100 years of GR

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Massless cosmic strings

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Plan

- cosmic strings and the Penrose limit;
- massless strings in a flat space: mathematical model and physical effects
- massless strings in de Sitter universe;
- massless strings in our Universe

Basic facts:

- worldsheet of a cosmic string is an extremal surface in a given background geometry (Nambu-Goto eqs);

$$X^{\mu} = X^{\mu}(\sigma, \tau)$$

extrinsic curvatures vanish

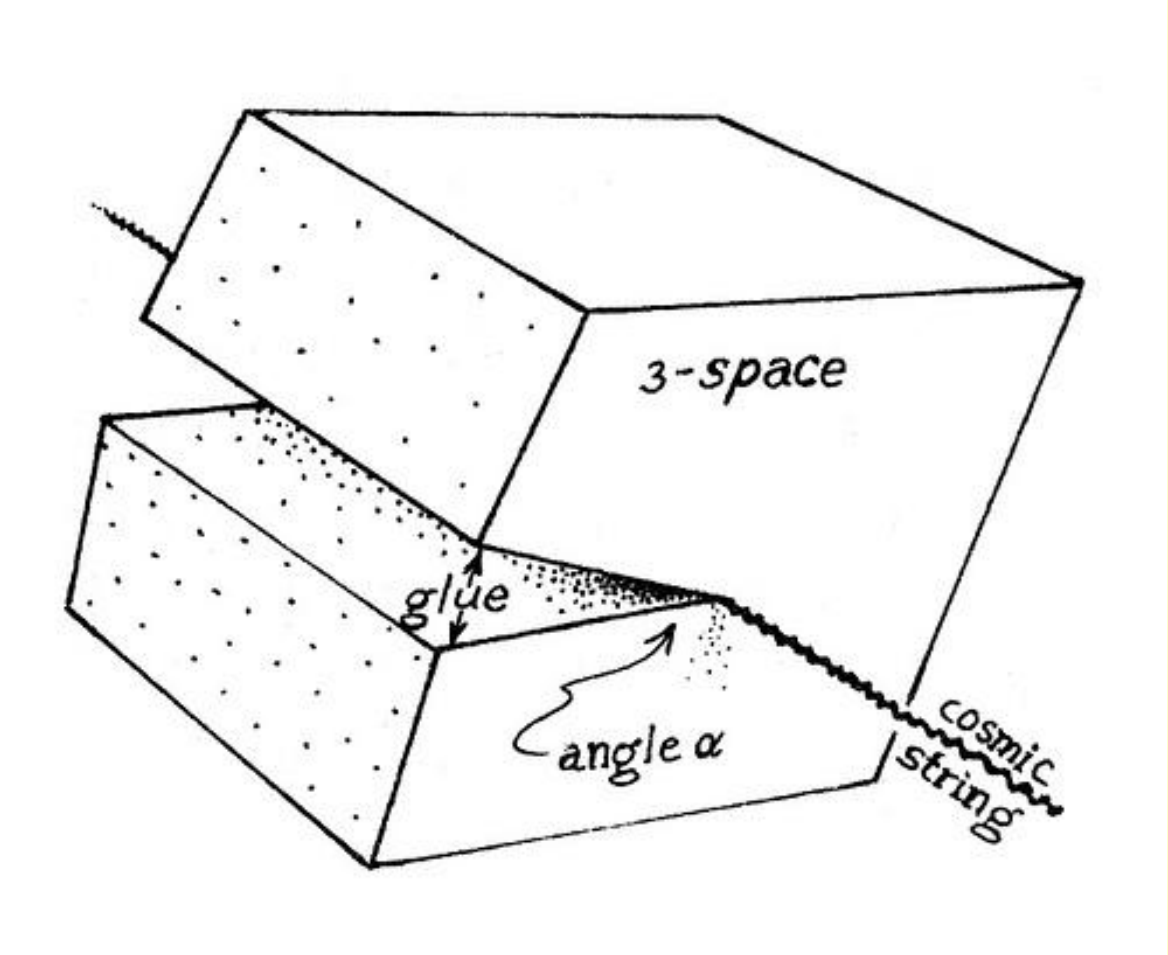
- gravitational effects of the string are global (no local changes in the geometry)

