Helmholtz International Summer School

Dense Matter in Heavy Ion Collisions and *Astrophysics*

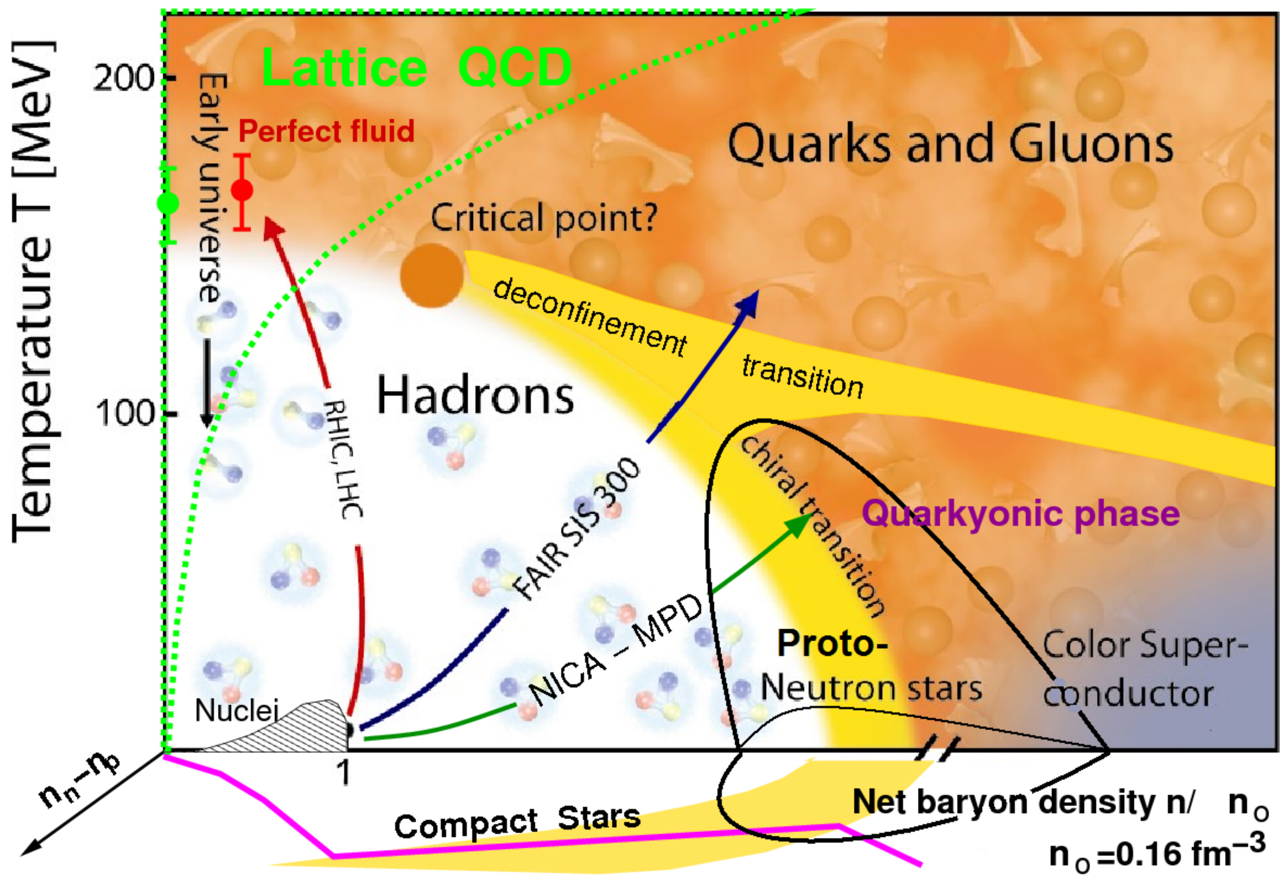
JINR, Dubna, Russia, August 28 – September 8, 2012

Spinodal Instabilities in Phase Transitions

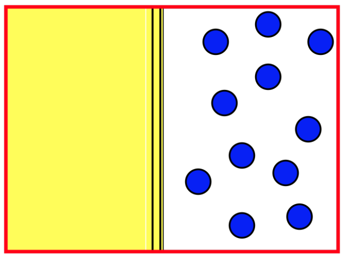
Jørgen Randrup (LBNL)

