Statistical description of extended particle systems

Nugzar Makhaldiani

Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Moscow Region, Russia
e-mail address: mnv@jinr.ru

1 Quantum field theory and Fractal calculus - universal language of fundamental physics

The extended particle processes, we will describe in terms of QFT. As a concrete model, we take relativistic scalar field model with lagrangian

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - g\varphi^n, \ \mu = 0, 1, ..., D-1 \tag{1}$$

In the case

$$n = \frac{2D}{D-2} \tag{2}$$

the coupling constant g is dimensionless, and the model is renormalizable. We take euklidian form of the QFT which unifies quantum and statistical physics problems. The main objects of theory are Green functions - correlation functions - correlators,

$$\begin{aligned} &G_m(x_1, x_2, ..., x_m) =< \varphi(x_1)\varphi(x_2)...\varphi(x_m) > \\ &= \int d\varphi(x)\varphi(x_1)\varphi(x_2)...\varphi(x_m)e^{-S(\varphi)} \end{aligned} \tag{3}$$

where $d\varphi-$ invariant measure

$$d(\varphi + a) = \varphi. \tag{4}$$

For gaussian actions,

$$S = S_2 = \int dxdy\phi(x)A(x,y)\phi(y) \tag{5}$$

the QFT is solvable,

$$\begin{aligned} G_m(x_1, ..., x_m) &= \frac{\delta^m}{\delta J(x_1)...J(x_m)} lnZ(J)|_{J=0}, \\ Z(J) &= \int d\varphi e^{-S_2 + J\cdot\varphi} = e^{\frac{1}{4}\int dxdyJ(x)A^{-1}(x,y)J(y)} \end{aligned} \tag{6}$$

Non trivial problem is to calculate correlators for non gaussian QFT

1.1 Renormdynamics

In quantum perturbation calculations, we find the following corrections to the classical lagrangian

$$\Delta L = (z-1)\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - (z_m - 1)\frac{m^2}{2}\varphi^2 - (z_g - 1)g\varphi^n. \tag{7}$$

Corrected, effective, lagrangian become

$$L + \Delta L = z\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - z_m\frac{m^2}{2}\varphi^2 - z_g g\varphi^n \tag{8}$$

We can restor the classical form of the lagrangian, by corresponding renormalization transformations,

$$\begin{aligned} &\varphi \Rightarrow z^{-1/2}\varphi \\ &m^2 \Rightarrow z_m^{-1} z m^2 \\ &g \Rightarrow z_g^{-1} z^{n/2} g \end{aligned} \tag{9}$$

In the infinitezimal form they define the following renormdynamic motion equations

$$\begin{aligned} &\frac{\mu d}{d\mu}g = \frac{d}{dt}g \equiv \dot{g} = \beta(g),\ t = \ln(\frac{\mu}{\mu_0}), \\ &\dot{m} = \eta(g)m, \\ &\dot{\varphi} = (\frac{\mu\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \eta(g)\frac{m\partial}{\partial m})\varphi \equiv D\varphi = -\frac{1}{2}\gamma(g)\varphi \end{aligned} \tag{10}$$

For correlators, renormdinamic equations are

$$(D + m\frac{\gamma(g)}{2})G_m = 0, \tag{11}$$

For renorminvariant quantities - renomintegrals of motion I,

$$\dot{I} = DI = 0, \tag{12}$$

Solution of the renormdynamic equation for coupling constant, $\bar{g}$, is given in the implicit form by the following integral

$$\int_g^{\bar{g}} \frac{dg}{\beta(g)} = \ln\frac{\bar{\mu}}{\mu} \equiv t \tag{13}$$

The mass parameter running is given as

$$m = \bar{m}\exp(-\int_\mu^{\bar{\mu}} \frac{d\mu}{\mu}\eta(g(\mu))), \tag{14}$$

the correlator (renorm)dynamics is given as

$$G_n(p;g,m,\mu) = \exp(\frac{n}{2}\int_{\mu}^{\bar{\mu}} \frac{d\mu}{\mu}\gamma(g(\mu)))\cdot G_n(p;\bar{g},\bar{m},\bar{\mu}) \qquad (15)$$

From dimensional considerations,

$$G_n(\lambda p;g,m,\mu) = \lambda^{nd_\varphi - D}\Phi(p;g,\frac{m}{\lambda},\frac{\mu}{\lambda}), \qquad (16)$$

so

$$(\lambda\frac{\partial}{\partial\lambda}+\mu\frac{\partial}{\partial\mu}+m\frac{\partial}{\partial m})G_n(\lambda p;...) = (nd_\varphi - D)G_n(\lambda p;...), \qquad (17)$$

Now we obtain the main equation of the scale dynamics of correlators

$$\begin{aligned}&\check{O}G_n(\lambda p;g,m,\mu) = 0,\\ &\check{O} \equiv \frac{\lambda\partial}{\partial\lambda} - \beta(g)\frac{\partial}{\partial g} - (\eta(g)-1)\frac{m\partial}{\partial m} - n(d_\varphi + \frac{\gamma(g)}{2}) + D,\end{aligned} \qquad (18)$$

1.2 Renormdynamics of observable quantities in high energy physics

Let us consider one particle semiinclusive distribution

$$F(p,n)=\frac{d\sigma_n}{\bar{d}p}=\frac{1}{(n-1)!}\int\prod_{i=1}^{n-1}\bar{d}p'_i\delta(p_1+p_2-p-\Sigma_{i=1}^{n-1}p'_i)$$
$$\cdot|G_{n+2}(p_1,p_2,p,p'_1,p'_2,...,p'_{n-1};g(\mu),m(\mu)),\mu)|^2,$$
$$\bar{d}p\equiv\frac{d^3p}{E(p)},\ E(p)=\sqrt{p^2+m^2}. \tag{19}$$

From renormdynamic equation

$$DG_{n+2}=\frac{\gamma}{2}(n+2)G_{n+2}, \tag{20}$$

We obtain

$$DF(p,n)=\gamma(n+2)F(p,n),$$
$$DF(p)=\gamma(<n>+2)F(p),$$
$$D<n^k(p)>=\gamma(<n^{k+1}(p)>-<n^k(p)><n(p)>),$$
$$DC_k=\gamma<n(p)>(C_{k+1}-C_k(1+k(C_2-1)))$$
$$F(p)\equiv\frac{d\sigma}{\bar{d}p}=\sum_n\frac{d\sigma_n}{\bar{d}p},\ <n^k(p)>=\frac{\sum_n n^k d\sigma_n/\bar{d}p}{\sum_n d\sigma_n/\bar{d}p}$$
$$C_k=\frac{<n^k(p)>}{<n(p)>^k} \tag{21}$$

1.3 Universal scaling relations for multi particle cross sections

From dimensional considerations, following combination of cross sections must be universal function (Koba, Nielsen, Olesen, 1972)

$$<n>\frac{\sigma_n}{\sigma}=\Psi(\frac{n}{<n>}), \tag{22}$$

similar relation for the inclusive cross sections is (Matveev, Sissakian, Slepchenko, 1975)

$$<n(p)>\frac{d\sigma_n}{\bar{d}p}/\frac{d\sigma}{\bar{d}p}=\Psi(\frac{n}{<n(p)>}) \tag{23}$$

Let us find explicit form of the universal functions from renormdynamic equations. From the definition of the moments we have,

$$C_k=\int_0^\infty dx x^k\Psi(x), \tag{24}$$

so they are independent from different parameters,

$$DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow$$
$$C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \quad (25)$$

Now we can invert momentum transform and find universal functions (Ernst, Schmitt, 1976; Slepchenko, Sissakian, N.M. 1978)

$$\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz},$$
$$C_2 = 1 + \frac{1}{c} \quad (26)$$

The value of parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n > = \frac{1}{\sqrt{c}} < n > . \quad (27)$$

1.4 Closed equation of renormdynamics for the generating function of the observables

Let us consider generating function of the topological crossections

$$F(h, g, m, \mu) = \Sigma_{n \geq 2} h^n \sigma_n,$$
$$\sigma_n = \frac{1}{n!}\frac{d^n}{dh^n} F|_{h=0}; \ \sigma = F|_{h=1};$$
$$< n > = \frac{d}{dh} \ln F|_{h=1}, ... \quad (28)$$

It is natural that for generating function we have closed renormdynamic equation (N.M. 1980)

$$(D - \gamma(\frac{h\partial}{\partial h} + 2))F = 0,$$
$$F(h, g, m, \mu) = F(\bar{h}, \bar{g}, \bar{m}, \bar{\mu}) \exp(2\int_{\mu}^{\bar{\mu}} \frac{d\mu}{\mu}\gamma(\bar{g}(\mu))),$$
$$\bar{h} = h \exp(\int_{\mu}^{\bar{\mu}} \frac{d\mu}{\mu}\gamma(\bar{g}(\mu))), \ \bar{m} = m \exp(\int_{\mu}^{\bar{\mu}} \frac{d\mu}{\mu}\eta(\bar{g}(\mu))),$$
$$\int_g^{\bar{g}} \frac{dg}{\beta(g)} = \ln\frac{\bar{\mu}}{\mu} \quad (29)$$

