Frontiers of Mathematical Physics

Integrable, nonintegrable and partially integrable dynamical systems in generalized gravity theories

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arXiv:1112.3023 (math-phys) (first part of a paper, on D=3, D=4 theories)

General properties and some spherical/cylindrical reductions, expansion near horizons, vecton – scalaron duality, topological portrait of Static (BH) - Cosmology solutions

arXiv:1011.2445 v1 (gr-qc) 1-dim. theory as a relativistic particle

arXiv:1008.2333 v1 (hep-th) attempt at a new general formulation of geom.

arXiv:1003.0782 v3 (hep-th) further generalizations, cosmological solutions.

arXiv:0812.2616 v2 (gr-qc) the first paper on new interpretation of **Einstein** 3 papers of **1926**; simplified model, static solutions, existence of horizons, non-integrability, approximate solutions by various power series expansions.

We briefly describe the simplest class of **affine theories of gravity in multidimensional space-times with symmetric connections** and their reductions to two-dimensional dilaton - vecton gravity field theories (DVG). The distinctive feature of these theories is the presence of an absolutely neutral massive (or tachyonic) vector field (vecton) with essentially nonlinear coupling to the dilaton gravity. We emphasize that the vecton field can be consistently replaced by a new effectively massive scalar field (scalaron) with an unusual coupling to dilaton gravity. Thus for treating this vecton - scalaron duality, one can use methods and results of dilaton gravity coupled to scalars in more complex vecton theories.

We present the DVG models derived by reductions of *D*=3 and *D*=4 affine theories and write *one dim. dynamical system simultaneously describing cosmological* ***and static states*** with different parameters: including singularities, horizons, tachyonic masses, and wrong-sign (phantom) kinetic terms.

Our approach is fully applicable to studying static and cosmological solutions in multidimensional theories (esp., by using the **scalaron - vecton duality**) as well as to general one-dimensional DSG models (DSG-1). The global structure of the solutions of integrable DSG-1 models can be usefully visualized by drawing their **`topological portraits'** resembling the phase portraits of dynamical systems and simply visualizing

static – cosmological duality

Content of the talk

Brief summary of affine models based on WEE ideas

Dimensional reduction to spherical and cylindrical configurations

Vecton – Scalaron equivalence in Dilaton Gravity $D = 2,1$

Unified treatment of **static** and **cosmological** solutions

Integrability *vs*. nonintegrability: new integrable models and

`Master Integral Equation' or MIE, in DVG and DSG

Partially integrable DSG (insufficient number of integrals)

Topological portraits (ideas and simplest examples)

Main principles (suggested by Einstein's approach)

1. **Geometry:** **dimensionless** *`action'* constructed of a ***scalar*** *density*; its variations give the geometry and main equations ***without complete specification of the analytic form of the Lagrangian .***

2. **Dynamics:** a concrete Lagrangian constructed of the ***geometric variables*** - homogeneous function of order D (e.g. , the square root of the determinant of the curvature) produces a physical **effective Lagrangian**.

3. **Duality** between the geometrical and physical variables and Lagrangians.

NB: This looks more artificial than the first two principles and works for rather special models (actually giving *exotic fields*, *tachyons* etc.

GEOMETRY OF SYMMETRIC CONNECTIONS

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + a^i_{jk}$$

$$\Gamma^i_{jk}[g] = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l})$$

$$r^i_{jkl} = -\gamma^i_{jk,l} + \gamma^i_{mk}\gamma^m_{jl} + \gamma^i_{jl,k} - \gamma^i_{ml}\gamma^m_{jk}$$

NONSYMMETRIC RICCI CURVATURE

$$r_{jk} = -\gamma^i_{jk,i} + \gamma^i_{mk}\gamma^m_{ji} + \gamma^i_{ji,k} - \gamma^i_{mi}\gamma^m_{jk}$$

Symmetric part of the Ricci curvature

$$s_{ij} \equiv \frac{1}{2}(r_{ij} + r_{j\,i})$$

Anti-symmetric part of the Ricci curvature

$$a_{ij} \equiv \frac{1}{2}(r_{ij} - r_{j\,i}) = \frac{1}{2}(\gamma^m_{jm,i} - \gamma^m_{im,j})$$

$$a_{ij,\,k} + a_{jk,i} + a_{ki,j} \equiv 0$$

VECTON: $\quad a_i \equiv a^m_{im}$

$$a_i \equiv \gamma^m_{mi} - \Gamma^m_{mi} \equiv \gamma_i - \partial_i \ln \sqrt{|g|}$$

$$a_{ij} \equiv -\frac{1}{2}(a_{i,j} - a_{j,i}) \equiv -\frac{1}{2}(\gamma_{i,j} - \gamma_{j,i})$$

$\alpha\beta$ - CONNECTION

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + \alpha(\delta^i_j\,\hat{a}_k + \delta^i_k\,\hat{a}_j) - (\alpha - 2\beta) g_{jk}\,\hat{a}^i$$

Weyl: $\beta = 0$

geo-Riemannian: $\alpha = 2\beta$.

