CP asymmetry is defined as follows

$$\varepsilon_i = \frac{\Gamma(N_i \to LH) - \Gamma(N_i \to \bar{L}\bar{H})}{\Gamma(N_i \to LH) + \Gamma(N_i \to \bar{L}\bar{H})}. \tag{11.195}$$

In the tree approximation the phases of Y do not enter into the expression for the decay probability and $\Gamma_0(N_i \to LH) = \Gamma_0(N_i \to \bar{L}\bar{H})$. In order to reveal CP violation we must take into account loop diagrams. The CP phases enter into the interference between the tree and loop diagrams. In the case of the hierarchical spectrum of N_i the most important contribution to the leptogenesis give decays of the lightest Majorana particles N_1. For the CP asymmetry we have

$$\varepsilon_1 = \frac{3}{16\pi} \sum_i \frac{\mathrm{Im}(YY^\dagger)^2_{1i}}{(YY^\dagger)_{11}} \frac{M_1}{M_i} \tag{11.196}$$

It is obvious from this expression that the CP asymmetry is due to phases of the nondiagonal elements $Y_{1l} Y^*_{il}$ $(i \neq 1)$. The decays of N_1's are out of equilibrium if the decay rate is smaller than the expansion rate at the time of the leptogenesis, i.e. the condition

$$\Gamma_{N_1} \lesssim H(T \simeq M_1). \tag{11.197}$$

is satisfied.

Taking into account that $\Gamma_{N_1} \simeq \frac{1}{8\pi}(YY^\dagger)_{11} M_1$ and that the Hubble constant is given by the relation $H(T) = 1.66\, g_*^{1/2} \frac{T^2}{M_P}$, from (11.197) we obtain the condition

$$\tilde{m}_1 = \frac{(YY^\dagger)_{11} v^2}{M_1} \lesssim 10^{-3}\ \mathrm{eV}. \tag{11.198}$$

The baryon asymmetry η_B is the product of the CP asymmetry ε_1, a wash out parameter η[9] and a factor which takes into account the sphaleron conversion of the lepton asymmetry into the baryon asymmetry. It was found that

$$\eta_B = -\frac{1}{103} \varepsilon_1 \eta. \tag{11.199}$$

Agreement with observations requires

$$M_1 \geq 10^9\ \mathrm{GeV}. \tag{11.200}$$

If we consider a one-flavor lepton asymmetry for the lightest neutrino mass we have the bound

[9] The calculation of the parameter η requires the numerical solution of the Boltzmann equation for leptogenesis. Approximately we have $\eta \simeq \frac{10^{-3}\mathrm{eV}}{\tilde{m}_1}$.

$$m_1 \leq 10^{-1}\ \text{eV}. \tag{11.201}$$

We know, however, from neutrino oscillation experiments that lepton mixing can be large. So it is natural to include all three flavors in the leptogenesis. Large flavor effects are possible. In particular, the estimate (11.201) is not valid in this case.

Let us also notice that in the case of the quasidegenerate spectrum of masses of heavy Majorana particles the picture of the leptogenesis becomes completely different. If $M_2 - M_1 \simeq \Gamma_N$ the lepton asymmetry is enhanced resonantly. The baryon asymmetry of the Universe could be explained for $M_i \simeq 1$ TeV.

In conclusion let us stress the following. The leptogenesis is an attractive possibility for the explanation of the puzzle of the baryon asymmetry of the Universe. It is a natural consequence of the seesaw mechanism of the neutrino mass generation which is commonly considered as the most plausible possibility of the explanation of the smallness of the neutrino masses. However, masses of heavy Majorana particles and Yukawa coupling constants are unknown parameters. Processes induced by the interaction (11.190), which drive leptogenesis, cannot be observed in a laboratory. Thus, it does not look possible to test leptogenesis in a model independent way.

The observation of the neutrinoless double β-decay would prove that massive neutrinos are Majorana particles. This observation will be a strong argument in favor of the seesaw mechanism and, consequently, leptogenesis.

