Dimensional Reduction of Gravity and Relation between Static States, Cosmologies and Waves

V. de Alfaro * and **A.T. Filippov** †

∗ *Dip. di Fisica Teorica and INFN, via Giuria 1, I-10125 Torino*

∗ *Accademia delle Scienze di Torino, via Maria Vittoria 3, I-10123 Torino*

+ *Joint Institute for Nuclear Research, Dubna, Moscow Region, RU-141980*

Abstract

We introduce generalized dimensional reductions of an integrable 1+1-dimensional dilaton gravity coupled to matter down to one-dimensional static states (black holes in particular), cosmological models and waves. An unusual feature of these reductions is the fact that the wave solutions depend on two variables – space and time. They are obtained here both by reducing the moduli space (available due to complete integrability) and by a generalized separation of variables (applicable also to nonintegrable models and to higher-dimensional theories). Among these new wave-like solutions we have found a class of solutions for which the matter fields are finite everywhere in space-time, including infinity.

These considerations clearly demonstrate that a deep connection exists between static states, cosmologies and waves. We argue that it should exist in realistic higher-dimensional theories as well. Among other things we also briefly outline the relations existing between the low-dimensional models that we have discussed here and the realistic higher-dimensional ones.

This paper develops further some ideas already present in our previous papers. We briefly reproduce here (without proof) their main results in a more concise form and give an important generalization.

1 Introduction

The recent observations of the acceleration of our Universe demonstrated the need to consider a wider multitude of cosmological models that possibly can explain the origin of the dark energy[1]. Although a model independent analysis of the present observational

*vda@to.infn.it

†filippov@theor.jinr.ru

[1]For a recent general review of the dark energy problem see [1]. Many cosmological models attempting to explain the origin of the dark energy are reviewed in [2], [3].

data [4] is compatible with the simplest assumption of supplementing the Einstein gravity by the Einstein cosmological constant term, this does not clarify the physical or geometrical origin of the dark energy. Many attempts to understand the dark energy origin are somehow related to an evaluation of the role of different scalar matter fields in gravity theory and of the (not necessarily constant) cosmological potential. They may originate from dimensional reductions of gravity or supergravity theories motivated by superstrings (for a general pre-dark-energy-era reviews of string motivated cosmological models see, e.g., [5], [6]).

One may also argue that a detailed understanding of nonperturbative features of even the standard gravity coupled to matter are not so well understood and should deserve a careful investigation in completely integrable models of gravity coupled to matter fields. For long time, many exact solution of gravity not coupled to matter field were found and studied (see a comprehensive review [7]) and, more recently, integrable models describing some important aspects of gravity theory were discovered (see, e.g., [8] - [11] and references therein). The first model that incorporated some most important properties of black holes coupled to many scalar matter fields [12] appeared 15 years ago and was rather popular as it allowed to understand some features of black hole evolution. In cosmology, only one-dimensional models were studied and their relation to the two-dimensional integrable models of gravity was not discussed. This may possibly be explained by the fact that the string-inspired CGHS model [12] is not realistic enough, having minimal coupling of scalar fields and constant cosmological potential (not corresponding to the cosmological constant in higher dimensions). As a result, the geometry of the two-dimensional space-time is trivial and cosmologies and waves in this model are not realistic at all. Much more realistic two-dimensional model were obtained from spherically symmetric gravity coupled to matter but they are not integrable. Realistic two-dimensional cosmological models were also inspired by string theory and some of them are related to integrable σ-models (see, e.g. [6]). However, they are not too well suited for considering black holes and other realistic static states since the cosmological potential in them is identically zero.

Thus, there exists a serious motivation to study more complex two-dimensional models of gravity coupled to matter fields and compare them to more realistic approximations that are able to describe real black holes, cosmologies and waves. This paper continues our studies of the dimensionally reduced description of high dimensional branes (black holes and other static states in particular), cosmological theories, and some waves coupled to gravity, which was presented in papers [13] - [18]. The first motive of this paper is to illustrate the different problems and approaches to their solution by a detailed consideration of the explicitly integrable N-Liouville model of gravity coupled to matter, which is close to some realistic theories and can give description of real black holes and cosmologies in one dimension. The second motive is to demonstrate that the most general approach to dimensional reduction from dimension (1+1) to (1+0) or (0+1) may be based on the properly formulated generalized separation of variables. We also will try to demonstrate more explicitly that a deep connection exists between black holes (or, more generally, static states of gravity coupled to matter), various cosmological models, and, possibly, the simplest waves of scalar matter coupled to gravity.

Some time ago we observed that a naive dimensional reduction from the (1+1)-

dimensional to the one-dimensional dilaton gravity does not produce the standard (FRW) cosmology. This fact was mentioned in [13], where we found it necessary to return to the 'parent' higher-dimensional spherically symmetric gravity (equivalent to the (1+1)-dimensional theory) and directly reduce it to the homogeneous and isotropic FRW cosmologies. This looks strange because the FRW cosmologies are spherically symmetric and the (1+1)-dimensional dilaton gravity describes all possible spherically symmetric solutions of the 'parent' theory. Thus the problem – how to obtain the standard cosmology directly from the dilaton gravity by some dimensional reduction – remained unsolved. Most misleading is the fact that the naive reduction produces the correct static state of pure gravity – the external part of the Schwarzschild black hole. The naive cosmology is simply the internal part of the black hole that has nothing to do with the FRW cosmology. The presence of matter fields significantly changes the picture – instead of the black hole there emerges a static state of matter having no horizon and the corresponding cosmology is one of the FRW cosmologies (in fact, it is the closed one). This result was obtained in an old paper [21] but nobody (including the author) appreciated its paradoxical meaning[2].

Attempting to understand this we proposed [14] to obtain the FRW cosmology by use of a more general dimensional reduction, in which the metric and dilaton of (effectively one-dimensional) black holes and cosmologies depend on two variables. Although in this consideration we still used the connection of the two-dimensional metric to the higher-dimensional one, we concluded that there may exist direct dimensional reductions from a two-dimensional theory to its one-dimensional descendants. Another lesson is that there may exist many different generalized dimensional reductions producing different static and cosmological solutions that are effectively one-dimensional but formally depend on two variables.

Note in passing that, to the best of our knowledge, this problem was never raised in the literature. This may be possibly explained by the fact that, for many years, the studies of black holes and those of cosmology were using very different approaches. The well known fact that the inner part of the Schwarzschild black hole is some cosmology was considered a curiosity devoid of serious meaning. While the theory of black holes was attempting to understand more and more complex theoretical models (from a spherically symmetric collapsing black hole to rotating axial ones), the basis of the cosmological theory was given by the FRW model and the mainstream cosmology mostly concentrated on astrophysical and observational aspects and their adaptation to modern physics. This is recently changing due to the dark energy problem and seems to be able to give a sufficient motivation to searches for a common background for all the three main objects of the modern gravity theory – static states (black holes, in particular), cosmological models and gravitational waves (or, waves of matter coupled to gravity).

In an unpublished paper [15] we tried to develop the above mentioned ideas in a more

[2]Note that the connection between the Schwarzschild black hole and some cosmology was first mentioned in [19]. This connection was implicitly touched upon in the papers [20]. A more general example demonstrating a close connection between static spherically symmetric states of scalar matter coupled to gravity and a spherically symmetric closed cosmology was presented in [21], although its meaning was not discussed in detail. At the same time, similar relations between some static p-branes and cosmologies were found, see [22] - [24].

systematic way. For example, we envisaged the idea that it is important to take into account the 'surface' (or 'total derivative') terms that appear in the process of dimensional reduction. We also observed that some standard 'gauge' ('coordinate') fixings may be dangerous in this context because they significantly restrict the number of possible dimensional reductions. However we were far from understanding the whole problem. For instance, in paper [15] we still considered only naive dimensional reductions of the integrable N-Liouville theory 'looking at, but not seeing' (so to speak) a new class of one dimensional solutions.

In [16] it was mentioned that the structure of the solutions of the integrable N-Liouville theory allows for a very simple interpretation of the dimensional reduction as a reduction in the moduli space. This immediately shows that in addition to static and cosmological solutions there exists a class of wave-like solutions. We attempted to carry on a more detailed investigation of these states in paper [18], which was not published mainly because the approach used in that paper seemed to be applicable to integrable models only.

Here we remake paper [18] all over again in the light of the results of [16] and [17] and relate the 'dynamical dimensional reduction' of integrable models to more general reductions, based on the generalized separation of variables (which can in principle be applied to nonintegrable theories as well). We also study in some detail the wave-like solutions that can be obtained in both approaches and discuss some nonsingular waves of scalar matter coupled to gravity.

2 General Relations between two-dimensional and one-dimensional Lagrangians

Let us consider a rather general two-dimensional Lagrangian

$$\mathcal{L}^{(2)} = \sqrt{-g}\,\{\varphi\, R + V(\varphi,\psi) + \sum_{m=3}^{N} Z_{mn}(\varphi,\psi)\,\nabla\psi_m\,\nabla\psi_n\,\}\,, \tag{1}$$

where g_{ij} is a generic (1+1)-dimensional metric with signature (-1,1), φ is the dilaton field, R is the Ricci curvature, $V(\varphi,\psi)$ is an effective ('cosmological') potential, and the coupling matrix of the scalar fields, $Z_{mn}(\varphi,\psi)$, is an arbitrary function of the dilaton fields and of the $(N-2)$ scalar massless fields ψ_n[3]. This theory may be integrable in two main cases: 1) when $V \equiv 0$ and 2) when all Z_{mn} are constants independent of the fields, while V is a special potential (see [16], [17]). If Z_{mn} has only negative eigenvalues we can transform ψ fields in the last case so as

$$Z_{mn} = -\delta_{mn}\,. \tag{2}$$

[3] In the models (1) derived by dimensional reductions of higher-dimensional (super)gravity theories the coupling matrix Z_{mn} is real, symmetric, and nondegenerate. We also suppose that all its eigenvalues are negative. The last condition may be violated if one introduces the so called 'phantom' fields, but here we do not consider such exotic models. The usually present dilaton kinetic term $W(\nabla\varphi)^2$ can be transformed away from the Lagrangian and we omit it here (see [14], [16]).

Let us concentrate on this case with $V(\varphi, \psi)$ given by

$$V = \sum_{n=1}^{N} g_n \exp q_n^{(0)}, \qquad q_n^{(0)} \equiv a_n \varphi + \sum_{m=3}^{N} \psi_m \, a_{mn} \,. \tag{3}$$

The theory is integrable if the parameters a_n, a_{nm} satisfy the condition

$$\sum_{l=3}^{N} a_{lm} a_{ln} - 2\,(a_m + a_n) = \gamma_n^{-1} \, \delta_{mn} \tag{4}$$

and Z_{mn} satisfy (2). When $m \neq n$ this is the pseudo orthogonality condition for the N-vectors $A_n \equiv \{a_{mn}\}$, $m = 1, ..., N$, $a_{1n} \equiv 1 + a_n$, $a_{2n} \equiv 1 - a_n$, i.e. (4) is equivalent to

$$A_n \cdot A_m \equiv -\, a_{1m} \, a_{1m} + \sum_{l=2}^{N} a_{lm} a_{ln} = 0. \tag{5}$$

For $m = n$ eq. (4) defines γ_n, the important parameters of the theory,

$$\gamma_n^{-1} = A_n \cdot A_n = -\,4\, a_n + \sum_{l=3}^{N} a_{ln}^2 \,. \tag{6}$$

For a detailed description of the properties of the parameters a_{mn}, γ_n satisfying (4), see [14] – [17] (there we showed that one and only one of the norms γ_n^{-1} is negative; usually we take $\gamma_1 < 0$).