Lecture I:	*Phase coexistence (equilibrium)*	TUE 11:30-12:30
Lecture II:	*Phase separation (non-equilibrium)*	THU 10:00-11:00
Seminar:	*Problem Solving, Discussion*	FRI 17:00-18:00
Lecture III:	*Nuclear collisions (fresh results)*	SAT 11:30-12:30

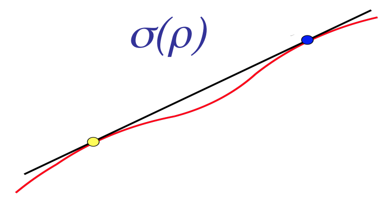
Possible (ρ, T) phase diagram of strongly interacting matter

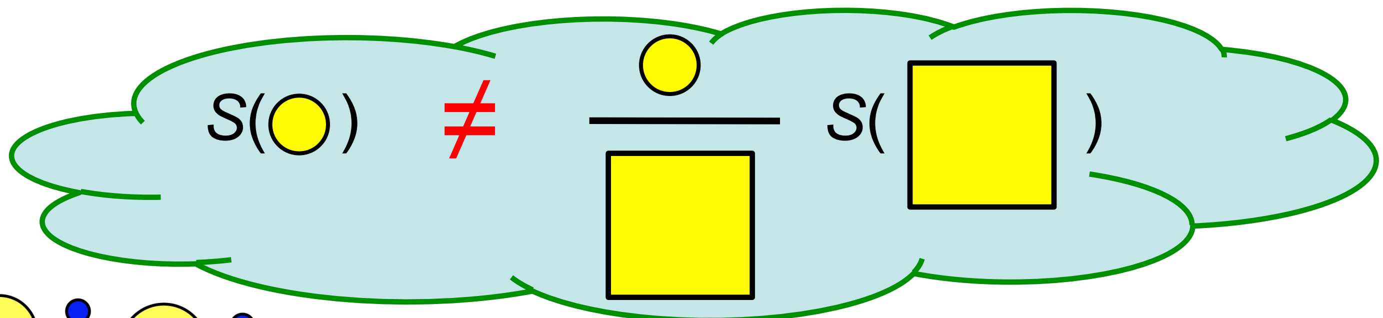


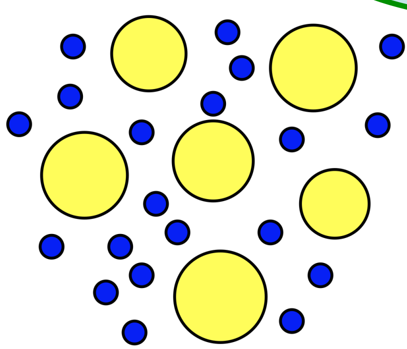
Lecture I: Phase coexistence (equilibrium)



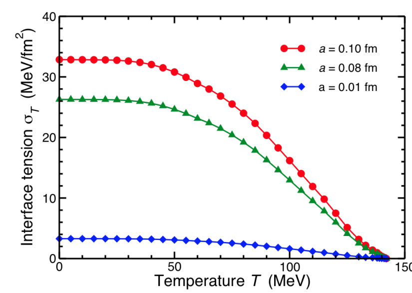
Thermodynamics:
large & uniform systems



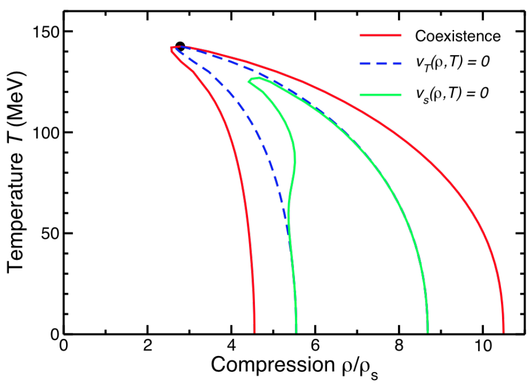




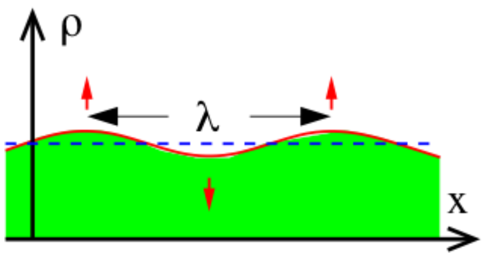
Small & non-uniform:
gradient effects,
interface tension



