Geometry of supersymmetric solutions in ten dimensions

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Introduction

Type II supergravity: one of the possible perturbative limits of string theory

$g_{\mu\nu}$ metric $\qquad$ ϕ 'dilaton' (e^ϕ = string coupling)

$H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ "gauge curvature" with 3 indices $\quad$ [3-form $H = dB$]

F_p : "RR" p-form $\qquad p = \begin{cases} 1,\ 3,\ 5,\ 7,\ 9 & \text{(IIA)} \\ 0,\ 2,\ 4,\ 6,\ 8 & \text{(IIB)} \end{cases}$

ψ_μ gravitino

λ dilatino

Supersymmetry: bosons [forces] $\longleftrightarrow$ fermions [matter]

Aim of this talk: characterize supersymmetric solutions

$$\delta g_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}$$

$$\delta\psi_\mu = \nabla_\mu \epsilon + \text{~~RR fluxes~~}$$

- Set $\psi = 0; \;\Rightarrow \delta g_{\mu\nu} = 0$
- Set RR $= 0 \Rightarrow \delta\psi_\mu = 0$

$$\nabla_\mu \epsilon = 0$$

"restricted holonomy"

Finding solutions is easiest in terms of forms

$$\underbrace{\bar{\epsilon}\gamma_{\mu_1\ldots\mu_k}}\epsilon$$

antisymmetric

Example:

$$M_{10} = \text{Mink}_4 \times M_6$$

J symplectic form

Ω holomorphic volume form

$$\nabla_m \epsilon = 0 \quad \Longleftrightarrow \quad \begin{cases} dJ = 0 \\ d\Omega = 0 \\ J \wedge \Omega = 0 \\ J^3 = i\Omega \wedge \bar{\Omega} \end{cases}$$

millions of Calabi-Yau's found this way

[Calabi'57, Yau'77]

$\delta g_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}$

$\delta\psi_\mu = \nabla_\mu \epsilon + \text{RR fluxes}$

- Set $\psi = 0; \;\Rightarrow \delta g_{\mu\nu} = 0$
- If we don't set $\mathrm{RR} = 0$:

⇓

messy system! no holonomy interpretation

But another interpretation exists!

[Graña, Minasian, Petrini, AT '05]

$M_{10} = \mathrm{Mink}_4 \times M_6$

supersymmetry $\overset{\text{(blood and tears)}}{\Longleftrightarrow}$

$$d\phi_- = 0$$
$$d\phi_+ = F$$

the space is 'generalized complex'

name given in [Hitchin '02]

similar equations also exist for

$M_{10} = \mathrm{AdS}_4 \times M_6$ [holographically dual to CFT3]

<u>But</u>:

- what about other spacetimes?

$(\text{black hole})_4 \times M_6$

$(\text{Lifschitz})_4 \times M_6$ [holographically dual to nonrelativistic CFT3]

...

- what about different dimensions?

$\mathrm{Mink}_d \times M_{10-d}$

$\mathrm{AdS}_d \times M_{10-d}$

...

In this talk, I will describe a solution to this problem:

A system in terms of forms equivalent to supersymmetry

1. Review of the system for vacuum solutions
2. The new system: algebraic equations
3. The new system: differential equations

I. Vacuum solutions

Type II theories: two supersymmetry generators $\epsilon_{1,2}$

$M_{10} = \text{Mink}_4 \times M_6$ ⇨ decompose: $\epsilon_{1,2} = \zeta_+ \otimes \eta_+^{1,2} + \text{c.c.}$

4d (ζ_+) 6d ($\eta_+^{1,2}$)

[in IIB, from now on]

forms: $\eta_+^{1,\dagger} \gamma_{mn} \eta_+^2$ etc.

a more efficient method:

$$(\eta_+^1 \otimes \eta_-^{2\,\dagger})_{\alpha\beta} = \eta^1_{+\,\alpha} \eta^{2\,*}_{-\,\beta}$$

is a bispinor

'Clifford map':

bispinors		forms
$\gamma^{m_1 \ldots m_k}$	$\longleftrightarrow$	$dx^{m_1} \wedge \ldots \wedge dx^{m_k}$

So $\phi_\pm \equiv \eta_+^1 \otimes \eta_\pm^{2\,\dagger}$ are forms

Algebraic constraints.

When can a pair of forms $\phi_\pm$ be written as $\eta^1_+ \otimes \eta^{2\,\dagger}_\pm$?