Note that the integrability does not depend on the values g_n of the coupling parameters. Some of them may even be zero[4]. Also, without changing integrability we may add to the theory (1) any number of free massless scalar fields. In all these cases we can write explicitly the general solution of the two-dimensional integrable theory defined by equations (1) - (4).

In this paper we mainly concentrate on the dimensional reduction of the (1+1)-dimensional integrable model to the one-dimensional integrable one. Before turning to this main subject let us first outline relations of these integrable models to realistic theories. To simplify presentation we mainly work in the light-cone (conformally flat) gauge, where the metric is

$$ds^2 = -\,4\, f(u,v)\, du\, dv \equiv -4\varepsilon\, e^{F(u,v)} du\, dv = e^F \,(dr^2 - dt^2\,)\,, \;\; \varepsilon = \pm 1\,, \tag{7}$$

with r and t defined as $r \equiv u - \varepsilon v$, $t \equiv u + \varepsilon v$. In the exponential representation of the metric in (7) the zeroes of $f(u,v)$ (horizons) may emerge when $F \to -\infty$. If we cross a horizon, f changes sign given by ε but the space-time representation remains unchanged due to the automatic changing of the relation between the space-time and the light-cone coordinates (resulting in $r \leftrightarrow$ t).

[4] If the theory (1) is obtained by dimensional reductions of a higher-dimensional supergravity theory on tori, the coupling constants g_n are negative. An exception emerges in the case of the spherical symmetry reduction that produces one positive term in the two-dimensional potential (3).

The naive reduction to one dimension (space or time) is obtained if we suppose that the metric, the dilaton, and the scalar matter fields depend a single variable $\tau = a(u) + b(v)$, which may be a space or a time coordinate. The reduced Lagrangian then can be written in the form

$$\mathcal{L}^{(1)} = -\frac{1}{l(\tau)} \Big(\dot{\varphi} \dot{F} + \sum_{m,n=3}^{N} Z_{mn} \dot{\psi}_m \dot{\psi}_n \Big) + l(\tau) \, \varepsilon \, e^F \, V(\varphi, \psi) \, . \tag{8}$$

If $Z_{mn} = -\delta_{mn}$, then eqs. (3), (4) also give the sufficient condition for its integrability.

However, the theory (8) may be integrable in other cases as well. For example, let us suppose that $Z_{mn} = -(1/\phi'(\varphi)) \, \delta_{mn}$, where $\phi'(\varphi) > 0$. Then

$$\mathcal{L}^{(1)} = \frac{1}{l(\tau) \, \phi'(\phi)} \Big(-\dot{\phi} \dot{F} + \sum_{n=3}^{N} \dot{\psi}_n^2 \Big) + [l(\tau) \, \phi'(\varphi)] \, \varepsilon \, e^F \, [V/\phi'(\varphi)] \, . \tag{9}$$

Introducing the new Lagrange multiplier $\bar{l}(\tau) \equiv l(\tau) \, \phi'(\phi)$ and defining

$$V(\varphi, \psi) \, / \phi'(\varphi) \equiv \bar{V}(\phi, \psi) \, ,$$

the new one-dimensional Lagrangian takes the form

$$\bar{\mathcal{L}}^{(1)} = \frac{1}{\bar{l}(\tau)} \Big(-\dot{\phi} \dot{F} + \sum_{n=3}^{N} \dot{\psi}_n^2 \Big) + \bar{l}(\tau) \, \varepsilon \, e^F \, \bar{V}(\phi, \psi) \, . \tag{10}$$

This model is integrable if the new potential $\bar{V}(\phi, \psi)$ can be written in the form (3) with the parameters $\bar{a}_n$, a_{mn} $(m \geq 3)$ satisfying the orthogonality conditions (4):

$$\bar{V}(\phi, \psi) = \sum_{n=1}^{N} g_n \exp \bar{q}_n^{(0)} \, , \quad \bar{q}_n^{(0)} = \bar{a}_n \, \phi + \sum_{m=3}^{N} \psi_m \, a_{mn} \, , \tag{11}$$

$$\sum_{l=3}^{N} a_{lm} \, a_{ln} - 2 \, (\bar{a}_m + \bar{a}_n) = \bar{\gamma}_n^{-1} \, \delta_{mn} \, . \tag{12}$$

Unlike the integrable one-dimensional model (8) obtained from the integrable two-dimensional model (1), both satisfying the integrability conditions (2) - (4), the new integrable model (10) is derived from the apparently nonintegrable two-dimensional theory. Indeed, to get the exponential in ϕ potential $\bar{V}$ we should start with the potential V non-exponential in φ and with the φ-dependent Z_{mn}. To better understand this subtle relation between integrability in dimensions one and two, consider a typical case $Z_{mn} = -\varphi \, \delta_{mn}$, $\varphi = \exp \phi$. Then equations (11), mean that (note that here $\bar{a}_n - 1 = a_n$):

$$V(\varphi, \psi) \equiv \phi'(\varphi) \, \bar{V}(\ln\varphi, \psi) = \sum_{n=1}^{N} g_n \, \varphi^{\bar{a}_n - 1} \exp\Big[\sum_{m=3}^{N} \psi_m a_{mn} \Big] \equiv$$

$$\equiv \sum_{n=1}^{N} g_n \, \varphi^{\bar{a}_n - 1} \, e^{-\bar{a}_n \varphi} \exp\Big[\bar{a}_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn} \Big] \, , \tag{13}$$

where $\bar{a}_n, a_{mn}$ satisfy the orthogonality relations (12).

Although the theory (1) with $Z_{mn} = -\varphi\,\delta_{mn}$ and the potential (13) is certainly not integrable we may try to approximate it by an integrable, theory in some narrow enough interval of φ [5]. First, we should approximate in this interval $Z_{mn} \approx -\varphi_0\,\delta_{mn}$, where $\varphi_0 > 0$. Second, we should either rescale the matter fields to have $Z_{mn} \approx -\delta_{mn}$ or simply divide all the terms in the Lagrangian (1) by φ_0 and then introduce $\bar{\varphi} \equiv \varphi/\varphi_0$ as the new dilaton field. The second idea looks more attractive because it preserves the pseudo orthogonality conditions for $\bar{a}_n$, a_{mn} and rescales only the coupling constants g_n. After these simple manipulations we can forget about rescaling and simply fix the scale by setting $\varphi_0 = 1$. Then, to get the integrable two-dimensional theory we suppose that $\bar{g}_n(\varphi) \equiv g_n \varphi^{\bar{a}_n - 1}\, e^{-\bar{a}_n \varphi}$ can be approximated by constants in some interval of φ around φ_0. The simplest approximation is to replace in this expression φ by φ_0. A more refined approximation may be possible if several parameters $\bar{a}_n$ are equal, say, $\bar{a}_n = \bar{a}$ for $n \leq n_0$. If $g_n = 0$ for $n_0 < n \leq N$ and thus the potential has n_0 terms, we can use a better approximation for the potential. The derivative of $g_n \varphi^{\bar{a}_n - 1}\, e^{-\bar{a}_n \varphi}$ vanishes if $\varphi = \varphi_1 \equiv (\bar{a}_n - 1)/\bar{a}_n$. This means that, if $|\varphi - \varphi_1|$ is not too large, the approximation of $\bar{g}_n(\varphi)$ by $\bar{g}_n(\varphi_1)$ is reasonable for $n \leq n_0$ and therefore

$$V(\varphi, \psi) = \sum_{n=1}^{n_0} g_n(\varphi) \exp\Big[\bar{a}\varphi + \sum_{m=3}^{N} \psi_m a_{mn}\Big] \approx \sum_{n=1}^{n_0} \bar{g}_n(\varphi_1)\, \exp\Big[\bar{a}\varphi + \sum_{m=3}^{N} \psi_m a_{mn}\Big]. \quad (14)$$

The simplest examples of one-dimensional integrable N-Liouville models that can be obtained by dimensional reductions of higher-dimensional theories are: $\bar{a}_1 = \bar{a}, \bar{a}_n = -\bar{a}, n > 1$ and $\bar{a}_1 = \bar{a}_2 = \bar{a}, \bar{a}_n = -\bar{a}, n > 2$. They can be derived from the models considered in detail in [25] (see also [5] and Appendix for more general models). To better understand the above discussion it is advisable to have a look at the model of ref. [25], which we present in our notation and without reproducing the higher-dimensional considerations. The two-dimensional Lagrangian can be written in the form (1) with $N = 4$, $Z_{mn} = -\varphi\,\delta_{mn}$ and with the potential (13), in which $\bar{a}_n, a_{mn}$ satisfy (12). The nonzero parameters of the model are:

$$\bar{a}_1 = \bar{a}_2 = -\bar{a}_3 = -\bar{a}_4 = 1; \;\; a_{31} = 2\sqrt{3}, \;\; a_{32} = 2/\sqrt{3}, \;\; a_{43} = -a_{44} = 2\,. \quad (15)$$

The origin of the items in the potential is the following: the first term is the curvature of the three-dimensional sphere, $g_1 = 6$ (in the flat limit this term vanishes), the second and the fourth terms originate from the reductions of the three-form in the ten-dimensional Lagrangian, the third is generated by the Abelian gauge field generated by the KK-reduction of the six-dimensional space-time to the five-dimensional one; ψ_4 is the KK scalar field and ψ_3 is the scale factor of the five-dimensional spherical reduction. The coupling constants g_2, g_3, g_4 are negative (note that $g_n/\gamma_n > 0$, $n = 1, ..., 4$). We can now write the one-dimensional Lagrangian (8) and transform it into the N-Liouville form

[5]Such an approximation might be used in considering time-dependent phenomena near static horizons of black holes or, alternatively, when deriving an approximate description of nonperturbative short-time inhomogeneities in cosmology.

(10). The described special model and its generalizations can be used as a 'laboratory' for confronting exact and approximate solutions.

We therefore see that in the 'realistic' case, $Z_{mn} = -\varphi\,\delta_{mn}$, the one-dimensional theory may be reducible to the N-Liouville model while the two-dimensional theory, from which it was obtained by the naive dimensional reduction, is not integrable. Although we know that the one-dimensional solutions must also solve the two-dimensional equations, the procedure of finding the corresponding two-dimensional functions requires some care because we used transformations of Lagrange multipliers (in this sense the reduction is not 'naive' at all!). Moreover, we do not know the complete set of solutions of the two-dimensional theory and thus cannot directly dimensional reduce of the solutions (as it is possible in the integrable case).