Lecture II: Phase separation (non-equilibrium)

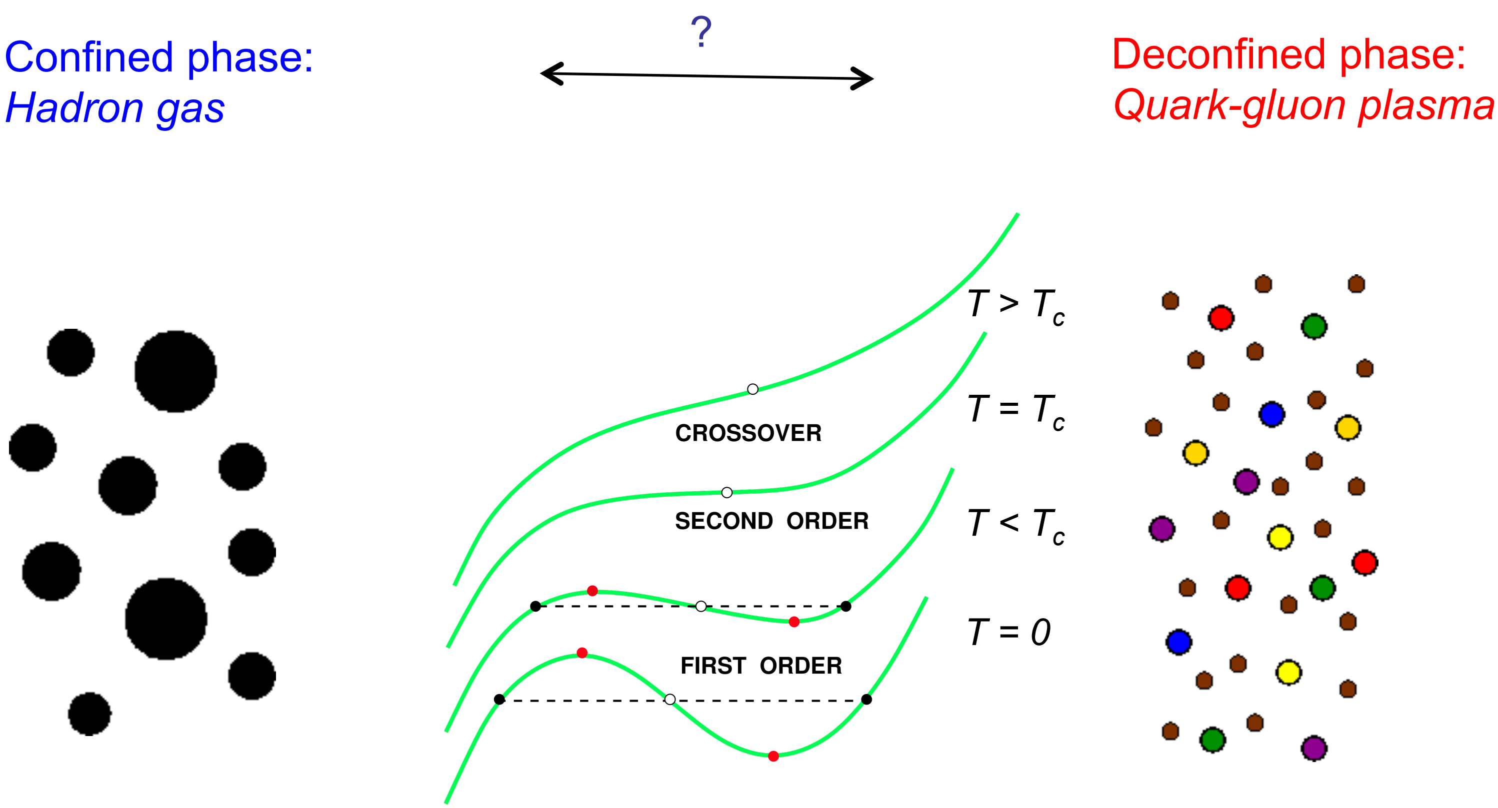


Equation of state:
interpolate between hadron gas and quark-gluon plasma



Spinodal instabilities:
fluid dynamics, growth rates

Equation of State



Hadron Gas versus *Quark-Gluon Plasma*

$$p^H = p_\pi + p_N + p_{\bar{N}} + p_w$$

Free pions, nucleons, and antinucleons:

$$p_\pi(T) = -g_\pi \int_{m_\pi}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 - \mathrm{e}^{-\beta\epsilon}]$$
$$p_N(T,\mu_0) = g_N \int_{m_N}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\epsilon-\mu_0)}]$$
$$p_{\bar{N}}(T,\mu_0) = g_N \int_{m_N}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\epsilon+\mu_0)}]$$

+ compressional energy density:

$$w(\rho) = \left[-A\left(\frac{\rho}{\rho_s}\right)^\alpha + B\left(\frac{\rho}{\rho_s}\right)^\beta\right]\rho$$
$$p_w(\rho) = \rho\partial_\rho w(\rho) - w(\rho)$$

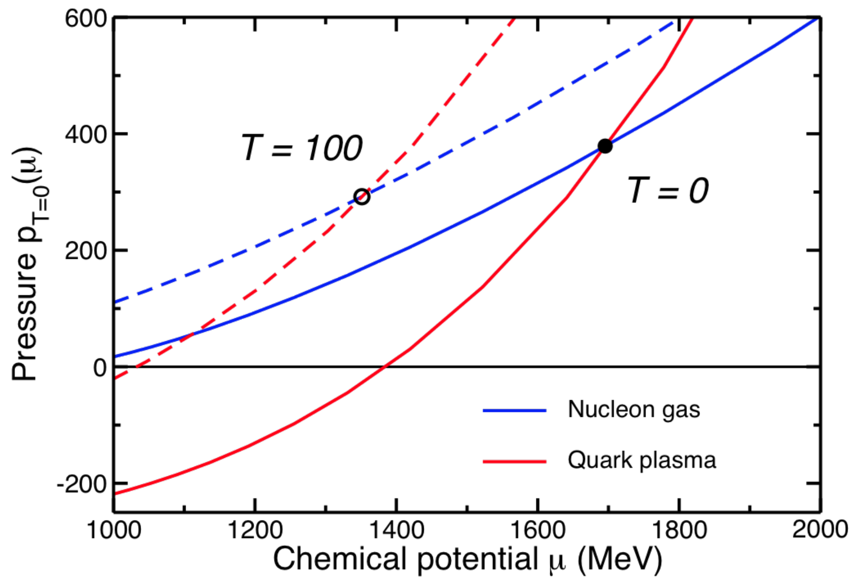
$$\mu = \mu_0 + \partial_\rho w = 3\mu_q$$

$$p^Q = p_g + p_q + p_{\bar{q}} - B$$

Free gluons, quarks, and antiquarks:

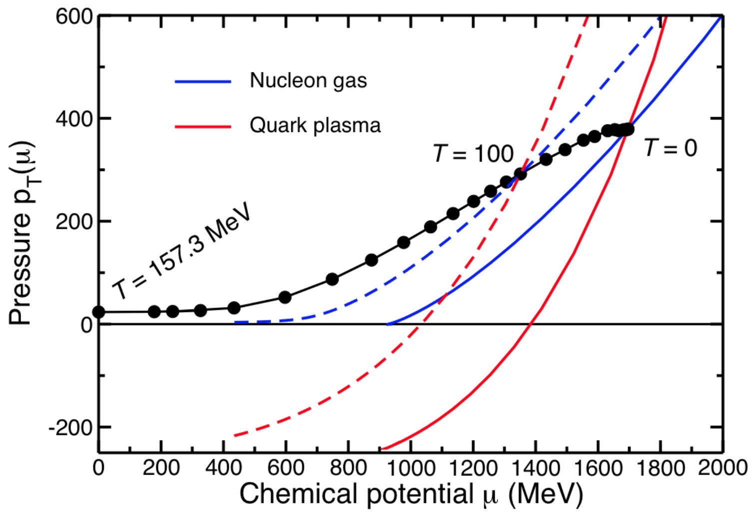
$$p_g = g_g \frac{\pi^2}{90} T^4$$
$$p_q + p_{\bar{q}} = g_q \left[\frac{7\pi^2}{360}T^4 + \frac{1}{12}\mu_q^2 T^2 + \frac{1}{24\pi^2}\mu_q^4\right]$$

