Relations between GPDs and TMDs: model results and beyond

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Outline of the talk

Introduction

- Generalized parton distributions (GPDs)
- Transverse momentum dependent PDFs (TMDs)
- Relations between GPDs and TMDs

Results of model calculations

- Results for relations of n^{th} type
- Limitations of model results
- Evidence for possible relations

Generalized TMDs (GTMDs)

- Definition and parametrization
- Implications on relations of n^{th} type

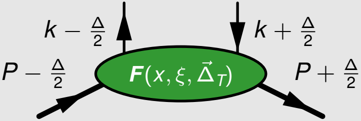
Summary

Based on: Meißner, Metz, Goeke; Phys. Rev. D **76** (2007) 034002

Generalized parton distributions (GPDs)

Definition of GPDs

GPDs appear in the QCD description of hard exclusive reactions and are defined by the **correlator**:



kinematical variables:
$x = \frac{k^+}{P^+}$, $\xi = -\frac{\Delta^+}{2P^+}$, $t = \Delta^2$

For **unpolarized quarks** this correlator can for instance be parametrized by two GPDs:

$$F^q(x, \xi, \vec{\Delta}_T; \lambda, \lambda')$$

$$= \frac{1}{2} \int \frac{dz^-}{2\pi}\, e^{ik\cdot z} \left\langle p', \lambda' \right| \bar{\psi}\left(-\tfrac{z}{2}\right) \gamma^+ \mathcal{W}_{\text{GPD}}\, \psi\left(\tfrac{z}{2}\right) \left| p, \lambda \right\rangle \Big|_{z^+ = \vec{z}_T = 0}$$

$$= \frac{1}{2P^+}\, \bar{u}(p', \lambda') \left[\gamma^+ H^q(x, \xi, t) + \frac{i\sigma^{+\mu}\Delta_\mu}{2M} E^q(x, \xi, t) \right] u(p, \lambda)$$

Generalized parton distributions (GPDs)

GPDs in impact parameter space

The Fourier transformed GPD correlator ($\vec{\Delta}_T \to \vec{b}_T$ for $\xi = 0$) leads to the **impact parameter space** representation of the GPDs. In the case of unpolarized quarks it is given by:

$$\mathcal{F}^q(x, \vec{b}_T; S) = \int \frac{d^2\vec{\Delta}_T}{(2\pi)^2}\, e^{-i\vec{\Delta}_T\cdot\vec{b}_T}\, F^q(x, 0, \vec{\Delta}_T; S)$$

$$= \mathcal{H}^q(x, \vec{b}_T^2) + \frac{\epsilon_T^{ij} b_T^i S_T^j}{M}\, \mathcal{E}'^q(x, \vec{b}_T^2)$$

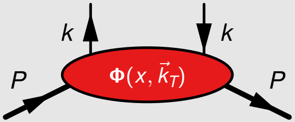
with $\mathcal{H}^q(x, \vec{b}_T^2) = \int \frac{d^2\vec{\Delta}_T}{(2\pi)^2}\, e^{-i\vec{\Delta}_T\cdot\vec{b}_T}\, H^q(x, 0, -\vec{\Delta}_T^2)$

$$\mathcal{E}'^q(x, \vec{b}_T^2) = \frac{\partial}{\partial \vec{b}_T^2} \int \frac{d^2\vec{\Delta}_T}{(2\pi)^2}\, e^{-i\vec{\Delta}_T\cdot\vec{b}_T}\, E^q(x, 0, -\vec{\Delta}_T^2)$$

Transverse momentum dependent PDFs (TMDs)