Einstein $\alpha = -\beta = \frac{1}{6}$

LINEAR TERMS in $s_{ij} - R_{ij}(g)$

$$(\alpha + \beta)(\nabla_i \hat{a}_j + \nabla_j \hat{a}_i) + (\alpha - 2\beta)\, g_{ij} \nabla_m \hat{a}^m$$

QUADRATIC TERMS in $s_{ij} - R_{ij}(g)$

$$\hat{a}_i \hat{a}_j \left[(\alpha - 2\beta)^2 - 3\alpha^2\right] + 2\, g_{ij} \hat{a}^2 (\alpha - 2\beta)(\alpha + \beta)$$

In addition to this dependence on the vecton, the generalized Einstein equations will depend on it through dynamics specified by the chosen Lagrangian

FROM GEOMETRY TO DYNAMICS

REQUIREMENTS TO LAGRANGIAN DENSITIES

1. IT IS INDEPENDENT OF DIMENSIONAL CONSTANTS.

2. ITS INTEGRAL OVER SPACE-TIME IS DIMENSIONLESS.

3. IT CAN DEPEND ON TENSOR VARIABLES HAVING
a DIRECT GEOMETRIC MEANING and
a NATURAL PHYSICAL INTERPRETATION.

4. THE RESULTING GENERALIZED THEORY MUST AGREE WITH ALL ESTABLISHED EXPERIMENTAL CONSEQUENCES OF EINSTEIN'S THEORY.

r_{ij}, s_{ij}, a_{ij}, and $a_k \equiv a^i_{ik}$ satisfy requirement **3**.

Einstein's choice is $\mathcal{L} = \mathcal{L}(s_{ij}, a_{ij})$

A simple nontrivial choice of a geometric Lagrangian density generalizing the Eddington – Einstein Lagrangian ,

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r} \ ,$$

is the following, depending on one dimensionless parameter:

$$\mathcal{L} = \mathcal{L}(s_{ij} + \nu a_{ij}) = \sqrt{-\det(s_{ij} + \nu a_{ij})}$$

$$\det(s_{ij}) < 0$$

When $\nu a_{ij} \rightarrow 0$ it will give Einstein's gravity with the cosmological constant.

Now we define (following Einstein) the metric and field densities by a Legendre-like transformation

$$\frac{\partial \mathbf{L}}{\partial s_{ij}} \equiv \mathbf{g}^{ij}\,, \qquad \frac{\partial \mathbf{L}}{\partial a_{ij}} \equiv \mathbf{f}^{ij}$$

dual to

$$s_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{g}^{ij}}\,, \qquad a_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{f}^{ij}}$$

$$\nabla_i^{\gamma}\, \mathbf{f}^{ki} = \partial_i \mathbf{f}^{ki} \equiv \mathbf{a}^k. \qquad \nabla_i^{\gamma}\, \mathbf{g}^{ik} = -\frac{D+1}{D-1}\hat{\mathbf{a}}^k$$

The main equation $\nabla_i^{\gamma}\, \mathbf{g}^{jk} = -\frac{1}{D-1}(\delta_i^j \hat{\mathbf{a}}^k + \delta_i^k \hat{\mathbf{a}}^j)$

for any dimension D

Defining the Riemann metric tensor g_{ij} by the equations

$$g^{ij}\sqrt{-g} = \mathbf{g}^{ij}\,, \quad g_{ij}\,g^{jk} = \delta_i^k$$

$$\nabla_i\, g_{jk} = 0, \quad \nabla_i\, g^{jk} = 0 \qquad \hat{a}^k \equiv \hat{\mathbf{a}}^k/\sqrt{-g}$$

we can **derive** the expression for the connection coefficients

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + \alpha_D\,[\,\delta^i_j\,\hat{a}_k + \delta^i_k\,\hat{a}_j - (D-1)\,g_{jk}\,\hat{a}^i]$$

$$\alpha_D \equiv [(D-1)(D-2)]^{-1}\,, \qquad \beta_D \equiv -[2(D-1)]^{-1}$$

We thus have derived the connection using a rather general dynamics!

Not using any particular form of the geometric Lagrangian!