Phase crossing:

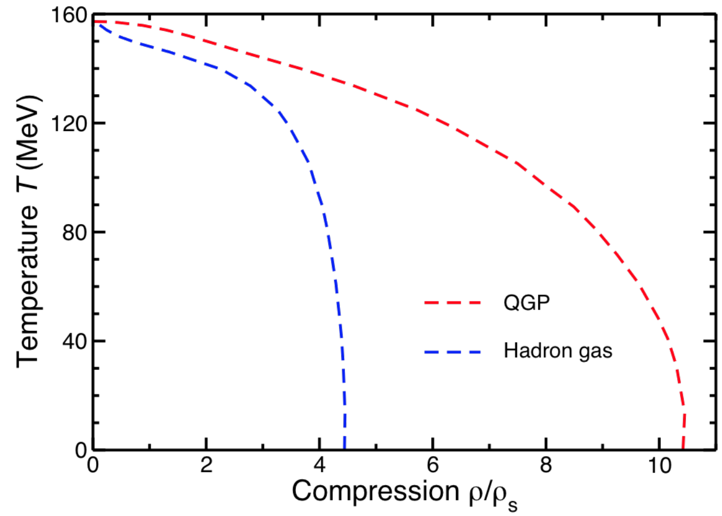


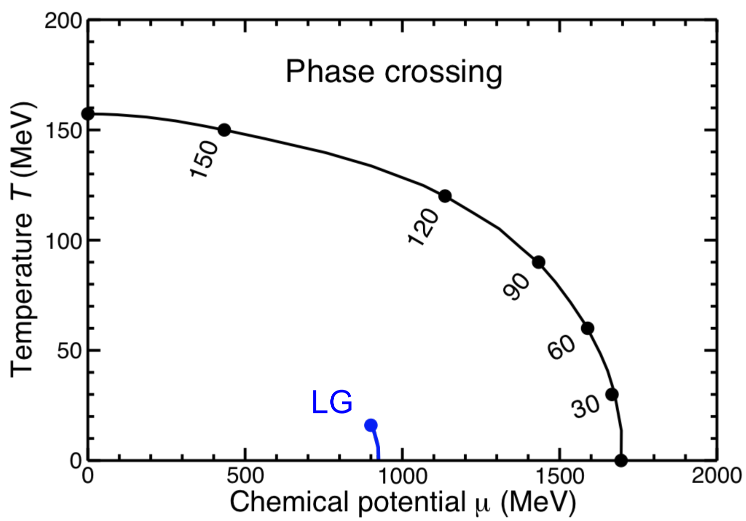
Hadron Gas versus Quark-Gluon Plasma

