Non-Ultralocality : Faddeev-Reshetikhin procedure and Pohlmeyer reduction

François DELDUC

ENS DE LYON

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With
M. MAGRO (ENS Lyon) and B. VICEDO (Hertfordshire)

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Plan

- Motivation

- Non-ultralocality, lattice Poisson bracket

- First steps of Faddeev-Reshetikhin procedure

- Symmetric space coset model

- Pohlmeyer reduction

- Extensions and conclusion

Motivation

Goal = Quantization of the $AdS_5 \times S^5$ superstring from first principles

→

Construct corresponding **Quantum** **Integrable** **Lattice** Model

Long-term goal...

- Non ultralocality :

$$\{\mathscr{L}_1, \mathscr{L}_2\} = [r_{12}, \mathscr{L}_1 + \mathscr{L}_2]\delta_{\sigma\sigma'} + [s_{12}, \mathscr{L}_1 - \mathscr{L}_2]\delta_{\sigma\sigma'} + 2s_{12}\delta'_{\sigma\sigma'}$$

delta prime leads to an ambiguity in the Poisson bracket of the monodromy

$$T(\lambda) = P\overleftarrow{\exp}\int \mathscr{L}(\sigma,\lambda)d\sigma$$

It is difficult to associate to the continuum model an integrable lattice model.

- Ultralocal model $(s_{12} = 0)$

Lattice Poisson bracket $\{T_1^n, T_2^m\} = [r_{12}, T_1^n T_2^m]\delta^{m,n}$

Monodromy $M = T^N T^{N-1} \ldots T^2 T^1$

$\mathrm{Tr} M^k$ in involution

Freidel-Maillet Quadratic Algebra

[Freidel-Maillet '91]

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

Jacobi identity

$$[a_{12}, a_{13}] + [a_{13}, a_{23}] + [a_{13}, a_{23}] = 0$$
$$[a_{12}, c_{13}] + [a_{12}, c_{23}] + [c_{13}, c_{23}] = 0$$

Integrability

$$a - d + b - c = 0$$

$\mathrm{Tr} M^k$ in involution

Continuum limit : $r = a + \frac{1}{2}(b-c) \qquad s = \frac{1}{2}(b+c)$

$$\{\mathscr{L}_1, \mathscr{L}_2\} = [r_{12}, \mathscr{L}_1 + \mathscr{L}_2]\delta_{\sigma\sigma'} + [s_{12}, \mathscr{L}_1 - \mathscr{L}_2]\delta_{\sigma\sigma'} + 2 s_{12} \delta'_{\sigma\sigma'}$$

Faddeev-Reshetikhin approach

[FR '86]

SU(2) Principal Chiral Model

Described by:

- Hamiltonian $H = \int d\sigma \, \mathrm{Tr}\big((j^0)^2 + (j^1)^2\big)$

- Canonical Poisson bracket

$$\{j_1^0(\sigma), j_2^0(\sigma')\} = [C_{12}, j_2^0(\sigma)]\delta_{\sigma\sigma'}$$
$$\{j_1^0(\sigma), j_2^1(\sigma')\} = [C_{12}, j_2^1(\sigma)]\delta_{\sigma\sigma'} - C_{12}\delta'_{\sigma\sigma'}$$
$$\{j_1^1(\sigma), j_2^1(\sigma')\} = 0$$

- Lax matrix $\mathscr{L}(\lambda) = \frac{1}{1-\lambda^2}(j^1 + \lambda j^0)$

⟶ Satisifies a non-ultralocal r/s algebra

FR Strategy = To **get rid** of Non-ultralocality

First steps of FR approach

1. Keep the **same** Lax matrix

2. Replace **canonical non-ultralocal** PB by the **ultralocal** PB

$$\{j_1^0(\sigma), j_2^0(\sigma')\}' = [C_{12}, j_2^0(\sigma)]\,\delta_{\sigma\sigma'}$$
$$\{j_1^0(\sigma), j_2^1(\sigma')\}' = [C_{12}, j_2^1(\sigma)]\,\delta_{\sigma\sigma'}$$
$$\{j_1^1(\sigma), j_1^1(\sigma')\}' = [C_{12}, j_2^0(\sigma)]\,\delta_{\sigma\sigma'}$$