Definition of TMDs

TMDs appear in the QCD description of hard semi-inclusive reactions and are defined by the **correlator**:



kinematical variables:
$x = \frac{k^+}{P^+}, \vec{k}_T^2$

For **unpolarized quarks** this correlator can for instance be parametrized by two TMDs:

$$\Phi^q(x, \vec{k}_T; S)$$
$$= \frac{1}{2} \int \frac{dz^-}{2\pi} \frac{d^2\vec{z}_T}{(2\pi)^2} e^{ik\cdot z} \langle P,S|\, \bar{\psi}\left(-\tfrac{z}{2}\right) \gamma^+ \mathcal{W}_{\rm TMD}\, \psi\left(\tfrac{z}{2}\right) |P,S\rangle \Big|_{z^+=0}$$
$$= f_1^q(x, \vec{k}_T^2) - \frac{\epsilon_T^{ij} k_T^i S_T^j}{M} f_{1T}^{\perp q}(x, \vec{k}_T^2)$$

Relations between GPDs and TMDs

Probability interpretation

GPDs in impact parameter space and TMDs have in common, that (up to some subtleties) both can be interpreted as **probability densities** of finding partons either at **transverse position $\vec{b}_T$** or with **transverse momentum $\vec{k}_T$**.

Comparison of correlators for unpolarized quarks

A **comparison of the correlators** for unpolarized quarks

$$\mathcal{F}^q(x, \vec{b}_T) = \mathcal{H}^q(x, \vec{b}_T^2) + \frac{\epsilon_T^{ij} b_T^i S_T^j}{M}\, \mathcal{E}'^q(x, \vec{b}_T^2)$$

$$\Phi^q(x, \vec{k}_T) = f_1^q(x, \vec{k}_T^2) - \frac{\epsilon_T^{ij} k_T^i S_T^j}{M}\, f_{1T}^{\perp q}(x, \vec{k}_T^2)$$

reveals a strong similarity for $\vec{b}_T \leftrightarrow \vec{k}_T$. One might therefore assume some **relations between GPDs and TMDs**:

$$\mathcal{H}^q \leftrightarrow f_1^q \qquad\qquad \mathcal{E}'^q \leftrightarrow -f_{1T}^{\perp q}$$

(Diehl, Hägler; Eur. Phys. J. C **44** (2005) 87)

Relations between GPDs and TMDs

Extension to all GPDs and TMDs

This method can **easily be extended** to all existing GPDs and TMDs for quarks as well as for gluons and yields the following relations:

rel. of 1st type:
(well known; valid model independently)

$$\mathcal{H}^{q/g} \leftrightarrow f_1^{q/g} \qquad \tilde{\mathcal{H}}^{q/g} \leftrightarrow g_{1L}^{q/g}$$

$$\left(\mathcal{H}_T^q - \frac{\vec{b}_T^2}{M^2}\Delta_b \tilde{\mathcal{H}}_T^q\right) \leftrightarrow \left(h_{1T}^q + \frac{\vec{k}_T^2}{2M^2} h_{1T}^{\perp q}\right)$$

rel. of 2nd type:
(partly known; only valid in model calculations so far)

$$\mathcal{E}'^{q/g} \leftrightarrow -f_{1T}^{\perp q/g} \qquad \left(\mathcal{E}_T^q + 2\tilde{\mathcal{H}}_T^q\right)' \leftrightarrow -h_1^{\perp q}$$

$$\left(\mathcal{H}_T^g - \frac{\vec{b}_T^2}{M^2}\Delta_b \tilde{\mathcal{H}}_T^g\right)' \leftrightarrow -\frac{1}{2}\left(h_{1T}^g + \frac{\vec{k}_T^2}{2M^2} h_{1T}^{\perp g}\right)$$

rel. of 3rd type:
(partly known; only valid in model calculations so far)

$$\tilde{\mathcal{H}}_T''^{q} \leftrightarrow \frac{1}{2} h_{1T}^{\perp q} \qquad \left(\mathcal{E}_T^g + 2\tilde{\mathcal{H}}_T^g\right)'' \leftrightarrow \frac{1}{2} h_1^{\perp g}$$

rel. of 4th type:
(entirely new; only valid in model calculations so far)

$$\tilde{\mathcal{H}}_T'''^{g} \leftrightarrow -\frac{1}{4} h_{1T}^{\perp g}$$

no relations:

