

## NOTE ON THE DIAGONALIZATION OF A QUADRATIC LINEAR FORM DEFINED ON THE SET OF SECOND QUANTIZATION FERMION OPERATORS

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The problem of diagonalization of a quadratic linear form on the set of second quantization fermion operators is discussed and two different cases of diagonalization are indicated. It is shown that the quadratic linear fermion form in the second case may behave as a many-particle operator contrary to the Bose statistics. A particular case of diagonalization of the quadratic linear form on two types of fermion operators is analysed using the Dirac matrix algebra.

### 1. Introduction

In paper [10] the diagonalization of the quadratic linear (q.l.) form on the set of second quantization operators  $a_k$  and  $a_k^+$

$$L = \sum_k (F_k a_k^+ a_k + \varphi_k a_k + a_k^+ \varphi_k^+), \quad L = L^+, \quad (1.1)$$

was considered. Here  $F_k$ ,  $\varphi_k$ ,  $\varphi_k^+$  are certain coefficients,  $F_k$  being a  $c$ -number. Attention has been drawn to the fact that when  $a_k$  and  $a_k^+$  satisfy the Bose commutation relations the q.l. form (1.1) is diagonalized simply by the canonical shift transformation. If the operators  $a_k$  and  $a_k^+$  satisfy the Fermi commutation relations, the diagonalizing problem requires special consideration which is just the matter of the present paper.

The expression (1.1) is a Hermitian form and, consequently, it may be diagonalized. However, as will be shown below, since the q.l. form may behave as a many-particle operator, the diagonalization is different for the two following cases.

*The first case* occurs when the linear fermion sources (i.e. the terms  $\varphi_k a_k + a_k^+ \varphi_k^+$ ) are introduced with the aid of anticommutation relations with the initial fields and, simultaneously, by self-anticommuting spinor fields  $\varphi_k$  and  $\varphi_k^+$ . Then the form (1.1) is, in gist, a bilinear fermion form and consequently may be diagonalized by the linear ca-

nonical transformation [2]. The linear sources in this case may be interpreted as interaction terms between the initial fermion field and the external spinor field. We note that recently a problem similar in some respect was investigated in paper [5]. It consists in diagonalizing the interaction Hamiltonian between a one-mode boson field and one-mode fermion.

*The second case* occurs when the quantities  $\varphi_k$  and  $\varphi_k^+$  in equation (1.1) are supposed to be  $c$ -numbers.

In the present note we shall discuss the diagonalization in both cases and investigate the possibility of its practical realization. It should be noted that these questions have not been considered in the monographs available (see e.g. [2]).

## 2. Diagonalization in the first case

Consider the q.l. form (1.1), where  $a_k$  and  $a_k^+$  satisfy the fermion commutation relations and suppose that  $\varphi_k$  and  $\varphi_k^+$  are equal to

$$\varphi_k = \eta_k f_k, \quad \varphi_k^+ = \eta_k^* f_k^+, \quad (2.1)$$

where  $\eta_k$  and  $\eta_k^*$  are  $c$ -numbers and  $f_k$  and  $f_k^+$  are spinors satisfying the following commutation relations:

$$\{f_k, f_l\} = 0; \quad \{f_k, f_k^+\} = 1, \quad \{f_k, f_l^+\} = 0 \quad \text{for } k \neq l, \quad (2.2)$$

Here  $\{\dots\}$  is an anticommutator. Besides that we need the commutation relations for the fermion fields  $a_k$  and  $f_k$ . We find them by requiring the terms in the sum (1.1) with distinct indices  $k$  to commute:

$$[F_k a_k^+ a_k + \varphi_k a_k + a_k^+ \varphi_k^+, F_l a_l^+ a_l + \varphi_l a_l + a_l^+ \varphi_l^+] = 0 \quad \text{for } k \neq l, \quad (2.3)$$

and hence

$$\{a_k, f_l^+\} = 0, \quad \{a_k^+, f_l^+\} = 0 \quad \text{for } k \neq l. \quad (2.4)$$

In addition, we require the fulfilment of the conditions (2.4) for  $k=l$ . This is a reasonable physical requirement because  $a_k$  and  $f_k$  are related to the distinct fermion fields.

