preamble

Some symmetries and integrability in generalized gravity theories

The topic

Creating the model: starting from

1. Phenomenology
2. Fundamental ideas

The second approach:

- Few ideas provide a variety of models
- Even the simplest models may be rather complicated

The topic: a long way from fundamental ideas to verification of models — need to distinguish the fruitful models as soon as possible.

DSG models and technique

Developing the technique which may take into account the general model properties.
The model is the effective Dilaton-Scalar-Gravity (DSG), which arises in generalized gravity theories:

$$\mathcal{L}_{eff} = \sqrt{-g^{(2)}}\left[\varphi R(g^{(2)}) + W(\varphi)(\nabla_\mu\varphi)^2 + \right. \\ \left. + Z_{ij}(\varphi;\psi)\nabla_\mu\psi_i\nabla^\mu\psi_j + X(\varphi;\psi)\right]. \tag{1}$$

Due to $D=2$ even vector fields can be dynamically replaced by scalar amplitudes. The 'static' and 'cosmological' states of these models depend only on the one variable

$$\mathcal{L} = -s\left[h^{-1}\dot{h}\dot{\varphi} + W(\varphi)\dot{\varphi}^2 + \right. \\ \left. + Z_{ij}(\varphi;\psi)\dot{\psi}_i\dot{\psi}_j\right] + s^{-1}hX(\varphi;\psi). \tag{2}$$

Linear symmetries

The DSG feature: non-linearity is shifted to the potential term. So, simplifying the kinetic term:

$$\mathcal{L} = A_{ij}\dot{q}^i\dot{q}^j - \mathcal{U}(q), \quad H = A_{ij}\dot{q}^i\dot{q}^j + \mathcal{U}(q) = 0, \tag{3}$$

with $A_{ij} = \mathrm{const}$. The Lagrange equations are

$$2A_{ij}\ddot{q}^j = -\partial_i\mathcal{U}(q) \quad \Rightarrow \quad \ddot{q}^k = -\frac{1}{2}(A^{-1})^{ki}\partial_i\mathcal{U}(q). \tag{4}$$

Contract with $(\lambda A_{kj} + B_{kj})q^j + c_k$, where $B_{kj} = -B_{jk}$, and use another DSG feature, vanishing Hamiltonian. Then

$$P = [(\lambda A_{kj} + B_{kj})q^j + c_k]\dot{q}^k = \mathrm{const} \quad \text{if} \tag{5}$$

$$\lambda\mathcal{U}(q) + \frac{1}{2}[\lambda q^i + (B_{kj}q^j + c_k)(A^{-1})^{ki}]\partial_i\mathcal{U}(q) = 0. \tag{6}$$

Hamiltonian formulation

Start with linear vector field $v^i = C^i_j q^j + c^i$ and the Hamiltonian $H = A^{ij}(q)p_i p_j + U(q) = 0$. If $C = \bar{C} + B$, with $\{\bar{C}, A\} = \lambda A$, then The Poisson bracket

$$\{H, \mathbf{pv}\} = -\lambda U - \partial_v U - (\mathbf{p}, \partial_v A(q)\mathbf{p}). \quad (7)$$

The conserving quantities $P = \mathbf{pv}$ form the algebra

$$\{\mathbf{pv}, \mathbf{pv}'\} = \mathbf{pv}'', \quad \text{where} \quad \mathbf{v}'' = [C', C]\mathbf{q} + (C'\mathbf{c} - C\mathbf{c}'). \quad (8)$$

Example. For symmetry along $v = \lambda\mathbf{q} + \mathbf{c}$ the Hamiltonian is

$$\lambda = 0: \ U(\mathbf{q}) = \Phi_1(\mathbf{q}_\perp), \quad \mathbf{q}_\perp = |\mathbf{c}|^2\mathbf{q} - (\mathbf{q}, \mathbf{c})\mathbf{c}; \quad (9)$$

$$\lambda \neq 0: \ U(\mathbf{q}) = |\mathbf{v}|^{-2}\Phi_2(\mathbf{v}/|\mathbf{v}|), \quad (10)$$

$$A^{ij}(\mathbf{q}) = \Phi_3(\mathbf{q}_\perp). \quad (11)$$

The criterion: The DSG models with symmetries are in favor.