Chapter 12
Conclusion and Prospects

After the discovery of neutrino oscillations and a very intensive period of studying this new phenomenon we can now formulate a clear program of future investigations of the neutrino mass and mixing problem. Future experiments are aimed to address the following fundamental problems:

1. What is the value of the angle θ_{13}?
2. Is *CP* invariance violated in the lepton sector? What is the value of the *CP* phase δ?
3. What is the character of the neutrino mass spectrum? Is it normal or inverted?
4. Are neutrinos with definite masses Majorana or Dirac particles?
5. What are the absolute values of the neutrino masses?
6. Do sterile neutrinos exist?

For the solution of the problem of CP violation and the problem of the neutrino mass spectrum the key-question is the value of the angle θ_{13}. Because this angle is small, as established by the CHOOZ experiment, we need to investigate very small effects beyond the leading approximation in order to measure the phase δ and to reveal the character of the neutrino mass spectrum. This will require many years of research, new technologies and challenging efforts.

We will start with a brief description of the neutrino program at the Japan Proton Accelerator Research Complex (J-PARC). J-PARC's high-intensity proton synchrotron is designed to accelerate protons up to an energy of 50 GeV. The initial proton energy is 30 GeV.

The first accelerator neutrino experiment at the J-PARC facility T2K (Tokai to Kamioka, distance 295 km) started in 2009. The major goal of the experiment is to determine the value of the parameter $\sin^2\theta_{13}$. This parameter will be measured through the observation of the oscillations $\nu_\mu \rightleftarrows \nu_e$. High precision measurements of the parameters Δm^2_{23} and $\sin^2\theta_{23}$ will also be performed in the experiment via the standard investigation of the disappearance channel $\nu_\mu \to \nu_\mu$. It is planned that the values of these parameters will be determined with the accuracies $\delta(\Delta m^2_{23}) < 10^{-4}\mathrm{eV}^2$ and $\delta(\sin^2\theta_{23}) \simeq 10^{-2}$.

The major background in the experiment on the search for the $\nu_\mu \to \nu_e$ transition comes from π^0 produced in the NC processes $\nu_\mu + N \to \nu_\mu + \pi^0 + \ldots$ and from

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the intrinsic ν_e's in the neutrino beam originating from decays of kaons and muons. These backgrounds will be studied in near detectors at a distance of 280 m from the target.

The T2K is an off-axis experiment with an off-axis angle of 2.5°. This allows to obtain a narrow-band beam[1] with an energy of about 0.7 GeV corresponding to the first oscillation maximum ($E_{1m} = \frac{\Delta m^2_{23} L}{2\pi}$). It is planned that the T2K experiment will run for five years with a beam power of the accelerator of 0.75 MW.[2] During this time an integrated proton power of $0.75\ \mathrm{MW} \times 5 \cdot 10^5$ s will be accumulated.

Within the first years of running the experiment, the data based on $(1\text{–}2)\mathrm{MW} \times 10^7\mathrm{s}$ integrated power could guide the future research plans. During this time the 3 σ sensitivity to the parameter $\sin^2 2\theta_{13}$ will be equal to $(5-3) \cdot 10^{-2}$, much lower than the CHOOZ upper bound ($\sin^2 2\theta_{13} \lesssim 2 \cdot 10^{-1}$). *If significant evidence in favor of the $\nu_\mu \to \nu_e$ transition will be found,* the preparation of an experiment on the search for the CP violation in the lepton sector would have highest priority. The CP violation can be investigated via

- a precise study of the energy spectrum of ν_e produced in the $\nu_\mu \to \nu_e$ transition,
- a comparison of probabilities of the $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions.

The experiment on the investigation of the CP violation in the lepton sector will require

- to increase the intensity of the neutrino beam,
- to build a new neutrino detector.

According to the KEK Roadmap the beam power of the J-PARC accelerator will be increased from 0.75 to 1.66 MW.

There are three different proposals for an experiment on the search for the CP violation at the J-PARC facility.

The Okinoshima long baseline experiment. In this experiment a huge 100 kiloton liquid Argon time projection chamber is planned to be used as a neutrino detector. This detector has a very good energy and spatial resolution which would allow to suppress the π^0 background (from the NC process $\nu_\mu + N \to \nu_\mu + \pi^0 + X$). The idea of the experiment is to measure the ν_e spectrum in the region of the first and the second oscillation maxima. In order to satisfy this requirement the baseline of the experiment has to be longer than 600 km. The experiment must be practically on-axis (in order to cover a wide energy range).

