Renormalization Group Approach

towards the QCD Phase Diagram

Bernd-Jochen Schaefer

Institut für Physik, Theoretische Physik

Karl-Franzens-Universität Graz

II. Some QCD applications

Helmholtz International Summer School

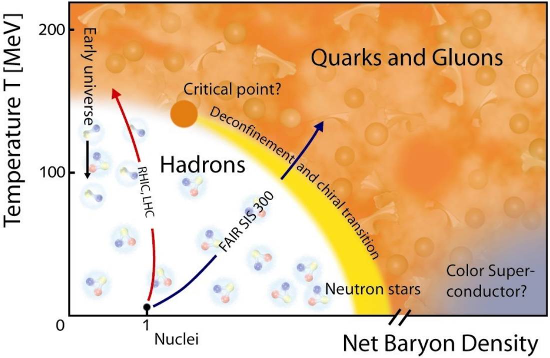
Dense Matter In Heavy Ion Collisions and Astrophysics

21$^{\mathrm{st}}$ Aug. - 1$^{\mathrm{st}}$ Sept, 2006

Dubna, Russia

Outline Part II

QCD Phase Diagram ($N_f = 3$)



lattice at $\mu = 0$: crossover
lattice at $\mu \neq 0$: "sign problem"

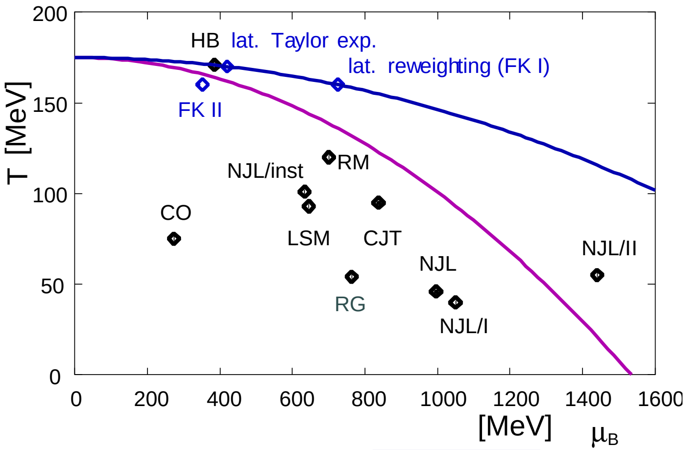
eff. models at $T = 0$: 1st-order
order parameter: $\langle \bar{q}q \rangle$

$\rightarrow \exists$ critical end point E (cf. as in water)

Where is the critical point? What is the size of crit. region?

model studies vs. lattice simulations

(**black** points) (lines & blue points)



Stephanov 2005

lattice:

- imaginary μ_B
- Taylor expansion around $\mu_B = 0$

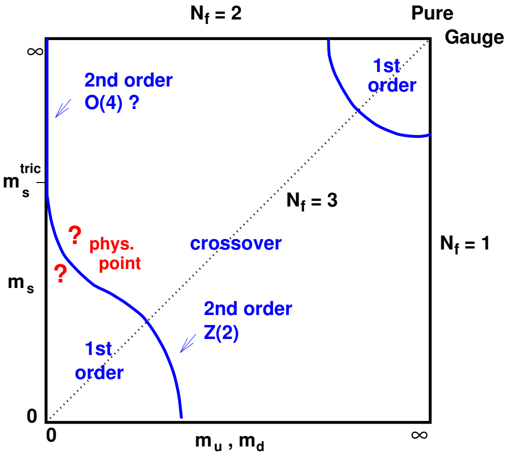
large m_q sensitivity?

if $m_s \to 0$
$\Rightarrow$ 1st-order

Mass Sensitivity (lattice, $N_f = 3, \mu_B = 0$)

$T_\chi^{N_f=2} \sim 175$ MeV

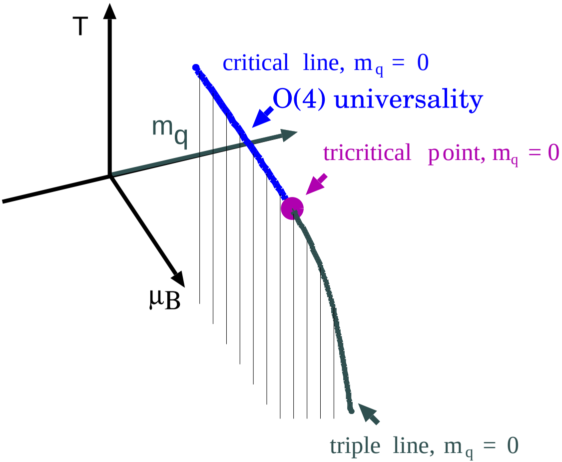
$T_d \sim 270$ MeV



$T_\chi^{N_f=3} \sim 155$ MeV

3D-view (T, μ_B, m_q) of 2-flavour phase diagram

Chiral limit: $(m_q = 0)$ $O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$



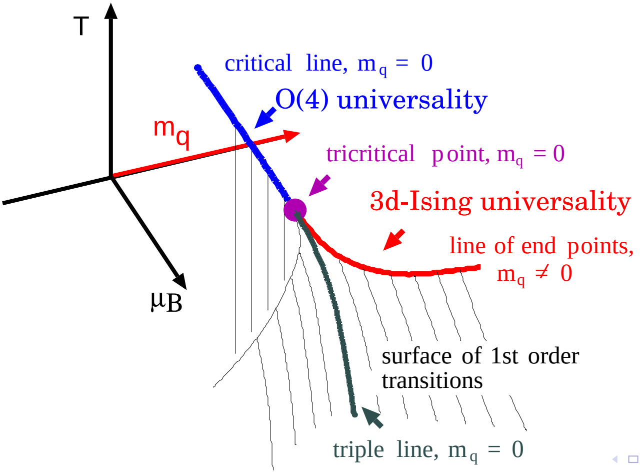
General properties

- chiral limit
  tricritical point (Gaussian fixed point)

3D-view (T, μ_B, m_q) of 2-flavour phase diagram

Chiral limit: $(m_q = 0)$ $O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$

$m_q \neq 0$: no symmetry remains $\longrightarrow$ only one critical mode σ (Ising) ($\vec{\pi}$ massive)



General properties

- chiral limit
  tricritical point
  (Gaussian fixed point)
- finite m_q
  critical endpoints
  (3D-Ising class)

Landau-Ginzburg approach

Landau-Ginzburg potential: expansion in order parameter $\vec{\phi} = (\sigma, \vec{\pi})$

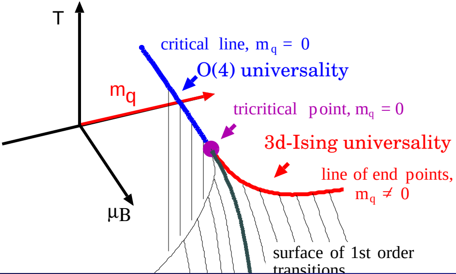
$$\Omega(T,\mu;\phi) \sim a(T,\mu)\vec{\phi}^{\,2} + b(T,\mu)\vec{\phi}^{\,4} + c\vec{\phi}^{\,6} + m\sigma \quad ; \quad c > 0$$

$m = 0$:

- 2$^{\text{nd}}$ order line: $a = 0,\ b > 0$
  4 fields massless $\rightarrow O(4)$ universality
- tricritical point: $b = 0$
  $a = b = 0 \Rightarrow$ mean-field exponent
- 1$^{\text{st}}$ order line: $b < 0$

$m \neq 0$:

- 2$^{\text{nd}}$ order line $\longrightarrow$ crossover
- tricritical point $\longrightarrow$ critical point
  end point of a 1$^{\text{st}}$ order line
  σ massless, $\vec{\pi}$ massive $\rightarrow$ Ising class
- 1$^{\text{st}}$ order line $\longrightarrow$ 1$^{\text{st}}$ order line



What are the sizes of the critical regions?