1.5 Explicit form of Generating function in the case KNO scaling

Let us find generating function in the case of KNO scaling. From the definition of Generating function and using topological cross section from KNO, we find

$$F(h) = \sum_n h^n \frac{\sigma}{< n >}\Psi(\frac{n}{< n >}) = \frac{\sigma}{< n >}\sum \Psi(\frac{n}{< n >})h^n$$
$$= \frac{\sigma}{< n >}\Psi(\frac{\delta}{< n >})\frac{h^2}{1-h}, \ \delta \equiv h\frac{d}{dh}, \ q^\delta f(h) = f(qh), \tag{30}$$

No we can find more concrete form of the generation function, with explisit form of KNO function,

$$(\frac{\delta}{< n >})^{c-1}\exp(-c\frac{\delta}{< n >})\frac{h^2}{1-h} = (\frac{\delta}{< n >}(\frac{q^2h^2}{1-qh})^{\frac{1}{c-1}})^{c-1}, \tag{31}$$

so

$$F(h)_{KNO} = \frac{c^c}{\Gamma(c)}\frac{\sigma}{< n >}(\frac{\delta}{< n >}(\frac{q^2h^2}{1-qh})^{\frac{1}{c-1}})^{c-1}$$
$$= \frac{c^c}{(c-1)^{c-1}\Gamma(c)}\frac{\sigma}{< n >^c}\frac{q^2h^2}{1-qh}(\frac{2-qh}{1-qh})^{c-1},$$
$$q \equiv \exp(-\frac{c}{< n >}), \ \delta \equiv \frac{hd}{dh} \tag{32}$$

Now, the question is: is $F(1) = \sigma$? Obviously not!? We expect that made approximations, and KNO scaling take place for high energy and multiplicity, so we can approximate the parameter q as

$$q = \exp(-\frac{c}{< n >}) = 1 - \frac{c}{< n >}. \tag{33}$$

Now,

$$F(h)|_{h=1} = A\sigma, \ A = \frac{1}{\Gamma(c)(c-1)^{(c-1)}} + O(1/ < n >) \tag{34}$$

$A = 1$, if, $c = 2$. This way, we defined KNO function without free parameter, and predict, that $c = 2$.

1.6 Fractal calculus

There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical and applied physics.

I would like to give a short history, background, and some applications of the of the Fractal calculus (FC). Some speculations on the fine structure constants and the prime numbers are given.

1.7 Real, p - adic and q - uantum fractal calculus

There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical and applied physics.

I would like to give a short history, background, and some applications of the Fractal calculus (FC). Some speculations on the fine structure constants and the prime numbers are given.

2 Real, p - adic and q - uantum fractal calculus

Every (good) school boy/girl knows what is

$$\frac{d^n}{dx^n} = \partial^n = (\partial)^n, \tag{35}$$

but what is its following extension

$$\frac{d^\alpha}{dx^\alpha} = \partial^\alpha \ , \quad \alpha \in \Re \ ? \tag{36}$$

2.1 Euler, ... Liouville, ... Holmgren, ...

Let us consider the integer derivatives of the monomials

$$\begin{aligned} \frac{d^n}{dx^n} x^m &= m(m-1)...(m-(n-1))x^{m-n}, \quad n \leq m, \\ &= \frac{\Gamma(m+1)}{\Gamma(m+1-n)} x^{m-n}. \end{aligned} \tag{37}$$

L.Euler (1707 - 1783) invented the following definition of the fractal derivatives,

$$\frac{d^\alpha}{dx^\alpha} x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \tag{38}$$

J.Liouville (1809-1882) takes exponent as a base function,

$$\frac{d^\alpha}{dx^\alpha}e^{ax} = a^\alpha e^{ax}. \tag{39}$$

J.H. Holmgren invented (in 1863) the following integral transformation,

$$D_{c,x}^{-\alpha}f = \frac{1}{\Gamma(\alpha)}\int\limits_c^x |x-t|^{\alpha-1}f(t)dt. \tag{40}$$

It is easy to show that

$$\begin{aligned} D_{c,x}^{-\alpha}x^m &= \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}(x^{m+\alpha}-c^{m+\alpha}), \\ D_{c,x}^{-\alpha}e^{ax} &= a^{-\alpha}(e^{ax}-e^{ac}), \end{aligned} \tag{41}$$

so, $c=0$, when $m+\alpha \geq 0$, in Holmgren's definition of the fractal calculus, corresponds to the Euler's definition, and $c=-\infty$, when $a>0$, corresponds to the Liouville's definition. Holmgren's definition of the fractal calculus reduce to the Euler's definition for finite c, and to the Liouvill's definition for $c=\infty$,

$$\begin{aligned} &D_{c,x}^{-\alpha}f = D_{0,x}^{-\alpha}f - D_{0,c}^{-\alpha}f, \\ &D_{\infty,x}^{-\alpha}f = D_{-\infty,x}^{-\alpha}f - D_{-\infty,\infty}^{-\alpha}f. \end{aligned} \tag{42}$$

We considered the following modification of the $c=0$ case (N.M. 2003,[1]),

$$\begin{aligned} D_{0,x}^{-\alpha}f &= \frac{|x|^\alpha}{\Gamma(\alpha)}\int\limits_0^1 |1-t|^{\alpha-1}f(xt)dt \\ &= \frac{|x|^\alpha}{\Gamma(\alpha)}B(\alpha,\ \partial x)f(x) = |x|^\alpha\frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)}f(x), \\ & f(xt) = t^{x\frac{d}{dx}}f(x). \end{aligned} \tag{43}$$

We can define also FC as

$$\begin{aligned} &D^\alpha f = (D^{-\alpha})^{-1}f = \frac{\Gamma(\partial x+\alpha)}{\Gamma(\partial x)}(|x|^{-\alpha}f), \\ &\partial x = \delta+1,\ \delta = x\partial \end{aligned} \tag{44}$$

For the Liouville's case,

$$D_{-\infty,x}^\alpha f = (D_{-\infty,x})^\alpha f = (\partial_x)^\alpha f, \tag{45}$$

$$\partial_x^{-\alpha}f = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt t^{\alpha-1}e^{-t\partial_x}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt t^{\alpha-1}f(x-t)$$

$$= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} dt(x-t)^{\alpha-1} f(t) = D_{-\infty,x}^{-\alpha} f. \tag{46}$$

As an example, let us consider integer derivatives, $\alpha = -n$,

$$\begin{aligned} D_{0x}^{n} f &= \frac{1}{x^n} \frac{\Gamma(\partial x)}{\Gamma(-n+\partial x)} f \\ &= x^{-n}(-n+\partial x)(-n+1+\partial x)...(-1+\partial x) f = ... \\ &= x^{-n}(-n+1+x\partial) x^{n-1} \partial^{n-1} f = \partial^n f = f^{(n)}. \end{aligned} \tag{47}$$

The integrals can be calculated as

$$D^{-n} f = (D^{-1})^n f, \tag{48}$$

where

$$\begin{aligned} D^{-1} f &= x \frac{\Gamma(\partial x)}{\Gamma(1+\partial x)} f = x \frac{1}{\partial x} f = x (\partial x)^{-1} f \\ &= (\partial)^{-1} f = \int_0^x dt f(t). \end{aligned} \tag{49}$$

As another example, let us consider Weierstrass C.T.W. (1815 - 1897) fractal function

$$f(t) = \sum_{n \geq 0} a^n e^{i(b^n t + \varphi_n)}, \quad a < 1, \quad ab > 1. \tag{50}$$

For fractals we have not integer derivatives,

$$f^{(1)}(t) = i \sum (ab)^n e^{i(b^n t + \varphi_n)} = \infty, \tag{51}$$

but the fractal derivative,

$$f^{(\alpha)}(t) = \sum (ab^{\alpha})^n e^{i(b^n t + \pi\alpha + \varphi_n)}, \tag{52}$$

when $ab^{\alpha} = a' < 1$, is another fractal (50).

2.2 p - adic fractal calculus

p-adic analog of the fractal calculus (40) ,

$$D_x^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{Q_p} |x-t|_p^{\alpha-1} f(t) dt, \tag{53}$$

where $f(x)$ is a complex function of the p-adic variable x, with p-adic $\Gamma-$ function

$$\Gamma_p(\alpha) = \int_{Q_p} dt |t|_p^{\alpha-1} \chi(t) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \tag{54}$$

was considered by V.S. Vladimirov [2].