Having in mind that the 'effectively one-dimensional solutions' formally depending on two variables are rather difficult to find in the nonintegrable theory, we first carefully study the integrable model,

$$\mathcal{L}^{(2)} = \sqrt{-g}\left\{\varphi R + V(\varphi,\psi) - \sum_{n=3}^{N}(\nabla\psi_n)^2\right\}, \tag{16}$$

with $V(\varphi,\psi)$ given by eqs. (2) - (4), and study the dimensional reduction of its solutions to static states, cosmologies and waves. In view of the connections between the one-dimensional and the two-dimensional theories in 'realistic' cases, one may hope that these considerations will help to find analogous reductions in realistic theories by using a generalized dimensional reduction (in Section 5, we give a simple example hinting that this should be possible). To substantiate this observation, we also show how the wave-like solutions of the N-Liouville theory (1) - (4) can be obtained with the aid of a generalized separation of variables in the equations of motion, without constructing the general two-dimensional solutions.

3 General solution of the N-Liouville model

We have shown in [13] – [17] that the equations of motion for the N-Liouville model defined by (1) – (4) greatly simplify if we write them in the light-cone (LC) coordinates (7) using the following new functions

$$q_n(u,v) \equiv F(u,v) + q_n^{(0)}(u,v) = \sum_{m=1}^{N}\psi_m a_{mn}\,, \tag{17}$$

where $\psi_1 \equiv \frac{1}{2}(F+\varphi)$, $\psi_2 \equiv \frac{1}{2}(F-\varphi)$ and a_{mn} are the coordinates of the N-vectors A_n defined above. Using (5), (6) it is easy to invert (17) and express ψ_n in terms of q_n (see [21], [16])

$$\psi_n = \sum_{m=1}^{N}\epsilon_n a_{nm}\gamma_m\, q_m \quad, \epsilon_1 = -1,\ \epsilon_{n>1} = 1\,. \tag{18}$$

Varying the Lagrangian (16) in ψ_n, φ, g_{ij} and then transforming the obtained equations to the LC coordinates, we can find N equations of motion for $\psi_{n\geq 1}$ and two constraints.

Expressing ψ_n in terms of q_n we thus obtain N Liouville equations[6],

$$\partial_u \partial_v q_n = \tilde{g}_n \exp q_n , \quad \tilde{g}_n \equiv \varepsilon g_n / \gamma_n \, , \tag{19}$$

and two constraints for n functions q_n,

$$\sum_{n=1}^{N} \gamma_n \Big[(\partial_i q_n)^2 - 2 \partial_i^2 q_n \Big] = 0. \tag{20}$$

The solution of the system of equations and constraints is easier to find if we define the new functions (motivated by the conformal properties of the Liouville equation)

$$X_n(u,v) \equiv \exp[\, - q_n(u,v)/2] \, . \tag{21}$$

Then the Liouville equations (19) are equivalent to

$$X_n \, \partial_u \partial_v X_n - \partial_u X_n \, \partial_v X_n = -\frac{1}{2} \tilde{g}_n \tag{22}$$

and their solution must satisfy the constraints

$$\sum_{n=1}^{N} \gamma_n \, X_n^{-1} \, \partial_i^2 X_n \; = \; 0, \;\; i = u, v. \tag{23}$$

The solution of these equation is described in detail in [13] – [17] and we will not repeat it. Here we only write a more general form of the solution that we need for a correct dimensional reduction[7].

Now let us proceed to writing the general solutions of the N-Liouville theory. Differentiating eq. (22) with respect to u and v we find that

$$\partial_u (X_n^{-1} \, \partial_v^2 X_n) \; = \; 0 \, , \qquad \partial_v (X_n^{-1} \, \partial_u^2 X_n) \; = \; 0 \, . \tag{24}$$

It follows that if X_n satisfies eq. (24) then there exist some 'potentials' $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$ such that

$$\partial_u^2 X_n \; - \mathcal{U}_n(u) \, X_n \; = \; 0, \qquad \partial_v^2 X_n \; - \; \mathcal{V}_n(v) \, X_n \; = \; 0 \, . \tag{25}$$

This motivates the introduction of two ordinary differential equations

$$a_n''(u) \; - \mathcal{U}_n(u) \, a_n(u) \; = \; 0, \qquad b_n''(v) \; - \; \mathcal{V}_n(v) \, b_n(v) \; = \; 0 \, . \tag{26}$$

It follows that the solutions of the Liouville equations, X_n, can be expressed in terms of the solutions of these equations: $a_n(u)$ and $b_n(v)$ can be taken arbitrary and the linearly

[6] In physically motivated models, e.g., in those obtained from higher-dimensional theories by dimensional reductions, the signs of g_n and of γ_n are often correlated so that the signs of nonvanishing coupling constants $\tilde{g}_n$ are the same for all n.

[7] The form used in the previous papers is essentially correct but was written in a somewhat misleading form that can provoke its incorrect use; this will become more evident below.

independent solutions are then found without knowledge of the potential[8]. However, this does not help to solve the constrains (23) that will be shown below to have the form

$$\sum_{n=1}^{N} \gamma_n \frac{a_n''(u)}{a_n(u)} = 0, \qquad \sum_{n=1}^{N} \gamma_n \frac{b_n''(v)}{b_n(v)} = 0\,, \tag{27}$$

where $a_n(u)$ and $b_n(v)$ are arbitrary solutions of eqs. (27). Although $2(N-1)$ of the potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$ and of the corresponding $a_n(u)$, $b_n(v)$ are thus arbitrary, the two remaining functions a, b satisfy the second order equations with arbitrary potentials that cannot be solved when γ_n are arbitrary numbers. In [13] – [17] we showed that using the fact that in the N-Liouville theory the sum of γ_n is zero the solution can be constructed. Below we slightly generalize this result[9].

Introducing the linearly independent solutions of eqs. (26), $a_n^{(1)}(u)$, $a_n^{(2)}(u)$ and $b_n^{(1)}(v)$, $b_n^{(2)}(v)$, normalized by requiring that their wronskians be

$$W\Big[a_n^{(1)}(u),\, a_n^{(2)}(u)\Big] = 1, \qquad W\Big[b_n^{(1)}(v),\, b_n^{(2)}(v)\Big] = 1\,, \tag{28}$$

we may show that the X_n satisfying eq. (24) should have the form

$$X_n(u,v) = a_n^{(i)}(u)\, C_{ij}^{(n)} b_n^{(j)}(v) \tag{29}$$

where $C_{ij}^{(n)}$ is a numerical matrix (and we sum over $i,j=1,2$). It is not difficult to check that X_n satisfies eqs. (22) if and only if

$$\det\, C_{ij}^{(n)} = -\frac{1}{2}\tilde{g}_n\,. \tag{30}$$

In [13] – [16] we used a truncated form of the solution obtained taking $C_{12}^{(n)} = C_{21}^{(n)} = 0$, $C_{11}^{(n)} = 1$ and $C_{22}^{(n)} = -\tilde{g}_n/2$. The general solution can be obtained from the truncated one by applying the linear transformations of the basic chiral fields $a_n^{(i)}(u)$, $b_n^{(j)}(v)$ that preserve the wronskians:

$$a_n^{(i)}(u) \to A_{ij}^{(n)}\, a_n^{(j)}(u), \quad b_n^{(i)}(v) \to B_{ij}^{(n)}\, b_n^{(j)}(v), \tag{31}$$

where $\det A_{ij}^{(n)} = 1 = \det B_{ij}^{(n)}$. Although the truncated representation of the solution can be, with due care, used in general considerations, the general solution (29) is more adequate in order to analyze the physical properties of the solutions and is especially important for further dimensional reductions as we shall see below.

As in the previous papers, we may choose $a_n^{(1)}(u) \equiv a_n(u)$ and $b_n^{(1)}(v) \equiv b_n(v)$, where $a_n(u)$ and $b_n(v)$ are arbitrary functions, and take

$$a_n^{(2)}(u) \equiv \bar{a}_n(u) \equiv a_n(u) \int \frac{du}{a_n^2(u)}\,, \qquad b_n^{(2)}(v) \equiv \bar{b}_n(v) \equiv b_n(v) \int \frac{dv}{b_n^2(v)}\,. \tag{32}$$

[8]This form for the general solution of the Liouville equation was mentioned in [26]. It group theoretical meaning and applications are discussed in [27], [28], where further references can be found.

[9]For some special potentials, the equations (26) can be solved even for arbitrary γ_n. Below we study in detail the case of constant potentials that can be considered as a sort of a generalized dimensional reduction of the N-Liouville theory with or without constraints.

This allows us to apply the considerations of the previous papers to the general solutions X_n. The crucial fact that makes it possible to solve the constraints in [13] – [16] is that they are equivalent to the constraints (27) and that the pseudo orthogonality conditions imply the identity

$$\sum_{n=1}^{N} \gamma_n \, = \, 0 \, . \tag{33}$$

Indeed, we have

$$\frac{\partial_u^2 X_n}{X_n} \, = \, \frac{[\partial_u^2 \, a_n^{(i)}(u)\,]\, C_{ij}^{(n)}\, b_n^{(i)}(v)}{a_n^{(i)}(u)\, C_n^{(ij)}\, b_n^{(j)}(v)} = \mathcal{U}_n(u) = \frac{a_n^{''}(u)}{a_n(u)} \, = \, \frac{\bar{a}_n^{''}(u)}{\bar{a}_n(u)},$$

and a similar expression holds for the functions $b_n(v)$, $\bar{b}_n(v)$. It follows that the constraints (23) have the form (27) that has allowed us to solve the constraints in [13] – [16]. Now applying the construction of these papers we can write the solution of the equations and constraints with the same expression for the basic functions a_n and b_n:

$$a_n(u) \, = \, |\sum \gamma_m \mu_m(u)|^{-1/2} \, \exp \int du \, \mu_n(u), \tag{34}$$

$$b_n(u) \, = \, |\sum \gamma_m \nu_m(v)|^{-1/2} \, \exp \int du \, \nu_n(v), \tag{35}$$

where the moduli functions $\mu_n(u)$, $\nu_n(v)$ satisfy the constraints[10]

$$\sum_{n=1}^{N} \gamma_n \, \mu_n^2(u) \, = \, 0 \, , \qquad \sum_{n=1}^{N} \gamma_n \, = \, 0 \, .\nu_n^2(v) \, . \tag{36}$$

Inserting (34), (35) into (29) we find q_n and then using the orthogonality relations the original fields ψ_n, ϕ, F can be written in terms of a_n, b_n (see [13] – [16]; these expressions are reproduced in the next Section).

As it has been shown in refs. [14], [16], the space of moduli $(\mu_1, \, ..., \mu_n)$, $(\nu_1, \, ..., \nu_n)$ may be reduced further by the two coordinate gauge fixing conditions. In [14], [16] it was suggested to introduce the new coordinates $(U, \, V)$ by writing

$$\sum \gamma_n \mu_n(u) \, = \, U'(u), \qquad \sum \gamma_n \nu_n(v) \, = \, V'(v) \, , \tag{37}$$

but in some problems a different choice of coordinate conditions may become more convenient.