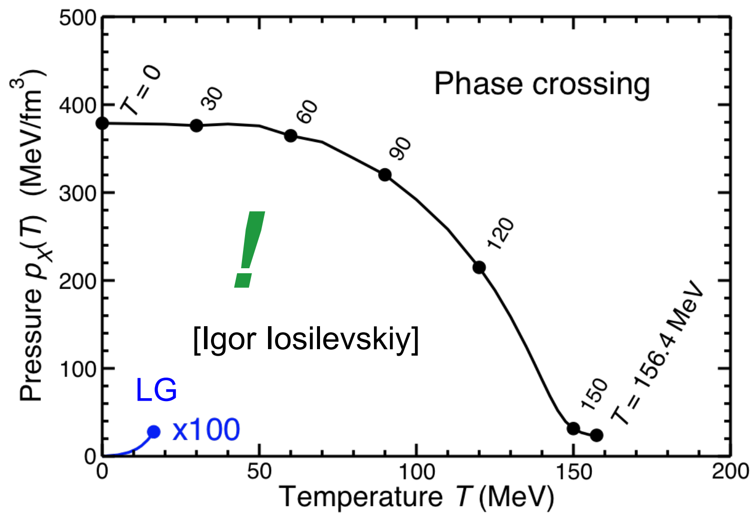
Phase crossing



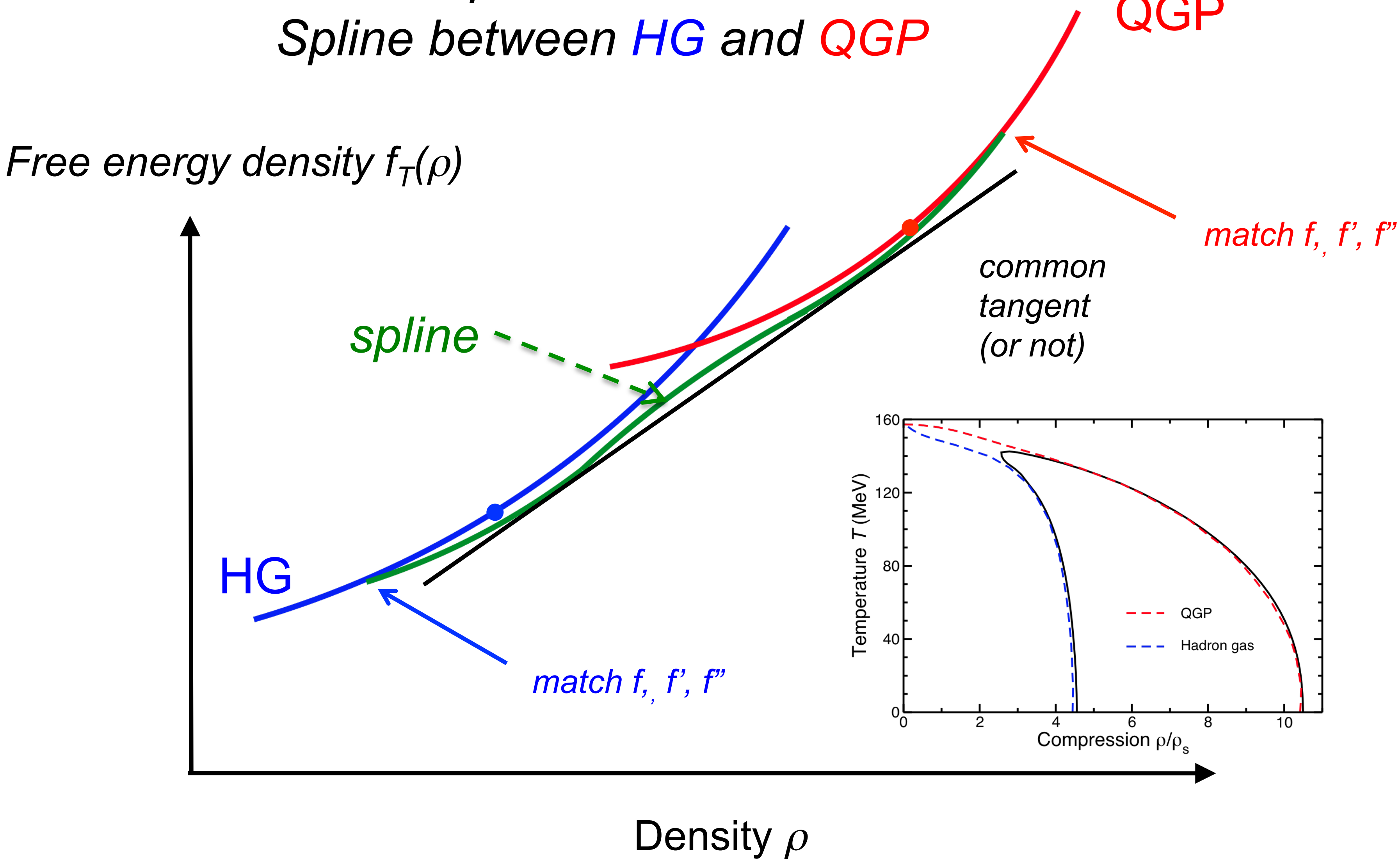
Phase "boundaries"