The following modification of p-adic FC is given in [1]

$$\begin{aligned} D_x^{-\alpha} f &= \frac{|x|_p^\alpha}{\Gamma_p(\alpha)} \int_{Q_p} |1-t|_p^{\alpha-1} f(xt) dt \\ &= |x|_p^\alpha \frac{\Gamma_p(\partial |x_p|)}{\Gamma_p(\alpha + \partial |x|_p)} f(x). \end{aligned} \tag{55}$$

Last expression is applicable for functions of type $f(x) = f(|x|_p)$.

2.3 Fractal qalculus

Another important mathematical structure is q-calculus (qalculus), [3]. The basic object of this calculus is q-derivative

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} = \frac{1 - q^{x\partial}}{(1-q)x} f(x), \tag{56}$$

where either $0 < q < 1$ or $1 < q < \infty$. In the limit $q \to 1, D_q \to \partial_x$.

Now we define the fractal q-calculus,

$$\begin{aligned} & D_q^\alpha f(x) = (D_q)^\alpha f(x) \\ & = ((1-q)x)^{-\alpha} (f(x) + \\ & \sum_{n \geq 1} (-1)^n \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} f(q^n x)). \end{aligned} \tag{57}$$

For the case $\alpha = -1$, we obtain the integral operator

$$\begin{aligned} D_q^{-1} f(x) &= (1-q)x(1-q^{x\partial})^{-1} f(x) \\ &= (1-q)x \sum_{n \geq 0} f(q^n x). \end{aligned} \tag{58}$$

In the case of $1 < q < \infty$, we can give a good analytic sense to these expressions for prime numbers $q = p = 2, 3, 5, ..., 29, ..., 137, ...$ This is an algebra-analytic quantization of the q-calculus and corresponding physical models. Note also, that p-adic calculus is the natural tool for the physical models defined on the fractal spaces like Bete lattice (or Brua-Tits trees, in mathematical literature) [4].

2.4 Fractal finite - difference calculus

Usual finite difference calculus (see, e.g. [5]) is based on the following (left) derivative operator

$$D_- f(x) = \frac{f(x) - f(x-h)}{h} = (\frac{1 - e^{-h\partial}}{h}) f(x). \tag{59}$$

We define corresponding fractal calculus as

$$D_-^\alpha f(x) = (D_-)^\alpha f(x). \tag{60}$$

In the case of $\alpha = -1$, we have usual finite difference sum as regularization of the Riemann integral

$$D_-^{-1} f(x) = h(f(x) + f(x-h) + f(x-2h) + ...). \tag{61}$$

(I believe that) the fractal calculus (and geometry) are the proper language for the quantume (field) theories, and discrete versions of the fractal calculus are proper regularizations of the fractal calculus and field theories, [6].

2.5 Hypergeometric functions

One of the interesting applications of the new calculus is the following representation of the hypergeometric functions

$$
\begin{aligned}
& F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \\
& \cdot \int_0^1 dt t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha} \\
= & \frac{\Gamma(\gamma)}{\Gamma(\beta)} D_{0,1}^{\beta-\gamma}(t^{\beta-1}(1-tx)^{-\alpha}) \\
= & \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dt t^{\beta+x\partial_x-1}(1-t)^{\gamma-\beta-1}(1-x)^{-\alpha} \\
= & \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{\Gamma(\beta+x\partial_x)}{\Gamma(\gamma+x\partial_x)}(1-x)^{-\alpha} \\
= & \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} x^n \\
& = \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{\Gamma(\beta+x\partial_x)}{\Gamma(\gamma+x\partial_x)} \frac{\Gamma(\alpha+x\partial)}{\Gamma(\alpha)} e^x,
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
& (\alpha)_\delta e^x = \frac{\Gamma(\alpha+x\partial)}{\Gamma(\alpha)} e^x = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1+x\partial_x} e^{-t} e^x \\
= & \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-t} e^{tx} = \sum_{n\geq 0} \frac{(\alpha)_n}{n!} x^n \\
= & \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-(1-x)t} = (1-x)^{-\alpha}.
\end{aligned} \tag{63}
$$

So

$$
\begin{aligned}
F(\alpha,\beta;\gamma;x) &= \frac{\Gamma(\alpha+x\partial)}{\Gamma(\alpha)} \frac{\Gamma(\beta+x\partial)}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma+x\partial)} e^x \\
&= \frac{(\alpha)_\delta(\beta)_\delta}{(\gamma)_\delta} e^x
\end{aligned} \tag{64}
$$

with obvious any parameter generalization

$$
\begin{aligned}
& F(\alpha_1,...\alpha_p;\beta_1,...\beta_q;x) = \\
& \frac{\Gamma(\alpha_1+x\partial)\cdots\Gamma(\alpha_p+x\partial)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)} \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\beta_1+x\partial)\cdots\Gamma(\beta_q+x\partial)} e^x \\
= & \sum_{n\geq 0} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!} x^n \\
= & \frac{(\alpha_1)_\delta\cdots(\alpha_p)_\delta}{(\beta_1)_\delta\cdots(\beta_q)_\delta} e^x
\end{aligned} \tag{65}
$$

2.6 n-dimensional hypergeometric functions

Let us consider the following generalization of the previous integral representation of the hypergeometric function (62)

$$
\begin{aligned}
&F(\alpha,\beta_1,\beta_2;\gamma;x_1,x_2) = \\
&\frac{\Gamma(\gamma)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma-\beta_1-\beta_2)} \cdot \\
&\int d^2 t t_1^{\beta_1-1} t_2^{\beta_2-1} (1-t_1-t_2)^{\gamma-\beta_1-\beta_2-1} (1-t_1x_1-t_2x_2)^{-\alpha} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\Gamma(\beta_2)} \frac{\Gamma(\beta_1+\delta_1)\Gamma(\beta_2+\delta_2)}{\Gamma(\gamma+\delta_1+\delta_2)} (1-t_1-t_2)^{-\alpha}; \\
&(1-t_1-t_2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty duu^{\alpha-1} e^{-u(1-t_1-t_2)} \\
&= \sum_{n_1,n_2\geq 0} (\alpha)_{n_1+n_2} t_1^{n_1} t_2^{n_2} \\
&= \frac{(\alpha)_{\delta_1+\delta_2}}{(\alpha)_{\delta_1}(\alpha)_{\delta_2}} (1-t_1)^{-\alpha}(1-t_2)^{-\alpha} \\
&= (\alpha)_{\delta_1+\delta_2} e^{t_1+t_2}
\end{aligned} \tag{66}
$$

So

$$
\begin{aligned}
&F(\alpha,\beta_1,\beta_2;\gamma;x_1,x_2) = \\
&\frac{(\alpha)_{\delta_1+\delta_2}}{(\alpha)_{\delta_1}(\alpha)_{\delta_2}} \frac{(\gamma)_{\delta_1}(\gamma)_{\delta_2}}{(\gamma)_{\delta_1+\delta_2}} F(\alpha,\beta_1;\gamma,x_1)F(\alpha,\beta_2;\gamma;x_2) \\
&= \sum_{n_1,n_2} \frac{(\alpha)_{n_1+n_2}(\beta_1)_{n_1}(\beta_2)_{n_2}}{(\gamma)_{n_1+n_2} n_1! n_2!} x_1^{n_1} x_2^{n_2}
\end{aligned} \tag{67}
$$

where

$$
1 \geq t_1+t_2 \geq 0,\ \ t_1,t_2 \geq 0,\ \ \delta \equiv x\partial \tag{68}
$$

It is obvious n-dimensional generalization of these formulas.

2.7 Lauricella Hypergeometric functions (LFs)

For LFs (see, e.g. [Miller,1977]), we find the following formulas

$$
\begin{aligned}
&F_A(a;b_1,...,b_n;c_1,...,c_n;z_1,...,z_n) \\
&= \frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n} \\
&= \frac{(a)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n) \\
&= T^{-1}(a)F^n \\
&= \Sigma_{m\geq 0} \frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}} \frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},
\end{aligned}
$$

$$
\begin{aligned}
&|z_1| + ... + |z_n| < 1;\\
&F_B(a_1, ..., a_n; b_1, ..., b_n; c; z_1, ..., z_n)\\
&= \frac{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ e^{z_1+...+z_n}\\
&= \frac{(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ F(a_1, b_1; c_1; z_1)...F(a_n, b_n; c_n; z_n)\\
&= T(c)F^n\\
&= \Sigma_{m\geq 0}\frac{(a_1)_{m_1}...(a_n)_{m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\\
&|z_1| < 1, ..., |z_n| < 1;\\
&F_C(a; b; c_1, ..., c_n; z_1, ..., z_n)\\
&= \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n}\\
&= \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}\ F(a_1, b_1; c_1; z_1)\\
&...F(a_n, b_n; c_n; z_n)\\
&= T^{-1}(a)T^{-1}(b)F^n = T^{-1}(b)F_A\\
&= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b)_{m_1+...+m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\\
&|z_1|^{1/2} + ... + |z_n|^{1/2} < 1;\\
&F_D(a; b_1, ..., b_n; c; z_1, ..., z_n)\\
&= \frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...\delta_n}}\ e^{z_1+...+z_n}\\
&= \frac{(a)_{\delta_1+...+\delta_n}(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(c)_{\delta_1+...\delta_n}}\ F(a_1, b_1; c_1; z_1)\\
&...F(a_n, b_n; c_n; z_n)\\
&= T^{-1}(a)T(c)F^n = T(c)F_A = T^{-1}(a)F_B\\
&= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\\
&|z_1| < 1, ..., |z_n| < 1.
\end{aligned}
\tag{69}
$$