The Okinoshima island (658 km from J-PARC, off-axis angle 0.8°) is an ideal place for the detector. Another option is South Korea (1,000 km baseline, off-axis angle 1°). If $\sin^2\theta_{13} \geq 2 \cdot 10^{-2}$ the phase δ can be measured in the proposed

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[1] Due to the kinematics of the decay $\pi^+ \to \mu^+ + \nu_\mu$, the energy of neutrinos emitted at a certain angle between the pion and the neutrino directions in the laboratory system depends rather weakly on the pion energy. The larger the off-axis angle the more monochromatic (and less intensive) a neutrino beam can be produced.

[2] $\mathrm{MW} = 10^6\ V \cdot A \simeq 6.24 \cdot 10^{15}$ GeV s^{-1}.

experiment with an accuracy of (20–30)°. The suggested duration of the experiment with ν_μ beam is 5 years. The antineutrino run can be considered as a next step.

The second proposal is the *J-PARC to Kamioka long baseline experiment*. This experiment is similar to T2K but with a Water Cerenkov detector much larger than Super-Kamiokande. This new detector will have a fiducial volume of 570 kiloton. The baseline of the experiment is 295 km and the off-axis angle 2.5°. Neutrino oscillations in the region of the first oscillation maximum will be studied. The effect of CP violation will be investigated by the measurement of the difference of neutrino oscillations in the $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ channels. An option with 2.2 years of ν_μ and 7.8 years of $\bar{\nu}_\mu$ running times is considered.

The third proposal is the *J-PARC to Kamioka and Korea long baseline experiment*. In this experiment, two 270 kiloton Water Cerenkov detectors, one in Kamioka (295 km baseline, 2.5° off-axis angle) and another one in Korea (1,000 km baseline, 1° off-axis angle) will be utilized. This will allow to study neutrino oscillations in the region of the first and the second oscillation maxima. It is planned to operate the experiment with a ν_μ beam (5 years) and with a $\bar{\nu}_\mu$ beam (5 years). Effects of the CP violation and neutrino mass hierarchy will be studied.

Another accelerator long baseline neutrino experiment of the next generation is the *NOvA experiment at Fermilab*. In this experiment, a neutrino beam from the Main Injector (120 GeV protons, beam power 0.7 MW) will be utilized. NOvA is an off-axis experiment (14 mrad off-axis) with a distance of 810 km between the neutrino source and the far neutrino detector. Two detectors will be used. The far detector (14 kiloton) is a liquid-scintillator detector with good electron identification and energy resolution. The near detector is a 222 ton detector.

In the NOvA experiment, $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ neutrino oscillations will be investigated. The inclusive channels $\nu_\mu \to \nu_\mu$ and $\bar{\nu}_\mu \to \bar{\nu}_\mu$ will also be studied in detail. It is planned to reach a sensitivity of $\sin^2 2\theta_{13} \simeq (1-2)\cdot 10^{-2}$. At the next step of the experiment, CP violation will be investigated and the neutrino mass hierarchy is planned to be revealed (via the matter effect).

The first data in the NOvA experiment with a 2.5 kiloton detector are expected in 2012. The full detector will be ready in 2014.

The major aim of neutrino oscillations experiments in the coming years will be the solution of the problem of the value of the parameter $\sin^2 2\theta_{13}$. This problem will be attacked by the T2K and NOvA long baseline accelerator experiments and new reactor experiments.

The *Double Chooz reactor experiment* is planned to start in 2010 with one detector (8.3 ton fiducial mass) at a distance of about 1 km from two reactors (8.6 GW_{th}). After a running time of 1.5 years the 90% CL limit $\sin^2 2\theta_{13} < 6\cdot 10^{-2}$ will be reached (if no neutrino oscillations will be observed). In 2011, phase II of the experiment with the far and near detectors will be started. After a running time of 3 years with two detectors the sensitivity limit $\sin^2 2\theta_{13} < 3\cdot 10^{-2}$ (90% CL) will be reached. In the *Daya Bay reactor experiment*, $\bar{\nu}_e$ from six reactors in China (17.4 GW_{th}) will be detected. The fiducial mass of the detectors is 80 ton. Two near detectors will be installed. It is planned that in 2011 the Daya Bay experiment will start data taking with all detectors. After 3 years of running time the sensitivity limit $\sin^2 2\theta_{13} < 1\cdot 10^{-2}$ (90% CL) will be reached.