In the mentioned papers it was shown that the pseudo orthogonality conditions imply that all γ_n except one should be positive, so we always choose $\gamma_1 < 0$. It follows then that $\mu_1(u)$ and $\nu_1(v)$ cannot have zeroes, due to the constraints[11]. Accordingly,

$$\int_{u_0}^{u} \mu_1(u) \, du \qquad \text{and} \qquad \int_{v_0}^{v} \nu_1(v) \, dv \tag{38}$$

[10] It is easy to write the potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$ in terms of the moduli but these complex formulas do not allow to find inverse expressions except the case of constant moduli treated in the next Section.

[11] This is not true if all μ_n or ν_n vanish. Then the constraints are trivially satisfied but this highly degenerate case is not interesting to discuss. More interesting is the case of imaginary moduli, to which our approach can be fully applied. We do not consider the corresponding solutions because they may have singularities for finite values of u and v. In this paper we concentrate on solutions that can be singular only at infinity in the (u, v)-space.

are monotonic functions and so they can be chosen as new coordinates, i.e. we may write

$$\int_{u_0}^{u} \mu_1(u)\, du \equiv U \quad \text{and} \quad \int_{v_0}^{v} \nu_1(v)\, dv \equiv V\,. \tag{39}$$

This is obviously equivalent to the choice

$$\mu_1(u) \equiv \nu_1(v) \equiv 1 \tag{40}$$

while using the original coordinates $(u,\, v)$.

A simple way to incorporate all the above properties of the moduli space is to introduce the unit vectors (see [14], [16])

$$\hat{\xi}_k(u) \equiv \hat{\gamma}_k\ \mu_k(u), \qquad \hat{\eta}_k(v) \equiv \hat{\gamma}_k\ \nu_k(v), \quad k = 2, 3, ..., N\,, \tag{41}$$

where $\hat{\gamma}_k \equiv \sqrt{\gamma_k/|\gamma_1|}$. From the known properties of the parameters γ_k and the constraints on μ_n and ν_n (including $\mu_1 = \nu_1 = 1$) we see that

$$\hat{\xi}^2 \equiv \sum_{k=2}^{N} \hat{\xi}_k^2 = 1, \quad \hat{\eta}^2 \equiv \sum_{k=2}^{N} \hat{\eta}_k^2 = 1, \quad \hat{\gamma}^2 \equiv \sum_{k=2}^{N} \hat{\gamma}_k^2 = 1\,. \tag{42}$$

The independent moduli $\hat{\xi}(u)$, $\hat{\eta}(v)$ belong to a sphere $S^{(N-2)}$ and the moduli space consists of pairs of continuous trajectories $(\hat{\xi}(u),\, \hat{\eta}(v))$ on $S^{(N-2)}$. Pairs of points $(\hat{\xi}^{(0)}$, $\hat{\eta}^{(0)})$ also define an interesting class of solutions that will be studied below in some detail[12]. The moduli $\hat{\xi}(u)$ and $\hat{\eta}(v)$ should be defined for all the real values of the coordinates $(u,\, v)$. These trajectories in $S^{(N-2)}$ may form closed curves or be infinite in both directions or even have one or two finite end points. If a trajectory, say $\hat{\xi}(u)$, has an end point $\hat{\xi}^{(0)}$, this means that $\hat{\xi}(u) \rightarrow \hat{\xi}^{(0)}$ for $u \rightarrow +\infty$ (similarly it can have $\hat{\xi}^{(0)}$ as initial point if $\xi(u) \rightarrow \hat{\xi}^{(0)}$ for $u \rightarrow -\infty$).

Using these simple properties one may try to construct a classification of trajectories by their topological and asymptotic properties. This is fairly obvious for $S^{(1)}$ and simple enough for $S^{(2)}$. In higher dimensions (starting with $S^{(3)}$) it may be rather difficult to find a reasonable classification of all possible trajectories. However, many physically interesting solutions are those defined by the points $(\hat{\xi}^{(0)},\, \hat{\eta}^{(0)})$. They, in fact, define some nontrivial dimensional reductions of the 1+1 dimensional solutions and may describe static black holes (more generally, static states of gravitating matter), cosmologies and various types of nonlinear waves coupled to gravity. Among these waves there may exist waves that are finite everywhere in space and time, including infinity. All these solutions will be one of the main subjects of the following discussion.

Let us first make some comments about the dimensional reduction, extending the discussion on this subject given in two previous papers [14], [16]. In the 'naive' reduction we simply reduce the dimension of space-time on which the dynamical functions (say,

[12] In our previous papers we mainly studied the solutions for which $\hat{\xi}^{(0)} = \hat{\eta}^{(0)}$. The static solution with two horizons are described by the solutions defined by $\hat{\xi}^{(0)} = \hat{\eta}^{(0)} = \hat{\gamma}$ (see [13]). These are very special examples of the solutions discussed in the next Section.

$q_n(u,\, v)$) are defined, by constructing a subclass of solutions dependent on one variable (e.g. $q_n(t) \equiv q_n(u-v)$). This naive reduction for the 1+1 dimensional dilaton gravity theory functions is discussed in detail in [14]. There we also argued that the naive reduction to variables dependent only on time does not give all possible cosmologies. It also looks as if the reduction from higher dimensions D to 1+0 is not equivalent to the reduction from dimension 1+1 to 1+0. In appendix 6.2 of paper [14] we show how to restore all possible homogeneous and isotropic cosmologies by considering a more general reduction of the 1+1 dimensional theory.

More general approaches to reductions use group-theoretical considerations (see e.g. [29] - [33] and references therein) but these ones are not always easy to apply directly to the reduction of the equations of motion. The generalized separation of variables introduced in [17] may give an even more general approach to dimensional reduction. In particular, when applied to theories of dimension 1+1, this approach allows one to reproduce all known black hole and cosmological solutions [17]. In addition, it works well when the dilaton gravity couples to any number of matter fields and gives (presumably) new solutions that, while essentially one-dimensional, depend on both variables, r and t.

Here we study the dimensional reduction in the integrable N-Liouville theory that gives a class of reduced solutions dependent on combinations of both variables. An example of this type of solution is given at the end of Section 3 of ref. [16]; it was obtained by dimensional reduction of the moduli space. The idea of this dimensional reduction (let us temporary call it a 'dynamical dimensional reduction') is very simple to formulate in terms of the moduli space described above. While the solutions of the 1+1 dimensional theory are given by the pairs of functions, $(\hat{\xi}(u),\, \hat{\eta}(v))$, the reduced solutions are given by the points $(\hat{\xi}^{(0)},\, \hat{\eta}^{(0)})$. They describe waves of scalar matter; special cases are given by static states (black holes in particular) and cosmologies. Therefore all these objects are unified in one more general class.

We shall study them in the next Section and will try to find out those ones that have matter fields remaining finite at both $r \to \infty$ and $t \to \infty$. Then we will demonstrate that instead of this dynamical dimensional reduction we can construct the same states by using the generalized separation of the space and time variables. This is a fairly simple exercise that does not require knowledge of the (1+1)-dimensional solutions. Thus it can in principle be applied to nonintegrable problems. We discuss this at the end of the paper using a very simple example.

4 States with constant moduli

Let us consider solutions with constant moduli μ_n, ν_n. Then $\mathcal{U}_n(u) = \mu_n$, $\mathcal{V}_n(u) = \nu_n$ and it is easy to write explicitly the basic chiral fields. If no moduli vanish[13] we can use the following set according to conditions (26):

$$a_n(u) = \frac{1}{\sqrt{2\,\mu_n}}\, e^{-\mu_n u}, \quad b_n(u) = \frac{1}{\sqrt{2\,\nu_n}}\, e^{-\nu_n u}, \tag{43}$$

[13]If some μ_n are allowed to vanish we may use different basic functions, e.g., $a_n = \cosh \mu_n$, $\bar{a}_n = \mu_n^{-1} \sinh \mu_n$, but we shall not discuss these subtleties here.

$$\bar{a}_n(u) = \frac{1}{\sqrt{2\,\mu_n}}\, e^{\mu_n u}, \quad \bar{b}_n(u) = \frac{1}{\sqrt{2\,\nu_n}}\, e^{\nu_n u}. \tag{44}$$

Then the general constant - moduli solution of eqs. (22) may be written as

$$X_n(u,v) = \frac{1}{2\sqrt{\mu_n \nu_n}} \Big[C_{11}^{(n)}\, e^{-\mu_n u - \nu_n v} + C_{22}^{(n)}\, e^{\mu_n u + \nu_n v} + + C_{12}^{(n)} e^{-\mu_n u + \nu_n v} + C_{21}^{(n)} e^{\mu_n u - \nu_n v} \Big], \tag{45}$$

where

$$C_{11}^{(n)}\, C_{22}^{(n)} - C_{12}^{(n)}\, C_{21}^{(n)} = -\frac{1}{2} \tilde{g}_n \,. \tag{46}$$

In general, when all the elements $C_{ij}^{(n)} \neq 0$, we may, without loss of generality, regard all the moduli to be non-negative, $\mu_n \geq 0$, $\nu_n \geq 0$. If they are not all vanishing, then $\mu_1 \neq 0$, $\nu_1 \neq 0$ due to the constraints (recall that $\gamma_1 < 0$ and $\gamma_k > 0$)

$$|\gamma_1| \mu_1^2 = \sum_{k=2}^{N} \gamma_k \mu_k^2, \quad |\gamma_1| \nu_1^2 = \sum_{k=2}^{N} \gamma_k \nu_k^2 \,. \tag{47}$$

To make our presentation more concise, in what follows we suppose that $\mu_n > 0$, $\nu_n > 0$. Recall also that we can choose the gauge in which $\mu_1 = \nu_1 = 1$. In this gauge we have $\mu_k \leq \sqrt{|\gamma_1|/\gamma_k}$, $k > 1$.

The matrix elements $C_{ij}^{(n)}$ may in general have different signs and thus $X_n(u,v)$ may have zeroes at some finite points (u_0, v_0) [14], $X_n(u_0,v_0) = 0$, and correspondingly $q_n(u_0,v_0) = -2 \ln |X_n(u_0,v_0)| = \infty$. In general, at these points the matter fields will be infinite too. In order to avoid these singularities we suppose that all $C_{ij}^{(n)} > 0$ [15]. Of course, special solutions with $C_{12}^{(n)} = C_{21}^{(n)} = 0$ or with $C_{11}^{(n)} = C_{22}^{(n)} = 0$ for all n are also of interest because, as will be clear in a moment, they may describe static or cosmological states. Note that our remarks on singularities are fully applicable to the general solution (29). To avoid singularities at finite points we in general should suppose that the basic chiral functions are positive and $C_{ij}^{(n)} \geq 0$ (some of them should not vanish).