Answer:

- introduce 'generalized Lorentz group' $\mathrm{SO}(6,6)$

generators:
- ordinary rotations $\omega^m{}_n$
- wedge by two-forms B_{mn}
- contraction with bi-vector β^{mn}

- then: if and only if the little group $\mathrm{Stab}(\phi_\pm)$ is $\mathrm{SU}(3) \times \mathrm{SU}(3)$

[in particular, $\phi_\pm$ are 'pure']

Why?

each η has a little group $\mathrm{Stab}(\eta) = \mathrm{SU}(3)$

In fact:

$$\phi_\pm \quad \longleftrightarrow \quad (g, B, \phi, \eta^{1,2})$$

antisymmetric potential — dilaton — internal metric

[Hitchin '02, Gualtieri '04]

Differential equations.

algebraic: most general solution is known!

supersymmetry $\iff$ $\begin{cases} \phi_\pm \text{ have little group } \mathrm{SU}(3)\times\mathrm{SU}(3) \\ d_H\phi_- = 0 \\ d_H\phi_+ = F \end{cases}$

[Graña, Minasian, Petrini, AT '05]

$$H = dB;\; d_H \equiv d - H\wedge$$

II. Ten dimensions: algebraic equations

We now want to generalize the system we just saw to any supersymmetric solution in 10d

[not just $\left.\begin{matrix}\mathrm{Mink}_4\\ \mathrm{AdS}_4\end{matrix}\right\} \times M_6$]

First question: what is the little group of a 10d spinor ϵ?

$$K^M = \bar\epsilon \gamma^M \epsilon \qquad \text{light-like vector: } K^M K_M = 0$$

ϵ also defines a 5-form:

$$\epsilon \otimes \bar\epsilon = K + \Omega_5 + *K$$

$\iota_K \Omega$ breaks SO(8) to Spin(7)

$$\mathrm{Stab}(K) = \mathrm{SO}(8) \ltimes \mathbb{R}^8$$

$$\boxed{\mathrm{Stab}(\epsilon) = \mathrm{Spin}(7) \ltimes \mathbb{R}^8}$$

With two spinors $\epsilon_{1,2}$, we can define bilinear

$$\Phi \equiv \epsilon_1 \otimes \bar{\epsilon}_2$$

Algebraic constraints.

When can a form Φ be written as $\epsilon_1 \otimes \bar{\epsilon}_2$?

- Again look at 'generalized Lorentz' $\mathrm{SO}(10, 10)$

From $\epsilon_{1,2}$, one would expect $(\mathrm{ISpin}(7))^2$

Unfortunately the right answer is: $\cap$

- if and only if $\mathrm{Stab}(\Phi)$ is $(\mathrm{Spin}(7)^2 \times \mathrm{Sl}(2, \mathbb{R})) \ltimes \mathrm{Heis}_{33}$

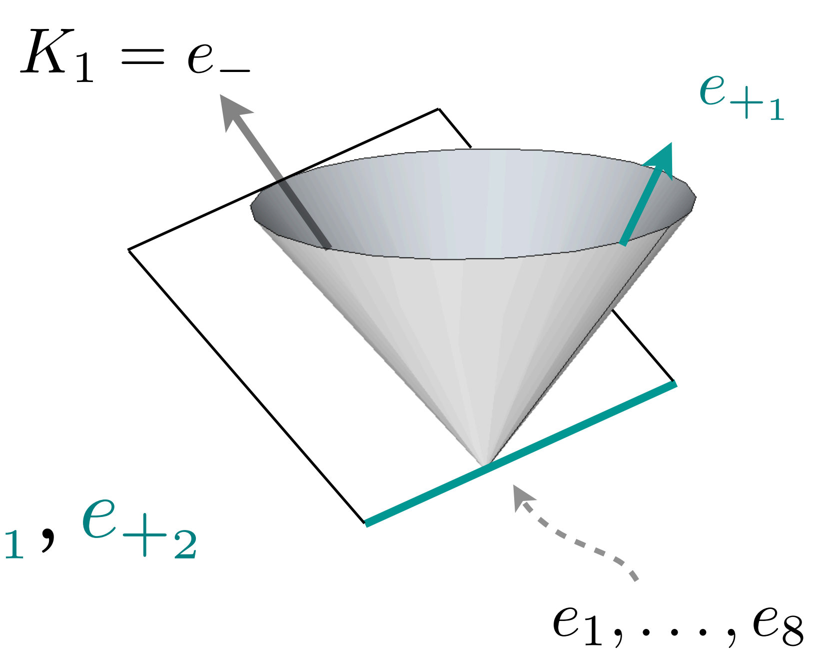
[Φ is not pure!]