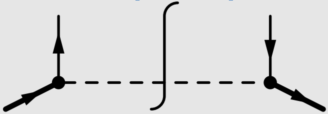
$$\tilde{\mathcal{E}}^{q/g}, \tilde{\mathcal{E}}_T^{q/g} \not\leftrightarrow g_{1T}^{q/g}, h_{1L}^{\perp q/g}$$

Results of model calculations

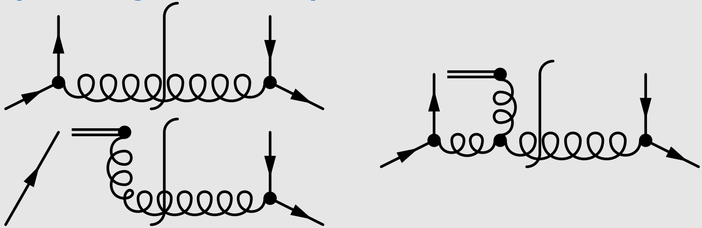
Models under consideration

To analyze the relations between GPDs and TMDs we performed **lowest order calculations** in two models:

- **scalar diquark spectator model of the nucleon**



- **quark target model in perturbative QCD**



Results of model calculations

Results for relations of 1st type

The relations of 1st type are **valid even model independently**, because the involved GPDs and TMDs can be reduced to the **same forward PDFs**:

$$q(x) = \int d^2\vec{b}_T\, \mathcal{H}^q(x, \vec{b}_T^2) = \int d^2\vec{k}_T\, f_1^q(x, \vec{k}_T^2)$$

This formula holds for **all relations of 1st type**.

Results for relation of 4th type

The relation of 4th type **becomes trivial** in lowest order model calculations, as the involved GPD and TMD vanish:

$$\tilde{\mathcal{H}}_T^{\prime\prime\prime g}(x, \vec{b}_T^2) = h_{1T}^{\perp g}(x, \vec{k}_T^2) = 0$$

Results of model calculations

Results for relations of 2nd type

By explicit model calculations, one finds:

$$\frac{h_2(n)}{2M^2\,(1-x)} \int d^2\vec{\Delta}_T \left(\frac{\vec{\Delta}_T^2}{2M^2}\right)^{n-1} E^q\left(x, 0, -\frac{\vec{\Delta}_T^2}{(1-x)^2}\right)$$

$$= \int d^2\vec{k}_T \left(\frac{\vec{k}_T^2}{2M^2}\right)^{n} f_{1T}^{\perp q}(x, \vec{k}_T^2) \qquad (\text{for } 0 \leq n \leq 1)$$

This formula holds for **all relations of 2nd type** with an **identical factor $h_2(n)$**, which only depends on n and on the used model.

In **impact parameter space**, $n = 1$ yields the most convenient form of the relation:

$$\int d^2\vec{b}_T\, \vec{\mathcal{I}}^q(x, \vec{b}_T)\, \frac{\epsilon_T^{ij} b_T^i S_T^j}{M}\, \mathcal{E}'^q(x, \vec{b}_T^2)$$

$$= -\int d^2\vec{k}_T\, \vec{k}_T\, \frac{\epsilon_T^{ij} k_T^i S_T^j}{M}\, f_{1T}^{\perp q}(x, \vec{k}_T^2)$$

(Burkardt, Hwang; Phys. Rev. D **69** (2004) 074032)

Results of model calculations

Results for relations of 3rd type

By explicit model calculations, one finds:

$$\frac{h_3(n)}{2M^2\,(1-x)^2}\int d^2\vec{\Delta}_T\left(\frac{\vec{\Delta}_T^2}{2M^2}\right)^{n-1}\tilde{H}_T^q\left(x,0,-\frac{\vec{\Delta}_T^2}{(1-x)^2}\right)$$
$$=\int d^2\vec{k}_T\left(\frac{\vec{k}_T^2}{2M^2}\right)^{n} h_{1T}^{\perp q}(x,\vec{k}_T^2)\qquad(\text{for } 0\leq n\leq 1)$$

This formula holds for **all relations of 3rd type** with an **identical factor $h_3(n)$**, which only depends on n and **not** on the used model.