With positive $C_{ij}^{(n)}$, it is natural to rewrite the solution as

$$X_n = \frac{1}{\sqrt{\mu_n \nu_n}} \Big\{ C_n^+ \cosh\Big(\mu_n u + \nu_n v + \delta_n^+\Big) + C_n^- \cosh\Big(\mu_n u - \nu_n v + \delta_n^-\Big) \Big\} \tag{48}$$

where we define $C_n^{\pm}$, $\delta_n^{\pm}$ by the equations

$$C_n^+ = C_{11}^{(n)}\, e^{\delta_n^+} = C_{22}^{(n)}\, e^{-\delta_n^+}, \;\; C_n^- = C_{12}^{(n)}\, e^{\delta_n^-} = C_{21}^{(n)}\, e^{-\delta_n^-}, \tag{49}$$

[14] In what follows we usually suppose that $-\infty < u < +\infty$, $-\infty < v < +\infty$).

[15] The existence of singularities is a typical feature of the Liouville equation that was carefully studied by different researchers in the past, see, e.g., [26], [34], [35]. As explained in Introduction, our purpose is to study minimally singular solutions and thus we allow for singularities at infinity only.

$$\delta_n^+ = \frac{1}{2}\ln\frac{C_{11}^{(n)}}{C_{22}^{(n)}}, \quad \delta_n^- = \frac{1}{2}\ln\frac{C_{12}^{(n)}}{C_{21}^{(n)}} \tag{50}$$

and, according to (46), we have

$$\left(C_n^+\right)^2 - \left(C_n^-\right)^2 = -\frac{1}{2}\tilde{g}_n \,. \tag{51}$$

Note the important property of this simple formula: $C_n^- = 0$ is only possible if $\tilde{g}_n < 0$ while we can take $C_n^+ = 0$ for $\tilde{g}_n > 0$ only. Recalling that $\tilde{g}_n \equiv \varepsilon g_n/\gamma_n$, where ε is the sign of the metric f, we infer the following fact: as $\gamma_1 < 0$ and $\gamma_k > 0$ for $k > 1$, the common sign of all $\tilde{g}_n$ is possible only if $g_1/g_k < 0$ (or $g_1 = 0$). This is important for constructing the standard one-dimensional solutions.

The static or cosmological solutions can be obtained from (48) by taking $\mu_n = \nu_n$ and $C_n^- = 0$ or $C_n^+ = 0$ for all n. Then the static states depend on the space coordinate defined as $r \equiv (u - \varepsilon v)$ while the cosmological states depend on the time coordinate $t \equiv (u + \varepsilon v)$ [16]. Note that in order to obtain all possible static and cosmological solutions from (45) we should allow not just positive matrix elements $C_{ij}^{(n)}$. Note also that in general one may take imaginary μ_n and ν_n keeping $X_n(u,v)$ real but in this paper we will not consider oscillating waves that necessarily have singularities at finite points of the (u,v) space.

For a better understanding of the physical meaning of these solutions we rewrite them as functions of r and t:

$$X_n = \frac{1}{\sqrt{\mu_n\nu_n}}\left\{C_n^+ \cosh\left(\lambda_n r + \bar{\lambda}_n t + \delta_n^+\right) + C_n^- \cosh\left(\lambda_n t + \bar{\lambda}_n r + \delta_n^-\right)\right\}, \tag{52}$$

where we use the notation $\lambda_n = (\mu_n + \nu_n)/2$ and $\bar{\lambda}_n = (\mu_n - \nu_n)/2$. We see that X_n are sums of two waves having the phase velocities $V_n^- \equiv \lambda_n/\bar{\lambda}_n$ and $V_n^+ \equiv \bar{\lambda}_n/\lambda_n$.

Now we turn to the study of the asymptotic behaviour of the fields ψ_m, φ, F for the class of solutions given by (48). First, we write and briefly discuss the general formulas. As it follows from equations (46), (47) of ref. [16], the expressions for the original fields in terms of X_n are the following:

$$\psi_m = -2\sum_{n=1}^{N} a_{mn}\gamma_n \ln|X_n(u,v)|, \quad m \geq 3, \tag{53}$$

$$\varphi = 4\sum_{n=1}^{N} \gamma_n \ln|X_n(u,v)|, \tag{54}$$

$$F = 4\sum_{n=1}^{N} a_n\gamma_n \ln|X_n(u,v)|, \tag{55}$$

and the curvature of the 1+1 dimensional space-time is

$$R = e^{-F}\,\partial_u\partial_v F = -\Pi_{n=1}^N |X_n(u,v)|^{-4a_n\gamma_n} \cdot \sum_{m=1}^{N} 2a_m\gamma_m\,\tilde{g}_m\,X_m^{-2} \tag{56}$$

[16]Of course, the real identification of the space and time coordinates in dimension (1+1) is only possible if we relate the two-dimensional metric to a higher-dimensional one, see a discussion in [14]. In what follows we usually take $r = u + v$ and $t = u - v$ having in mind that all formulas are essentially symmetric under the substitution $r \leftrightarrow t$

where we have used eq. (22).

As we emphasized above, we suppose that the solutions have no singularities for finite points (u,v). In general, they have singularities when the variables (u,v) (or (r,t)) are infinite. However, for some choices of moduli the infinities may cancel out. For example, if all the functions X_n are infinite and the quantities $\ln|X_n(u,v)|$ have the same asymptotic behaviour, i.e.

$$\ln|X_n(u,v)| = \ln|X(u,v)| + f_n(u,v),$$

where $f_n(u,v)$ are asymptotically finite, then ψ_m and φ may be finite and $F \to -\infty$ (this means that $|f| = e^F \to 0$), as it can be seen from the relations (see [14], [16])

$$\sum_{n=1}^{N} \gamma_n \; = \; 0; \quad \sum_{n=1}^{N} a_{mn}\gamma_n = 0, \;\; m \geq 3; \quad 4\sum_{n=1}^{N} a_n\gamma_n = -2\,. \tag{57}$$

It follows that also R in this case is finite. In fact, as it has been shown in [14], [16], this behaviour may be realized for dimensionally reduced, static (depending only on r) solutions, which for $r \to \pm\infty$ have horizons as $|f| \to 0$.

Now let us turn to the general dimensionally reduced solution (48) with constant μ_n and ν_n. We look for minimally singular solutions that, being finite at space and/or time infinity, can be regarded as localized in some weak sense. To study the singularities at infinity of the solutions with constant moduli, we can use the general formulas (53) - (55) to find their asymptotic behaviour for $r \to \infty$ and/or $t \to \infty$. It is not difficult to find the following asymptotics

$$\ln\,|X_n(u,v)|_{r\to\pm\infty} = \pm(\lambda_n\, r + \bar{\lambda}_n\, t\, + \delta_n^+) + \ln\frac{C_n^+}{2\sqrt{\mu_n\nu_n}} \;+\; ...\,, \tag{58}$$

$$\ln\,|X_n(u,v)|_{t\to\pm\infty} = \pm(\lambda_n\, t + \bar{\lambda}_n\, r + \delta_n^-) + \ln\frac{C_n^-}{2\sqrt{\mu_n\nu_n}} \;+\; ....\tag{59}$$

The omitted terms are exponentially small in the two cases:
1) when $r \to \pm\infty$ and t is finite or also $t \to \pm\infty$ but $|t|/|r| \leq 1-\epsilon$, $\epsilon > 0$ may be an arbitrary small number (see (58)), or
2) when $t \to \pm\infty$ and r is finite or also $r \to \pm\infty$ but $|r|/|t| \leq 1-\epsilon$ (eq. (59)). We see that the fields themselves are in general infinite at infinity. However their first derivatives are finite at infinity and thus everywhere; moreover, as the moduli are bounded from above, the first derivatives are also bounded. The second derivatives and $\exp q_n$ are localized in space and time being exponentially small at infinity.

To summarize, if we consider the condition for finiteness of the expressions for ψ_m, φ, F we may insert in (58), (59) only the divergent term,

$$-\, q_n^\infty/2 = \ln|X_n^\infty| \;=\; \lambda_n\,|\tau|, \quad \text{for } |\tau| \to \infty \tag{60}$$

where $\tau = r \to \pm\infty$ (and t is finite), or $\tau = t \to \pm\infty$ (and r is finite). The cases when $|r| \to \infty$ and $|t| \to (1-\epsilon)r$ or $|t| \to \infty$ and $|r| = |t|(1-\epsilon)$ can be treated similarly.

Thus we see that the divergent parts of the fields (53), 54), (55) and of the curvature (56) are

$$\psi_m^\infty \;= -\,2\sum_{n=1}^{N} a_{mn}\gamma_n\lambda_n\,|\tau|, \quad 3\leq m\leq N\,, \tag{61}$$

$$\varphi^\infty \;= 4\sum_{n=1}^{N} \gamma_n\lambda_n\,|\tau|\,, \tag{62}$$

$$F^\infty \;= 4\sum_{n=1}^{N} a_n\gamma_n\lambda_n\,|\tau|. \tag{63}$$

$$R^\infty \;= -\exp\Big[-2|\tau|\Big(2\sum_{n=1}^{N} a_n\gamma_n\lambda_n - \lambda_{\rm min}\Big)\Big]\,, \tag{64}$$

where $\lambda_{\rm min}$ is the minimum value of λ_n for $1\leq n\leq N$. In order to have finite fields $\psi_m,\ \varphi,\ F$, the expressions ψ_m^∞, φ^∞ and F^∞ must vanish[17]. R will be 0 or ∞ depending on the sign of the expression in the round brackets; it will be finite if that expression is zero. For any given $N-$ Liouville model the parameters a_{mn}, a_n and γ_n are fixed and we are free to choose only the parameters λ_n that should satisfy the constraints described above. Of course, not all of these constraints can be satisfied at the same time.

Suppose that we have solved, e.g., the equations $\psi_m^\infty = 0$ and found λ_n. Recalling that $\mu_n = \lambda_n + \bar\lambda_n$, $\nu_n = \lambda_n - \bar\lambda_n$, we immediately can see that in order to satisfy the constraints (47) λ_n (and $\bar\lambda_n$) should be somehow restricted. The easiest way to find this restriction is to use the unit vectors (41) that are now constant. Recalling that

$$\hat\xi_k = \hat\gamma_k\,(\lambda_k + \bar\lambda_k)\,, \qquad \hat\eta_k = \hat\gamma_k\,(\lambda_k - \bar\lambda_k), \qquad k = 2,...,N, \tag{65}$$

we define the vectors

$$\hat\lambda_k^+ \;= \hat\gamma_k\lambda_k = \frac{1}{2}(\hat\xi_k + \hat\eta_k)\,, \quad \hat\lambda_k^- = \hat\gamma_k\bar\lambda_k = \frac{1}{2}\,(\hat\xi_k - \hat\eta_k)\,, \qquad k = 2,...,N \tag{66}$$

which satisfy two conditions (normalization and orthogonality)

$$\hat\lambda_+^2 \,+\, \hat\lambda_-^2 \,=\, 1, \quad \hat\lambda_+\,\cdot\,\hat\lambda_- \,=\, 0\,. \tag{67}$$

It is easy to see that these two conditions are equivalent to the constraints (47) .