A bilinear hermitian form on the set of second quantization fermion operators can be diagonalized by the linear canonical transformation [2]. We note that the conditions (2.4) permit to consider the operator

$$F a^+ a + \eta f a + \eta^* a^+ f^+ \quad (2.5)$$

only. In order to diagonalize it we use a linear canonical transformation of the fermion operators  $a$  and  $f$  to the new operators  $b$  and  $g$ , respectively

$$\begin{aligned} a &= x_1 b + x_2 g^+, & a^+ &= x_1^* b^+ + x_2^* g, \\ f^+ &= y_1 b + y_2 g^+, & f &= y_1^* b^+ + y_2^* g. \end{aligned} \quad (2.6)$$

Here the coefficients  $x_{1,2}$  and  $y_{1,2}$  have the form

$$x_{1,2} = \frac{\lambda_{1,2}}{\sqrt{|\eta|^2 + \lambda_{1,2}^2}}, \quad y_{1,2} = \frac{\eta}{\sqrt{|\eta|^2 + \lambda_{1,2}^2}}, \quad (2.7)$$

where

$$\lambda_{1,2} = \frac{1}{2}(F \pm \sqrt{4|\eta|^2 + F^2}) \quad (2.8)$$

are the eigenvalues of the matrix which is formed from the coefficients of the equations of motion for the operators  $a$  and  $f^+$ . The quantities

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad (2.9)$$

are the eigenvectors of this matrix. If one applies the canonical transformation (2.6) to (2.5), the q.l. form (2.5) reduces to the following expression

$$\lambda_1 b^+ b + \lambda_2 g g^+ \quad (2.10)$$

Thus, the problem of diagonalizing the q.l. form is solved for this case.

### 3. Diagonalization in the second case

This case is more difficult and therefore, to begin with, we assume that the index  $k$  in the sum (1.1) takes only one value. The physical meaning of this assumption is that no dispersion occurs. We note that the problem of diagonalizing the q.l. form (1.1) without dispersion has been considered for this case in paper [4] with the aid of both the canonical transformation and the two-time, temperature-dependent Green functions [8], [9].

The operators  $a^+$  and  $a$  are defined by the manner in which they act on the vectors  $|0\rangle$  and  $|1\rangle$ :

$$\begin{aligned} a|0\rangle &= 0, & a^+|0\rangle &= |1\rangle, \\ a|1\rangle &= |0\rangle, & a^+|1\rangle &= 0, \end{aligned} \quad (3.1)$$

and any operator in this space can be expressed as a linear combination of the four linear independent operators  $a, a^+, a^+a, 1$ . Let us introduce a unitary operator  $U$  which diagonalizes the q.l. form and such that  $UaU^+ = b, Ua^+U^+ = b^+$ . Then we have

$$\begin{aligned} b &= \tau_1 a^+ a + \tau_2 a + \tau_3 a^+, \\ b^+ &= \tau_1 a^+ a + \tau_3^* a + \tau_2^* a^+, \\ b^+ b &= (\tau_1^2 + |\tau_2|^2 + |\tau_3|^2) a^+ a + \tau_1 \tau_3^* a + \tau_1 \tau_3 a^+, \end{aligned} \quad (3.2)$$

where  $\tau_1, \tau_2, \tau_3$  are certain coefficients and  $\tau_1^* = \tau_1$ . The resolution of the system (3.2) with respect to  $a^+$  and  $a$  enables to express  $\tau_1, \tau_2, \tau_3$  in terms of the parameters  $F, \varphi$  and  $\varphi^*$  of the problem. The characteristic equation in this case is

$$\lambda^2 - |f|^2 = 0, \quad \lambda = \pm |f|. \quad (3.3)$$

If the characteristic roots are known, one can evaluate the eigenvectors of the matrix (1.1) in the basis (3.1) and, consequently, find the diagonalizing matrix [1]. We note that the fermion operators without index are, in principle, the Pauli operators [6].