In the *RENO reactor experiment*, $\bar{\nu}_e$ from six reactors in Korea (16.4 $\mathrm{GW_{th}}$) will be detected. Two detectors at distances of 150 and 1,500 m will be utilized. The fiducial mass of the far detector is 15.4 ton. It is planned to start the experiment in 2010. After three years of running time the sensitivity limit of the experiment will be $\sin^2 2\theta_{13} < 2 \cdot 10^{-2}$ (90% CL).

Thus, during 3–5 years after the start of a new generation of reactor and long baseline accelerator experiments, a sensitivity of a few 10^{-2} for the parameter $\sin^2 2\theta_{13}$ is expected. During these years a decision on a new neutrino facility presumably will be taken. There are three options for the future neutrino facility: Superbeams, Neutrino Factory and Beta Beams. We will describe briefly these facilities. *Superbeams* are high intensity neutrino beams from pion decays. The technology for a multi-MW proton accelerator, the source of the superbeam, is the most developed one. There are, however, several problems concerning experiments searching for $\nu_\mu \rightleftarrows \nu_e$ and $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_e$ oscillations with beams from pion decays:

- Detection of electrons require huge low density underground detectors.
- Background from admixture of electron neutrinos produced in kaon and muon decays and background from π^0 produced in NC neutrino processes.
- In order to have comparable neutrino and antineutrino statistics, the antineutrino run must last much longer than the neutrino run. In fact, the intensity of an antineutrino beam (produced in π^--decays) is significantly lower than the intensity of a neutrino beam (produced in π^+-decays); the cross section of the interaction of antineutrinos with nucleons is about two times smaller than the neutrino cross section.

At a *Neutrino Factory* the phase space of muons, produced in decays of low-energy pions, will be compressed in order to create a muon beam. Then muons will be accelerated up to a proper energy and injected into a storage ring with long straight sections. Neutrinos produced in the decay $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$ or in the decay $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$ will be detected. With neutrinos produced in μ^+-decay (μ^--decay), it is possible to investigate the transition $\nu_e \to \nu_\mu$ ($\bar{\nu}_e \to \bar{\nu}_\mu$) (assuming that it is possible to distinguish μ^- (μ^+) produced in the CC process $\nu_\mu + N \to \mu^- + X$ ($\bar{\nu}_\mu + N \to \mu^+ + X$) from μ^+ (μ^-) produced by non-oscillated $\bar{\nu}_\mu$'s (ν_μ's) in the process $\bar{\nu}_\mu + N \to \mu^+ + X$ ($\nu_\mu + N \to \mu^- + X$)). The comparison of the probabilities of the transitions $\nu_e \to \nu_\mu$ and $\bar{\nu}_e \to \bar{\nu}_\mu$ is a direct way to study the CP violation in the lepton sector.

A Neutrino Factory requires a few-GeV proton accelerator with a very high ($\sim$4 MW) beam power. The construction of the Neutrino Factory is a challenge for accelerator physics. Several R&D projects aiming to develop a cooling system, to study target problems, the problems of a high beam power proton accelerator, etc., are going on all over the world. A Neutrino Factory presumably is the best choice for the determination of precise values of the parameter $\sin^2 2\theta_{13}$ and of the CP phase δ and the best facility for the determination of the character of the neutrino mass spectrum. In the *Beta Beam* facility, electron neutrinos and antineutrinos will be produced in decays in flight of β-radioactive ions. It was proposed to use beams of radioactive ^{6}He and ^{18}Ne ions as sources of $\bar{\nu}_e$ and ν_e, respectively (in another proposal it was proposed to use ^{8}Li and ^{8}B ions). Beta Beams are pure sources of ν_e and $\bar{\nu}_e$. These beams are an ideal tool for the study of the CP violation in the lepton sector. Like in the case of the Neutrino Factory there are no systematic errors connected with the source. However, neutrino energies at the Beta Beam facility are typically much lower than neutrino energies in the case of the Neutrino Factory. The most difficult problem of the Beta Beam facility is the problem of the ion-production yield.