Using a simple dimensional reduction to the dimension 1+1 (similar to spherical or cylindrical reductions in the metric case) we easily derive the important relation between geom. and phys.

$$\mathcal{L} = -\frac{1}{2}\sqrt{|\det(s + \lambda^{-1} a)|} = -2\Lambda\sqrt{|\det(\mathbf{g} + \lambda\mathbf{f})|} = \mathcal{L}^*$$

Λ having the dimension L^{-2}

Using the above definitions, → we can then write the **generalized Einstein eqs.**

$$s_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{g}^{ij}}\,, \qquad a_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{f}^{ij}}$$

In **dimension D** we can similarly derive the relation

$$\mathcal{L}^* \equiv \sqrt{-\det(s_{ij} + \nu a_{ij})} \sim \sqrt{-g}\ [\det(\delta_i^j + \lambda f_i^j)]^{1/(D-2)}$$

The generalized Einstein–Eddington-Weyl model in dimension D

$$\mathcal{L}_{eff} = \sqrt{-g}\left[-2\Lambda\,[\det(\delta_i^j + \lambda f_i^j)]^{1/(D-2)} + R(g) + c_a\, g^{ij} a_i a_j\right]$$

Restoring the dimensions and expanding the root term up to the second order in the vector and scalar fields

$$\mathcal{L}_{eff} \cong \sqrt{-g}\left[R[g] - 2\Lambda - \kappa\left(\frac{1}{2}F_{ij}F^{ij} + \mu^2 A_i A^i + g^{ij}\partial_i\psi\,\partial_j\psi + m^2\psi^2\right)\right]$$

$$A_i \sim a_i,\; F_{ij} \sim f_{ij},\; \kappa \equiv G/c^4$$

NB: $\partial_i\psi$ Is proportional to F_{ij} for i < 4, j=4

Dimensional reductions of

$$\mathcal{L}_{\rm ph} = \sqrt{-g}\left[-2\Lambda\left[\det(\delta_i^j + \lambda f_i^j)\right]^\nu + R(g) + c_a\, g^{ij} a_i a_j\right]$$

Spherical reduction of the theory

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij}\, dx^i\, dx^j \; + \; \varphi^{2\nu}\, d\Omega_{D-2}^2(k)$$

$$\mathcal{L}_D^{(2)} = \sqrt{-g}\left[\varphi R(g) + k_\nu\, \varphi^{1-2\nu} + \frac{1-\nu}{\varphi}(\nabla\varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2\varphi\, \mathbf{a}^2\right]$$

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda\varphi\left[1 + \frac{1}{2}\lambda^2\mathbf{f}^2\right]^\nu \qquad \mathbf{f}^2 \equiv f_{ij} f^{ij} \qquad \nu \equiv (D-2)^{-1}$$

Weyl rescaling

$$g_{ij} = \hat{g}_{ij}\, w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu} \qquad \mathbf{f}^2 = w^2\, \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w\, \hat{\mathbf{a}}^2$$

Cylindrical reductions: Kaluza from D=4 to D=2

$$ds_4^2 = (g_{ij} + \varphi\,\sigma_{mn}\,\varphi_i^m \varphi_j^n)\,dx^i dx^j + 2\varphi_{im}\,dx^i dy^m + \varphi\,\sigma_{mn}\,dy^m dy^n$$

$$\sigma_{22} = e^{\eta}\cosh\xi, \quad \sigma_{33} = e^{-\eta}\cosh\xi, \quad \sigma_{23} = \sigma_{32} = \sinh\xi$$

$$\left[\varphi R(g) + \frac{1}{2\varphi}(\nabla\varphi)^2 + V_{\text{eff}}(\varphi,\xi,\eta) - \frac{\varphi}{2}[(\nabla\xi)^2 + (\cosh\xi)^2\,(\nabla\eta)^2]\right]$$

$$V_{\text{eff}}(\varphi,\xi,\eta) = -\frac{\cosh\xi}{2\varphi^2}\left[Q_1^2 e^{-\eta} - 2Q_1Q_2\tanh\xi + Q_2^2 e^{\eta}\right]$$

$$\mathcal{L}_W^{(2)} = \sqrt{-g}\left[\varphi R(g) - \frac{Q_1^2}{2\varphi^{5/2}}\,e^{-\eta} - \frac{\varphi}{2}\,(\nabla\eta)^2\right]$$

$$\mathcal{L}^{(2)}_{DW} = \sqrt{-g}\left[\varphi R(g) + k_\nu\, \varphi^{-\nu} - 2\Lambda\varphi^{\nu}\left[1 + \frac{1}{2}\lambda^2\varphi^{2(1-\nu)}\mathbf{f}^2\right]^{\nu} - m^2\varphi\,\mathbf{a}^2\right]$$

3-dimensional theory

$$\mathcal{L}^{(2)}_{3} = \sqrt{-g}\left[\varphi R(g) - 2\Lambda\varphi - \lambda^2\Lambda\varphi\,\mathbf{f}^2 - m^2\varphi\,\mathbf{a}^2\right]$$