$$ds^2 = -dt^2 + dz^2 + \alpha^2 r^2 d\varphi^2 + dr^2 \quad ,$$

$$0 \le \varphi < 2\pi$$

$$\alpha = 1 - 4\mu G, \quad 0 < \alpha < 1$$

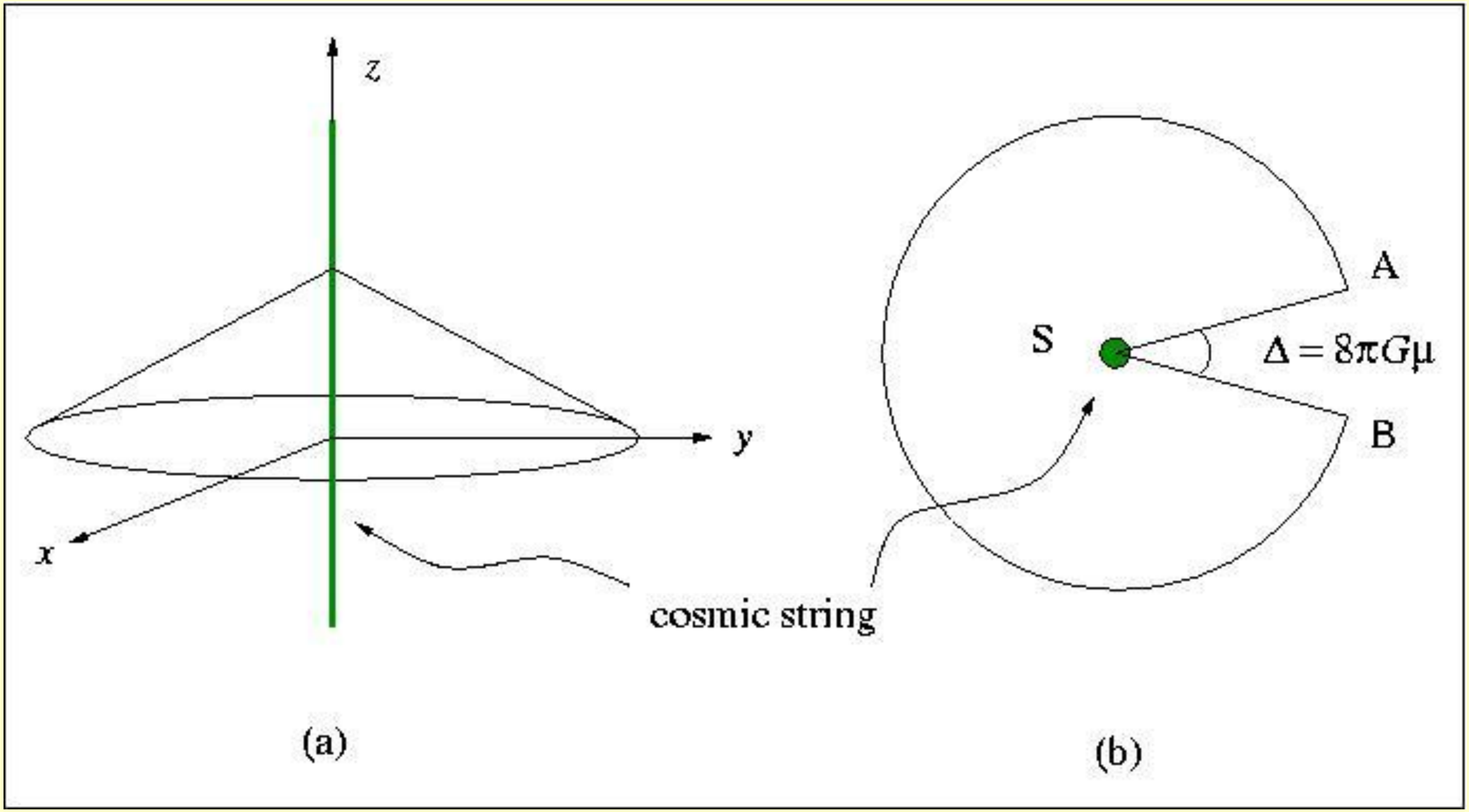
μ is the string tension, when the string is at rest