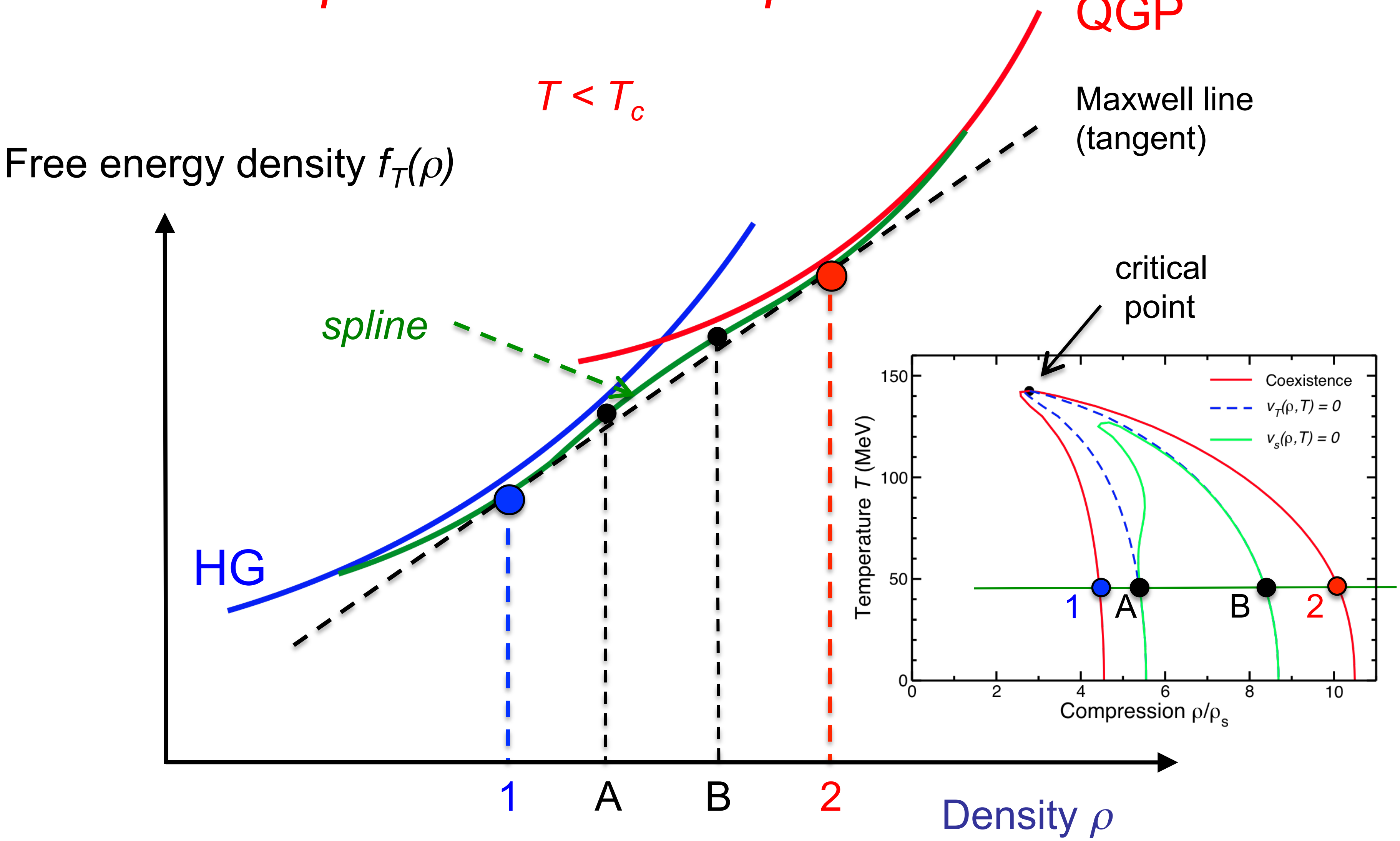




Equation of state: Spline between HG and QGP



Equation of State: spline



Thermodynamic relations

$$\sigma(\varepsilon,\rho) \qquad \pi = \sigma - \beta\varepsilon - \alpha\rho = p/T$$