It is interesting problem to find integral representations of these hypergeometric functions. As a first step in this direction, note that previous example of two dimensional hypergeometric function was LFs - F_D for $n = 2$, so for general n, we have

$$
\begin{aligned}
&F_D(a, b_1, ...b_n; c; x_1, ..., x_n) =\\
&\frac{\Gamma(c)}{\Gamma(b_1)...\Gamma(b_n)\Gamma(c - b_1 - ... - b_n)}\cdot\\
&\int d^n t t_1^{b_1-1}...t_n^{b_n-1}(1 - t_1 - ... - t_n)^{c-b_1-...-b_n-1}\cdot\\
&(1 - t_1x_1 - ... - t_nx_n)^{-a}
\end{aligned}
\tag{70}
$$

Next step is to find operators that transform one LF to another and express them in terms of integral transformation of the FC. We have

$$F_N = T_{ND}F_D, \;\; N = A, B, C,$$
$$T_{AD} = T^{-1}(c) = \frac{(c)_{\delta_1+...+\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}$$
$$T_{BD} = T(a) = \frac{(a_1)_{\delta_1}...(a_n)_{\delta_n}}{(a)_{\delta_1+...\delta_n}}$$
$$T_{CD} = T^{-1}(c)T^{-1}(b) \tag{71}$$

These operators reduce to the following one

$$(a)_\delta f(x) = \frac{\Gamma(a+\delta)}{\Gamma(a)}f(x)$$
$$= \frac{1}{\Gamma(a)}\int_0^\infty dt t^{a-1}e^{-t}f(tx);$$
$$(a)_\delta^{-1} = \frac{\Gamma(a)}{\Gamma(\delta+1)}x^{1-a}D^{1-a};$$
$$(a)^{-1}_{\delta_1+...+\delta_n} = (a+\delta_1+...+\delta_{n-1})^{-1}_{\delta_n}...(a)^{-1}_{\delta_1} \tag{72}$$

Monomials

$$\Psi(x_1, ..., x_n) = x_1^{m_1}...x_n^{m_n}, \tag{73}$$

are eigenfunctions of the operator $T(a)$

$$T(a)\Psi(x_1, ..., x_n) =$$
$$\frac{\Gamma(a)}{\Gamma(a_1)...\Gamma(a_n)}\frac{\Gamma(a_1+m_1)...\Gamma(a_n+m_n)}{\Gamma(a+m_1+...+m_n)}\Psi \tag{74}$$

For generalized Euler B-function we have

$$B_n(a_1, ..., a_n) = \frac{\Gamma(a_1)...\Gamma(a_n)}{\Gamma(a_1+...+a_n)}$$
$$= \int_0^1 dt_1...dt_n t_1^{a_1-1}...t_n^{a_n-1}\delta(t_1+...+t_n-1), \tag{75}$$

so, for analytic functions $f(x_1, ..., x_n)$, when $a = a_1 + ... + a_n$,

$$T(a)f = \frac{1}{B_n(a_1, ..., a_n)}\int_0^1 dt^n t_1^{a_1-1}...t_n^{a_n-1}$$
$$\cdot\delta(t_1+...+t_n-1)f(t_1x_1, ..., t_nx_n). \tag{76}$$

Using this B_n- bien transformation, we obtain an integral formula for F_B

$$F_B(a_1, ..., a_n; b_1, ..., b_n; c; z_1, ..., z_n) = \frac{\Gamma(a_1+a_2+...+a_n)}{\Gamma(a_1)\Gamma(a_2)...\Gamma(a_n)}$$

$$
\begin{aligned}
&\cdot \int_0^1 d^n t t_1^{a_1-1} ... t_n^{a_n-1} \delta(t_1+t_2+...+t_n-1) \\
&\cdot F_D(a_1+...+a_n; b_1,...,b_n; c; t_1x_1,...,t_nx_n) \\
&= \frac{\Gamma(a_1+...+a_n)\Gamma(c)}{\Gamma(a_1)...\Gamma(a_n)\Gamma(b_1)...\Gamma(b_n)\Gamma(c-b_1-...-b_n)} \\
&\cdot \int_0^1 dt^n t_1^{a_1-1} ... t_n^{a_n-1} \delta(t_1+t_2+...+t_n-1) \\
&\cdot \int_0^1 d^n y y_1^{b_1-1} ... y_n^{b_n-1} (1-y_1-...-y_n)^{c-b_1-...-b_n} \\
&\cdot (1-t_1x_1y_1-...-t_nx_ny_n)^{-(a_1+...+a_n)} \qquad (77)
\end{aligned}
$$

2.8 Riemann ζ - function

Let us consider Riemann ζ - function,

$$
\zeta = \sum_{n\geq 1} n^{-s} \qquad (78)
$$

Note, that

$$
x\partial x^n = nx^n, \ (x\partial)^{-s} x^n = n^{-s} x^n, \ n \geq 1, \qquad (79)
$$

so

$$
\begin{aligned}
&(x\partial)^{-s} \sum_{n\geq 1} x^n = \sum_{n\geq 1} \frac{x^n}{n^s} \equiv \zeta(s,x) = (H)^{-s} \frac{x}{1-x}, \\
&H = x\partial \qquad (80)
\end{aligned}
$$

than

$$
H^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tH}, \ q^H f(x) = f(qx), \qquad (81)
$$

so

$$
\begin{aligned}
\zeta(s) = \zeta(s,1) &= \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-t}x}{1-e^{-t}x}|_{x=1} \\
&= \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{1}{e^t-1}. \qquad (82)
\end{aligned}
$$

Now we consider following finite sum

$$
\begin{aligned}
Z_N(\alpha) &\equiv \sum_{n=1}^N e^{\alpha n} = \sum_{n=1}^N q^n = \frac{q(1-q^N)}{1-q} \\
&= q[N]_q, \quad q = e^\alpha. \qquad (83)
\end{aligned}
$$

Using the Liouville's case of the fractal calculus, we obtain a compact representation of the following Riemann ζ - function's motivated finite sums

$$
\zeta_N(s) = \sum_{n=1}^N n^{-s} = \frac{d^\beta}{d\alpha^\beta} Z_N(\alpha)|_{\alpha\to 0}, \quad \beta = -s \qquad (84)
$$

3 Field theory applications of FC

Let us consider the following action

$$S = \frac{1}{2}\int\limits_{Q_v} dx\Phi(x)D_x^\alpha\Phi,\ v = 1, 2, 3, 5, ... \qquad (85)$$

Q_1 is real number field, Q_p, p - prime, are p-adic number fields. In the momentum representation

$$S = \frac{1}{2}\int\limits_{Q_v} du\tilde{\Phi}(-u)|u|_v^\alpha\tilde{\Phi}(u), \qquad (86)$$

$$\begin{aligned}\Phi(x) &= \int\limits_{Q_v} du\chi_v(ux)\tilde{\Phi}(u),\\ D^{-\alpha}\chi_v(ux) &= |u|_v^{-\alpha}\chi_v(ux).\end{aligned} \qquad (87)$$

The statistical sum of the corresponding quantum theory is

$$\begin{aligned}Z_v &= \int d\Phi e^{-\frac{1}{2}\int\Phi D^\alpha\Phi} = det^{-1/2}D^\alpha\\ &= (\prod_u |u|_v)^{-\alpha/2}.\end{aligned} \qquad (88)$$

3.1 String theory applications

For (symmetrized, 4-tachyon) Veneziano amplitude we have (see, e.g. [?])

$$\begin{aligned}B_s(\alpha,\beta) &= B(\alpha,\beta) + B(\beta,\gamma) + B(\gamma,\alpha)\\ &= \int_{-\infty}^{\infty} dx|1-x|^{\alpha-1}|x|^{\beta-1},\\ &\quad \alpha+\beta+\gamma = 1\end{aligned} \qquad (89)$$

For the p-adic Veneziano amplitude we take

$$B_p(\alpha,\beta) = \int_{Q_p} dx|1-x|_p^{\alpha-1}|x|_p^{\beta-1} \qquad (90)$$

Now we obtain the N-tachyon amplitude using fractal calculus. We consider the dynamics of particle given by multicomponent generalization of the action (85), $\Phi \to x^\mu$. For the closed trajectory of the particle passing through N points, we have

$$A(x_1, x_2, ..., x_N) = \int dt \int dt_1 ... \int dt_N \delta(t - \Sigma t_n)$$

$$v(x_1,t_1;x_2,t_2)v(x_2,t_2;x_3,t_3)...v(x_N,t_N;x_1,t_1)$$
$$= \int dx(t)\Pi(\int dt_n\delta(x^\mu(t_n)-x_n^\mu))exp(-S[x(t)])$$
$$= \int \Pi(dk_n^\mu\chi(k_nx_n))\tilde{A}(k)exp(-S), \qquad (91)$$

where

$$\tilde{A}(k) = \int dxV(k_1)V(k_2)...V(k_N)exp(-S),$$
$$V(k_n) = \int dt\chi(-k_nx(t)) \qquad (92)$$

-vertex function.