$$ds^2 = -dt^2 + dz^2 + r^2 d\bar{\varphi}^2 + dr^2 \quad ,$$

$$0 \leq \bar{\varphi} < 2\pi\alpha$$

Deficit angle



Conical singularities and stress-energy tensor of a string

$$R_{\mu\nu} = 2\pi(1-\alpha)\delta^{(2)}(x-X)\sum_i n^i{}_\mu n^i{}_\nu$$

n^i is a pair of normal vectors to the string worldsheet, $(n^i \cdot n^j) = \delta^{ij}$

$$R = 4\pi(1-\alpha)\delta^{(2)}(x-X) = 16\pi\mu G\delta^{(2)}(x-X)$$

introduce a pair of vectors tangent to the string worldsheet, $(l \cdot n^j) = 0,$

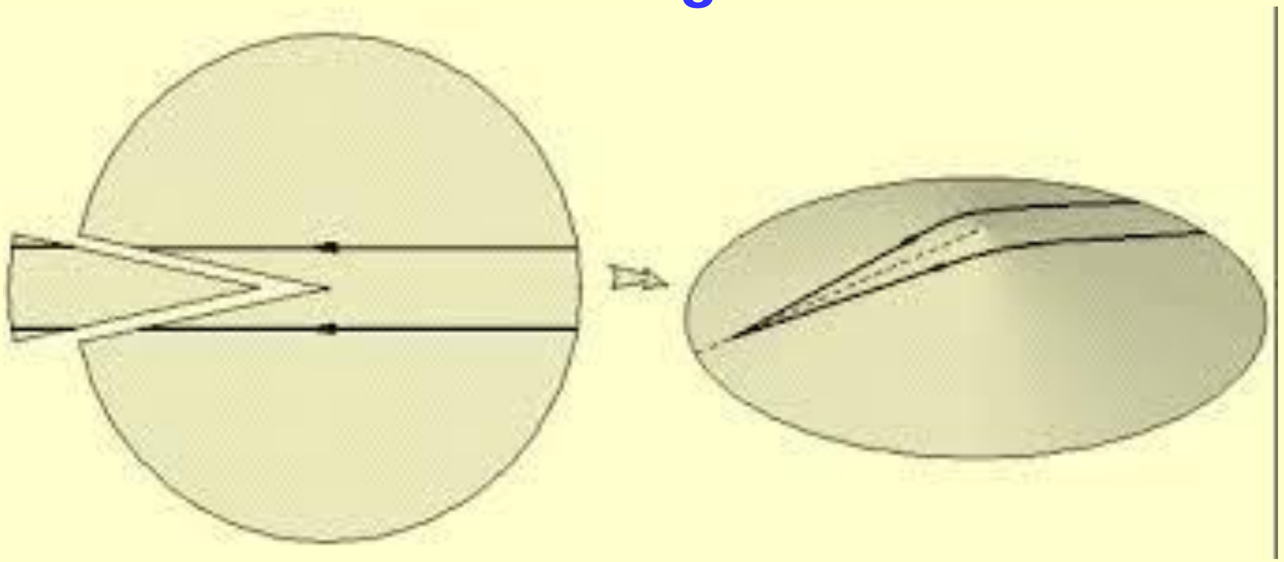
l is directed along the string axis, U_μ - 4-velocity, $U^2 = -1,$