3. Find Hamiltonian H' such that $(H', \{\cdot,\cdot\}')$ has **same classical dynamics** as $(H, \{\cdot,\cdot\})$

Degeneracy of ultralocal bracket

- A priori, look for H' s.t. $\forall f, \quad \{H', f\}' = \{H, f\}$
- But ultralocal PB is degenerate !

$$T_{\pm\pm} = \mathrm{Tr}\left[(j^0 \pm j^1)^2\right] \text{ are Casimirs } i.e.\ \{h, T_{\pm\pm}\}' = 0 \quad \forall h$$

↓

1. Only possible to reproduce **Reduction** of PCM dynamics defined by setting Casimirs to constants

2. Can be done in a **consistent way** because these quantities are chiral/antichiral

Reduction of conformal symmetry

→ Hamiltonian H' for reduced dynamics

Symmetric space F/G σ-model

- Phase Space : pair (A,Π) of fields taking value in $f=\mathrm{Lie}(G)$
- Automorphism $\sigma, \sigma^2=1,\ f=f^{(0)}+f^{(1)},\ f^{(0)}=g=\mathrm{Lie}(G)$
- Lax matrix

$$\mathscr{L}=A^{(0)}+\frac{1}{2}(\lambda^{-1}+\lambda)A^{(1)}+\frac{1}{2}(1-\lambda^2)\Pi^{(0)}+\frac{1}{2}(\lambda^{-1}-\lambda)\Pi^{(1)}$$

belongs to the twisted loop algebra $\hat{f}^{\sigma}=\oplus_n(\lambda^{2n}f^{(0)}\oplus\lambda^{2n+1}f^{(1)})$

- Canonical Poisson bracket constructed from
  - an R-matrix $R=P_{\geq 0}-P_{<0}$
  - a function $\varphi(\lambda)=4\lambda/(1-\lambda^2)^2$

The bracket is of (r, s) type, with

$$r = \frac{1}{2}(R + \varphi R \varphi^{-1}), \quad s = \frac{1}{2}(R - \varphi R \varphi^{-1})$$

In the spirit of the Faddeev-Reshetikhin procedure : modify the Poisson bracket

If one keeps the same R-matrix but take $\varphi(\lambda) = 1$, one would find $s = 0$, however this leads to a fully degenerate bracket.

The closest possibility is $\varphi(\lambda) = \lambda^{-1}$

which leads to a Poisson bracket with $s = P_0$

This Poisson bracket is not ultralocal, but its non ultralocality is confined to the part of the loop algebra which is independent of the spectral parameter λ

Mild non ultralocality

New bracket is already known :

M. Semenov-Tian-Shansky, A. Sevostyanov (1995)

have shown that this bracket is the continuum limit of a Freidel-Maillet (abcd) type lattice bracket.

Extra data :

Solution of mCYBE on $f^0 = g$ denoted α

Then

$$a = r + \alpha, \; b = -s - \alpha, \; c = -s + \alpha, \; d = r - \alpha$$

New bracket

$$\{A^{(0)}_{\underline{1}}(\sigma), A^{(0)}_{\underline{2}}(\sigma')\}' = -[C^{(00)}_{\underline{12}}, 2A^{(0)}_{\underline{2}}(\sigma) + \Pi^{(0)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{12}}\delta'_{\sigma\sigma'},$$

$$\{A^{(0)}_{\underline{1}}(\sigma), A^{(1)}_{\underline{2}}(\sigma')\}' = -[C^{(00)}_{\underline{12}}, A^{(1)}_{\underline{2}}(\sigma) + \Pi^{(1)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'},$$

$$\{A^{(0)}_{\underline{1}}(\sigma), \Pi^{(0)}_{\underline{2}}(\sigma')\}' = 0,$$

$$\{A^{(0)}_{\underline{1}}(\sigma), \Pi^{(1)}_{\underline{2}}(\sigma')\}' = -[C^{(00)}_{\underline{12}}, A^{(1)}_{\underline{2}}(\sigma) + \Pi^{(1)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'},$$

$$\{A^{(1)}_{\underline{1}}(\sigma), A^{(1)}_{\underline{2}}(\sigma')\}' = -[C^{(11)}_{\underline{12}}, \Pi^{(0)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'},$$