$\rightarrow$ Ginzburg criterion

Ginzburg criterion

Ginzburg criterion: size of crit. region ↔ break down of mean-field theory

Landau-Ginzburg potential for 2nd order phase transition

$$\Omega(T,\mu;\phi) \sim d(\vec{\nabla}\phi)^2 + a' t \phi^2 + b\phi^4 \qquad ; \qquad t = (T - T_c)/T_c$$

⇒ Ginzburg-Levanyuk temperature τ_{GL}

For $t < \tau_{GL}$ fluctuations are important

$$|t| \sim \frac{T_c^2}{a' d^3} b^2 \equiv \tau_{GL} \sim m_q^{4/5} \sim m_\pi^2$$

Ginzburg criterion

Ginzburg criterion: size of crit. region↔ break down of mean-field theory

Landau-Ginzburg potential for 2$^{\mathrm{nd}}$ order phase transition

$$\Omega(T,\mu;\phi) \sim d(\vec{\nabla}\phi)^2 + a' t \phi^2 + b\phi^4 \qquad ; \qquad t = (T - T_c)/T_c$$

⇒ Ginzburg-Levanyuk temperature τ_{GL}

For $t < \tau_{GL}$ fluctuations are important

$$|t| \sim \frac{T_c^2}{a' d^3} b^2 \equiv \tau_{GL} \sim m_q^{4/5} \sim m_\pi^2$$

but this criterion useless

- size depends on microscopic dynamics
- universality not applicable

e.g. O(2) class

He^4 λ-transition: $\tau_{GL} \sim 10^{-15}$

$O(2)$ spin model: $\tau_{GL} \sim 0.3$

⇒ size of crit. region shrinks as $m_q \to 0$ ($\tau_{GL} \sim b^2$)

Non-trivial critical region suppression

Suppression of size of crit. region,
where non-trivial critical behavior sets in
also observed in other models

Critical region suppression ($\mu = 0$)

- Yukawa theory with spon. χ SB — Rosenstein et al. 1994
- Gross-Neveu model (large-N) — Kocic, Kogut 1995
- MC simulations confirm these results

Outline

RG Approaches

$\Gamma_k[\phi]$ scale dependent effective action ; $t = \ln(k/\Lambda) \quad \rightarrow \quad k\partial_k \equiv \partial_t$

1. Exact RG

ERG (average effective action) [Wetterich]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\, \partial_t R_k \left(\frac{1}{\Gamma_k^{(2)} + R_k} \right) \quad ; \quad \Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}$$

2. Proper-time RG

PTRG [Liao]

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left[\partial_t f_k(\Lambda^2 \tau) \right] \mathrm{Tr} \exp\left(-\tau \Gamma_k^{(2)} \right)$$

3. other approximations

Proper-time RG

- effective action Γ (one-loop level):

$$\Gamma[\phi] = S_{\text{class}} + \frac{1}{2}\text{Tr}\ln S^{(2)}_{\text{class}} \qquad ; \qquad S^{(2)}_{\text{class}} = \frac{\delta^2 S_{\text{class}}}{\delta\phi\delta\phi}$$

Proper-time RG

- effective action Γ (one-loop level):

$$\Gamma[\phi] \quad = \quad S_{\text{class}} + \frac{1}{2} \text{Tr} \ln S^{(2)}_{\text{class}} \qquad ; \qquad S^{(2)}_{\text{class}} \quad = \quad \frac{\delta^2 S_{\text{class}}}{\delta\phi\delta\phi}$$

- Schwinger proper-time representation:

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left[\partial_t f_k(\Lambda^2 \tau)\right] \text{Tr} \exp\left(-\tau S^{(2)}_{\text{class}}\right) \qquad ; \qquad t = \ln\left(\frac{k}{\Lambda}\right)$$

regulator $f_k \sim \Gamma(d/2, \tau k^2)$

Proper-time RG

- effective action Γ (one-loop level):

$$\Gamma[\phi] \quad = \quad S_{\text{class}} + \frac{1}{2} \text{Tr} \ln S^{(2)}_{\text{class}} \qquad ; \qquad S^{(2)}_{\text{class}} \quad = \quad \frac{\delta^2 S_{\text{class}}}{\delta\phi\delta\phi}$$

- Schwinger proper-time representation:

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left[\partial_t f_k(\Lambda^2 \tau)\right] \text{Tr} \exp\left(-\tau S^{(2)}_{\text{class}}\right) \qquad ; \qquad t = \ln\left(\frac{k}{\Lambda}\right)$$

- RG improvement: $S^{(2)}_{\text{class}} \to \Gamma^{(2)}_k$

regulator $f_k \sim \Gamma(d/2, \tau k^2)$

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left[\partial_t f_k(\Lambda^2 \tau)\right] \text{Tr} \exp\left(-\tau \Gamma^{(2)}_k\right)$$

Outline

Flow Equations

- ansatz for Γ_k in the UV: quark-meson model

$$\Gamma_{k=\Lambda} = \int d^4x \left\{ \bar{q}[\not{\partial} + g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + V(\sigma^2 + \vec{\pi}^2) \right\}$$

Flow Equations

- ansatz for Γ_k in the UV: quark-meson model

$$\Gamma_{k=\Lambda} = \int d^4x \left\{ \bar{q}[\not\partial + g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + V(\sigma^2+\vec{\pi}^2) \right\}$$

flow for grand canonical potential BJS, J.Wambach

$$\partial_t\Omega_k(T,\mu;\phi) = \frac{k^4}{12\pi^2}\left[\frac{3}{E_\pi}\coth\left(\frac{E_\pi}{2T}\right) + \frac{1}{E_\sigma}\coth\left(\frac{E_\sigma}{2T}\right) - \frac{2N_cN_f}{E_q}\left\{\tanh\left(\frac{E_q-\mu}{2T}\right) + \tanh\left(\frac{E_q+\mu}{2T}\right)\right\}\right]$$

$$E_\pi^2 = 1 + 2\Omega_k'/k^2\,, \qquad E_\sigma^2 = 1 + 2\Omega_k'/k^2 + 4\phi^2\Omega_k''/k^2\,, \qquad E_q^2 = 1 + g^2\phi^2/k^2$$

$$\phi \sim \langle\bar{q}q\rangle\,, \qquad \Omega_k' = \partial\Omega_k/\partial\phi \qquad \text{etc}$$

- quark fluctuations: chiral symmetry breaking
- meson fluctuations: chiral symmetry restoration

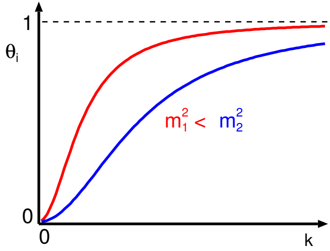
Threshold Functions

- crucial ingredients of flow equations e.g. here for vacuum flow

$$\partial_t \Omega_k(0,0) = \frac{k^4}{12\pi^2}\Big(3\underbrace{\frac{1}{\sqrt{1+m_\pi^2/k^2}}}_{\theta_\pi} + \underbrace{\frac{1}{\sqrt{1+m_\sigma^2/k^2}}}_{\theta_\sigma} - 4N_cN_f\underbrace{\frac{1}{\sqrt{1+m_q^2/k^2}}}_{\theta_q}\Big)$$

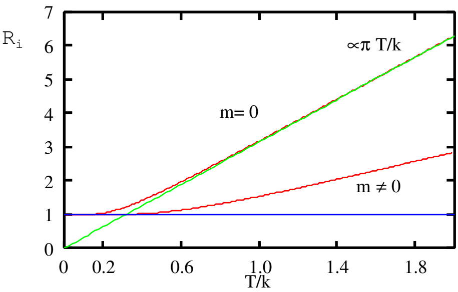
→ describe smooth decoupling of massive modes during evolution

$$\theta_i = \frac{1}{\sqrt{1+m_i^2/k^2}} \qquad i = \pi, \sigma, q$$



$$R_i = \frac{\theta_i(T,\mu)}{\theta_i(0,0)}$$

bosonic case

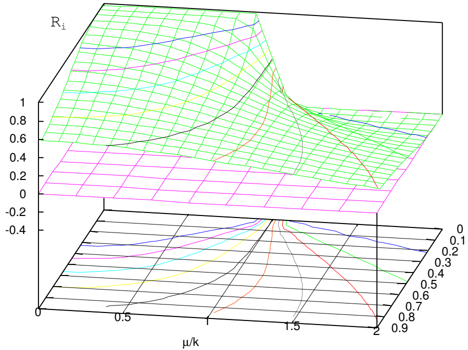


linearly increase in T/k

$(3+1)$-Dim. $\longrightarrow$ 3-Dim.