$$\sum_i n^i{}_\mu n^i{}_\nu + l_\mu l_\nu - U_\mu U_\nu = g_{\mu\nu}$$

$$T_{\mu\nu} = \frac{1}{8\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = \rho(U_\mu U_\nu - l_\mu l_\nu)$$

$$\rho = \mu\delta^{(2)}(x-X)$$

Lensing effects



'cut' coodinates

$$ds^2 = -dt^2 + dz^2 + r^2 d\bar{\varphi}^2 + dr^2 \quad ,$$

$$\bar{x} = r\cos\bar{\varphi}\,, \quad \bar{y} = r\sin\bar{\varphi}\,, \; 0 \le \bar{\varphi} < 2\pi\alpha$$

$$ds^2 = -dt^2 + dz^2 + d\bar{x}^2 + d\bar{y}^2$$

'continuous' coodinates

$$ds^2 = -dt^2 + dz^2 + \alpha^2 r^2 d\varphi^2 + dr^2 \quad ,$$

$$x = r\cos\varphi\,, \quad x = r\sin\varphi\,, \; 0 \le \varphi < 2\pi$$

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 + (\alpha^2 - 1)\frac{(xdy - ydx)^2}{x^2 + y^2}$$

geodesics are

straight lines

curved lines

The effect

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 + (\alpha^2 - 1)\frac{(xdy - ydx)^2}{x^2 + y^2}$$

$$x = r\cos\varphi, \quad y = r\sin\varphi, \qquad \varphi = \alpha^{-1}\bar{\varphi}$$

$\bar{\varphi}$ is a usual polar anle

geodesics moving along x orthogonally to the string change $\bar{\varphi}$

$$\Delta\bar{\varphi}_+ = -\pi \;, \text{ if } y>0 \quad , \qquad \Delta\bar{\varphi}_- = \pi \;, \text{ if } y<0$$

in the 'smooth coordinates'

$$\Delta\varphi_+ = -\alpha^{-1}\pi \simeq \text{-}(1+4\mu\text{G})\pi, \text{ if } y>0$$

$$\Delta\varphi_- = \alpha^{-1}\pi \simeq (1+4\mu\text{G})\pi, \text{ if } y<0$$

$$|\Delta\varphi_+ - \Delta\varphi_-| \simeq 2\pi(1+4\mu\text{G})$$

in the 'smooth coordinates' parallel geodesics passing the string from different sides rotate with respect to each other by the angle $\Delta\varphi \simeq 8\pi\mu G$

Moving string and the Penrose limit (Aichelburg-Sexl boost)

use 'continuous' coodinates

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 + (\alpha^2 - 1)\frac{(xdy - ydx)^2}{x^2 + y^2}$$

go to the frame where the string moves with the velocity v along the x-direction

$$x = \gamma(x' - vt') \ , \ \gamma = \frac{1}{\sqrt{1 - v^2}} \qquad (c = 1)$$

the Penrose limit (an 'ultra boost'): v approaches the velocity of light, $\mu \to 0$ (massless string) with a finite energy (per length)

$$E \equiv \lim_{v \to 1} (\gamma\mu)$$

Metric in the Penrose limit

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 + (\alpha^2 - 1)\frac{(xdy - ydx)^2}{x^2 + y^2}$$

$$\gamma(\alpha^2 - 1) \rightarrow -8GE$$

$$-dt^2 + dx^2 = -dudv \quad , \quad u = t' - x' \quad , \quad v = t' + x'$$

$$\gamma^{-1}x \rightarrow -u \quad ,$$

$$\frac{\gamma}{x^2 + y^2} \rightarrow \frac{\pi}{|y|}\delta(u)$$

$$ds^2 = -dudv + dz^2 + dy^2 - 8\pi GE|y|\delta(u)du^2$$

Israel, Barrabes, Hogan (2002), van de Meent (2012)