Up to now we have discussed prospects for neutrino oscillation experiments.[3] In conclusion, we will make a few remarks on future measurements of the absolute neutrino masses and on the search for neutrinoless double β-decay. The Mainz and Troitsk experiments on the measurement of the endpoint part of the electron spectrum of the ^{3}H-decay obtained the bound

$$m_\beta < 2.2 \text{ eV}. \tag{12.1}$$

The next-generation experiment *KATRIN* (Karlsruhe Tritium Neutrino Experiment) is under preparation and is expected to start in 2011/2012. The sensitivity of the experiment to the neutrino mass m_β after a running time of about 5–6 years will be 0.2 eV. If no effect of the neutrino mass will be observed this sensitivity corresponds to the upper bound

$$m_\beta < 0.2 \text{ eV} \quad (90\% \text{ CL}). \tag{12.2}$$

In the case that an effect of the neutrino mass will be found, this sensitivity corresponds to $m_\beta = 0.3(0.35)$ eV at 3(5) σ significance.

The present cosmological bound on the sum of neutrino masses from WMAP and SDSS observations is

$$\sum_i m_i \leq 0.7 \text{ eV}. \tag{12.3}$$

In the future, with a more precise CMB temperature measurement (PLANCK satellite), with measurements of the weak gravitational lensing and with other data the sensitivity to the sum of the neutrino masses will be improved to 0.1 eV. This sensitivity is much better than the sensitivity of the terrestrial experiments. It is necessary, however, to stress that cosmological bounds are model dependent.

New experiments on the search for neutrinoless double β-decay are under preparation. The crucial issue for these experiment is to reach the required background suppression. The *GERDA* experiment with 18 kg of enriched ^{76}Ge will start in 2010. After a running time of 2 years, the limit $T_{1/2}(^{76}\text{Ge}) > 2 \cdot 10^{25}$ y (90% CL) will

[3] For a detailed discussion of future neutrino facilities and future neutrino oscillation experiments, see Proceedings of the workshop "European Strategy for Future Neutrino Physics" CERN, Yellow Report 2009.

be reached. In the next phase of the experiment with 40 kg of enriched ^{76}Ge, a sensitivity of $T_{1/2}(^{76}\mathrm{Ge}) \simeq 2.5 \cdot 10^{26}$ y is expected. A third phase of the experiment with 500–1,000 kg of enriched ^{76}Ge is considered.

The *cryogenic experiment CUORE* is presently under construction at the Gran Sasso laboratory. The detector will consist of 988 cubic natural TeO_2 crystals (750 g) arranged in 19 towers. The completion of the setup is expected in 2012. After a running time of 5 years, a sensitivity of $T_{1/2}(^{130}\mathrm{Te}) \simeq 2.1 \cdot 10^{26}$ y is expected.

EXO is a large challenging project (1–10 ton of enriched ^{136}Xe). The tagging of Ba produced in the decay $^{136}\mathrm{Xe} \to ^{136}\mathrm{Ba} + e + e$ would allow to suppress background. The prototype experiment with 200 kg of enriched ^{136}Xe (without Ba tagging) is under preparation at present. It is expected to reach a sensitivity of $T_{1/2}(^{136}\mathrm{Xe}) \simeq 1 \cdot 10^{26}$ y after two years of running.

In the *Super-NEMO experiment*, the $0\nu\beta\beta$-decay of ^{82}Se (100 kg of ^{82}Se spread among 20 modules) will be searched for. In 2011 the run with one module will be completed.

In the $SNO+$ *experiment*, the $0\nu\beta\beta$-decay of enriched ^{150}Nd (50–500 kg in low-activity scintillator) will be searched for.

In the *KamLAND* detector, the search for the $0\nu\beta\beta$-decay will start in 2011 with 200–400 kg of enriched ^{136}Xe and will be continued in 2013 with 1 ton of enriched ^{136}Xe. After a running time of 5 years, a sensitivity of $T_{1/2}(^{136}\mathrm{Xe}) \simeq 10^{26}$ y is expected.

In these new experiments on the search for the $0\nu\beta\beta$-decay it is planned to reach the sensitivity to the effective Majorana mass $m_{\beta\beta} \simeq$ a few 10^{-2} eV. Such values correspond to the inverted hierarchy of neutrino masses.