Motion equation

$$D^\alpha x^\mu - i\Sigma k_n^\mu\delta(t-t_n) = 0, \qquad (93)$$

in the momentum representation

$$|u|^\alpha\tilde{x}^\mu(u) - i\Sigma_n k_n^\mu\chi(-ut_n) = 0 \qquad (94)$$

have the solution

$$\tilde{x}^\mu(u) = i\Sigma k_n^\mu\frac{\chi(-ut_n)}{|u|^\alpha}, \quad u\neq 0, \qquad (95)$$

the constraint

$$\Sigma_n k_n = 0, \qquad (96)$$

and the zero mod $\tilde{x}_n^\mu(0)$, which is arbitrary. Integration in (91) with respect to this zero mod gives the constraint (96). On the solution of the equation (93)

$$x^\mu(t) = iD_t^{-\alpha}\Sigma_n k_n^\mu\delta(t-t_n)$$
$$= \frac{i}{\Gamma(\alpha)}\Sigma_n k_n^\mu|t-t_n|^{\alpha-1}, \qquad (97)$$

the action (85) takes value

$$S = -\frac{1}{\Gamma(\alpha)}\Sigma_{n<m}k_nk_m|t_n-t_m|^{\alpha-1},$$
$$\tilde{A}(k) = \int \Pi_{n=1}^N dt_n exp(-S) \qquad (98)$$

In the limit, $\alpha\to 1$, for p-adic case we obtain

$$x^\mu(t) = -i\frac{p-1}{p\ lnp}\Sigma_n k_n^\mu ln|t-t_n|,$$
$$S[x(t)] = \frac{p-1}{p\ lnp}\Sigma_{n<m}k_nk_m\ ln|t_n-t_m|,$$

$$\tilde{A}(k) = \int \Pi_{n=1}^{N} dt_n \Pi_{n<m} |t_n - t_m|^{\frac{p-1}{p\ lnp} k_n k_m}. \tag{99}$$

Now in the limit $p \to 1$ we obtain the proper expressions of the the real case

$$\begin{aligned}
&x^\mu(t) = -i\Sigma_n k_n^\mu ln|t - t_n|, \\
&S[x(t)] = \Sigma_{n<m} k_n k_m \ ln|t_n - t_m|, \\
&\tilde{A}(k) = \int \Pi_{n=1}^{N} dt_n \Pi_{n<m} |t_n - t_m|^{k_n k_m}.
\end{aligned} \tag{100}$$

Note that, by fractal calculus and vector generalization of the model (85), fundamental string amplitudes were obtained in N.M. 1988, [7].

3.2 Field theory and condensed state physics applications

In various applications of the renormgroup (RG) method [8, 9], for fractal space-time with dimension $d = n - 2\varepsilon$, the RG $\beta-$function is

$$\beta(\alpha, \varepsilon) = \beta(\alpha) - \varepsilon\alpha. \tag{101}$$

For any given α (and corresponding scale a), there is the value of ε (fractal dimension), with

$$\beta(\alpha, \varepsilon) = 0, \quad \varepsilon = \beta(\alpha)/\alpha, \quad d = n - 2\beta(\alpha)/\alpha. \tag{102}$$

In the region of scales with linear $\beta(\alpha)$ function, we have self similar fractal space-time, or corresponding physical fields live on the fractal subspace of the space-time manifold.

One loop perturbative value of $\beta(\alpha)$ in quantum electrodynamics [10] is

$$\beta(\alpha) = \beta_1 \alpha^2, \quad \beta_1 = \frac{1}{3\pi}, \tag{103}$$

so

$$\varepsilon = \frac{1}{3\pi}\frac{1}{137} = 77.5 \times 10^{-5}, \quad d = 3.998 < 4. \tag{104}$$

For one loop perturbative QCD,

$$\beta_1 = -\frac{9}{4\pi}, \tag{105}$$

and if we take, e.g., α=0.1,

$$d = 4.001 > 4. \tag{106}$$

Fractal dimension of the subspace occupied by a physical field $\varphi(x)$ can be measured by correlation function

$$< \varphi(x)\varphi(y) > \sim |x - y|^{2-d}. \tag{107}$$

For macroscopic phenomena (e.g. Ball Lightning) we can consider mathematical models of fractals in 4 dimensional space-time manifolds. For hadronic phenomena (like Fair Ball) we should consider more than 4 dimensional manifolds. One of the possible mechanisms of the sonoluminescence (see e.g. [12]) can be the dimensional phase transition with decreasing scale of the babbles from the phase with $d < 4$, to the phase with $d > 4$.

3.3 Standard Model of Fundamental Interections and Beyond

After the unification of electro-magnetic and weak interactions in Gleshow-Weinberg-Salam model with gauge group $U(1) \times SU(2)$, [11] a next step is a unification of electro-magnetic, weak and strong interactions (with gauge group $SU(3)$) in a Grand Unification Theory (GUT), with a simple group G with one coupling constant g, [11]. A bridge between the electro-magnetic scale 100 GeV and grand unification scale $10^{16} GeV$ (Planck scale $10^{19} GeV$) can be provided by supersymmetry [13]. At the scale of unification M, the $U(1), SU(2)$ and $SU(3)$ coupling constants are equal to g. In Supersymmetric Generalization of the Standard Model of Fundamental Interactions [15],

$$\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1.0 \tag{108}$$

Note that, in this interval, the only prime number is 29. Our proposal is that the (ultraviolet asymptotic) value α_{uv}^{-1} is 29.0.., which corresponds to the (infrared asymptotic) value of the electro-magnetic fine structure constant $\alpha^{-1} = 137.0...$

3.4 Low energy unification

According to the resent (non-) perturbative calculations in QCD, strong coupling constant rise from perturbative to the maximum value of order 2 and then decrease to the value of order 1 at the scales reached in the lattice calculations. In QED with magnetic monopoles, we have Dirac quantization of the electric -e, and magnetic -g, charges, $eg = 1$, so, at the self-dual point, $e = g = 1$,

$$\alpha_e = \alpha_g = \frac{1}{4\pi} \simeq 0.1 \tag{109}$$

We propose, that at the self-dual point, and corresponding energy scale, we have low energy unification of the strong and electro-magnetic coupling constants. It is very interesting to test this possibility with nonperturbative calculations.

Besides that, nonperturbative $\beta-$functions define corresponding fractal dimensions according to equation (102).

3.5 Some observations on the coupling constants values

For the dual-symmetric (inverse) value of the electro-magnetic(-strong) interaction fine structure constants low energy unification, we have

$$\begin{aligned} &\alpha_D^{-1} = 4\pi = 12.57 \simeq 13 \\ &= 2^2 + 3^2 = | \pm 2 \pm 3i|^2 = | \pm 3 \pm 2i|^2 \end{aligned} \tag{110}$$

For the corresponding higher energy unification value, we have

$$\begin{aligned} &\alpha_{GUT}^{-1} = 29 \\ &= 2^2 + 3^2 + 4^2 = | \pm 2i \pm 3j \pm 4k|^2 = ... \end{aligned} \tag{111}$$

where i, j, k are quaternionik unit vectors.

For the electromagnetic fine structure coupling constant, we have

$$\alpha^{-1} = 137.0... \tag{112}$$

For the corresponding twin prime number, we have

$$\begin{aligned} &139 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \\ &= | \pm 2i \pm 3j \pm 4k \pm 5l \pm 6n \pm 7m|^2 \end{aligned} \tag{113}$$

where i, j, k, l, n, m are oktvian unit vectors.