some more details

$$\frac{\gamma}{x^2+y^2} \to \frac{\pi}{|y|}\delta(u) \qquad ?$$

a hint

$$\underbrace{\left[\partial_x^2 + \frac{1}{y^2}\partial_y(y^2\partial_y)\right]}_{\text{radial part of 4-D Laplacian}} \frac{1}{x^2+y^2} = -4\pi^2\frac{1}{y^2}\delta(y)\delta(x)$$

$$\partial_x^2 + \frac{1}{y^2}\partial_y(y^2\partial_y) \to \underbrace{\frac{1}{y^2}\partial_y(y^2\partial_y)}_{\text{radial part of 3-D Laplacian}}$$

$$\left[\frac{1}{y^2}\partial_y(y^2\partial_y)\right]\frac{1}{|y|} = -4\pi\frac{1}{y^2}\delta(y)$$

therefore, $$\frac{\gamma}{x^2+y^2} \to \frac{\pi}{|y|}\delta(u)$$

Stress-energy tensor of a massless string

$$T_{\mu\nu} = \rho(U_\mu U_\nu - l_\mu l_\nu)$$

$$\rho = \mu\delta(y)\delta(x)$$

the Penrose limit:

$$x \to \gamma u \quad , \quad \gamma\delta(x) \to \delta(u) \quad ,$$

$$\gamma\mu \to E \quad , \quad \gamma^2\rho = \tilde{\rho} \quad ,$$

$$\gamma^{-1}U_\mu \to u_\mu \quad , \quad u^2 = 0$$

$$T_{\mu\nu} = \tilde{\rho}\, u_\mu u_\nu \quad ,$$

$$\tilde{\rho} = E\delta(y)\delta(u)$$

u_μ is the velocity of the string

Do the metric and the stress-energy correspond each other?

$$ds^2 = -dudv + dz^2 + dy^2 - 8\pi GE|y|\delta(u)du^2$$

regularization

$\delta(u) \to \chi(u)$ a smooth function

$$ds^2 = -dudv + dz^2 + dy^2 - 8\pi GE|y|\chi(u)du^2$$

exact solution to the problem

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = \tilde{\rho}\, u_\mu u_\nu \quad , \quad \tilde{\rho} = E\delta(y)\chi(u)$$

$\chi(u)$ is a 'profile' of a family of strings moving one after another

Deviation of geodesics by the moving string

Let $O(\Delta\varphi)$ be a mutual rotation of parallel geodesics after passing the string from different sides (in the frame, where the string is at rest)

Let $L(v)$ be the Lorentz boost to the frame where the string is moving with the velocity v.

The mutual rotation of the parallel geodesics by the moving string is

$$O'(\Delta\varphi) = L(v)O(\Delta\varphi)L^{-1}(v)$$

The boost is

$$v \to 1, \quad \gamma\Delta\varphi \to 8\pi GE \equiv \lambda$$

The mutual rotation of the parallel geodesics by the massless string is

$$\lim_{v\to 1} O'(\Delta\varphi) = M(\lambda)$$

If a string moves between an observer and a test particle (light)

with the initial velocity u^{μ}, the obsever will register the particle (light)

with a changed velocity $\bar{u}^{\mu} = M^{\mu}{}_{\nu}(\lambda)u^{\nu}$

$$\bar{u}^{u} = u^{u}$$

$$\bar{u}^{v} = u^{v} + \varepsilon(y)\lambda u^{y} + \frac{\lambda^{2}}{4}u^{u}$$

$$\bar{u}^{y} = u^{y} + \varepsilon(y)\frac{\lambda}{2}u^{u} \quad ,$$

$$\bar{u}^{z} = u^{z}$$

Parabolic subgroup of the Lorentz group

$\varepsilon(y)$ is the sign function

one has an Abelian subgroup of the Lorentz group - a hyperbolic subgrop

$$M(\lambda_1)M(\lambda_2) = M(\lambda_1 + \lambda_2)$$

Properties of parabolic transformations

1. $M^{\mu}{}_{\nu}(\lambda)u^{\nu} = u^{\mu}$, where u^{μ} is the 4-velocity of the string,

the string world-sheet is left invariant; **'null rotations'**

2. $M(\lambda)$ determines transformation of components of a vector under

a parallel transport around the string ($M(\lambda)$ is a holonomy);