Appendix A
Diagonalization of a Hermitian Matrix. The Case 2 × 2

Let us consider a hermitian operator H. Eigenstates and eigenvalues of the operator H are given by the equation

$$H\ |i\rangle = E_i\ |i\rangle\ . \tag{A.1}$$

We will assume that the states $|i\rangle$ are normalized. We have

$$\langle i|k\rangle = \delta_{ik}. \tag{A.2}$$

Let $|\alpha\rangle$ be a full system of normalized and orthogonal states:

$$\langle \alpha'|\alpha\rangle = \delta_{\alpha'\alpha}. \tag{A.3}$$

In the α-representation we have

$$\sum_{\alpha} \langle \alpha'|H|\alpha\rangle\ \langle \alpha|i\rangle = E_i\ \langle \alpha'|i\rangle \tag{A.4}$$

Equation (A.4) has nonzero solution if the condition

$$\mathrm{Det}(\mathcal{H} - E) = 0 \tag{A.5}$$

is satisfied. This equation determines the eigenvalues of the matrix $\mathcal{H}$.

It is obvious from (A.2) and (A.3) that

$$\sum_{\alpha} \langle i|\alpha\rangle\ \langle \alpha|k\rangle = \delta_{ik} \tag{A.6}$$

and

$$\sum_{i} \langle \alpha'|i\rangle\ \langle i|\alpha\rangle = \delta_{\alpha'\alpha}\ . \tag{A.7}$$

From (A.7) and (A.4) we find

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$$\langle \alpha' | H | \alpha \rangle = \sum_i \langle \alpha' | i \rangle \; E_i \; \langle i | \alpha \rangle. \tag{A.8}$$

Let us determine a matrix U by the following relation

$$U_{\alpha i} = \langle \alpha | i \rangle. \tag{A.9}$$

From (A.6) and (A.7) follows that U is a unitary matrix:

$$U \; U^{\dagger} = 1, \quad U^{\dagger} \; U = 1. \tag{A.10}$$

Equation (A.4) in the matrix form can be written as follows

$$\mathcal{H} = U \; E \; U^{\dagger}, \tag{A.11}$$

where $E_{ik} = E_i \; \delta_{ik}$.

Two sets of basic vectors $|\alpha\rangle$ and $|i\rangle$ are connected by the unitary transformation. In fact we have

$$|\alpha\rangle = \sum_i |i\rangle \; \langle i|\alpha\rangle = \sum_i U^*_{\alpha i} |i\rangle, \quad |i\rangle = \sum_\alpha |\alpha\rangle \; \langle \alpha | i\rangle = \sum_\alpha |\alpha\rangle \; U_{\alpha i} \; . \tag{A.12}$$

Let us now consider 2×2 real, symmetrical matrix $\mathcal{H}$ with trace equal to zero. We have

$$\mathcal{H} = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}, \tag{A.13}$$

where a and b are real quantities. For the eigenstates and the eigenvalues of the matrix $\mathcal{H}$ we have the following equation

$$\mathcal{H} \; u_i = E_i \; u_i. \tag{A.14}$$

For the eigenvalues E_i from (A.5) we find from (A.5)

$$E_{1,2} = \mp \sqrt{a^2 + b^2}. \tag{A.15}$$

Further we have

$$\mathcal{H} = O \; E \; O^T, \tag{A.16}$$

where O is a real orthogonal 2×2 matrix ($O^T \; O = 1$) and

$$E = \sqrt{a^2 + b^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.17}$$

The matrix O has the following general form

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{A.18}$$

From (A.16), (A.17) and (A.18) we find the following equations for the angle θ

$$a = \sqrt{a^2 + b^2}\ \cos 2\theta, \quad b = \sqrt{a^2 + b^2}\ \sin 2\theta\ . \tag{A.19}$$

From these relations we find

$$\tan 2\,\theta = \frac{b}{a}\ , \quad \cos 2\,\theta = \frac{a}{\sqrt{a^2 + b^2}}\ . \tag{A.20}$$

Let us notice two extreme cases

1. The nondiagonal element b is equal to zero.
   In this case $\theta = 0$ (there is no mixing) and $E_{1,2} = \mp a$
2. The diagonal element a is equal to zero.
   In this case $\theta = \pi/4$ (maximal mixing) and $E_{1,2} = \mp b$