Vecton – Scalaron DUALITY

$$ds^2 = -4\,h(u,v)\,du\,dv\,, \qquad \sqrt{-g} = 2h \qquad f^n_{uv} \equiv a^n_{u,v} - a^n_{v,u}$$

$L\,/2h$ = $\varphi R + V(\varphi,\psi) + X(\varphi,\psi;\mathbf{f}^2_n)$ $\qquad -2\mathbf{f}^2_n = (f^n_{uv}/h)^2$

$L'\,/2h$ = $\varphi R + V(\varphi,\psi) + X_{\text{eff}}(\varphi,\psi;q_n)$ $\qquad q_n(u,v) \equiv h^{-1}X_n\,f^n_{uv}$

Effective action on `mass shell'; f – from eq, & $\qquad$ & $\quad X_n \equiv \dfrac{\partial X}{\partial\,\mathbf{f}^2_n}$

$$X_{\rm eff}(\varphi,\psi;q_n) = X(\varphi,\psi;\bar{\mathbf{f}}_n^2) + \sum q_n(u,v)\sqrt{-2\,\bar{\mathbf{f}}_n^2}$$

where: $2\,\bar{\mathbf{f}}_n^2 = -(q_n/\bar{X}_n)^2$ $\qquad \bar{X}_n \equiv \frac{\partial}{\partial\,\bar{\mathbf{f}}_n^2}X(\varphi,\psi;\bar{\mathbf{f}}_n^2)$

$$\partial_u(h^{-1}X_n f^n_{uv}) = -Z_n a^n_u \;=\; \partial_u\, q_n(u,v)$$

This defines a^n_u in terms of $q_n(u,v)$ and (φ,ψ)

$$X_{\rm eff} = -2\Lambda\sqrt{\varphi}\left[1+q^2/\lambda^2\Lambda^2\varphi^2\right]^{\frac{1}{2}}$$ for ***D*** = 4

$V = 2k\varphi^{-\frac{1}{2}}\,, \quad \bar{Z} = -1/m^2\varphi$ **N.B**: normally, ***Z*** ~ to dilaton φ

$$X_{\rm eff}(\varphi;\,q(u,v)) = -q^2/\lambda^2\Lambda\varphi - 2\Lambda\varphi$$ ***D*** = 3

The result: we can study **DSG** instead of **DVG**

General **dilaton gravity** coupled to massless vectors and eff. massive scalars

$$\mathcal{L}^{(2)} = \sqrt{-g}\,[\,U(\varphi)R(g) + V(\varphi) + W(\varphi)(\nabla\varphi)^2 + \\ + X(\varphi, \psi, F^2_{(1)}, ..., F^2_{(A)}) + Y(\varphi, \psi) + \sum_n Z_n(\varphi, \psi)(\nabla\psi_n)^2\,]\,.$$

Dilaton gravity **dual** to vecton gravity with **massless Abelian vector fields,** Weyl frame

$$\mathcal{L}^{(2)}_{\rm eff} = \sqrt{-g}\left[\varphi R + V(\varphi, \psi) + X_{\rm eff}(\varphi, \psi; q) + \sum Z(\varphi, \psi)(\nabla\psi)^2\right].$$

Dilaton – Scalar Gravity (DSG) **dual** to **massive vecton gravity** in Weyl frame

$$\mathcal{L}_{\rm dsg} = \sqrt{-g}\left[\varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2 + \sum Z(\varphi, \psi)(\nabla\psi)^2\right]$$

A general theory of **HORIZONS** in DSG

L' /2h = $\varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2$ (omitting normal scalars)

Consider **STATIC** solutions that normally **have horizons** when there are **no scalars**

All the equations can be derived from the Hamiltonian (constraint)

H = $\dot{\varphi}\,\dot{h}/h + hU + \bar{Z}\,\dot{q}^2 + Z\,\dot{\psi}^2$ **(= 0** in the end)

Without the scalars the EXACT solutions is: $h = C_0^2\,[N_0 - N(\varphi)]$

where $N(\varphi) \equiv \int U(\varphi) d\varphi$ $C_0 \tau = \int d\varphi\,[N_0 - N(\varphi)]^{-1}$

There is always a horizon, i.e. $h \rightarrow 0 \text{ for } \varphi \rightarrow \varphi_0$

Horizons are classified into:

:

regular **simple**, regular **degenerate**, **singular**

We find a gen. sol. near horizon as locally convergent power series in: $\tilde{\varphi} \equiv \varphi - \varphi_0$

$$h = \sum h_n \tilde{\varphi}^n, \qquad \chi = \sum \chi_n \tilde{\varphi}^n, \qquad q = \sum q_n \tilde{\varphi}^n, \qquad \chi(\varphi) \equiv \dot{\varphi}$$

$$h_0 = \chi_0 = 0 \qquad q_0 \neq 0 \qquad \tilde{\varphi} \equiv \varphi - \varphi_0$$

The equations for these functions are **not integrable** and we do not know exact solutions of the recurrence relations

Practically the **same equations** are applicable to studies of the **cosmological models** with vecton. The best chance to **test the theory is in cosmology**

However, we can show that the global picture cannot be found without knowledge of horizons connecting static and cosmological solutions.