$\rightarrow$ dimensional reduction for large T automatically included

fermionic case

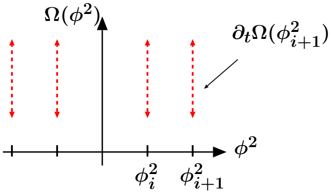


fermions are always suppressed
no zero Matsubara mode

Solving Flow Equations

two possibilities

1.) Solve coupled flow eqs on ϕ^2 grid:



2.) Taylor expansion around ϕ_0^2:

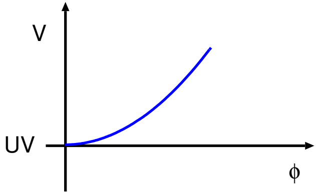
$$\Omega(\phi^2) = \sum_{n=0}^{N} a_n(\phi^2 - \phi_0^2)^n$$

- Initial condition at e.g. $\Lambda = 500$ MeV tree-level parameterization

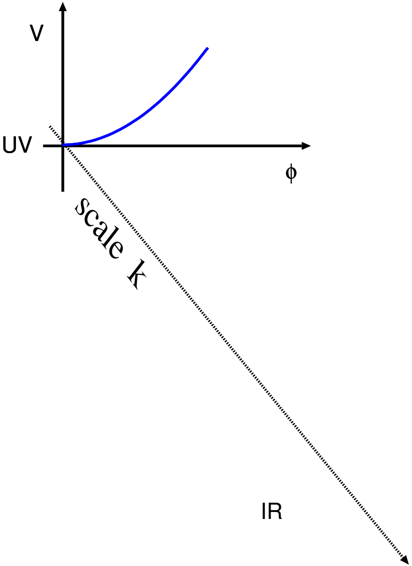
$$V_\Lambda = \frac{1}{4}\lambda_\Lambda(\phi^2)^2 \qquad \text{symmetric potential}$$

- Fixed UV parameterization (λ_Λ) such to reproduce $\phi_0 \equiv f_\pi \sim 93$ MeV in the IR

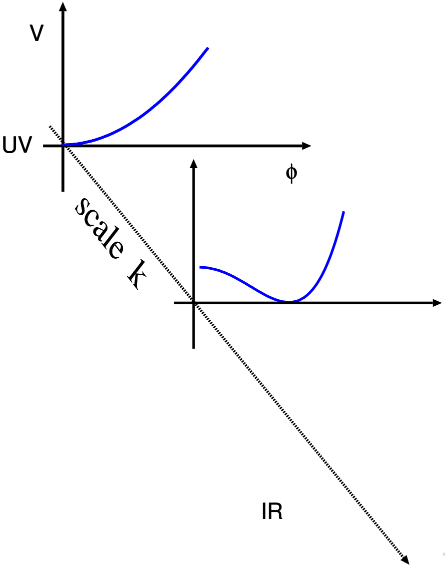
Scale Evolution of the Potential (Vacuum)



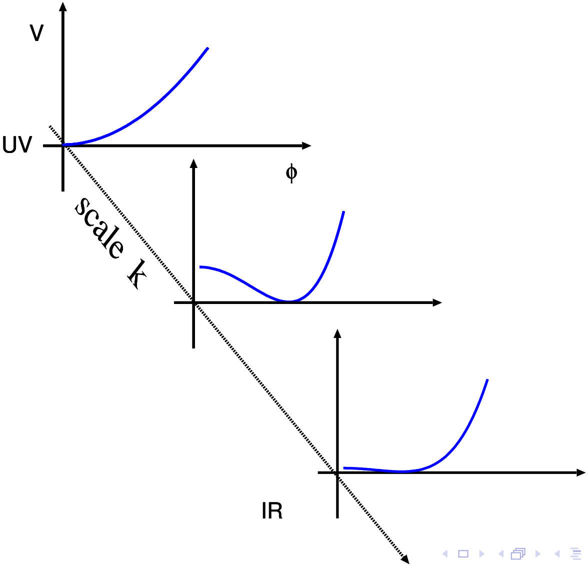
Scale Evolution of the Potential (Vacuum)



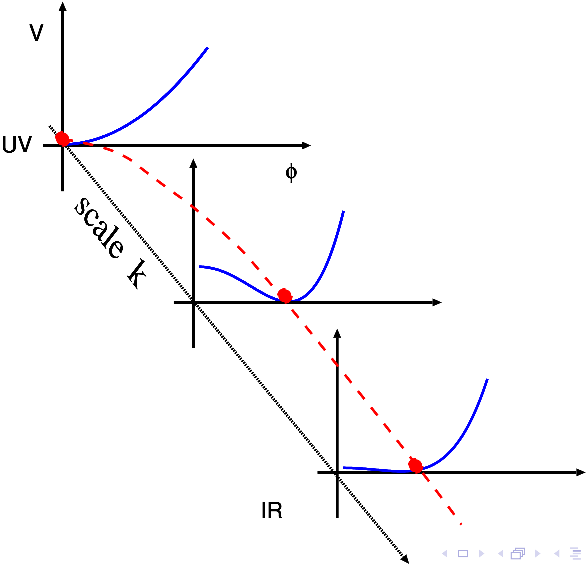
Scale Evolution of the Potential (Vacuum)

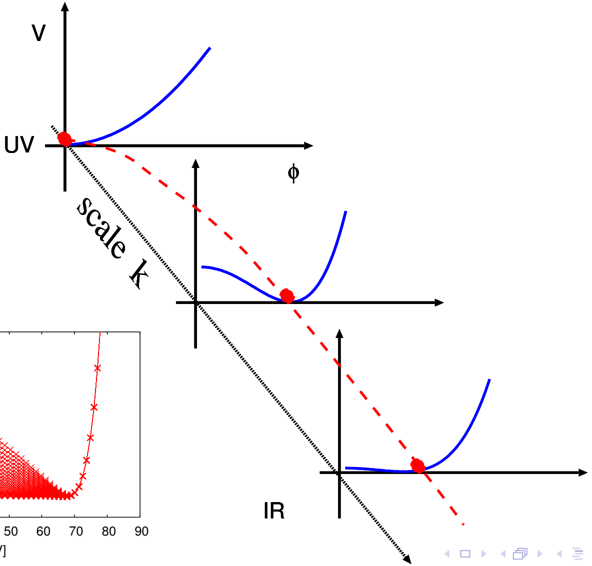


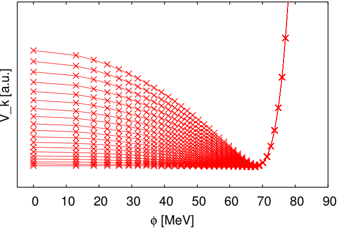
Scale Evolution of the Potential (Vacuum)

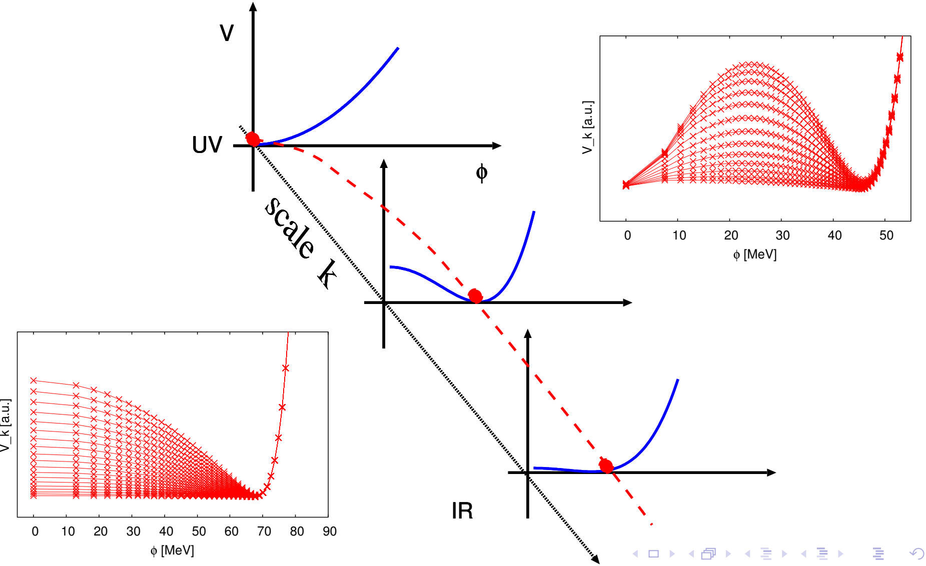


Scale Evolution of the Potential (Vacuum)