In **impact parameter space**, $n=1$ yields the most convenient form of the relation:

$$\int d^2\vec{b}_T\,\vec{b}_T^2\,\tilde{\mathcal{H}}_T^{\prime\prime q}(x,\vec{b}_T^2)=\frac{1}{2}\int d^2\vec{k}_T\,\vec{k}_T^2\,h_{1T}^{\perp q}(x,\vec{k}_T^2)$$

Results of model calculations

Limitations of model results

So far all relations have only been confirmed using **lowest order calculations** in simple spectator models. However, it is likely that the relations are **not valid in general**.

Even in model calculations they will **probably break down**

- if one takes into account **higher orders** or
- if one uses **more complicated models**.

Evidence for possible relations

Although the relations are probably not valid in general, it is still possible, that they are **at least approximately valid**.

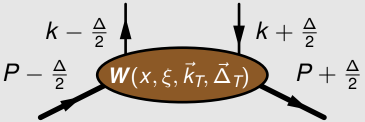
Evidence for this is, that the relations

- provide an **intuitive picture of the Sivers effect** and
  (Burkardt, Phys. Rev. D **66** (2002) 114005)
- are **compatible with data and lattice calculations**.
  (M. Göckeler *et al.* [QCDSF Collaboration], Phys. Rev. Lett. **98** (2007) 222001)

Generalized TMDs (GTMDs)

Definition of GTMDs

It is possible to **combine the concepts** of GPDs and TMDs into a single object that is given by the **correlator**:



kinematical variables:

$$x = \frac{k^+}{P^+},\ \xi = -\frac{\Delta^+}{2P^+},\ \vec{k}_T,\ \vec{\Delta}_T$$

For **unpolarized quarks** and a **spinless target** this correlator can for instance be parametrized by one GTMD:

$$\begin{aligned} &W^q(x, \xi, \vec{k}_T, \vec{\Delta}_T) \\ &= \frac{1}{2}\int \frac{dz^-}{2\pi}\,\frac{d^2\vec{z}_T}{(2\pi)^2}\, e^{ik\cdot z}\,\langle p' |\, \bar{\psi}(-\tfrac{z}{2})\,\gamma^+\,\mathcal{W}_{\mathrm{GTMD}}\,\psi(\tfrac{z}{2})\,| p\rangle\Big|_{z^+=0} \\ &= F_1^q(x, \xi, \vec{k}_T^2, \vec{k}_T\cdot\vec{\Delta}_T, \vec{\Delta}_T^2) \end{aligned}$$

Generalized TMDs (GTMDs)

Parametrization of the GTMD correlator

Altogether, one finds **16 quark GTMDs** for a spinless target, which are **complex functions** of x, ξ, $\vec{k}_T^2$, $\vec{k}_T \cdot \vec{\Delta}_T$, and $\vec{\Delta}_T^2$:

twist-2:
(4 GTMDs)

$$W^{q[\gamma^+]} = F_1^q \qquad W^{q[\gamma^+\gamma_5]} = \frac{i\epsilon_T^{ij} k_T^i \Delta_T^j}{M^2}\, \tilde{G}_1^q$$

$$W^{q[i\sigma^{j+}\gamma_5]} = \frac{i\epsilon_T^{ij} k_T^i}{M}\, H_1^{k,q} + \frac{i\epsilon_T^{ij} \Delta_T^i}{M}\, H_1^{\Delta,q}$$