It is important to use local language. BUT! The physics can not be completely understood without global picture.

Main differential equations

$$\psi' = E(\xi)\,, \qquad H' = -E^2 H(\xi)\,;$$

$$\chi' = -Z(\varphi)\,U(\varphi,\psi)\,H\,, \qquad \eta' = -\frac{1}{2}Z(\varphi)\,U_\psi(\varphi,\psi)\,H\,.$$

$$\psi' \equiv \frac{d\psi}{\partial\xi}\,, \qquad U_\psi \equiv \frac{\partial U}{\partial\psi}\,, \qquad \xi \equiv \int d\varphi\, Z^{-1}(\varphi)\,, \qquad Z(\varphi) = 1/\xi'(\varphi)\,.$$

$$E(\xi) \equiv \frac{\eta(\xi)}{\chi(\xi)}\,, \qquad H(\xi) \equiv \frac{h(\xi)}{\chi(\xi)}\,. \qquad \frac{d\eta}{d\chi} = \frac{U_\psi}{2U}\,, \qquad \frac{d\ln H}{d\psi} = -E\,.$$

$$\psi(\xi) = \psi_0 + \int_{\xi_0}^{\xi} E(\bar\xi) \equiv \mathcal{I}\{E;\xi\}\,,$$

Solutions in terms of **one function *E***

Basic solutions

$$H(\xi) = H_0 \exp\int_{\xi_0}^{\xi} E^2(\bar\xi) \equiv H_0\,\exp\mathcal{I}\{E^2;\xi\}\,,$$

$$\chi(\xi) = \chi_0 - \mathcal{I}_1\{E;\xi\}\,, \qquad \eta = \eta_0 - \mathcal{I}_2\{E;\xi\}\,,$$

$$\mathcal{I}_1\{E;\xi\} = -H_0 \int_{\xi_0}^{\xi} d\bar{\xi}\, Z[\varphi(\bar{\xi})]\; U[\varphi(\bar{\xi}, \mathcal{I}\{E;\bar{\xi}\}]\; e^{\mathcal{I}\{E^2;\bar{\xi}\}}\,.$$

$$\mathcal{I}_2\{E;\xi\} = -\frac{1}{2} H_0 \int_{\xi_0}^{\xi} d\bar{\xi}\, Z[\varphi(\bar{\xi})]\; U_{\psi}[\varphi(\bar{\xi}), \mathcal{I}\{E;\bar{\xi}\}]\; e^{\mathcal{I}\{E^2;\bar{\xi}\}}$$

THE MASTER INTEGRAL EQUATION

$$E(\xi) = \frac{\eta_0 - \mathcal{I}_2\{E;\xi\}}{\chi_0 - \mathcal{I}_1\{E;\xi\}}$$

$$U = u(\varphi)v(\psi) \Rightarrow \frac{U_\psi}{2U} = v_\psi(\psi)/2v(\psi) = g_1 \quad \text{if} \quad v(\psi) = e^{2g_1\psi}$$

Then $\mathcal{I}_2 = \mathcal{I}_1$ and we find the second-order diff. eq. for $E(\xi)$

It is integrable if, in addition, $Z(\varphi)\,u(\varphi) = g\,e^{g_2\,\xi(\varphi)}$.

Without it we have a 'partial integrability as $\eta = g_1\chi + C_0$.

This is a generalization of the model in ATF `96

If $C_0 = 0$, one can analytically derive the solutions depending on three parameters.