DSG models with linear symmetries

Now let us turn back to the system (2) and try to find the Hamiltonians with linear symmetries. DSG feature: the metric $h = \varepsilon e^F$ contributes in a unique way into Hamiltonian.
Case $C = 0$, the *shift* symmetry along the constant vector field. Let the kinetic term depend on φ and $\psi_k,\ k = 1..K$. Consider the constant vector $\vec{\beta} \in \mathbb{R}^{N-K}$ which just have zeros in it's corresponding k coordinates. The symmetry along $\mathbf{c} = (\alpha, 0, \beta^1, \ldots, \beta^N)$ arises for the potential

$$e^F U(\varphi;\psi) = e^{F - \vec{\psi}\vec{\beta}\alpha/|\vec{\beta}|^2} X(\varphi; \{\psi_{i_k}\}, \{\vec{\beta}^{\perp}_{(m)}\vec{\psi}\}), \qquad (12)$$

where $\vec{\beta}^{\perp}_{(m)}$ are $N - K - 1$ arbitrary linear independent constant vectors, belonging to the the orthogonal complement to $\vec{\beta}$ in $\mathbb{R}^{N-K}$. Due to the combination with metric the exponential potentials have additional symmetries.

DSG configurations with one scalar mode

Examples

In case of one scalar mode we may consider the case $C \neq 0$, the symmetry along linear and not constant vector field, i.e. deformations.
The dynamical system with the Hamiltonian constraint reads

$$\mathcal{L} = \dot{F}\dot{\xi} + \dot{\psi}^2 - e^F U(\xi, \psi), \quad H = \dot{F}\dot{\xi} + \dot{\psi}^2 + e^F U(\xi, \psi) = 0, \tag{13}$$

For example, the spherical dimensional reduction from $D = 3$ in Einstein gravity with **ordinary scalar field** implies that $Z(\varphi) = \varphi$, so $\varphi = e^\xi$ and the effective potential obtains the exponential factor, $U = U_0 e^{2\xi}\psi^2$. The spherical dimensional reduction in affine generalization of gravity leads to the effective **scalaron field** with inverse coupling to dilaton: $Z = 1/\varphi$, thus $\varphi = \sqrt{2\xi}$, and the effective potential obtains the power-like factor, $U = U_0\psi^2/\xi$.

Linear symmetries

The antisymmetric matrix can be parameterized as $\bar{B}_{kj} = \epsilon_{kjl} b^l$ due to the three-dimensional coordinate space. Thus we arrive to the following conditions for the scalar-dilaton potential term, $U(\xi, \psi)$:

$$\begin{aligned} & b_2 U_\psi = (2b_3 - \lambda) U, \\ & \left[(\lambda/2 + b_3)\xi - b_2\psi + c_1\right] U_\xi + \frac{1}{2}\left[\lambda\psi - b_1\xi + c_3\right] U_\psi = \\ & = -(b_1\psi + c_2 + \lambda) U. \end{aligned} \tag{14}$$

The multiplicative potentials are in favor. Examples:

- $U = e^{g_1 \xi}\Phi(\psi) \Rightarrow P = \dot{F} - g_1\dot{\xi}$
- $U = e^{g_2 \psi}\Phi(\xi) \Rightarrow P = 2\dot{\psi} - g_2\dot{\xi}$
- $U(\xi, \psi) = U_0 \xi^{g_1} \psi^{2g_2} \Rightarrow P = \xi\dot{F} - (1 + g_1 + g_2)\dot{\xi} + \psi\dot{\psi}.$

Scalaron $X = \mu\psi^2/\varphi + \Lambda\varphi \to U = \mu\psi^2/2\xi + \Lambda$ still suits!

Massless variant

Suppose that the scalar mode is in some approach integrated out, providing the system

$$\mathcal{L} = -\dot{F}\dot{\xi} + e^F U(\xi) + V(\xi), \quad H = \dot{F}\dot{\xi} + e^F U(\xi) + V(\xi) = 0. \tag{15}$$

The equation for the metric reads

$$\ddot{F} = -e^F U_\xi - V_\xi. \tag{16}$$

When the potentials are linear, $U = g_1\xi + g_3$, $V = g_2\xi + g_4$, it can be easily integrated, being multiplied by $\dot{F}$:

$$\frac{1}{2}\dot{F}^2 = C_1 - g_1 e^F - g_2 F \equiv R(F). \tag{17}$$

Hence, a non-linear symmetry may arise in DSG.

Conclusion

- The various DSG models can be easily tested on possessing simple symmetries, involving linear, bilinear and non-linear integrals.
- Starting from the fundamental ideas one may advance in obtaining the effective models with such symmetries, which provide at least a partial integrability.
- In context of the affine generalization of gravity, the DSG model of linear scalaron field does possess such symmetries, which allows the further advance in verification and search for the applications to this theory.