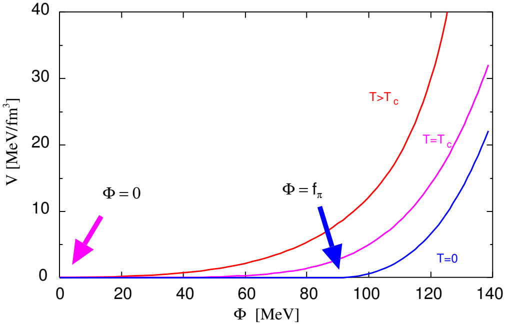

V
UV
ϕ
scale k
IR


V_k [a.u.]
0
10
20
30
40
50
60
70
80
90
ϕ [MeV]

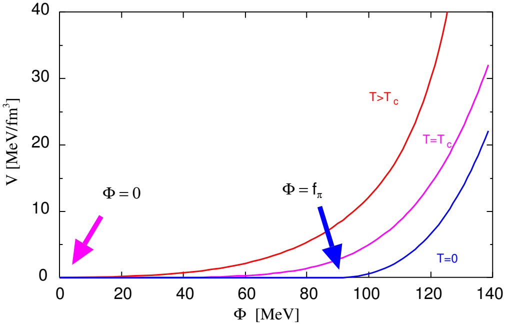


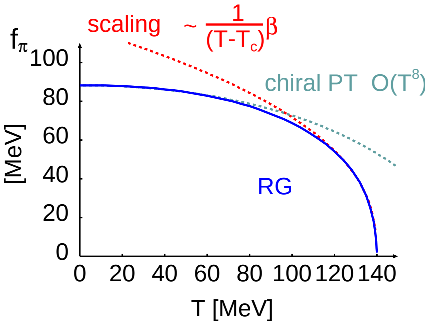
V
UV
ϕ
scale k
IR
V_k [a.u.]
0
10
20
30
40
50
ϕ [MeV]
V_k [a.u.]
0
10
20
30
40
50
60
70
80
90
ϕ [MeV]

- Evolved potential for different temperatures:



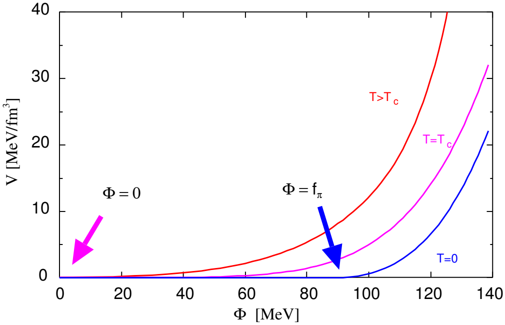
- Evolved potential for different temperatures:

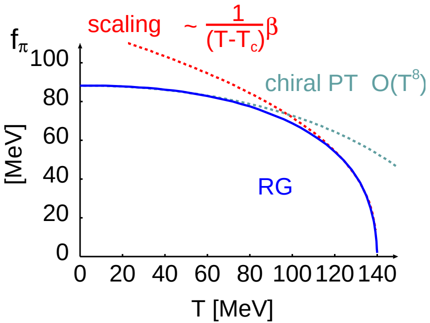




Chiral Phase Transition at Finite T & $\mu = 0$

- Evolved potential for different temperatures:





- chiral perturbation theory expansion: [Gerber, Leutwyler '89]

$$\frac{\langle\bar{q}q\rangle_T}{\langle\bar{q}q\rangle_0} = 1 - \frac{T^2}{8f_\pi^2} - \frac{T^4}{384 f_\pi^4} - \frac{T^6}{288 f_\pi^6}\ln\frac{\Lambda_q}{T} + \mathcal{O}(T^8)\,; \quad \Lambda_q = (470 \pm 110)\ \text{MeV}$$

Critical Exponents

- parameterize singular behavior of Ω near the phase transition
- depend only on internal symmetries and dimension of the system

$$\phi_0(k \to 0) \sim \left| \frac{T_c - T}{T_c} \right|^{\beta} \quad ; \quad \xi = (T - T_c)^{-\nu} \quad ; \quad \dots$$

Critical Exponents

- parameterize singular behavior of Ω near the phase transition
- depend only on internal symmetries and dimension of the system

$$\phi_0(k \to 0) \sim \left| \frac{T_c - T}{T_c} \right|^{\beta} \quad ; \quad \xi = (T - T_c)^{-\nu} \quad ; \quad \ldots$$

- altogether 6 exponents but 4 scaling relations

		η	δ	ν	β
$N = 4$	lattice	0.0254(38)	4.851(22)	0.7479(90)	0.3836(46)
	ϵ	0.03(1)	4.82(6)	0.73(2)	0.38(1)
	PTRG	0.037	4.79	0.78	0.40
$N = 100$	PTRG	0.0025	4.99	0.99	0.49
	large-N	0	5	1.0	0.5

Critical Exponents

- exponent ν for arbitrary N (here for $O(N)$-model)

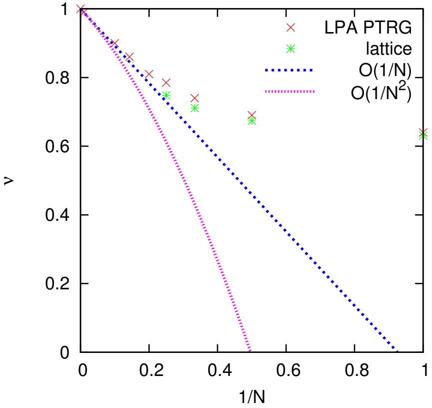
Critical Exponents

- exponent ν for arbitrary N (here for $O(N)$-model)
- comparison with $1/N$-expansion in N^{2}LO

$$\nu = 1 - 4\left(\frac{8}{3\pi^2}\frac{1}{N}\right) + \frac{112 - 27\pi^2}{6}\left(\frac{8}{3\pi^2}\frac{1}{N}\right)^2 + \mathcal{O}(1/N^3)$$

- exponent ν for <u>arbitrary N</u> (here for $O(N)$-model)
- comparison with $1/N$-expansion in N^{2}LO

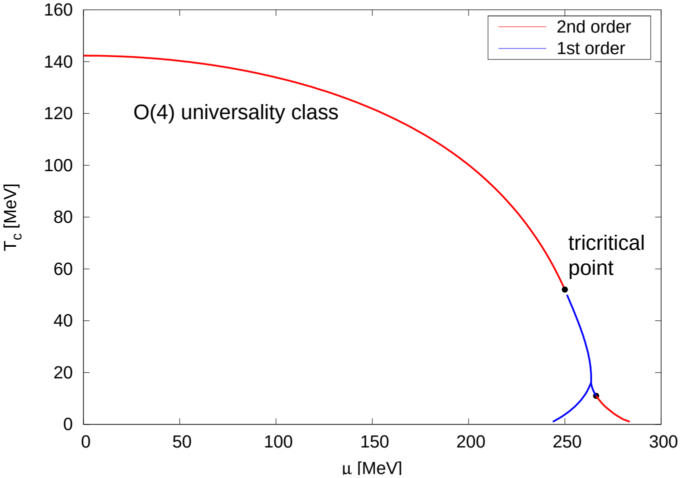
$$\nu = 1 - 4\left(\frac{8}{3\pi^2}\frac{1}{N}\right) + \frac{112-27\pi^2}{6}\left(\frac{8}{3\pi^2}\frac{1}{N}\right)^2 + \mathcal{O}(1/N^3)$$



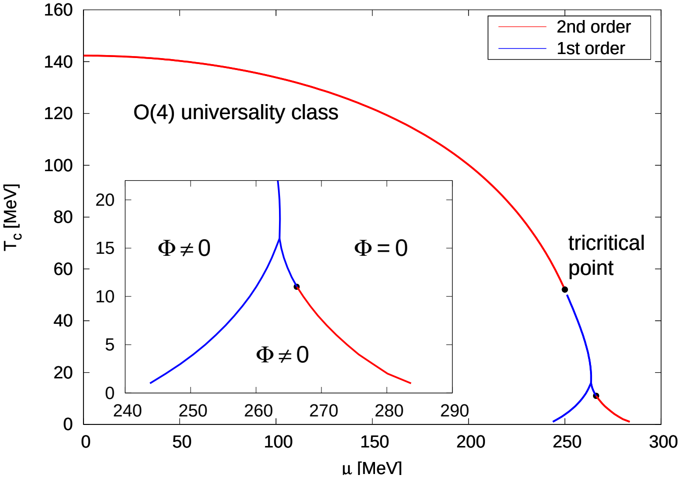
K.I. Aoki, Morris et al.