With this condition, we solve the essentially nonlinear system using **effective iterations of the MIE** from ***E* = *g*_1**

Gauge independent dim. reduction: 2 to 1

$$ds_2^2 = e^{2\alpha(t,r)}dr^2 - e^{2\gamma(t,r)}dt^2$$

$$\epsilon \mathcal{L}_{\rm V}^{(1)} = e^{\epsilon(\alpha-\gamma)}\varphi\left[\dot{\psi}^2 - 2\dot{\alpha}_\epsilon \frac{\dot{\varphi}}{\varphi} - (1-\nu)\left(\frac{\dot{\varphi}}{\varphi}\right)^2\right] -$$

\epsilon = -1 for static solutions

$$\cdot\, e^{\epsilon(\gamma-\alpha)}\mu_v\,\varphi\, a_\epsilon^2 + \epsilon\, e^{\alpha+\gamma}\left[V + X(\mathbf{f}^2)\right]$$

$$\mathcal{L}_a \equiv h\, X(\mathbf{f}^2) - \tilde{\mu}\,\varphi\, a^2\,; \qquad \tilde{\mu} = e^{\epsilon(\gamma-\alpha)}\mu_v\,, \quad h = \epsilon\, e^{\alpha+\gamma}\,.$$

$$-\lambda^2\left(\frac{\dot{a}}{h}\right)^2 = -y^2\,, \quad p_a \equiv \frac{\partial \mathcal{L}_a}{\partial \dot{a}} = \lambda\frac{\partial X}{\partial y}\,, \Rightarrow -2q\,, \quad a \Rightarrow p/2$$

$$\mathcal{H}_a(p_a\,, a) \equiv p_a\,\dot{a} - \mathcal{L}_a = -h\left[X - y\frac{\partial X}{\partial y}\right] + \tilde{\mu}\,\varphi\, a^2$$

$$l_\epsilon^{-1}\left[\varphi\,\dot\psi^2 - 2\dot\alpha_\epsilon\,\dot\varphi - (1-\nu)\,\frac{\dot\varphi^2}{\varphi} + \frac{\dot q^2}{m^2\varphi}\right]$$

Scalaron Lagr. and eff. potential

$$+\,l_\epsilon\,\epsilon\,e^{2\alpha_\epsilon}\,U(\varphi,\psi,q)$$

$$X_{\rm eff} = X - y\frac{\partial X}{\partial y} = -2\Lambda\varphi\,\frac{x}{y}\left[1-(1-2\nu)y^2\right]$$

$$y = x\left[1-(1-\nu)\,x^2+\ldots\right]$$

$$1-y^2 = |x|^{-\sigma} - (\sigma/2)\,|x|^{-2\sigma} + \qquad\qquad \sigma\equiv 1/(1-\nu)$$

$$y = x\,(1-y^2)^{1-\nu} \qquad\qquad x\equiv q/(-2\nu\lambda\Lambda\,\varphi)$$

$$X_{\rm eff} = -2\Lambda\varphi\,[1 + q^2/4\nu\lambda^2\Lambda^2\varphi^2 \; + O(x^4)]$$

Universal asympt.

$$X_{\rm eff} = -2\sqrt{q^2/\lambda^2} + O(|x|^{-\sigma})$$

$$X_{\rm eff}/(-2\Lambda\varphi) \equiv v_\nu(x)$$

Reduced potential

$$\mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1}\,[\,\dot{\bar{q}}^2 - \xi\,\dot{\alpha}_\epsilon\,] - \bar{l}_\epsilon\,2\Lambda h\,\varphi^{\nu-1}\,v_\nu\,(mx)$$

$$v_\nu(x) = 1 + \nu x^2 + O(x^4)$$

Monotonic concave function

$$v_\nu(x) = 2\nu x\left[1 + \frac{1-\nu}{2\nu}x^{-\sigma} + \right.$$

$$\mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1}\,[\,\dot{\bar{q}}^2 - \dot{\xi}\,\dot{\alpha}_\epsilon\,] - \bar{l}_\epsilon\, 2\Lambda h\, \varphi^{\nu-1}\, v_\nu\,(mx)$$

Hamiltonian: $$p_q^2/4 - p_\alpha p_\xi \;+\; 2\Lambda h\, \varphi^{\nu-1}\, v_\nu\,(mx) \;=\; 0$$

$$\dot{\bar{q}} = \bar{l}_\epsilon\, p_q/\,2\,, \quad \dot{\xi} = -\,\bar{l}_\epsilon\, p_\alpha\,, \quad \dot{\alpha} = -\,\bar{l}_\epsilon\, p_\xi$$

D = 2: $$2\Lambda h\, v_1\,(mx) \equiv hV\,(m^2 q^2/\xi)$$

$$\dot{p}_q = \;-2hV'\, m^2\bar{q}/\xi\,, \quad \dot{p}_\xi = hV'\, m^2\bar{q}^2/\xi^2$$

$$\dot{p}_\alpha = -2hV(m^2\bar{q}^2/\xi) \qquad\qquad p_q^2 - 4p_\alpha\, p_\xi + 4hV = 0$$

$$\bar{q}\, p_q + 2\,\xi\, p_\xi - p_\alpha = \;C_0$$

Hamiltonian

Integral

Simple linear PDE for V leads to integrals

As distinct from the standard Einstein theory, the generalized one is **not integrable** even in dimension one (static states and cosmologies). Therefore, in addition to the above solutions we need a global information on the system, which we may attempt to present as

topological portrait.