Using these definitions we can construct the solutions as follows. First take any $(N-2)$-dimensional vector $\hat\lambda_+$ with the norm ≤ 1 and then take any vector orthogonal to $\hat\lambda_+$ and having the norm $\sqrt{1-\hat\lambda_+^2}$. This gives the general solution. However, if λ_k are restricted by some additional conditions (e.g., finiteness of ψ_n), satisfying the condition $\hat\lambda_+^2\,\leq 1$ is a

[17]The condition of finiteness of the fields ψ_n is physically natural but not absolutely necessary. Also, in general, we need not require the fields ϕ and F to be finite at infinity. As the first derivatives of all the fields are finite everywhere, including infinity, the corresponding energy and momentum densities are finite everywhere for any choice of the moduli.

nontrivial problem. Indeed, if we have derived λ_k from some equations, we should check that

$$\hat{\lambda}_+^2 \equiv \sum_{k=2}^N \hat{\gamma}_k^2 \lambda_k^2 \leq 1 \,, \tag{68}$$

and we shall see in a moment that this condition is really restrictive. If we have checked it, then the rest is as stated above – we take as $\hat{\lambda}_-$ any vector orthogonal to $\hat{\lambda}_+$ and having the norm $\sqrt{1-\hat{\lambda}_+^2}$. Then, according to our general construction, the vectors

$$\hat{\xi} = \hat{\lambda}_+ + \hat{\lambda}_- \,, \quad \hat{\eta} = \hat{\lambda}_+ - \hat{\lambda}_- \tag{69}$$

define the solution satisfying the constraints (47).

Consider first the equations for λ_n given by the conditions $\psi_m^\infty = F^\infty = \varphi^\infty = 0$. Using the identities (57) we find that the conditions $\psi_m^\infty = 0$, $F^\infty = 0$, $\varphi^\infty = 0$ give respectively the following equations for $y_k \equiv \gamma_k(\lambda_k - 1)$:

$$\sum_{k=2}^N a_{mk} y_k = 0 \,, \qquad m = 3, ..., N \,, \tag{70}$$

$$\sum_{k=2}^N y_k = 0 \,, \tag{71}$$

$$\sum_{k=2}^N a_k y_k = \frac{1}{2} \,. \tag{72}$$

In general all these equations have no solution, as we have N linear equations for $N-1$ parameters y_k. The homogeneous equations (70), (71) always allow for the zero solution $y_k = 0$, which gives $\mu_n = \nu_n = 1$ (as $\mu_1 = \nu_1 = 1$). This corresponds to the static solution with two horizons (see [14], [16]).

Note that eqs. (70), (71) may have nontrivial ($y_k \neq 0$) solutions if the determinant of the matrix of these equations is zero. Of course, this is possible only for some special systems and we do not consider this possibility in general. Note only that for $N = 3$ (a single scalar field) this is impossible because for $a_{33} = a_{32}$ both γ_2^{-1} and γ_3^{-1} should vanish (see Appendix to [16]). The analysis of the case $N = 4$ is much more cumbersome. It shows that for any linearly independent set of the vectors A_n satisfying the conditions (5) the system (70) has only the trivial solution $y_k \equiv 0$. We believe that this statement is true for any N but, at the moment, cannot rigorously prove it.

The combination of the equations (70) and (72) usually has a solution because the determinant of these equations is normally nonvanishing. For $N = 3$ it is easy to find this solution and to show that, under certain restrictions, it defines the moduli satisfying the conditions (68). This is demonstrated at the end of this Section. We do not treat the general case $N > 3$. Although it is easy to find the solution of the inhomogeneous linear system (70) and (72), it is difficult to analytically derive the general restrictions on a_n, a_{mn} under which the solution satisfies (68).

Thus, let us first consider a somewhat simpler problem of finding the solutions of the system (70) which satisfy the restriction (68). Suppose (without loss of generality) that not all a_{mN} vanish and that the square matrix a_{mk} $(2 < k < N-1,\, 3 < m < N)$ is nondegenerate. Then we can, in general, solve the inhomogeneous system,

$$\sum_{k=2}^{N-1} a_{mk}\, y_k \;=\; -\, a_{mN}\, y_N\,, \qquad m = 3, ..., N\,, \tag{73}$$

and express y_k $(k = 2, ..., N-1)$ in terms of a more or less arbitrarily chosen y_N and of the parameters of the system (of course, we can in principle choose any y_k instead of y_N). The corresponding parameters λ_n are

$$\lambda_1 = 1, \qquad \lambda_k \;=\; 1 + \frac{y_k}{\gamma_k}, \quad k = 2, ..., N\,. \tag{74}$$

According to (68) some of the y_k should be negative (if $y_k < 0$ for all k then obviously $\lambda_+^2 < 1$ because $\gamma_k > 0$). Moreover, as

$$\hat{\lambda}_+^2 = \frac{1}{|\gamma_1|}\sum_{k=2}^{N}\gamma_k\lambda_k^2 = \frac{1}{|\gamma_1|}\sum_{k=2}^{N}\gamma_k\left(1+\frac{y_k}{\gamma_k}\right)^2 = 1 + \frac{1}{|\gamma_1|}\sum_{k=2}^{N}\left(\frac{y_k^2}{\gamma_k} + 2y_k\right) \tag{75}$$

the parameters y_k should satisfy the sufficiently stringent inequality

$$\sum_{k=2}^{N}\left(\frac{y_k^2}{\gamma_k} + 2y_k\right) \;<\; 0. \tag{76}$$

As follows from the previous considerations, this inequality is equivalent to the fundamental constraints (47) and is independent of any equations for y_k. The moduli λ_n related to y_k by (74) and defining any solution (52) ($\bar{\lambda}_n$ are defined by (66) (67)) should satisfy this inequality.

Returning to the system (73), we note that applying (76) to its solution gives a quadratic inequality for the arbitrary parameter y_N but the implicit dependence of this simple inequality on the parameters a_{mn} defining the model is so complex that it is rather difficult to make any general statement on the solution of this problem[18].

This can be seen even in the $N = 3$ case. The model is defined by three arbitrary parameters: a_{31}, a_{32}, a_{33}. The other parameters (a_n, γ_n) can be found by using the pseudo orthogonality conditions. In this case we have only one equation (70) which gives

$$y_2 = -\frac{a_{33}}{a_{32}}\, y_3. \tag{77}$$

Now, if $a_{33}/a_{32} < 0$, we choose $y_3 < 0$ that automatically gives $y_2 < 0$ and therefore $\hat{\lambda}_+^2 < 1$. To find the general solution consider the curve in the (λ_2, λ_3) plane defined by the condition $\hat{\lambda}_+^2 = 1$, i.e.

$$\hat{\lambda}_+^2 \;\equiv\; \hat{\gamma}_2^2\lambda_2^2 \;+\; \hat{\gamma}_3^2\lambda_3^2 \;=\; 1. \tag{78}$$

[18]It is not difficult to find the solution for any pseudo orthogonal system of parameters a_{mn}. However, a general analytic solution of the equations is hardly possible.

As both λ_2 and λ_3 depend only on y_3 they are related by the equation obtained by excluding this dependence[19]

$$\lambda_2\, a_{32}\, \hat{\gamma}_2^2 \,+\, \lambda_3\, a_{33}\, \hat{\gamma}_3^2 \,=\, a_{31}\,. \tag{79}$$

The points $(\lambda_2 > 0, \lambda_3 > 0)$ that are inside the ellipse (78) and belong to the straight line (79) give us all possible vectors $\hat{\lambda}_+$ satisfying required conditions. To find all possible values of μ_n, ν_n corresponding to this vector we take all possible $\hat{\lambda}_-$ orthogonal to $\hat{\lambda}_+$ with the norm $\hat{\lambda}_-^2 = 1 - \hat{\lambda}_+^2$ and thus find $\bar{\lambda}_n$. The solution obtained for the N-Liouville problem gives asymptotically finite scalar matter fields as we explained earlier.

We mentioned above that it is possible to find a solution of the equations (70), (72) satisfying the restriction (68), but the restriction requires that the parameters a_{mn} obey certain conditions. This statement is easy to illustrate by considering the simplest non-trivial case $N = 3$. Denoting $a_{3i} \equiv \alpha_i$ and recalling that (see [16] and Appendix)

$$a_i = \alpha_i(\alpha_j + \alpha_k) - \alpha_j \alpha_k\,, \quad \text{where} \quad (ijk) = (123)_{\text{cyclic}}\,,$$

it is easy to find the solution

$$y_2 \,=\, \frac{\alpha_3}{8a_1(\alpha_2 - \alpha_3)}\,, \qquad y_3 \,=\, -\frac{\alpha_2}{8a_1(\alpha_2 - \alpha_3)}\,. \tag{80}$$

Using the expressions for γ_i given in [16] (see also Appendix), one can show that y_2/γ_2 and y_3/γ_3 are negative if $\alpha_2 < \alpha_1 < \alpha_3$, $\alpha_2 < 0$, $\alpha_3 > 0$ (or, if $\alpha_3 < \alpha_1 < \alpha_2$, $\alpha_3 < 0$, $\alpha_2 > 0$). It follows that the condition $\lambda_+^2 < 1$ is fulfilled and we have the solution satisfying all the necessary conditions. In the special case $N = 3$ discussed in Appendix we have $a_1 = -a_2 = -a_3 = a$, $\alpha_1 = 0$, $\alpha_2 = \alpha < 0$, $\alpha_3 = -4a/\alpha > 0$ (if $a > 0$). Simple calculations give $y_2/\gamma_2 = y_3/\gamma_3 = -\frac{1}{2}$ and thus $\lambda_+^2 = \frac{1}{4}$. More general cases can be treated similarly.

The algorithm requested to find similar solutions for $N \geq 4$ is conceptually clear but in practice it is impossible to give general enough statements for generic parameters a_{mn}. Once the parameters are known we can study the asymptotic properties of the solutions. However, it is not clear whether it is possible to give a reasonable a priori classification of them. The same can be said about the possible asymptotic behavior of the curvature R. The singular part of the curvature is

$$R^{\infty} = \exp\Big[-2|\tau|\Big(2\sum_{k=2}^{N} a_k y_k + \frac{y_{\bar{n}}}{\gamma_{\bar{n}}}\Big)\Big]\,, \tag{81}$$

where $\bar{n}$ is the number of the minimal λ_n (note that $y_{\bar{n}}$ should necessarily be negative). In the simplest case $N = 3$ one can write an explicit expression of the expression in round brackets in terms of the three free parameters a_{31}, a_{32}, a_{33} and thus find when R^{∞} is zero, infinite or finite; the last condition is possible only when there exists one relation between the three parameters (in fact, a quadratic equation for a_{31}).

[19]To obtain this we should recall the identities (57) and the definition $\hat{\gamma}_k = \sqrt{\gamma_k/|\gamma_1|}$ introduced previously.

5 Separation of variables - simple examples

Now we obtain the solutions with constant moduli avoiding the complete explicit solution in dimension 1+1. Instead we employ the generalized separation of variables. Let us first try to separate the variables r and t in the Liouville equations (19). If we simply write $r = u + v$, $t = u - v$ for all n, we obviously end up with one of the naive reductions, for which $q_n = q_n(r)$ or $q_n = q_n(t)$.