3.6 p-adic deformations of classical theories

The electromagnetic fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \tag{114}$$

contains (unifies) three fundamental quantities, electron charge e, Plank's constant $\hbar$, and light velocity c. We usually take units as $\hbar = 1,\ c = 1$, but we can take as well, $e = 1,\ c = 1,\ \hbar = 137,0...$ or $e = 1,\ \hbar = 1,\ c = 137.0...$ In this system of units, $\hbar = 137.0...$(or $c = 137.0...$) and quantum perturbation theory may be p-adic convergent(, correspondingly, relativistic theory, considered as an expansion in powers of $c = 137$, is p-adic theory). In this mathematical sense, the quantum theory is p-adic (phase or part of the unified) theory. Real phase or part is usual classical (relativistic) theory. The same consideration on the scale (level) of GUT, in units, $g = 1,\ c = 1$, gives a p-adic convergent quantum theory with $\hbar = 29$. We can consider relativistic and quantum theories as p-adic deformations of classical theory.

4 Solution of the Schrödinger equation in terms of the Feynman continual integrals

A quantum system can be described by corresponding Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi >= \hat{H}|\psi >, \tag{115}$$

where $|\psi >= |\psi(t) >$ is the state vector from the state Hilbert space and

$$\hat{H} = H(\hat{p}, \hat{x}) \tag{116}$$

is an operator-Hamiltonian. In the case of nonrelativistic particle the operator is

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{117}$$

the fundamental bracket is

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{118}$$

The configuration space form of Eq. (115) is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t), \tag{119}$$

where

$$\psi(x,t) =< x|\psi(t) > \tag{120}$$

and Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x). \tag{121}$$

In the momentum space, we have

$$i\hbar \frac{\partial \psi(p,t)}{\partial t} = \hat{H}\psi(p,t), \tag{122}$$

where

$$\psi(p,t) =< p|\psi(t) > \tag{123}$$

and Hamiltonian is

$$\hat{H} = \frac{p^2}{2m} + V(i\frac{d}{dp}). \tag{124}$$

With proper normalization,

$$
\begin{aligned}
&1 =< \psi(t)|\psi(t) >=< \psi(t)| \int dx|x >< x|\psi(t) > \\
&= \int dx\psi^*(x,t)\psi(x,t) = \int dx\ \rho(x,t) = 1 \\
&1 =< \psi(t)| \int dp|p >< p|\psi(t) > \\
&= \int dp\ \psi^*(p,t)\psi(p,t) = \int dp\ \rho(p,t) >= 1,
\end{aligned}
\tag{125}
$$

where

$$\rho(x,t) = \psi^*(x,t)\psi(x,t) \tag{126}$$

and

$$\rho(p,t) = \psi^*(p,t)\psi(p,t) \tag{127}$$

are the probability densities of finding the particle at the point x and p correspondingly.

The formal solution of Eq. (115) is

$$|\psi(t) >= U(t)|\psi_0 >, \tag{128}$$

where

$$U(t) = exp\ (-\frac{i}{\hbar}t\hat{H}). \tag{129}$$

The main steps made (maid in Qt) are the following:

$$U(t) = (U^{1/N})^N = (U_T U_V)^N + O(1/N), \tag{130}$$

where

$$
\begin{aligned}
U^{1/N} &= U(\tau) = exp\ (-\theta\hat{H}) = exp\ (-\theta\hat{T})exp\ (-\theta\hat{V}) + O(1/N^2) \\
&= U_T U_V + O(1/N^2), \quad \theta = \frac{i}{\hbar}\tau, \quad \tau = \frac{t}{N}.
\end{aligned}
\tag{131}
$$

Then, for a corresponding matrix element we have (see Appendix)

$$< x_{n+1}|U(\tau)|x_n >\sim exp\ (\theta(\frac{m}{2}(\frac{x_{n+1}-x_n}{\tau})^2 - V(x_n))) + O(1/N^2) \tag{132}$$

and

$$< x_{out}|U(t)|x_{in} >= \int dx_0 dx_1 dx_2 ... dx_N < x_{out}|x_N >< x_N|U(\tau)|x_{N-1} > .$$

$$
\begin{aligned}
&. < x_{N-1}|U(\tau)|x_{N-2} > ... < x_1|U(\tau)|x_0 >< x_0|x_{in} > \\
&\sim \int dx_1 dx_2 ... dx_{N-1} exp\,(\frac{i}{\hbar}\tau \sum_{n=0}^{N-1}(\frac{m}{2}(\frac{x_{n+1}-x_n}{\tau})^2 - V(x_n))) + O(1/N), \\
&\ x_0 = x_{in}, \quad x_N = x_{out}.
\end{aligned} \tag{133}
$$

This finite dimensional integral representation of the matrix element is in the ground of the functional (continual) integral formulation of the quantum theory.

5 Density of states description of the statistical systems

If we take the trace of the transition amplitude

$$< out|exp(-\frac{i}{\hbar}\hat{H})|in >, \tag{134}$$

$$\begin{aligned}
&\sum_{s_n} < s_n|U(t)|s_n >= \sum_{n,m} < s_n|E_m >< E_m|s_n > exp(-\frac{i}{\hbar}E_m) \\
&= \sum_{E_m} N(E_m)e^{-\beta E_m} = \int_0^{\infty} dE\rho(E)e^{-\beta E} = \rho(E_c)e^{-\beta E_c} = e^{-\beta F} = Z(\beta), \\
&F = E_c - TS, \quad S = ln\rho(E_c), \quad \partial S/\partial E = T^{-1} = \beta
\end{aligned} \tag{135}$$

For density of states we have

$$\begin{aligned}
\rho(E) &= \sum_{E_m} N(E_M)\delta(E - E_M) = \sum_m \delta(E - E_m) = tr(E - \hat{H}) \\
&= \frac{1}{2\pi i\hbar}\int_{-\infty}^{+\infty} dte^{\frac{i}{\hbar}tE} tr(e^{-\frac{i}{\hbar}t\hat{H}})
\end{aligned} \tag{136}$$

Note, that in this type of expressions, we clearly see the origin of the energy - time uncertainty relation, $\Delta E\Delta t \sim 2\pi\hbar$.

If there are besides of the energy other integrals of motion, $N_1, ..., N_k$,

$$\rho(E, N_1, ..., N_k) = tr(\delta(E - \hat{H})\ \delta(N_1 - \hat{N}_1)\ ...\ \delta(N_k - \hat{N}_k)) \tag{137}$$

5.1 High energy, Barion number, Temperature,..., approximation

Let us take the hamiltonian form of the functional integral

$$tre^{-\frac{i}{\hbar}t\hat{H}} = \int_{x(0)=x(t)} \frac{dxdp}{2\pi\hbar} e^{\frac{i}{\hbar}\int_0^t (p\dot{x} - H(p,x))} \tag{138}$$

For higher energy, small time and static approximation we have

$$\begin{aligned}
\rho(E) &= \frac{1}{2\pi i\hbar}\int_{-\infty}^{+\infty} dte^{\frac{i}{\hbar}tE} tr(e^{-\frac{i}{\hbar}t\hat{H}}) \\
&= \frac{1}{2\pi i\hbar}\int_{-\infty}^{+\infty} dte^{\frac{i}{\hbar}tE}\int \frac{dxdp}{2\pi\hbar} e^{-\frac{i}{\hbar}tH(p,x)} = \int \frac{dxdp}{2\pi\hbar}\delta(E - H(p,x)) \\
&= \int \frac{dx}{2\pi\hbar}\frac{2m}{\sqrt{2m(E - V(x))}}
\end{aligned} \tag{139}$$

The number of states

$$N = \int dE\rho(E) = \int \frac{dxdp}{2\pi\hbar} \tag{140}$$

Exercise: Show that for harmonic oscillator,

$$
\begin{aligned}
H(p,x) &= \frac{p^2}{2m} + \frac{kx^2}{2}, \\
\rho(E) &= \sqrt{\frac{m}{k}} = \omega^{-1}; \\
N(E) &= \int_0^E dE\rho(E) = \frac{E}{\hbar\omega}
\end{aligned}
\tag{141}
$$

5.2 Exactly solvable model with maximal temperature

As we have seen, at high energy and temperature, we have classical statistical description

$$
Z(\beta) = \int \frac{d^Dxd^Dp}{(2\pi)^D} e^{-\beta H(p,x)} = (\frac{m}{2\pi\beta})^{D/2} \int d^Dx e^{-\beta V(x)} \tag{142}
$$

For potential of the form

$$
\begin{aligned}
V(x) &= 0, \ 0 \le |x| \le a, \\
&= b \cdot ln\frac{|x|}{a}, \ |x| > a,
\end{aligned}
\tag{143}
$$

we have

$$
Z(\beta) = (\frac{m}{2\pi\beta})^{D/2} \frac{\Omega_D}{D} \frac{a^D\beta b}{\beta b - D} \tag{144}
$$

So the temperature of the system is restricted by condition

$$
T = \beta^{-1} < b/D \equiv T_H \tag{145}
$$

Statistical energy of the system is

$$
\begin{aligned}
E = E_c &= -\frac{\partial lnZ(\beta)}{\partial\beta} \\
&= \frac{D/2-1}{\beta} + \frac{1}{\beta - D/b}
\end{aligned}
\tag{146}
$$

The case when $E = 0$ corresponds to a self supporting, non expending, no collapsing state of the system at the temperature

$$
\beta_N = \frac{2}{b}(\frac{D}{2} - 1) = \frac{D}{b} - \frac{2}{b} < \beta_H \tag{147}
$$

Note, that normal temperature is positive, corresponds to the stable state, for $D > 2$, and decrees with dimension, from infinity to zero. So it is easy (easier)to crate higher dimensional normal states. The volume of the system is

$$V_c = \frac{\int dxdpVe^{-\beta H}}{\int dxdpe^{-\beta H}} = \frac{\Omega_D}{D}\frac{\int_0^a dxx^{2D-1} + \int_a^\infty dxx^{2D-1}(\frac{a}{x})^{b\beta}}{Z(\beta)}$$
$$= \frac{\Omega_D}{D}a^D\frac{b\beta - D}{2(b\beta - 2D)},\ V_N = \frac{V(a)}{D+2} \tag{148}$$

The volume has positive value, for $T > T_2 = b/D$ or $T < T_1 = b/2D$, $T_N > T_2$.