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twist-3:
(8 GTMDs)

$$W^{q[1]} = \frac{M}{P^+}\left[E_2^q\right] \qquad W^{q[\gamma_5]} = \frac{M}{P^+}\left[\frac{i\epsilon_T^{ij} k_T^i \Delta_T^j}{M^2}\, \tilde{E}_2^q\right]$$

$$W^{q[\gamma^j]} = \frac{M}{P^+}\left[\frac{k_T^j}{M}\, F_2^{k,q} + \frac{\Delta_T^j}{M}\, F_2^{\Delta,q}\right]$$

$$W^{q[\gamma^j\gamma_5]} = \frac{M}{P^+}\left[\frac{i\epsilon_T^{ij} k_T^i}{M}\, G_2^{k,q} + \frac{i\epsilon_T^{ij} \Delta_T^i}{M}\, G_2^{\Delta,q}\right]$$

$$W^{q[i\sigma^{ij}\gamma_5]} = \frac{M}{P^+}\left[i\epsilon_T^{ij} H_2^q\right] \qquad W^{q[i\sigma^{+-}\gamma_5]} = \frac{M}{P^+}\left[\frac{i\epsilon_T^{ij} k_T^i \Delta_T^j}{M^2}\, \tilde{H}_2^q\right]$$

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twist-4:
(4 GTMDs)

$$W^{q[\gamma^-]} = \frac{M^2}{(P^+)^2}\left[F_3^q\right] \qquad W^{q[\gamma^-\gamma_5]} = \frac{M^2}{(P^+)^2}\left[\frac{i\epsilon_T^{ij} k_T^i \Delta_T^j}{M^2}\, \tilde{G}_3^q\right]$$

$$W^{q[i\sigma^{j-}\gamma_5]} = \frac{M^2}{(P^+)^2}\left[\frac{i\epsilon_T^{ij} k_T^i}{M}\, H_3^{k,q} + \frac{i\epsilon_T^{ij} \Delta_T^i}{M}\, H_3^{\Delta,q}\right]$$

Generalized TMDs (GTMDs)

Implications on relations of 1st type

The **model-independent validity** of the relations of 1st type can be seen directly, because the involved GPDs and TMDs are **limiting cases of the same GTMDs**:

$$\int d^2\vec{k}_T \,\mathrm{Re}\Big[F_1^q(x,0,\vec{k}_T^2,0,0)\Big]$$

$$= \int d^2\vec{b}_T\, \mathcal{H}^q(x,\vec{b}_T^2) = \int d^2\vec{k}_T\, f_1^q(x,\vec{k}_T^2)$$

Implications on relations of 3rd and 4th type

There are **no new implications** on the relations of 3rd and 4th type from the GTMDs for unpolarized targets, because these relations appear only in the case of transverse target polarization.

Generalized TMDs (GTMDs)

Implications on relations of 2nd type

For the GPDs and TMDs in the relations of 2nd type

$$\frac{h_2(n)}{2M^2\,(1-x)} \int d^2\vec{\Delta}_T \left(\frac{\vec{\Delta}_T^2}{2M^2}\right)^{n-1} \left(E_T^q + 2\tilde{H}_T^q\right) = \int d^2\vec{k}_T \left(\frac{\vec{k}_T^2}{2M^2}\right)^{n} h_1^{\perp q} \qquad (\text{for } 0 \le n \le 1)$$

the corresponding **expressions through GTMDs** are:

$$\left(E_T^q + 2\tilde{H}_T^q\right) = -2 \int d^2\vec{k}_T \,\mathrm{Re}\left[\frac{1}{2}\left(\frac{k_T^1}{\Delta_T^1} - \frac{k_T^2}{\Delta_T^2}\right) H_1^{k,q} + H_1^{\Delta,q}\right]$$

$$h_1^{\perp q} = \mathrm{Im}\left[H_1^{k,q}\right]\Big|_{\xi=\vec{\Delta}_T=0}$$

This result supports the assumption, that **in general the relations of 2nd type are not valid**, because the real and imaginary part of the GTMDs are independent functions.

Summary

- A **comparison of the correlators** for GPDs and TMDs yields possible relations.
- All these relations have been **confirmed using lowest order calculations** in simple spectator models.
- It is likely that the relations are **not valid in general**. Nevertheless, it is still possible, that they are **at least approximately valid**.
- The **results from GTMDs** support the assumption, that the relations are **not valid in general**.