One can consider the situation in the same way when the index  $k$  in equation (1.1) is equal to  $k=1$  and  $k=2$ .

$$L = F_1 a_1^\dagger a_1 + \varphi_1 a_1 + \varphi_1^* a_1^\dagger + F_2 a_2^\dagger a_2 + \varphi_2 a_2 + \varphi_2^* a_2^\dagger. \quad (3.4)$$

In the basis

$$|00\rangle = |0\rangle|0\rangle, \quad |10\rangle = |1\rangle|0\rangle, \quad |01\rangle = |0\rangle|1\rangle, \quad |11\rangle = |1\rangle|1\rangle \quad (3.5)$$

any operator can be expressed as a linear combination of the 16 operators

$$\begin{aligned} b = UaU^\dagger = & c_1 a_1^\dagger a_1 + c_2 a_1 + c_3 a_1^\dagger + c_4 a_2^\dagger a_2 + c_5 a_2 + c_6 a_2^\dagger + \\ & + c_7 a_1 a_2 + c_8 a_2^\dagger a_1 + c_9 a_1^\dagger a_2 + c_{10} a_2^\dagger a_1 + c_{11} a_1 a_2^\dagger a_2 + \\ & + c_{12} a_1^\dagger a_2^\dagger a_2 + c_{13} a_2^\dagger a_1^\dagger a_1 + c_{14} a_2 a_1^\dagger a_1 + c_{15} a_1^\dagger a_1 a_2^\dagger a_2 + c_{16}. \end{aligned} \quad (3.6)$$

The matrix  $\bar{L}$  in the basis (3.5) has the form

$$\bar{L} = \begin{bmatrix} 0 & f_1 & f_2 & 0 \\ f_1^* & F_1 & 0 & -f_2 \\ f_2^* & 0 & F_2 & f_1 \\ 0 & -f_2^* & f_1^* & F_1 + F_2 \end{bmatrix}. \quad (3.7)$$

and then the characteristic equation is

$$\begin{aligned} \lambda^4 - 2(F_1 + F_2)\lambda^3 + \{(F_1 + F_2)^2 + F_1 F_2 - 2(|\varphi_1|^2 + |\varphi_2|^2)\}\lambda^2 + \\ + (F_1 + F_2)(-F_1 F_2 + 2(|\varphi_1|^2 + |\varphi_2|^2)\lambda + |\varphi_1|^2 + |\varphi_2|^2) - \\ - (F_1 + F_2)(|\varphi_1|^2 F_2 + |\varphi_2|^2 F_1) = 0. \end{aligned} \quad (3.8)$$

In the general case, one can see that for  $k > 1$  the q.l. form defined on the set of fermion operators may not be a one-particle operator, i.e. may not have the form  $L' = \sum_\alpha \varepsilon_\alpha b_\alpha^\dagger b_\alpha$ .

In certain cases the terms  $a_1^\dagger a_1 a_2^\dagger a_2$  may be excluded. The criterion for this will be given in the next section. A more general situation, when  $k > 2$ , may be considered in a similar way [7].

#### 4. The use of the Dirac matrix algebra

The diagonalization of the q.l. forms (2.5) and (3.4) defined on two types of fermion operators can be simplified if one uses the Clifford algebra [3] of the Dirac matrices  $\Gamma_{\alpha\beta}$  which are expressed through the Kronecker product of the Pauli matrices  $\sigma_\alpha$  [1]

$$\Gamma_{\alpha\beta} = \sigma_\alpha \cdot \sigma_\beta, \quad \alpha, \beta = 0, 1, 2, 3, \quad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.1)$$

Any four-dimensional matrix  $\bar{C}$  can be written as

$$\bar{C} = \sum_{\alpha, \beta} g_{\alpha\beta} \Gamma_{\alpha\beta}, \quad \text{where} \quad g_{\alpha\beta} = \frac{1}{4} \text{Sp} \{ \bar{C} \cdot \Gamma_{\alpha\beta} \}. \quad (4.2)$$

Let us introduce the vacuum state and the basis in the following way

$$a_1|00\rangle=0, \quad a_2|00\rangle=0, \quad \langle 00|00\rangle=1, \quad (4.3)$$

$$\left. \begin{aligned} |00\rangle &= |0\rangle, & |01\rangle &= a_2^+ |00\rangle = |2\rangle, \\ |10\rangle &= a_1^+ |00\rangle = |1\rangle, & |11\rangle &= a_1^+ a_2^+ |00\rangle = |3\rangle. \end{aligned} \right\} \quad (4.4)$$