This Normal state can not be reached by rising continually temperature of the system. In the corresponding realistic models, e.g. for a heavy nucleus, we can obtain such a state in high energy collisions. To this kind of physical states may be ascribed ball lightning. Probably famous Tunguska Event was a Big ball lightning explosion.

5.3 n-field or toy standard model of condensed state and particle physics

One of the most popular nonlinear model of the higher energy physics and condensed state physics the n-field model, is given by the following lagrangian

$$L = \frac{1}{2}\partial_\mu n^a \partial^\mu n^a, \; n^2 = n_1^2 + n_2^2 + ... + n_N^2 = 1,$$
$$a = 1, 2, ..., N; \; \mu = 0, 1, ..., D. \tag{149}$$

The interaction parameter, coupling constant, can be invented in this way. First we consider a constraint with this parameter,

$$n^2 = \frac{1}{\alpha}, \tag{150}$$

than invent this constraint in the lagrangian by lagrange multiplier field, and make scaling transformation of n-field,

$$L = \frac{1}{2}\partial_\mu n^a \partial^\mu n^a + \lambda(n^2 - \frac{1}{\alpha}) \Rightarrow \frac{1}{\alpha}(\frac{1}{2}\partial_\mu n^a \partial^\mu n^a + \lambda(n^2 - 1)). \tag{151}$$

Now we can escribe the distinguished values of the fine structure constants

$$\alpha^{-1} = 13; \; 29; \; 139, \tag{152}$$

to the $N = 2, 3, 6$ and corresponding values of n-fields

$$13 = 2^2 + 3^2, \; 29 = 2^2 + 3^2 + 4^2, \; 139 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \tag{153}$$

It maybe better also to consider lattice dynamics with the following lagrangian and corresponding lattice action

$$L = \frac{1}{2}(n(k+\hat{\mu}) - n(k)))^2 + \lambda(n^2 - \frac{1}{\alpha})$$
$$\Rightarrow -\sum_{k,\mu} n(k+\hat{\mu})n(k) + \lambda(k)(n(k)^2 - \frac{1}{\alpha}) \tag{154}$$

We can further extend the constraint therm in the action inventing three different lagrange multipliers

$$S = -\sum_{k,\mu} n(k+\hat{\mu})n(k) + \lambda_1(k)(n_1(k)^2 + n_2(k)^2 - 13))$$
$$+\lambda_2(k)(n_3(k)^2 - 4^2)$$
$$+\lambda_3(k)(n_4(k)^2 + n_5(k)^2 + n_6(k)^2 - 10(10+1)))$$
$$\Rightarrow -\frac{1}{\alpha_1}(\sum_{k,\mu,a=1,2} n_a(k+\hat{\mu})n_a(k) + \lambda_1(k)(n_1(k)^2 + n_2(k)^2 - 1))$$

$$
\begin{aligned}
&-\frac{1}{\alpha_2}(\sum_{k,\mu} n_4(k+\hat{\mu})n_4(k) + \lambda_2(k)(n_4(k)^2-1)) \\
&-\frac{1}{\alpha_3}(\sum_{k,\mu,a=4,5,6} n_a(k+\hat{\mu})n_a(k) \\
&+\lambda_3(k)(n_4(k)^2+n_5(k)^2+n_6(k)^2-1)) \qquad (155)
\end{aligned}
$$

In this form, the action contains also other possibilities of discrete dynamics (phase) including

$$
n_1^2 = {\alpha_1}^{-1} = 4,\ n_2^2+n_3^2 = {\alpha_2}^{-1} = 25 \qquad (156)
$$

In some GUT models, this value of α_2 is preferable as unification point

5.4 Soliton solutions

Let us consider the following deformation of the $N = 3$ n-field model (Leese, 1991; Bogolubsky; N.M. 1997)

$$L = \partial_\mu n^a \partial^\mu n^a - m^2(1 - n_3^2),\ n^2 = n_1^2 + n_2^2 + n_3^2 = 1,$$
$$a = 1, 2, 3;\ \mu = 0, 1, ..., D. \quad (157)$$

It is convenient to introduce new field variable

$$z = \frac{n_1 + in_2}{1 + n_3},$$
$$n_1 = \frac{\bar{z} + z}{1 + |z|^2},\ n_2 = \frac{i(\bar{z} - z)}{1 + |z|^2},\ n_3 = \frac{1 - |z|^2}{1 + |z|^2}. \quad (158)$$

The n-field given by this expression is on the unit sphere. The new complex variable z is not constrained any more. The lagrangian in z variable takes form

$$L = 4\frac{|\partial_\mu z|^2 - m^2|z|^2}{(1 + |z|^2)}. \quad (159)$$

Corresponding motion equation is

$$(1 + |z|^2)z_\mu{}^\mu - 2\bar{z}z_\mu z^\mu + m^2(1 - |z|^2)z = 0. \quad (160)$$

For radially symmetric solutions the motion equation takes the following form

$$(1 + |z|^2)(\partial_\tau^2 - \partial_\eta^2)z - 2\bar{z}((\partial_\tau z)^2 - (\partial_\eta z)^2) + m^2 r^{2(D-1)}(1 - |z|^2)z = 0 \quad (161)$$

where

$$\tau = \frac{t}{r^{D-1}},\ \eta = \frac{r^{2-D} - r_0^{2-D}}{2 - D},\ D \neq 2$$
$$\tau = \frac{t}{r},\ \eta = ln\frac{r}{r_0},\ D = 2. \quad (162)$$

If we consider variable mass parameter, so that

$$m(r)r^{D-1} = m_0 r_0^{D-1} = const, \quad (163)$$

dependence of the motion equation on the dimension of the space becomes implicit through the variables τ and η. So, the D-dimensional problem reduces, for example, to one-dimensional one. Now we have the following particle-like solution of our motion equation

$$z(\tau, \eta) = e^{i\omega\tau} z(\eta),$$
$$z(\eta) = e^{\pm b\eta},\ b = \sqrt{m_0^2 r_0^{2(D-1)} - \omega^2}. \quad (164)$$

In the case of $D = 2$,

$$z(t,r) = e^{i\frac{\omega}{r}t}(\frac{r}{r_0})^{\pm b}; \tag{165}$$

when $D = 3$,

$$z(t,r) = e^{i\frac{\omega}{r^2}t}e^{\pm b(\frac{1}{r_0}-\frac{1}{r})} \tag{166}$$

We can also consider $0 < D < 1$,

$$z(t,r) = e^{i\omega r^{1-D}t}e^{\pm b\eta}. \tag{167}$$

For $D = 1$, $\eta = x \in (-\infty, +\infty)$.