3. transformation of coordinates:

$$\bar{x} = \left(1 - \frac{\lambda^2}{4}\right)x + \frac{\lambda^2}{4}t + \varepsilon(y)\lambda y$$

$$\bar{t} = -\frac{\lambda^2}{4}x + \left(1 + \frac{\lambda^2}{4}\right)t + \varepsilon(y)\lambda y$$

$$\bar{y} = y + \varepsilon(y)\frac{\lambda}{2}(t - x) \quad , \quad \bar{z} = z$$

Physical effects: spatial mutual movements

1. mutual movement of coordinate systems on different sides of the string:

$\bar{x}^{\mu}$ moves w.r.t. x^{μ} with the coordinate velocity

$$v_x = \frac{\lambda^2}{4+\lambda^2}, \quad v_y = -\varepsilon(y)\frac{2\lambda}{4+\lambda^2}, \; |v| = \frac{\lambda}{\sqrt{4+\lambda^2}} \simeq 4\pi GE$$

- two observers which are at rest w.r.t. each other start to move toward each other,

if a massless string passes between them;

- images of a star (in the lensing effect) move toward each other;

Physical effects: blue-shift of gamma spectra

2. gamma quantum 4-momentum:

$$\bar{k}^{\mu} = M^{\mu}_{\ \nu} k^{\nu} \quad , \qquad k^2 = \bar{k}^2 = 0$$

$$\bar{\omega} = \bar{k}^t = \left(1 + \frac{\lambda^2}{4}\right)\omega - \frac{\lambda^2}{4} k^x + \varepsilon(y)\lambda k^y =$$

- $\bar{\omega} = \omega$, if the quantum moves strictly in the same direction as the string,
- $\bar{\omega} = \left(1 + \frac{\lambda^2}{2}\right)\omega$, if the quantum moves strictly in the opposite direction,
- $\bar{\omega} = \left(1 + \lambda + \frac{\lambda^2}{4}\right)\omega$, if the quantum moves perpendicularly to the

direction of the string motion

Constructing space-time around a massless string

spacetime of a massless cosmic string is obtained from $\mathbb{R}^{1,3}$:

consider 2 half planes $\mathbb{R}_+, \mathbb{R}_-$ defined by equations

$$a_\pm u + b_\pm y = 0 \quad , \quad u \geq 0$$

require that under the action of the parabolic group $\mathbb{R}_+ \to \mathbb{R}_-$

$$a_- = a_+ + \frac{\lambda}{2} b_+ \quad , \quad b_- = b_+$$

It can be shown that intersection of such planes

$$\mathbb{R}_+ \cap \mathbb{R}_- = \text{string world-sheet} \quad u = y = 0$$

Constructing space-time around a massless string: cut and glue

- cut a domain of $\mathbb{R}^{1,3}$ between half planes $\mathbb{R}_+$ and $\mathbb{R}_-$
- glue along the cuts

new geometry, $\mathbb{R}^{1,3}_\lambda$, has the following properties:

1. $\mathbb{R}^{1,3}_\lambda$ is regular everywhere except the string world-sheet $u = y = 0$

and it is locally $\mathbb{R}^{1,3}$

2. $\mathbb{R}^{1,3}_\lambda$ is invariant under the action of the parabolic transformations;

3. construction of $\mathbb{R}^{1,3}_\lambda$ does not depend on the choice of a pair of cuts;

4. parallel geodesics experience a mutual (parabolic) transformation.

thank you for attention