In this basis we have

$$a_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_1^+ = (a_1)^+, \quad a_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_2^+ = (a_2)^2. \quad (4.5)$$

From (4.5) it follows that

$$\begin{aligned} \Gamma_{01} &= a_1 + a_1^+, & \Gamma_{10} &= (1 - 2a_1^+ a_1)(a_2^+ + a_2), \\ \Gamma_{02} &= i(a_1^+ - a_1), & \Gamma_{20} &= i(1 - 2a_1^+ a_1)(a_2^+ - a_2), \\ \Gamma_{03} &= (1 - 2a_1^+ a_1), & \Gamma_{30} &= 1 - 2a_2^+ a_2, \\ \Gamma_{13} &= a_2 + a_2^+, & \Gamma_{31} &= (a_1^+ + a_1)(1 - 2a_2^+ a_2), \\ \Gamma_{23} &= i(a_2^+ - a_2), & \Gamma_{32} &= i(a_1^+ - a_1)(1 - 2a_2^+ a_2), \\ \Gamma_{12} &= i(a_1^+ + a_1)(a_2^+ + a_2), & \Gamma_{11} &= (a_1^+ - a_1)(a_2^+ + a_2), \\ \Gamma_{22} &= -(a_1^+ + a_1)(a_2^+ - a_2), & \Gamma_{21} &= i(a_1^+ - a_1)(a_2^+ - a_2), \\ & & \Gamma_{33} &= (1 - 2a_1^+ a_1)(1 - 2a_2^+ a_2). \end{aligned} \quad (4.6)$$

In the basis (4.4) the q.l. form (3.4) is represented by the matrix

$$\bar{L} = \begin{bmatrix} L_{00} & \dots & L_{03} \\ \dots & \dots & \dots \\ L_{30} & \dots & L_{33} \end{bmatrix} \quad (4.7)$$

which is diagonalized by the unitary matrix  $U$

$$\bar{L} = U^{-1} L U = \begin{bmatrix} E_0 & & & 0 \\ & E_1 & & \\ & & E_2 & \\ 0 & & & E_3 \end{bmatrix}. \quad (4.8)$$

The transformed operator is

$$b_j = U^+ a_j U, \quad b_j^+ = U^+ a_j^+ U, \quad j=1, 2. \quad (4.9)$$

Now, expressing the matrices of the operators  $U, L', b, b^+$  in terms of  $\Gamma_{\alpha\beta}$  one can find how these operators are represented through the initial operators. After diagonalization, the q.l. form (3.4) is written as a linear combination of the values  $\Gamma_{00}, \Gamma_{03}, \Gamma_{30}, \Gamma_{33}$ . One can point out a simple rule indicating when terms of the type  $a_1^+ a_1 a_2^+ a_2$  are excluded, by putting the coefficients for  $\Gamma_{33}$  equal to zero

$$C_{33} = \frac{1}{4} \text{Sp}(\Gamma_{33} L') = 0.$$

If the ground state is chosen properly, the equality  $E_0 + E_3 = E_1 + E_2$  holds.

### 5. Concluding remarks

The diagonalization of the q.l. form defined on the second quantization fermion operators is thus more complicated compared with the Bose statistics. In practice, the diagonalization differs largely in the two cases mentioned.

In the first case the linear terms may be interpreted as an interaction of the initial fermion field with an external spinor field and the q.l. form is easily diagonalized by a linear canonical transformation.

In the second case the diagonalization is more complicated since the fermion operators with different indices anticommute and, consequently, the whole sum must be diagonalized simultaneously. This fact can be interpreted as the presence of a certain interaction in the system.

Such a distinction between the Bose and Fermi systems is associated with the symmetry properties of their wave functions. For a Bose system the addition of a particle leads to the displacement of the one-particle energies only, while in the case of Fermi statistics the whole system must be reconstructed. Moreover, the one-particle energies may be nonadditive. Thus, one can say that the q.l. fermion form may be a many-particle operator unlike it is in the Bose statistics. It is also a lead in distinguishing between various types of diagonalizations. This results in different diagonalization procedures.

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