Note that, near conformal dimension, $D = 2 + \epsilon$, we can interpret variable mass as renormalized one,

$$m(r) = m_0(\frac{r}{r_0})^{\epsilon} = zm_0,\ z = 1 + \epsilon ln(\frac{r}{r_0}) + ... \tag{168}$$

5.5 Sphaleron solutions

Let us consider another modification of the n-field model

$$L = \partial_\mu n^a \partial^\mu n^a - 2m^2(1-n_3),\ n^2 = n_1^2 + n_2^2 + n_3^2 = 1,$$
$$a = 1,2,3;\ \mu = 0,1,...,D. \tag{169}$$

The lagrangian and motion equation, in z-field formulation are

$$L = 4\frac{|\partial_\mu z|^2 - m^2(1+|z|^2)|z|^2}{(1+|z|^2)},$$
$$(1+|z|^2)z_\mu{}^\mu - 2\bar{z}z_\mu z^\mu - m^2(1+|z|^2)z = 0. \tag{170}$$

For radially symmetric solutions the motion equation takes form

$$(1+|z|^2)(\partial_\tau^2 - \partial_\eta^2)z) - 2\bar{z}((\partial_\tau z)^2 - (\partial_\eta z)^2) - m^2 r^{2(D-1)}(1+|z|^2)z = 0. \tag{171}$$

Motion equation, for variable mass of the type

$$mr^{D-1} = m_0 r_0^{D-1} = b = const, \tag{172}$$

has the following static solution (N.M. 2008)

$$z = -sh(b\eta),\ n_1(\eta) = -\frac{2sh(b\eta)}{ch^2(b\eta)},\ n_2 = 0,\ n_3(\eta) = \frac{2}{ch(b\eta)} - 1 \tag{173}$$

where

$$\tau = \frac{t}{r^{D-1}},\ \eta = \frac{r^{2-D} - r_0^{2-D}}{2-D},\ D \neq 2$$
$$\tau = \frac{t}{r},\ \eta = ln\frac{r}{r_0},\ D = 2. \tag{174}$$

5.6 Energy of the Sphaleron

The energy functional is

$$H = \int_{r_0}^{r} dx^D 4\frac{|\partial_t z|^2 + |\partial_x z|^2 + m^2(1+|z|^2)|z|^2}{(1+|z|^2)}, \tag{175}$$

For the sphaleron solution we obtain

$$E_{Sph} = 4\Omega_D \int_{r_0}^{r} dr r^{D-1} 2m^2(r) = \frac{8D}{D-2} V_0 m_0^2 (1 - (\frac{r_0}{r})^{D-2}),$$
$$V_0 = V_D(r_0) = \frac{\pi^{\frac{D}{2}}}{\Gamma(1+\frac{D}{2})} r_0^D. \tag{176}$$

For $D = 2$,

$$E = 16\pi m_0^2 r_0^2 \cdot ln(\frac{r}{r_0}) \tag{177}$$

For $D < 2$, For the sphaleron solution we obtain

$$E_{Sph} = \frac{8D}{2-D} V_0 m_0^2 ((\frac{r}{r_0})^{2-D} - 1) \tag{178}$$

6 Strings at higher energy, density, temperature

We can use for the thermodynamic description of the strings the formalism developed in previous sections (N.M. 1987)

$$Z(\beta) = tre^{\beta\hat{H}} = tr\int_0^\infty dE\delta(E-\hat{H})e^{-\beta E}$$
$$= \int_0^\infty dE\rho(E)e^{-\beta E},$$
$$\rho(E) = tr\delta(E-\hat{H}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dte^{itE}tre^{-it\hat{H}}, \tag{179}$$

where

$$tre^{-it\hat{H}} = \int_{\Phi_0=\Phi_t} d\Phi e^{iS(\Phi)}, \tag{180}$$

Φ is string field.

In leading order in perturbative theory, string Hamiltonian, H, is direct sum of the free - field Hamiltonians for each particle degree of freedom (of the single - string Fock space).

The spectrum of a string (in perturbation theory) at a given high energy has exponential degeneracy. For a single string of energy E the density of states grows as

$$\varrho(E) \sim e^{\beta_H E}, \tag{181}$$

where $\beta_H \sim l_s$ is of the order of the string length scale. Its entropy is

$$S(E) = \ln\rho(E) \sim \beta_H E; \tag{182}$$

effective temperature

$$\frac{1}{T} = \frac{\partial S}{\partial E} \sim \beta_H \tag{183}$$

String theory manifests upper bound on the value of the temperature of the string gas - Hagedorn temperature.

A limiting temperature was first observed in the dual theory of hadrons and the first physical interpretation was offered in the QCD theory of hadrons. Here instead of being an actual limiting temperature its presence suggests a change in the relevant degrees of freedom. A change resulting in a phase transition from composite objects to their constituents. The Hagedorn temperature in hadronic systems is related to a deconfinement transition into new phase - Gluquar(k) in which hadron liberate their quark-gluon constituents.

This QCD analogy is a recurrent theme when thinking about the fundamental strings of gravity and their possible constituents.

Appendix

1. In the main text of this paper we used the following relation:

$$e^{\varepsilon A}e^{\varepsilon B}e^{-\varepsilon A}e^{-\varepsilon B} = e^{\varepsilon^2[A,B]} + O(\varepsilon^3). \tag{184}$$

The relation

$$(1+\varepsilon A)(1+\varepsilon B)(1+\varepsilon A)^{-1}(1+\varepsilon B)^{-1} = (1+\varepsilon^2[A,B]) + O(\varepsilon^3)$$
$$= e^{\varepsilon^2[A,B]} + O(\varepsilon^3), \tag{185}$$

also maybe useful.

2. For coordinate and momentum state vectors, correspondingly $|x>$ and $|p>$,

$$\hat{x}|x>= x|x>, \quad \hat{p}|p>= p|p>,$$
$$<p|x>= \psi_x(p) = \frac{1}{\sqrt{2\pi\hbar}}\ exp\ (-\frac{i}{\hbar}px),\ \hat{x}\psi_x(p) = i\hbar\frac{\partial}{\partial p}\psi_x(p) = x\psi_x(p),$$
$$<x|y>= \int dp <x|p><p|y>= \int \frac{dp}{2\pi\hbar}\ exp\ (\frac{i}{\hbar}p(x-y))$$
$$= \delta(x-y). \tag{186}$$

Now we calculate the following matrix element

$$<x_{n+1}|\ exp\ (-a\hat{p}^2)|x_n> \ = \ \int d^Dp <x_{n+1}|p><p|x_n>exp\ (-ap^2)$$
$$= \int \frac{d^Dp}{(2\pi\hbar)^D}\ exp\ (i\frac{p(x_{n+1}-x_n)}{\hbar} - ap^2) \ = \ \frac{A^D}{(2\pi\hbar)^D}\ exp\ (-\frac{(x_{n+1}-x_n)^2}{4a\hbar^2}), \tag{187}$$

where in the case of the quantum mechanics of the particle, (131)

$$a = i\frac{t}{2m\hbar N} \tag{188}$$

$$A = \int dp\ exp\ (-ap^2) = \sqrt{\frac{\pi}{a}}. \tag{189}$$

References

[1] Nugzar Makhaldiani, Adelic Universe and Cosmological constant, *Communications of the JINR*, Dubna **E2-2003-215**; arXiv:hep-th/0312291, 2003.

[2] V.S. Vladimirov, *Russian Math. Surveys* **43** 19 (1988).

[3] G. Gasper, M. Rahman, *Basic Hypergeometric Series,* Cambridge Univ. Press, Cambridge, 1990.

[Miller,1977] W. Miller, Jr. *Symmetry and Separation of Variables,* Addison-Wesley PC, London, 1977.

[4] N.V. Makhaldiani, On Dynamics of Space Dimension, *Communications of the JINR*,
Dubna **P2-90-525** 1990.

[5] M. Shashkov, *Conservative Finite-Difference Methods on General Grids.* Boca Raton, CRC Prtess, 1996.

[6] N. Makhaldiani, work in progress.

[7] N.V. Makhaldiani, Number Fields Dynamics and the Compactification of Space Problem
in the Unified Theories of Fields and Strings, *Communications of the JINR*,
Dubna **P2-88-916** 1988.

[8] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields,* New York, 1959.

[9] K.G. Wilson and J. Kogut, *Phys. Rep.* **12C** 75 (1974).

[10] S. Weinberg, *The Quantum Theory of Fields, Volum I - Foundations,* Cambridge Univ. Press, 1995.

[11] S. Weinberg, *The Quantum Theory of Fields, Volum II - Modern Applications,*
Cambridge Univ. Press, 1996.

[12] B.P. Barber et al, *Phys. Rep.* **281(2)** 66 (1997).

[13] S. Weinberg, *The Quantum Theory of Fields, Volum III - Supersymmetry,* Cambridge Univ. Press, 2000.

[14] W. de Boer, C. Sander,*Global Electroweak Fits and Gauge Coupling Unification,* hep-ph/0307049.

[15] D.I. Kazakov, *Supersymmetric Generalization of the Standard Model of Fundamental Interactions, Textbook,* Dubna, JINR, 2004.

[16] *F. Berezin, M. Shubin,* Schrödinger equation, Kluwer, Dordrecht, 1991.

[17] *R.P. Feynman, A.R. Hibbs,* Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965.

[18] L. D. Faddeev, L. A. Takhtajan, *Hamiltonian methods in the theory of solitons* (Springer, Berlin, 1990).

[19] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishere, New York, 1987).

[20] A.M. Kosevich, B.A. Ivanov, A.S.. Kovalev,
Phys. Rep. **190** 117 (1990).

[21] S. Weinberg, *The Quantum Theory of Fields, Volum 3 - Supersymmetry,* Cambridge Univ. Press, 2000.

[22] W. de Boer, C. Sander,*Global Electroweak Fits and Gauge Coupling Unification,* hep-ph/0307049.