Chiral Phase Diagram for $N_f = 2$

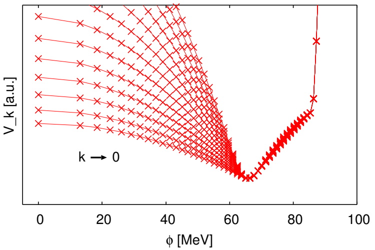
$O(4) \sim SU(2) \times SU(2)$ chiral limit



$O(4) \sim SU(2) \times SU(2)$ chiral limit no spinodal lines

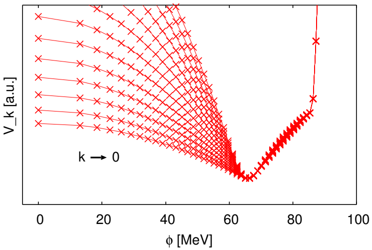


A Second Phase Transition



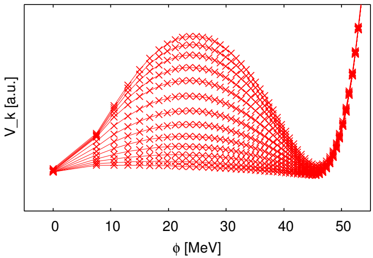
here e.g.
$T = 6$ MeV,
$\mu_q = 254.1$ MeV
$\rightarrow$ convex potential

A Second Phase Transition

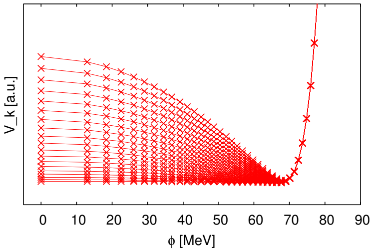


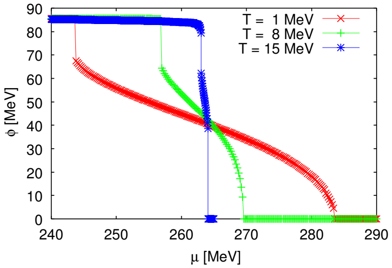
here e.g.
$T = 6$ MeV,
$\mu_q = 254.1$ MeV
$\rightarrow$ convex potential

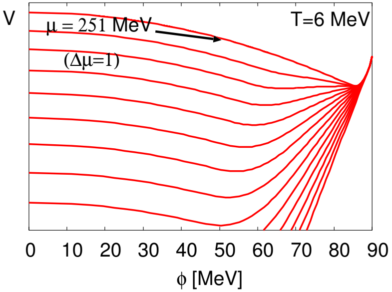
first-order phase transition



second-order phase transition







Order parameter $\phi_T(\mu)$:

- tricritical point:
  $T_t \sim 52$ MeV, $\mu_t \sim 251$ MeV
- splitting point:
  $T \sim 17$ MeV $\mu_q \sim 263$ MeV
- gap larger if $m_{q,vac}$ larger

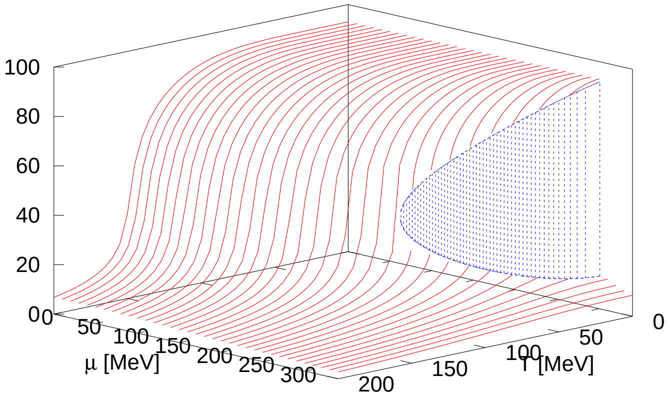
IR evolved potential:

here e.g.
$T = 6$ MeV fixed and
$\mu_q \geq 251$ MeV

Finite Quark Masses

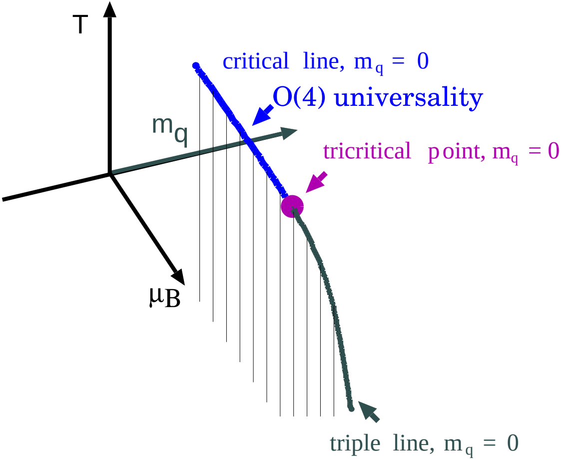
order parameter: $\phi(T, \mu)$

- 2nd-order transition $\rightarrow$ crossover
- shift of "T_c"
- shift tricritical point $\rightarrow$ critical



3D-view (T, μ_B, m_q) of 2-flavour phase diagram

Chiral limit: $(m_q = 0)$ $O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$



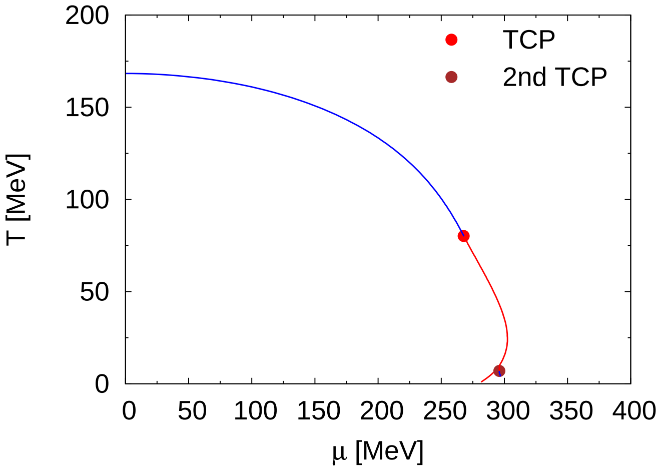
General properties

- chiral limit
  tricritical point
  (Gaussian fixed point)

Phase Diagram $m_q \sim 370$ MeV

TCP: $T_c \sim 80.2$ MeV

2. 'TCP': $T_c \sim 8$ MeV

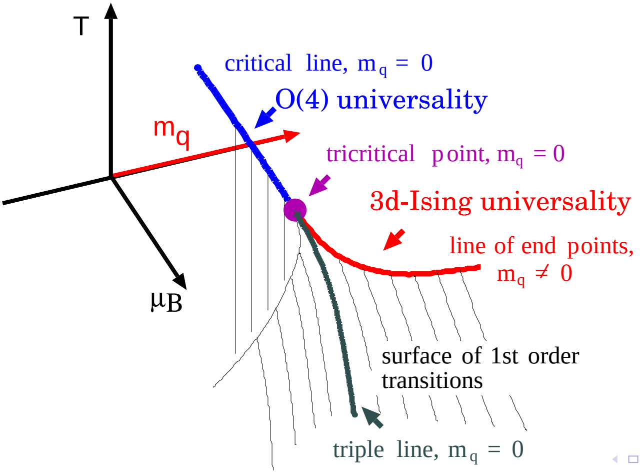


[BJS,J.Wambach]