However, taking into account our experience with the solutions of the N-Liouville model, we may try to use a less naive approach and write different r_n, t_n for different n:

$$r_n = \mu_n u + \nu_n v, \quad t_n = \mu_n u - \nu_n v\,. \tag{82}$$

Denoting by 'prime' and 'dot' the differentiations in r_n and t_n we obtain

$$q_n^{''} - \ddot{q}_n = \tilde{G}_n \exp q_n\,, \qquad \tilde{G}_n \equiv \frac{\tilde{g}_n}{\mu_n \nu_n}\,. \tag{83}$$

Now we separate the variables r_n and t_n by writing

$$q_n = \xi_n(t_n) + \eta_n(r_n) \tag{84}$$

and then differentiate the resulting equation in r_n or t_n. Thus we immediately find that the following two separations are possible

$$\eta_n^{''}(r_n) - C_n \exp \eta_n(r_n) = 0\,, \qquad C_n \equiv \tilde{G}_n \exp \xi_n = \text{const}\,, \tag{85}$$

$$\ddot{\xi}_n(t_n) - \bar{C}_n \exp \xi_n(t_n) = 0\,, \qquad \bar{C}_n \equiv \tilde{G}_n \exp \eta_n = \text{const}\,. \tag{86}$$

This separation gives only half of the solutions with constant moduli in the N-Liouville model. But now one can guess that the separation procedure will give different results if we apply it to X_n instead of q_n.

Indeed, let us try to separate the same variables (82) in equation (22) by using the same Ansatz (84) for X_n:

$$X_n(u,v) = \xi_n(t_n) + \eta_n(r_n)\,. \tag{87}$$

Differentiating the resulting equation

$$\Big(\xi_n(t_n) + \eta_n(r_n)\Big)\Big(\eta_n^{''}(r_n) - \ddot{\xi}_n(t_n)\Big) + \eta_n^{'2}(r_n) - \dot{\xi}_n^2(t_n) = -\frac{1}{2}\tilde{G}_n \tag{88}$$

in r_n and then in t_n, we find that

$$\eta_n^{'}(r_n)\,\dddot{\xi}_n(t_n) - \eta_n^{'''}(r_n)\,\dot{\xi}_n(t_n) = 0, \tag{89}$$

and therefore ξ_n and η_n satisfy the equations

$$\ddot{\xi}_n(t_n) - C_n \xi_n(t_n) = A_n, \tag{90}$$

$$\eta_n^{''}(r_n) - C_n\, \eta_n(r_n) = B_n\,. \tag{91}$$

where A_n, B_n and C_n are arbitrary constants. So we obtain the solutions,

$$\xi_n(t_n) = -A_n/C_n + \tilde{C}_n^- \,\mathrm{ch}\Big(\sqrt{C_n}\,(t_n + \delta_n^-)\Big)\,, \tag{92}$$

$$\eta_n(r_n) = -B_n/C_n + \tilde{C}_n^+ \,\mathrm{ch}\Big(\sqrt{C_n}\,(r_n + \delta_n^+)\Big), \tag{93}$$

with arbitrary integration constants $\tilde{C}_n^{\pm}$, $\delta_n^{\pm}$. Substituting these functions into eqs. (88), we find that these arbitrary constants must satisfy the equations

$$C_n \left((\tilde{C}_n^+)^2 \,-\, (\tilde{C}_n^-)^2 \right) \,=\, -\frac{\tilde{g}_n}{2\mu_n\nu_n}\,, \qquad A_n \,+\, B_n \,=\, 0\,. \tag{94}$$

We thus see that X_n given by eqs. (87), (92) - (94) coincides with (48) - (51) if we take $C_n = 1$ and $\tilde{C}_n^{\pm} = C_n^{\pm}/\sqrt{\mu_n\nu_n}$ (or, $C_n = 1/(\mu_n\nu_n)$ and $\tilde{C}_n^{\pm} = C_n^{\pm}$).

In these considerations we did not use the integrability of the N-Liouville model and one may expect that our approach can be applied to other nonlinear systems, not necessarily integrable. Indeed, in this way we treated spherically symmetric static states and cosmologies that are described by generally nonintegrable equations (see [17]). We thus expect that solutions similar to those with constant moduli in the integrable N-Liouville theory may exist also in nonintegrable realistic (1+1)- dimensional theories. Presumably they can be derived by using some sort of generalized separation of variables. A serious analysis of this subject will be attempted elsewhere while here we only give a very simple, almost trivial example.

Recalling the discussion of 'realistic' potentials in Section 2 let us consider a single scalar field coupled to the dilaton gravity with the potentials

$$V \,=\, g\,\varphi^{1+a}\,e^{\lambda\psi}, \quad Z = -\varphi\,. \tag{95}$$

Introducing the notation $\varphi \equiv e^{\phi}$ we write the main equations of motion in the form [20]

$$\partial_u\partial_v\phi \,+\, \partial_u\phi\,\partial_v\phi \,+\, \varepsilon g\, e^{F+a\phi+\lambda\psi} \,=\, 0, \tag{96}$$

$$2\partial_u\partial_v\psi + \partial_u\phi\partial_v\psi + \partial_v\phi\partial_u\psi - \varepsilon g\lambda e^{F+a\phi+\lambda\psi} = 0, \tag{97}$$

$$\partial_i^2\phi + (\partial_i\phi)^2 - \partial_i F\partial_i\phi + (\partial_i\psi)^2 = 0, \quad i = u, v. \tag{98}$$

Separations of variables are most convenient if we have bilinear equations for the unknown functions. In our present example eqs. (98) are bilinear and we can write one more bilinear equation combining (96) with (97). Defining $\Psi = \psi + \lambda\phi/2$, we have

$$2\partial_u\partial_v\Psi + \partial_u\Psi\partial_v\phi + \partial_v\Psi\partial_u\phi = 0. \tag{99}$$

The only equation containing the exponential is now eq. (96). We shall not attempt to give a complete study of these equations but simply write its very special solution

$$\phi = \alpha\,u + \beta\,v, \qquad \Psi = \mu\,(\alpha\,u - \beta\,v), \tag{100}$$

$$F = C_0 - \alpha\,u(a - \frac{\lambda^2}{2} + \mu\lambda) - \beta v\,(a - \frac{\lambda^2}{2} - \mu\lambda)\,, \tag{101}$$

[20]As always, we use the 'Weyl frame' and the light-cone coordinates for the Lagrangian (1).

where α, β are arbitrary parameters, $C_0 = \ln(-\alpha\beta/g)$ and $\mu^2 \equiv \lambda^2/4 - 1 - a$.

Defining $r \equiv u + v$, $t \equiv u - v$, we can rewrite these solutions in the wave-like (or, pulse-like) form

$$\phi = \frac{\alpha+\beta}{2}\Big(r + V_1 t\Big), \tag{102}$$

$$\Psi = \frac{1}{2}\Big[(\mu - \frac{\lambda}{2})\alpha - (\mu + \frac{\lambda}{2})\beta\Big]\,(r + V_2 t), \tag{103}$$

$$F = -\frac{1}{2}\Big[\alpha(a - \frac{\lambda^2}{2} + \lambda\mu) + \beta(a - \frac{\lambda^2}{2} - \lambda\mu)\Big]\,(r + V_3 t) \tag{104}$$

where the velocities of the pulses are

$$V_1 = \frac{\alpha - \beta}{2}, \tag{105}$$

$$V_2 = \frac{(\mu - \frac{\lambda}{2})\alpha + (\mu + \frac{\lambda}{2})\beta}{(\mu - \frac{\lambda}{2})\alpha - (\mu + \frac{\lambda}{2})\beta}, \tag{106}$$

$$V_3 = \frac{\alpha(a - \frac{\lambda^2}{2} + \mu\lambda) - \beta(a - \frac{\lambda^2}{2} - \mu\lambda)}{\alpha(a - \frac{\lambda^2}{2} + \mu\lambda) + \beta(a - \frac{\lambda^2}{2} - \mu\lambda)}\,. \tag{107}$$

Choosing the gauge $\alpha = \beta$ ('static' ϕ) we have

$$V_1 = 0, \quad V_2 = -\frac{2\mu}{\lambda}, \quad V_3 = -\frac{2\mu}{\lambda}\Big(1 - \frac{2a}{\lambda^2}\Big)^{-1}. \tag{108}$$

The model in which we have found this solution belongs to the class of two-dimensional theories obtained by standard dimensional reductions from higher-dimensional theories. We thus see that the wave - like solutions can be met not only in the integrable models with $N = 1$ but also in quite realistic and apparently nonintegrable theories. So one may hope that the connection between static, cosmological and wave - like solutions is a general feature of the theory of gravity [21].

6 Conclusion and outlook

The concrete results of this paper are as follows. We have generalized the analytic expression for the solution of the equations and constraints of the (1+1)-dimensional N-Liouville theory, (29), (30). We have introduced a very convenient gauge (40), in which the main physical properties of the solutions are particularly transparent, and then derived the representation of the solution in terms of moduli functions $\hat{\xi}(u) \in S^{(N-2)}$, $\hat{\eta}(v) \in S^{(N-2)}$. The reduction of the moduli space to the space of points $(\hat{\xi}^{(0)}, \hat{\eta}^{(0)})$, belonging to $S^{(N-2)}$ introduces an interesting new class of solutions of the N-Liouville theory, (45), (46). When $\hat{\xi}^{(0)} = \hat{\eta}^{(0)}$, these solutions can be interpreted as either static states or as cosmologies. When $\hat{\xi}^{(0)} \neq \hat{\eta}^{(0)}$ the corresponding solutions depend on both variables r and t.

[21] In [17] we demonstrated that the duality relation between static states and cosmology is present in the case of spherically reduced gravity coupled to scalar matter fields.

In general, the solutions have singularities at finite space-time points $(t,\ r)$ and at infinity. However there exists a subclass of solutions having no singularities at finite points, (48), (51). A solution of this (nonsingular) subclass may be usually infinite at the space and/or time infinity, but we showed that some of them are finite both for $r \to \pm\infty$ and for $t \to \pm\infty$. For the solutions to be finite at the space and time boundaries we have formulated explicit constraints that are simple to solve; however, they cannot be solved in general and so we cannot derive a general formula for the solution of these constraints in an arbitrary N-Liouville model. We therefore constructed an algorithm for solving this problem and illustrated it by a very simple example for $N = 3$.

The solutions without singularities at finite points have finite densities of energy and momentum and the shape of their first derivatives is step - like, similar to some topological solitons. So, they resemble solitons but their energy is not localized in space and for this reason we cannot call them solitons. The curvature R for these solutions is finite at finite points and may diverge at infinity (this depends on all the parameters defining the solution). Thus, in spite of their apparent simplicity, the solutions defined by constant moduli are rather complex objects strongly interconnected to the geometry of the space-time.

We believe that very similar waves can be obtained in realistic, nonintegrable theories. There they should be looked for with the aid of the generalized separation of variables. We demonstrated here that the wave solutions of the integrable (1+1)-dimensional theory can be obtained without any knowledge of its integrability; we also gave an example of the simplest solutions in a nonintegrable theory.