$$\{A^{(1)}_{\underline{1}}(\sigma), \Pi^{(0)}_{\underline{2}}(\sigma')\}' = 0,$$

$$\{A^{(1)}_{\underline{1}}(\sigma), \Pi^{(1)}_{\underline{2}}(\sigma')\}' = [C^{(11)}_{\underline{12}}, \Pi^{(0)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'},$$

$$\{\Pi^{(0)}_{\underline{1}}(\sigma), \Pi^{(0)}_{\underline{2}}(\sigma')\}' = 0,$$

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$$\{\Pi^{(1)}_{\underline{1}}(\sigma), \Pi^{(1)}_{\underline{2}}(\sigma')\}' = -[C^{(11)}_{\underline{12}}, \Pi^{(0)}_{\underline{2}}(\sigma)]\,\delta_{\sigma\sigma'}.$$

Reduction

1) $\Pi^{(0)}$ is in the center of the new bracket

coset model : $\Pi^{(0)}$ is a gauge constraint

$$\Rightarrow \Pi^{(0)} = 0$$

2) $A^{(1)} + \Pi^{(1)}$ becomes central

$\Rightarrow A^{(1)} + \Pi^{(1)} = T_+$ $\quad$ T_+ constant matrix in $f^{(1)}$.

coset model : partial gauge fixing + conformal invariance

3) $\mathrm{tr}(A^{(1)} - \Pi^{(1)})^n$ are central

$$\Rightarrow A^{(1)} - \Pi^{(1)} = gT_+g^{-1},\ g \in G$$

coset model : use of conformal invariance

→ Pohlmeyer reduction

Pohlmeyer reduction

One ends up with fields $g \in G$ and $A^{(0)} \in f^{(0)} = \mathrm{Lie}(G)$

The new Poisson bracket becomes

$$\{g_{\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} = 0,$$
$$\{g_{\underline{1}}(\sigma), A_{\underline{2}}(\sigma')\} = -2g_{\underline{1}}(\sigma)C^{(00)}_{\underline{12}}\delta_{\sigma\sigma'},$$
$$\{A_{\underline{1}}(\sigma), A_{\underline{2}}(\sigma')\} = -2[A_{\underline{1}}(\sigma), C^{(00)}_{\underline{12}}]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{12}}\delta'_{\sigma\sigma'}$$

Final model

- WZW model with a potential term
- Subgroup $H \subset G$ is gauged

$$h \in H \Rightarrow hT_+h^{-1}, \quad g \to hgh^{-1}$$

- Lax pair $\mathscr{L} = A^{(0)} + \frac{1}{2}\lambda^{-1}g^{-1}T_+g - \frac{1}{2}\lambda T_+$
- New Hamiltonian can be found

- Extensions

  String on $AdS_5 \times S^5$: coset model

$$SU(2,2|4)/SO(1,4) \times SO(5)$$

  - defined by an order 4 automorphism

  - contains odd fields

  The method fully extend to this case :

- One chooses a Poisson bracket with mild non ultralocality
- The reduced model is the Pohlmeyer reduction first obtained by Grigoriev and Tseytlin (2008)
- The reduced model has the same number of degrees of freefom as the original coset model

Lattice bracket

Choose α to be the standard solution of the mCYBE

$$\alpha = \Sigma_{\beta\in\Delta^+}(E_\beta \otimes E_{-\beta} - E_{-\beta} \otimes E_\beta)$$

Then the four objects (a,b,c,d) appearing in the lattice bracket are related by

$$b = -a(\lambda = 0),\ c = a(\lambda = \infty),\ d = a + b - c$$

→ There is only one basic object $a(\lambda)$ which is a classical r-matrix associated with a twisted loop algebra

→ The matrices b and c satisfy the classical Yang-Baxter equation

This allows for a quantization of the (a,b,c,d) structure

FD, M. Magro, B. Vicedo 1212.0894

Conclusion

Appealing structure which brings hope that one may be able

to quantize from first principles

at least the Pohlmeyer reduction of the superstring !

There is still a lot of work to be done.

The first step would be to construct an explicit realization of the classical lattice model