3D-view (T, μ_B, m_q) of 2-flavour phase diagram

Chiral limit: $(m_q = 0)$ $O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$

$m_q \neq 0$: no symmetry remains $\longrightarrow$ only one critical mode σ (Ising) ($\vec{\pi}$ massive)

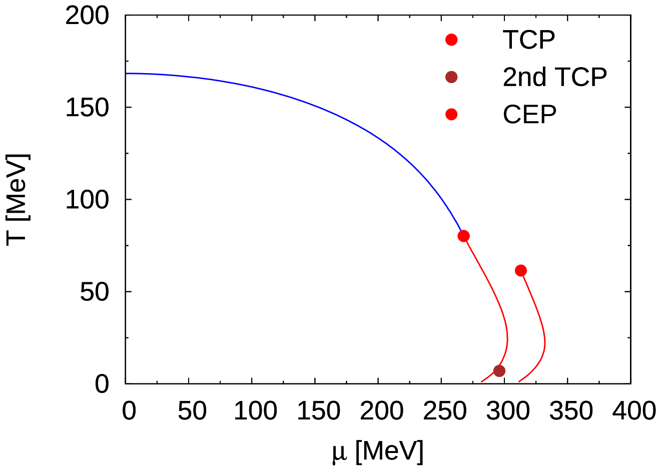


General properties

- chiral limit
  tricritical point
  (Gaussian fixed point)
- finite m_q
  critical endpoints
  (3D-Ising class)

Phase Diagram $m_q \sim 370$ MeV

TCP: $T_c \sim 80.2$ MeV 2. 'TCP': $T_c \sim 8$ MeV CEP: $T_c \sim 61.5$ MeV



[BJS,J.Wambach]

Susceptibilities

- quark number density: $n_q(T,\mu) = -\frac{\partial \Omega(T,\mu)}{\partial \mu}$
- quark number susceptibility: $\chi_q(T,\mu) = -\frac{\partial^2 \Omega(T,\mu)}{(\partial \mu)^2}$

Susceptibilities

- quark number density: $n_q(T,\mu) = -\frac{\partial\Omega(T,\mu)}{\partial\mu}$
- quark number susceptibility: $\chi_q(T,\mu) = -\frac{\partial^2\Omega(T,\mu)}{(\partial\mu)^2}$

- near critical point: $\chi_q \sim |g - g_c|^{-\epsilon} \quad ; \quad g = T, \mu$
- isothermal compressibility $\kappa_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)\Big|_{T,N} = \frac{\chi_q}{n_q^2}$

  $\rightarrow$ if χ_q is large then system is easy to compress

  $\Rightarrow$ interaction attractive (or weakly repulsive)

Susceptibilities

- quark number density: $n_q(T,\mu) = -\frac{\partial \Omega(T,\mu)}{\partial \mu}$
- quark number susceptibility: $\chi_q(T,\mu) = -\frac{\partial^2 \Omega(T,\mu)}{(\partial \mu)^2}$

  - near critical point: $\chi_q \sim |g - g_c|^{-\epsilon} \quad ; \quad g = T, \mu$
  - isothermal compressibility $\kappa_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)\Big|_{T,N} = \frac{\chi_q}{n_q^2}$

    $\rightarrow$ if χ_q is large then system is easy to compress

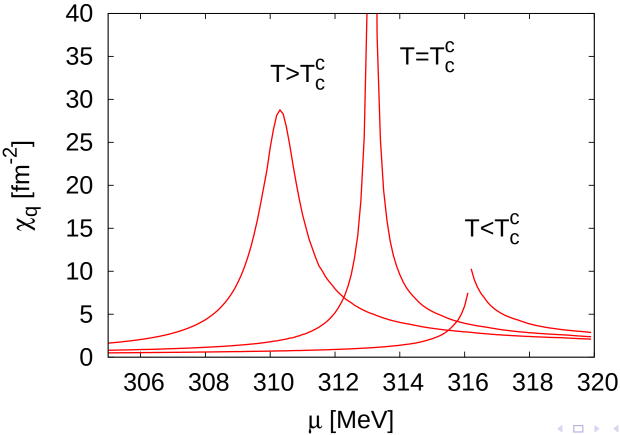
    $\Rightarrow$ interaction attractive (or weakly repulsive)

- scalar susceptibility: $\chi_\sigma(T,\mu) = 1/m_\sigma^2(T,\mu)$

  - zero-momentum projection of scalar propagator
  - encodes all fluctuations of order parameter

Quark-number susceptibility $\chi_q(T, \mu)$

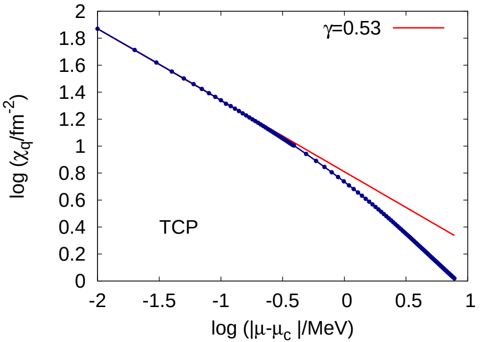
- diverges only at CEP
- finite everywhere else
- height decreases for decreasing μ towards T-axis
- For T below CEP: discontinuous $\rightarrow$ 1st order



Critical Exponents

$$\chi_q \sim |\mu - \mu_c|^{-\gamma}$$

TCP: $\gamma = 0.5$ (Gaussian)



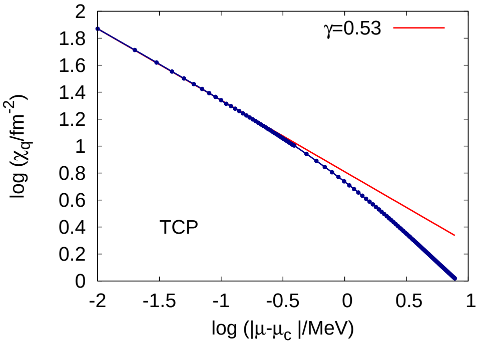
Critical Exponents

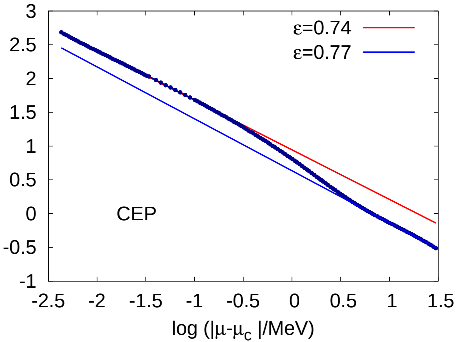
$$\chi_q \sim |\mu - \mu_c|^{-\gamma}$$

Mean field: $\epsilon = 2/3$

TCP: $\gamma = 0.5$ (Gaussian)

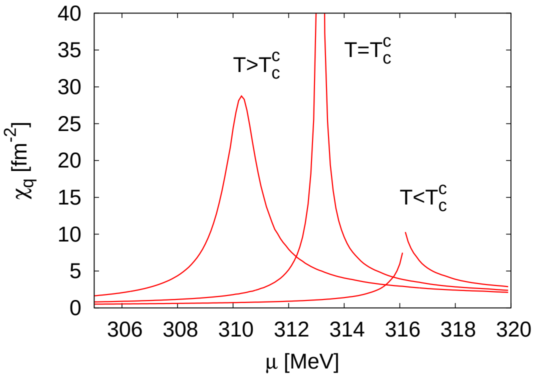
CEP: $\epsilon = 0.78$ (3D Ising)





Quark-number susceptibility $\chi_q(T,\mu)$

- diverges only at CEP
- finite everywhere else
- height decreases for decreasing μ towards T-axis
- For T below CEP: discontinuous $\rightarrow$ 1st order



define ratio:

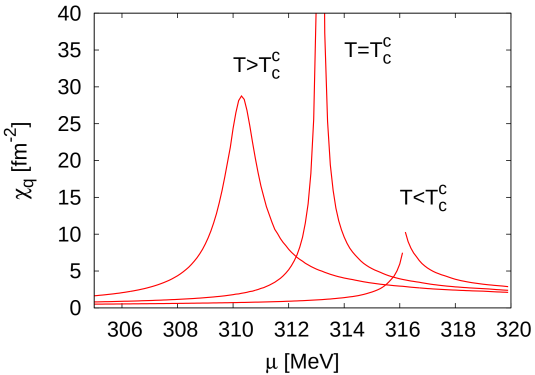
$$R_q := \chi_q / \chi_q^{\text{free}}$$

massless free quark gas

$$\chi_q^{\text{free}}(T,\mu) = N_c N_f \left(\frac{\mu^2}{\pi^2} + \frac{T^2}{3} \right)$$

Quark-number susceptibility $\chi_q(T,\mu)$

- diverges only at CEP
- finite everywhere else
- height decreases for decreasing μ towards T-axis
- For T below CEP: discontinuous $\rightarrow$ 1st order



define ratio:

$R_q := \chi_q / \chi_q^{\text{free}}$

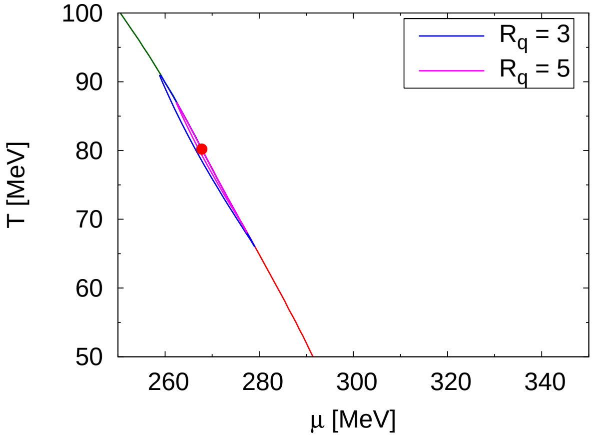
massless free quark gas

$$\chi_q^{\text{free}}(T,\mu) = N_c N_f \left(\frac{\mu^2}{\pi^2} + \frac{T^2}{3} \right)$$

e.g. $R_q = 3$ or $R_q = 5$

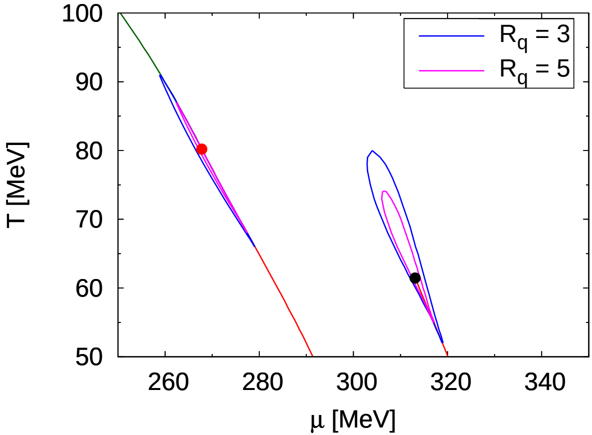
Critical Region with Quark Number Susceptibility χ_q

- size of crit. region shrinks as $m_q \to 0$



Critical Region with Quark Number Susceptibility χ_q

- size of crit. region shrinks as $m_q \to 0$



Let's compare...

the RG results to a mean-field approximation

Mean-field approximation

$N_f = 2$ Quark-Meson model

$$\mathcal{L} = \bar{q}\,(i\partial\!\!\!/ - g(\sigma + i\gamma_5\vec{\tau}\vec{\pi}))\,q + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

partition function:

$$\mathcal{Z}(T,\mu) = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}\sigma\mathcal{D}\vec{\pi}\,\exp\left\{ i\int_0^{1/T} dt d^3x\,(\mathcal{L} + \mu\bar{q}\gamma_0 q)\right\}.$$

Mean field approx.: $\sigma \to \langle\sigma\rangle \equiv \phi, \pi \to \langle\pi\rangle = 0$, integrate quark/antiquarks

Grand canonical potential

$$\Omega(T,\mu) = -\frac{T\ln\mathcal{Z}}{V} = \frac{\lambda}{4}(\langle\sigma\rangle^2 - v^2)^2 - c\langle\sigma\rangle + \Omega_{\bar{q}q}(T,\mu)$$

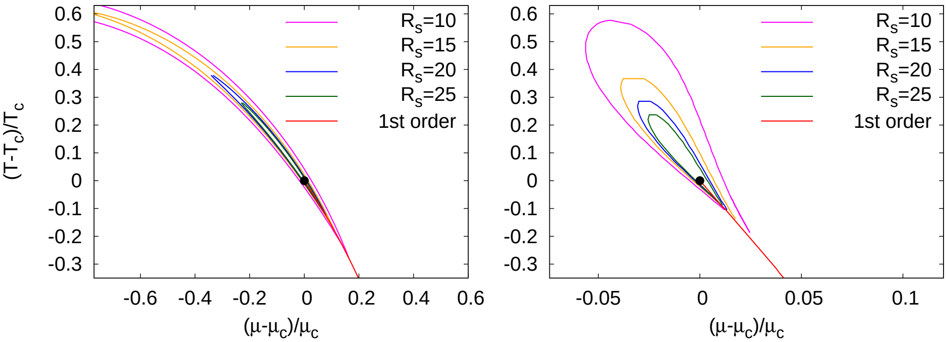
with

$$\Omega_{\bar{q}q}(T,\mu) = -2N_cN_fT\int\frac{d^3k}{(2\pi)^3}\left\{\ln(1+e^{-(E_q-\mu)/T}) + \ln(1+e^{-(E_q+\mu)/T})\right\}$$

Comparison with scalar χ_σ : MF↔RG

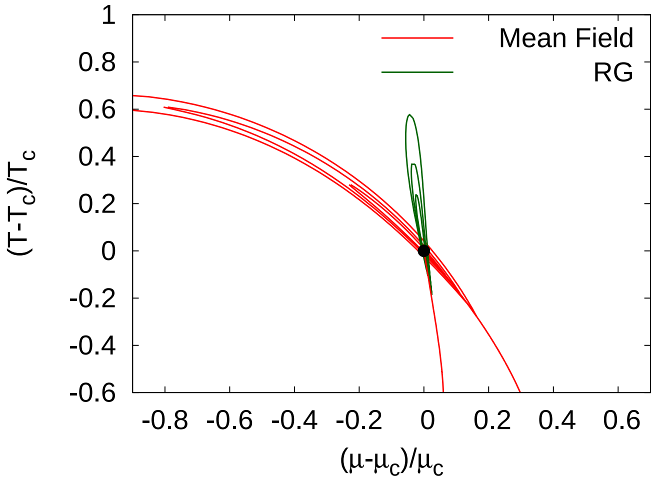
$$R_s = m_\sigma^2(0,0)/m_\sigma^2(T,\mu)$$

Mean field RG



critical region with RG more compressed

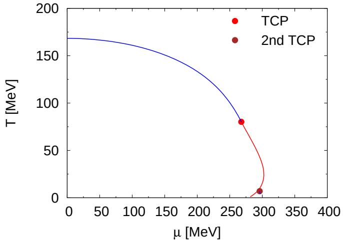
Comparison with scalar χ_σ : MF↔RG



Outline

Summary Part II

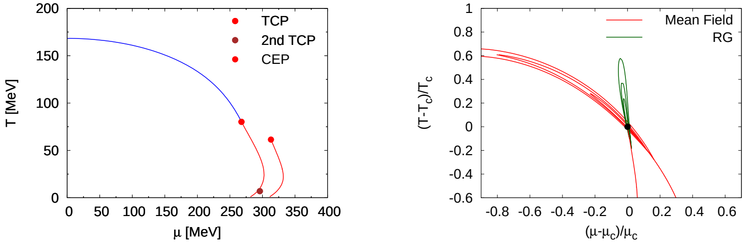
- ▷ proper-time RG → transparent physics, analytical threshold fkts. . .
- ▷ in phase diagram two TCP's (chiral limit) and CEP found