Appendix B
Diagonalization of a Complex Matrix

Let us consider a complex $n \times n$ matrix M. It is obvious that $M\,M^{\dagger}$ is the hermitian matrix with positive eigenvalues. In fact, we have

$$M\,M^{\dagger}\,|i\rangle = a_i\,|i\rangle, \tag{B.1}$$

where

$$a_i = \langle i|\,MM^{\dagger}|i\rangle = \sum_k |\langle i|\,M\,|k\rangle|^2 > 0. \tag{B.2}$$

Let us put $a_i = m_i^2$. We have

$$MM^{\dagger} = U\,m^2\,U^{\dagger}, \tag{B.3}$$

U is a unitary matrix and $(m^2)_{ik} = m_i^2\,\delta_{ik}$.

The matrix M can be presented in the form

$$M = U\,m\,V^{\dagger}, \tag{B.4}$$

where $m = +\sqrt{m^2}$ and[1]

$$V^{\dagger} = m^{-1}\,U^{\dagger}\,M \tag{B.5}$$

The matrix V is an unitary matrix. In fact, from (B.5) we find

$$V = M^{\dagger}\,U\,m^{-1}. \tag{B.6}$$

From (B.3), (B.5) and (B.6) we have

[1] We assumed that all eigenvalues of the matrix $M\,M^{\dagger}$ are different from zero. Thus, the diagonal matrix m^{-1} does exists.

$$V^\dagger\, V = m^{-1}\, U^\dagger\, M\, M^\dagger\, U\, m^{-1} = m^{-1}\, U^\dagger\, U\, m^2\, U^\dagger\, U\, m^{-1} = 1 \tag{B.7}$$

Thus, we have shown that a complex $n \times n$ nonsingular matrix M can be diagonalized by the bi-unitary transformation (B.4). Equation (B.4) is used in the procedures for the diagonalization of quark, lepton and neutrino mass terms.

Appendix C
Diagonalization of a Complex Symmetrical Matrix

Let us consider a complex matrix M which satisfies the condition

$$M = M^T. \tag{C.1}$$

We have shown in Appendix B that any complex matrix M can be presented in the form

$$M = V\, m\, W^{\dagger}, \tag{C.2}$$

where V and W are unitary matrices and $m_{ik} = m_i\, \delta_{ik},\ m_i > 0$. From (C.2) we obtain the relation

$$M^T = W^{\dagger\ T}\, m\, V^T. \tag{C.3}$$

From (C.2) and (C.3) we have

$$M\, M^{\dagger} = V\, m^2\, V^{\dagger}, \quad M^T\, M^{T\ \dagger} = W^{\dagger\ T}\, m^2\, W^T. \tag{C.4}$$

Taking into account that M is a symmetrical matrix, we obtain the following relation

$$V\, m^2\, V^{\dagger} = W^{\dagger\ T}\, m^2\, W^T. \tag{C.5}$$

From this relation we find

$$W^T\, V\, m^2 = m^2\, W^T\, V. \tag{C.6}$$

Thus, the commutator of the matrix $W^T\, V$ and the diagonal matrix m^2 is equal to zero. We assume that $m_i \neq m_k$ for all $i \neq k$. From (C.6) it follows in this case that the matrix $W^T\, V$ is diagonal. Taking also into account that $W^T\, V$ is a unitary matrix we conclude that $W^T\, V$ is a diagonal phase matrix

$$W^T\, V = S(\alpha), \tag{C.7}$$

where

$$S_{ik}(\alpha) = e^{i\,\alpha_i}\,\delta_{ik}. \tag{C.8}$$

From (C.8) we obtain the following relation

$$W^{\dagger} = S^{*}(\alpha)\,V^{T}. \tag{C.9}$$

Finally from (C.2) and (C.9) we find

$$M = U\,m\,U^{T}, \tag{C.10}$$

where $U = V\,S^{*}\left(\frac{\alpha}{2}\right)$ is an unitary matrix.

Thus, we have proved that any symmetrical $n \times n$ matrix can be presented in the form (C.10). The relation (C.10) is used in the procedure of the diagonalization of the Majorana mass terms.

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