We try to demonstrate that the portrait must include **both static and cosmological** solutions, and that the most important info is in the structure of horizons. Actually, it is not less important for cosmologies than for static states. We prefer to use the **local language** and do not use the term Black Hole which should be reserved for real physical objects

For the moment, the idea can be explained only on integrable systems and only on the plane. For nonintegrable systems

we need **3D portraites**

Rather a general integrable models, ATF `96

Massless scalar field INTEGRAL

$$V = W(\bar{g}_4 w^2 - \bar{g}_1), \quad Z^{-1} = W(\bar{g}_3 + \bar{g}_2 \log w)$$

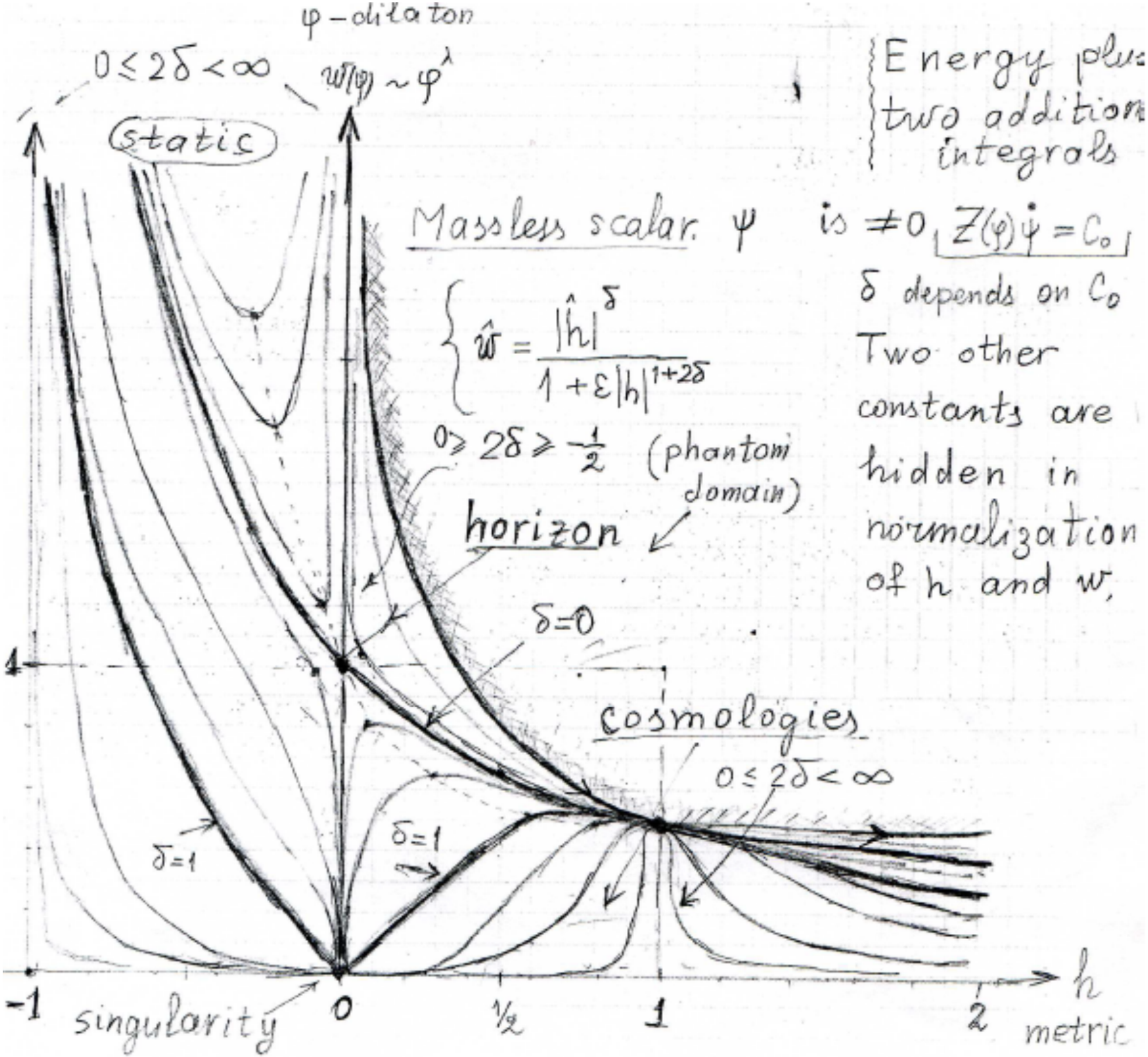
$$(F/W)^2 + 4\bar{g}_1 h + 2\bar{g}_2 C_0^2 \log h = \bar{C}_1$$