- I listed all possible solutions to this constraint.
- Stab($\Phi = \epsilon_1 \otimes \bar{\epsilon}_2$) bigger than Stab($\epsilon_1$) $\times$ Stab(ϵ_2)

$\Downarrow$

Φ does not determine a metric!

each ϵ defines an "incomplete vielbein";



Possibility: complete them with vectors e_{+_1}, e_{+_2}

$(\Phi, e_{+_1}, e_{+_2}) \mapsto$ metric

II. Ten dimensions: differential equations

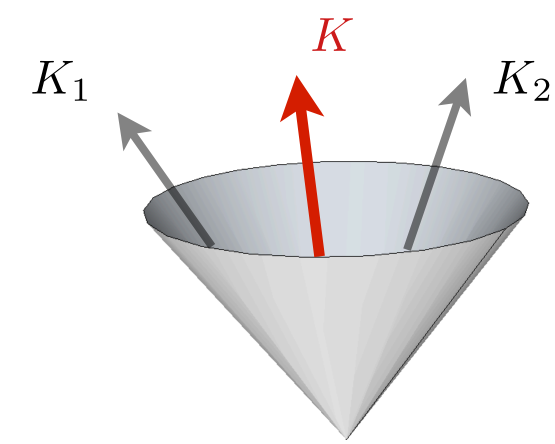
two lightlike vectors

$$K_i^M = \bar{\epsilon}_i \gamma^M \epsilon_i$$

timelike or lightlike

$$K = \tfrac{1}{2}(K_1 + K_2)$$

$$\tilde{K} = \tfrac{1}{2}(K_1 - K_2)$$



two old equations:

$$L_K g = 0$$

$$\iota_K H = d\tilde{K}$$

[K is Killing vector]

$$\Rightarrow \quad L_K H = \{d, \iota_K\} H = \iota_K dH = 0$$

[K preserves H]

[Hackett-Jones, Smith '04; Koerber, Martucci '07; Figueroa-O'Farrill, Hackett-Jones, Moutsopoulos '07]

What about Φ?

$$L_K g = 0 \qquad \iota_K H = d\tilde{K}$$

$$\boxed{d_H \Phi = (\tilde{K} \wedge + \iota_K) F}$$

[AT '11]

This equation reproduces the ones for vacua:

$$M_{10} = \mathbf{Mink}_4 \times M_6$$

$$\Phi = \mathrm{Re}[(v + i * v) \wedge \phi_+ + v \wedge w \wedge \phi_-]$$

$$K = v$$

$$\tilde{K} = 0$$

constant vectors on Mink_4

$$\Rightarrow \begin{cases} d\phi_- = 0 \\ d\phi_+ = F \end{cases}$$

✓

$$\text{supersymmetry} \iff \begin{cases} L_K g = 0 \qquad \iota_K H = d\tilde{K} \\ \boxed{d_H \Phi = (\tilde{K}\wedge + \iota_K) F} \\ (\psi \otimes \overline{\epsilon_2} e_{+_2}, d_H(\Phi e_{+_2}) - F) = 0 \\ (e_{+_1}\epsilon_1 \otimes \overline{\psi}, d_H(\Phi e_{+_2}) - F) = 0 \end{cases} \quad \forall \psi$$

[AT '11]

main result!

[(,) natural inner product]

- no Ansatz necessary
- same system in IIA and IIB
- no covariant derivatives

- forms $e_{+_1}\epsilon_1 \otimes \overline{\psi}$ should be characterized without spinors
- appearance of e_+ (inevitable?)

It is possible to apply it now to 4d susy solutions

[work in progress with my student Dario Rosa]

- one can apply similar methods to $\mathcal{N}=2$, $d=4$ supergravity
  - a 4d spinor describes an $\mathbb{R}^2$ structure: again an "incomplete vielbein"
  - we need again to introduce e_+'s
- but when K is timelike, $\exists$ nicer system

$$\begin{cases} dv_{ij} = \ldots \\ v_{ij} \cdot dq^u = \ldots \end{cases}$$

four 1-forms (v_{ij})

hyperscalars (q^u)

- we are now putting the 10d system in the same form
  - when K is timelike, we traded the ugly 'pairing equations' for $dK = \ldots$

Conclusions

- We saw how supersymmetry in type II string theory can be expressed in terms of forms.

- There is room for improvement, but potentially the system we found is more fundamental than the one for 4d compactifications.

- It still uses the geometry of $T \oplus T^*$, but it does not involve generalized complex structures or pure spinors.