In future, we plan to concentrate on the derivation of waves, similar to those found here, in realistic nonintegrable theories. Simple enough and interesting waves that should be studied from the point of view advocated here are cylindrical waves [36] and plane waves [37]. We also hope to find a closer connection between static states, cosmologies and waves in completely realistic theories. Some physical connections between waves and black holes were discussed in literature (see, e.g. [38] and references therein). A possible role of gravitational waves in cosmology was also studied in some detail (see, e.g. [39] and [40]). At first sight, our model results may look not so directly related to these investigations but being conceptually and technically very simple they can help to find a new systematic approach to a deeper understanding of these complex phenomena.

7 Appendix

Here we give parameters of a few typical N-Liouville models, some of which can be obtained by dimensional reduction of the higher-dimensional supergravity theories. We first construct the general $N = 4$ model. Using the formulas of Appendix to [16] we can immediately write the general expressions for a_i and γ_i $(i = 1, 2, 3)$ in the general $N = 4$ case. Introducing the notation $a_{3n} \equiv \alpha_n$, $a_n \equiv \beta_n$ $(n = 1, ..., 4)$ we have

$$4a_i = \alpha_i(\alpha_j + \alpha_k) - \alpha_j\alpha_k \, + \, \beta_i(\beta_j + \beta_k) - \beta_j\beta_k \,, \tag{109}$$

$$\gamma_i = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k) \, + \, (\beta_i - \beta_j)(\beta_i - \beta_k) \,, \tag{110}$$

where α_i, β_i are arbitrary parameters and $(ijk) = (123)_{\rm cyclic}$. The unknown parameters a_4, α_4, β_4 can be derived by applying the general procedure of ref. [16]. In the $N = 4$ case it gives three inhomogeneous linear equations for these three parameters. The solution is

$$a_4 = -\frac{1}{\Delta}[a_1(\alpha_2\beta_3 - \alpha_3\beta_2) + a_2(\alpha_3\beta_1 - \alpha_1\beta_3) + a_3(\alpha_1\beta_2 - \alpha_2\beta_1)]\,, \quad (111)$$

$$\alpha_4 = +\frac{2}{\Delta}[a_1(\beta_2 - \beta_3) + a_2(\beta_3 - \beta_1) + a_3(\beta_1 - \beta_2)]\,, \quad (112)$$

$$\beta_4 = -\frac{2}{\Delta}[a_1(\alpha_2 - \alpha_3) + a_2(\alpha_3 - \alpha_1) + a_3(\alpha_1 - \alpha_2)]\,, \quad (113)$$

$$\Delta = [\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)]\,. \quad (114)$$

Alternatively, we can use (109), (110) and the 'sum rules' (57) to derive γ_4, a_4, α_4, β_4 in terms of the previously found a_i, γ_i $(i = 1, 2, 3)$.

For $N = 5$, deriving the parameters of the models becomes much more cumbersome. Moreover, the general expressions become useless for practical analytical computations, even in the simple problems treated in the main body of our paper. Fortunately enough, realistic N-Liouville theories obtained from higher-dimensional theories often have much simpler structure than in the general case. We mentioned above the important examples:

$$\text{I.}\ \ -a_1 = a_2 = ... = a_N = -a\,; \quad \text{II.}\ \ -a_1 = -a_2 = a_3 = ... = a_N = -a\,. \quad (115)$$

To find the expressions for the parameters a_{mn} of these models it is convenient to use simple geometrical considerations instead of applying the general procedure. With this aim, we write the vectors A_n introduced in Section 2 in the form

$$A_n \equiv (1 + a_n\,, 1 - a_n\,, \vec{A}_n)\,, \quad \vec{A}_n \equiv (a_{3n}\,, ..., a_{Nn})\,, \quad (116)$$

where $\vec{A}_n$ are euclidean $(N-2)$-vectors that should satisfy the restrictions following from eqs. (5). For the models (115), one can explicitly solve this problem for arbitrary N but here we only write the result for $N = 4$. For the type I models:

$$\begin{aligned} \vec{A}_1 &= (0, 0),\ \ \gamma_1^{-1} = -4a; \quad \vec{A}_2 = (\alpha, 0),\ \ \gamma_2^{-1} = \alpha^2 + 4a; \\ \vec{A}_3 &= (-4a/\alpha,\ \beta); \qquad \gamma_3^{-1} = (\alpha^2\beta^2 + 4a\alpha^2 + 16a^2)/\alpha^2\,; \\ \vec{A}_4 &= (-4/\alpha, -4a(\alpha^2 + 4a)/\alpha^2\beta); \quad \gamma_4 = -(\gamma_1 + \gamma_2 + \gamma_3). \end{aligned} \quad (117)$$

For the type II models:

$$\begin{aligned} \vec{A}_1 &= (\alpha, 0),\ \ \gamma_1^{-1} = \alpha^2 - 4a; \quad \vec{A}_2 = (4a/\alpha\,, 0), \quad \gamma_2^{-1} = 4a(4a - \alpha^2)/\alpha^2; \\ \vec{A}_3 &= (0, \beta);\ \ \gamma_3^{-1} = \beta^2 + 4a\,; \quad \vec{A}_4 = (0, -4a/\beta),\ \ \gamma_4^{-1} = 4a(\beta^2 + 4a)/\beta^2. \end{aligned} \quad (118)$$

Using these results the reader can easily derive the vectors $\vec{A}_n$ for the type I and type II $N = 5$ models and generalize these formulas to arbitrary N.

The examples (115) are interesting not only because of their simplicity. As we mentioned in Section 2, they are generic in the context of dimensional reduction of some higher-dimensional supergravity theories (see, e.g. the model of ref. [25] the parameters of which are given in (15)).

8 Acknowledgments

One of the authors (A.T.F.) very much appreciates the support of the Department of Theoretical Physics of the University of Turin and of INFN (Section of Turin).

This work was supported in part by the Russian Foundation for Basic Research (Grant No. 06-01-00627-a).

References

[1] T. Padmanabhan, Dark Energy: mystery of millennium, astro-ph/0603114.

[2] E.J. Copeland, M. Sami, and S. Tsujikawa, Dynamics of dark energy, hep-th/0603057.

[3] S. Nojiri and S. Odintsov, Introduction to modern gravity and gravitational alternative for dark energy, hep-th/0601213.

[4] V. Sahni and A. Starobinsky, Reconstructing dark energy, astro-ph/0610026.

[5] J.E. Lidsey, D. Wands and E.J. Copeland, **Phys. Rep. 337** (2000) 343.

[6] M. Gasperini and G. Veneziano, **Phys. Rept. 373** (2003) 1.

[7] H. Stefani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlt, Exact solutions of the Einstein field equations, Cambridge University Press, Cambridge, 2003.

[8] V.A. Belinskii and V.E. Zakharov, **Sov. Phys. JETP 48** (1978) 985.

[9] D. Maison, **Phys. Rev. Lett. 41** (1978) 521.

[10] H. Nicolai, D. Korotkin, and H. Samtleben, Integrable classical and quantum gravity, hep-th/9612065 (Cargése lectures).

[11] G. Alekseev, **Theor. Math. Phys. 143** (2005) 120.

[12] C. Callan, S. Giddings, J. Harvey and A. Strominger, **Phys. Rev. D 45** (1992) 1005.

[13] V. de Alfaro and A.T. Filippov, Integrable Low Dimensional Theories describing Higher Dimensional Branes, Black Holes, and Cosmologies, hep-th/0307269.

[14] V. de Alfaro and A.T. Filippov, Integrable Low Dimensional Models for Black Holes and Cosmologies from High Dimensional Theories, **Mem. Acc. Sci. Torino**, in press, hep-th/0504101.

[15] V. de Alfaro and A.T. Filippov, Black Holes and Cosmological Solutions in Various Dimensions, 2004, unpublished.

[16] A.T. Filippov, Integrable models of 1+1 Dimensional Dilaton Gravity Coupled to Scalar Matter, **Theor. Math. Phys. 146** (2006) 95; hep-th/0505060.

[17] A.T. Filippov, Some Unusual Dimensional Reductions of Gravity: Geometric Potentials, Separation of Variables, and Static - Cosmological Duality, hep-th/0605276.

[18] V. de Alfaro and A.T. Filippov, Dynamical Dimensional Reduction, 2005, unpublished.

[19] J.R. Oppenheimer and H. Snyder, **Phys. Rev. D55** (1939) 374.

[20] M. Cavaglià, V. de Alfaro and A.T. Filippov, **IJMPD 4** (1995) 661; **IJMPD 5** (1996) 227; **IJMPD 6** (1996) 39.

[21] A.T. Filippov, **MPLA 11** (1996) 1691; **IJMPA 12** (1997) 13.

[22] A. Lukas, B.A. Ovrut and D. Waldram, **Phys. Lett. B393** (1997) 65.

[23] F. Larsen and F. Wilczek, **Phys. Rev. D55** (1997) 4591.

[24] H. Lü, S. Mukherji and C.N. Pope, **Int. J. Mod. Phys. A14** (1999) 4121.

[25] Y. Kiem, C.Y. Lee and D. Park, **Phys. Rev. D57** (1998) 2381.

[26] G.D. Dzhordzhadze, A. Pogrebkov and M.K. Polivanov, On the solutions with singularities of the Liouville equation, preprint IC/78/126, ICTP, Trieste, 1978.

[27] J.L. Gervais, **IJMPA 6** (1991) 2805.

[28] A.N. Leznov and M. Saveliev, Group - Theoretical Methods for Integration of Non Linear Dynamical Systems, Birkhäuser, Boston 1992.

[29] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, **Nucl. Phys. B 527** (1998) 142.

[30] P. Fré, A. Sorin, Nucl. Phys. B733 (2006) 33.

[31] K. Stelle, BPS Branes in Supergravity, hep-th/9803116.

[32] T. Mohaupt, **Class. Quant. Grav. 17** (2000) 3429.

[33] V.D. Ivashchuk and V.N. Melnikov, **Class. Quant. Grav. 18** (2001) R87.

[34] G.D. Dzhordzhadze, A. Pogrebkov and M.K. Polivanov, **Theor. Math. Phys. 40** (1979) 706; **J. Phys. A: Math. Gen. 19** (1986) 121.

[35] E. D'Hoker and R. Jackiw, **Phys. Rev. D 26** (1982) 3517; **Phys. Rev. Lett. 50** (1983) 1719.

[36] A. Einstein and N. Rosen, **J. Franklin Inst. 223** (1937) 43; N. Rosen, **Phys. Zs. der Zowjetunion 12** (1937) 366.

[37] P. Szekeres, **J. Math. Phys 13** (1972) 286.

[38] I.Ya. Aref'eva, K.S. Viswanathan, and I.V. Volovich, **Nucl. Phys. B452** (1995) 346.

[39] A. Feinstein, K.E. Kunze and M.A. Vazquez-Mozo, **Class. Quant. Grav. 17** (2000) 3599.

[40] V. Bozza and G. Veneziano, **JHEP 0010** (2000) 035.