Summary Part II

- ▷ proper-time RG → transparent physics, analytical threshold fkts. . .
- ▷ in phase diagram two TCP's (chiral limit) and CEP found
- ▷ size of critical region via susceptibilities $\chi_q(T,\mu)$ and $\chi_\sigma(T,\mu)$

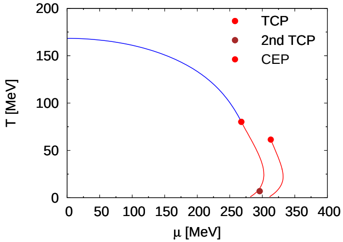
  ⟶ "compressed" with fluctuations

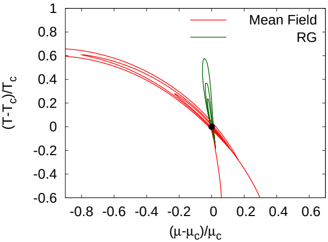


Summary Part II

- ▷ proper-time RG → transparent physics, analytical threshold fkts. . .
- ▷ in phase diagram two TCP's (chiral limit) and CEP found
- ▷ size of critical region via susceptibilities $\chi_q(T,\mu)$ and $\chi_\sigma(T,\mu)$

  $\longrightarrow$ "compressed" with fluctuations





- ▷ critical exponents consistent w/ $3d$ Ising universality class at CEP

Literature

- Books on RG
  - Binney et al., "The Theory of Critical Phenomena", 1993
  - Le Bellac, "Quantum and Statistical Field Theory", 1995
- Reviews on RG
  - Bagnuls, Bervillier; Phys. Rept. 348 (2001) 91
  - Berges, Tetradis, Wetterich; Phys. Rept. 363 (2002) 223
  - Polonyi; Central Eur. J. Phys. 1 (2004) 1
  - Pawlowski; hep-th/0512261

Thanks to...

the Organizers of this school for invitation
&
Jan M. Pawlowski

Appendix

Momentum-shell integration

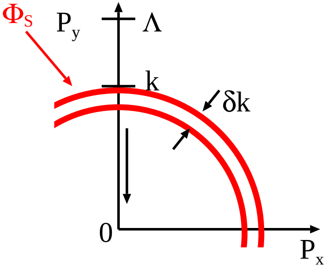
- for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}}, k]}{\partial k}$$

Momentum-shell integration

▷ for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}}, k]}{\partial k}$$

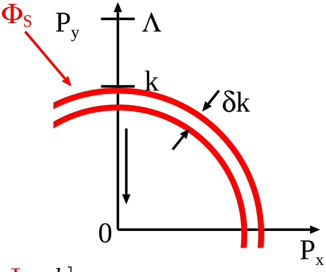


Momentum-shell integration

▷ for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

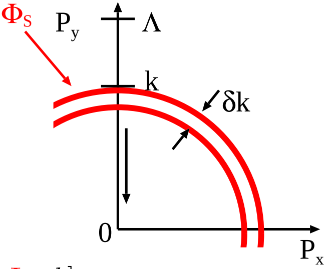
$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}}, k]}{\partial k}$$

▷ change of eff. action $\partial S_{k,\text{eff}}$ when $k \to k - \delta k$



$$e^{-S_{k,\text{eff}}[\Phi_{\text{low}}, k - \delta k]} = \int \mathcal{D}\Phi_S \; e^{-S_{k,\text{eff}}[\Phi_{\text{low}} + \Phi_S, k]}$$

Momentum-shell integration



- for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

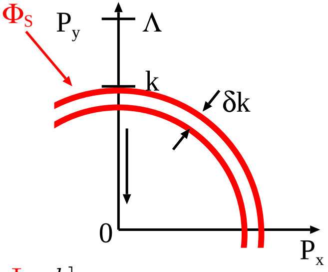
$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}}, k]}{\partial k}$$

- change of eff. action $\partial S_{k,\text{eff}}$ when $k \to k - \delta k$

$$e^{-S_{k,\text{eff}}[\Phi_{\text{low}}, k-\delta k]} = \int \mathcal{D}\Phi_S \; e^{-S_{k,\text{eff}}[\Phi_{\text{low}} + \Phi_S, k]}$$

$$= e^{-S_{k,\text{eff}}[\Phi_{\text{low}}, k]} \int \mathcal{D}\Phi_S e^{-\int_p' \frac{1}{2}\Phi_S \frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\Phi_S + \mathcal{O}(\delta k^2)}$$

Momentum-shell integration



- for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}}, k]}{\partial k}$$

- change of eff. action $\partial S_{k,\text{eff}}$ when $k \to k - \delta k$

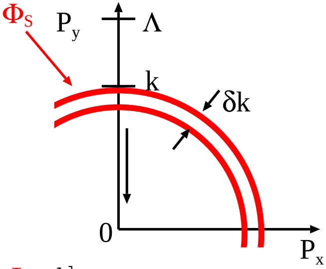
$$e^{-S_{k,\text{eff}}[\Phi_{\text{low}}, k - \delta k]} = \int \mathcal{D}\Phi_S \; e^{-S_{k,\text{eff}}[\Phi_{\text{low}} + \Phi_S, k]}$$

$$= e^{-S_{k,\text{eff}}[\Phi_{\text{low}}, k]} \int \mathcal{D}\Phi_S e^{-\int_p' \frac{1}{2}\Phi_S \frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\Phi_S + \mathcal{O}(\delta k^2)}$$

- only first order δk must be kept to find flow:

$$\Rightarrow S_{k,\text{eff}}[\Phi_{\text{low}}, k - \delta k] = S_{k,\text{eff}}[\Phi_{\text{low}}, k] + \frac{1}{2}\text{tr}' \ln\left(\frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\right)$$

Momentum-shell integration



- for flow of $S_{k,\text{eff}}$ decrease boundary cutoff k

$$\longrightarrow \frac{\partial S_{k,\text{eff}}[\Phi_{\text{low}},k]}{\partial k} = \frac{1}{2}\text{tr}'\ln\left(\frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\right)$$

- change of eff. action $\partial S_{k,\text{eff}}$ when $k \to k-\delta k$

$$e^{-S_{k,\text{eff}}[\Phi_{\text{low}},k-\delta k]} = \int \mathcal{D}\Phi_S\, e^{-S_{k,\text{eff}}[\Phi_{\text{low}}+\Phi_S,k]}$$

$$= e^{-S_{k,\text{eff}}[\Phi_{\text{low}},k]}\int \mathcal{D}\Phi_S e^{-\int_p' \frac{1}{2}\Phi_S \frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\Phi_S + \mathcal{O}(\delta k^2)}$$

- only first order δk must be kept to find flow:

$$\Rightarrow S_{k,\text{eff}}[\Phi_{\text{low}},k-\delta k] = S_{k,\text{eff}}[\Phi_{\text{low}},k] + \frac{1}{2}\text{tr}'\ln\left(\frac{\partial^2 S_{k,\text{eff}}}{\partial\Phi\partial\Phi}\right)$$

Solutions to exercises

solution to exercises

- modified Legendre transform

$$\Gamma_k[\phi] = -\ln Z_k[j] + j\phi - \Delta S_k[\phi]$$

$$= -\ln \int \mathcal{D}\chi e^{-S[\chi] - \Delta S_k[\chi] + j\chi} - \ln e^{-j\phi + \Delta S_k[\phi]}$$

$$= -\ln \int \mathcal{D}\chi e^{-S[\chi]-\Delta S_k[\chi]+j\underbrace{(\chi - \phi)}+\Delta S_k[\phi]}$$

$$= -\ln \int \mathcal{D}\tilde{\chi} e^{-S[\tilde{\chi}+\phi]-\Delta S_k[\tilde{\chi}+\phi]+j\tilde{\chi}+\Delta S_k[\phi]}$$

$$= -\ln \int \mathcal{D}\tilde{\chi} e^{-S[\tilde{\chi}+\phi]+j\tilde{\chi}-\Delta S_k[\tilde{\chi}+\phi]+\Delta S_k[\phi]}$$

$$= -\ln \int \mathcal{D}\tilde{\chi} e^{-S[\tilde{\chi}+\phi]+j\tilde{\chi}-\Delta S_k[\tilde{\chi}]+(j_k-j)\tilde{\chi}}$$