$$W = (1-\nu)/\phi,\; w = \phi^{1-\nu},\; Z = -\gamma\phi$$

$$w = \frac{|h|^\delta}{|1 + \varepsilon|h|^{1+2\delta}|}\,.$$

$$\bar{g}_1 = \bar{g}_2 = 0$$

$$w(\phi),\; w'/w \equiv W/U'$$



φ – dilaton
0 ≤ 2δ < ∞
w(φ) ~ φ^λ
static
Energy plu
two additional
integrals
Massless scalar ψ is ≠ 0
Z(φ)ψ̇ = C₀
δ depends on C₀
ŵ = |ĥ|^δ / (1 + ε|h|^(1+2δ))
Two other constants are hidden in normalization of h and w.
0 ≥ 2δ ≥ −1/2 (phantom domain)
horizon
δ=0
cosmologies
0 ≤ 2δ < ∞
δ=1
δ=1
singularity
-1
0
½
1
2
h
metric

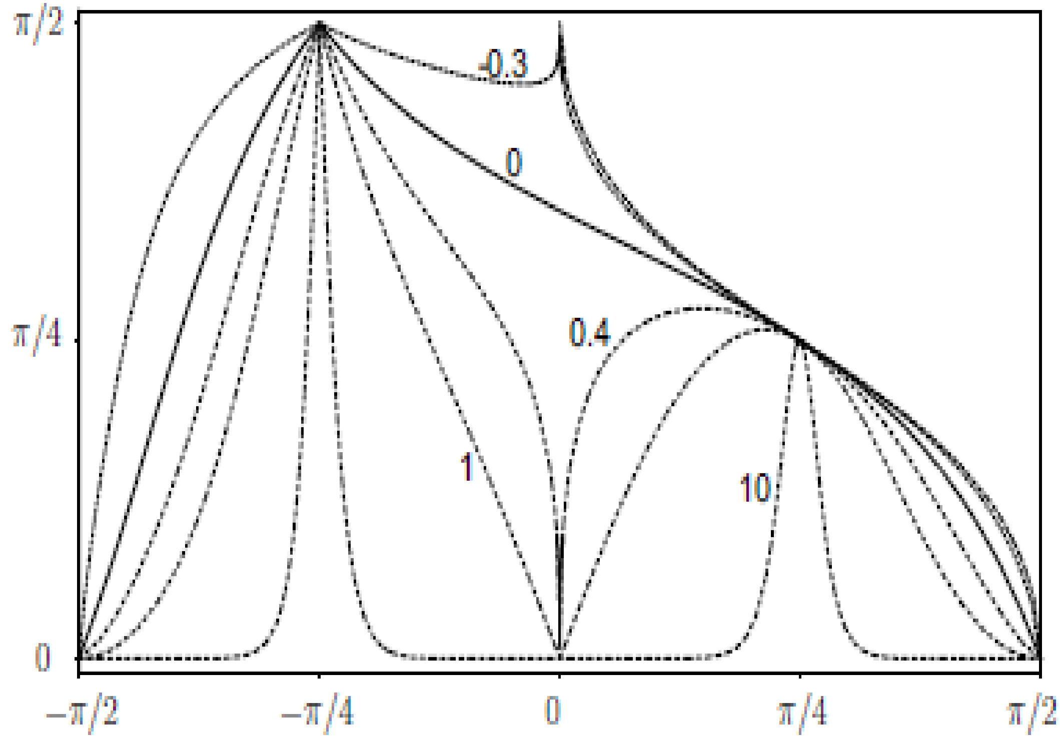


Figure 1: Topological portrait $\arctan w(x)$ of the dilaton-scalar configuration. Values of δ are given on plot. Solid line — separatrix with zero δ, dashed line — separatrix with arbitrary δ.

Picture by E.Davydov

Though the **MIE** provides us with an apparently ***new*** approach to solving most difficult global problems (e.g., a transition to **chaotic behavior**?) the considered examples are, of course, simply `warming up' ones.

Crucial thing is to learn of how to find (partial) **portraits** in **not integrable** case (necessary **3D portraits!)**

A very important field for further studies is to generalize our **S-C duality** to a possible **S-C-Waves triality**. It was uncovered in the integrable *N*-Liouville models by VdA and ATF *(effectively one-dimensional waves of matter)* but a generalization to non-integrable case looks difficult although really important for cosmological applications.

Effects of *nonlinear Lagrangians* are being studied (like in `B-I cosmology')

Vecton dark matter can be produced in ***strong*** *gravitational fields* only. **Quantum gravity** is necessary!

Inflation and ***dark matter*** are crucial things to study and test the theory in cosmological models

THE

END