$$\beta = \partial_\varepsilon \sigma(\varepsilon,\rho) = \sigma_\varepsilon = 1/T$$

$$\alpha = \partial_\rho \sigma(\varepsilon,\rho) = \sigma_\rho = -\mu/T$$

$$f = \varepsilon - T\sigma$$

$$\mu_T(\rho) = \partial_\rho f_T(\rho) \qquad p_T(\rho) = \rho\partial_\rho f_T(\rho) - f_T(\rho)$$

$$\sigma_T(\rho) = -\partial_T f_T(\rho) \qquad \varepsilon_T(\rho) = f_T(\rho) - T\partial_T f_T(\rho)$$

$$h_T(\rho) = p_T(\rho) + \varepsilon_T(\rho) = \rho\partial_\rho f_T(\rho) - T\partial_T f_T(\rho)$$ *Enthalpy density*

$$v_s^2 = \frac{\rho}{h}\left(\frac{\partial p}{\partial \rho}\right)_s = -\frac{T}{h}[h^2\sigma_{\varepsilon\varepsilon} + 2h\rho\sigma_{\varepsilon\rho} + \rho^2\sigma_{\rho\rho}]$$ *Isentropic sound speed*

$$v_T^2 = \frac{\rho}{h}\left(\frac{\partial p}{\partial \rho}\right)_T = -\frac{\rho}{h}\frac{\rho T}{\sigma_{\varepsilon\varepsilon}}[\sigma_{\varepsilon\varepsilon}\sigma_{\rho\rho} - \sigma_{\varepsilon\rho}^2]$$ *Isothermal sound speed*

Ideal fluid dynamics without conserved charges

$\eta, \zeta, \kappa = 0$

Energy-momentum tensor: $T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}$ $u^\mu = (\gamma, \gamma\mathbf{v})$

$$0 = \partial_\mu T^{\mu\nu} \begin{cases} \nu = 0: \quad 0 = \partial_\mu T^{\mu 0} = \partial_t(\varepsilon + pv^2)\gamma^2 + \partial_i(\varepsilon + p)\gamma^2 v^i & \text{E} \\ \nu = i: \quad 0 = \partial_\mu T^{\mu i} = \partial_t(\varepsilon + p)\gamma^2 v^i + \partial_j(\varepsilon + p)\gamma^2 v^j v^i + \partial^i p & \text{M} \end{cases}$$

Non-relativistic flow (v << 1):

$$\begin{cases} \nu = 0: \quad \partial_t \varepsilon = -\partial_i(\varepsilon + p)v^i & \text{E} \\ \nu = i: \quad \partial_t(\varepsilon + p)v^i = -\partial^i p & \text{M} \end{cases}$$

$$\partial_t E - \partial_i M: \quad \boxed{\partial_t^2 \varepsilon(x) = \partial_i \partial^i p(x)}$$

Sound equation

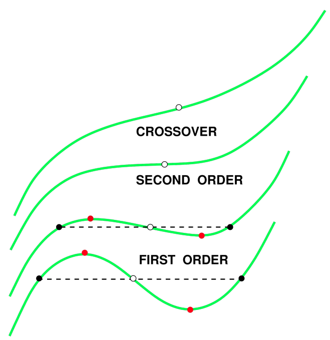
Equation of state: $p_0(\varepsilon)$

$$p(x) = p_0(\varepsilon(x)) \;\Rightarrow\; \partial_i \partial^i p(x) = \frac{\partial p_0(\varepsilon)}{\partial \varepsilon} \partial_i \partial^i \varepsilon(x)$$

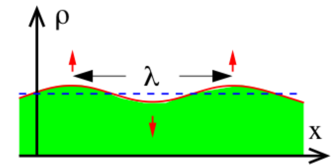
$$\boxed{\partial_t^2 \varepsilon = v_s^2 \nabla^2 \varepsilon}$$

$$v_s^2 \equiv \partial_\varepsilon p_0$$

(sound speed)^2



Evolution of small disturbances



Small disturbance in a uniform stationary fluid

$$\varepsilon(x,t) \;=\; \varepsilon_0 \;+\; \delta\varepsilon(x,t) \;,\; \delta\varepsilon \ll \varepsilon_0$$

=> $v << 1$

First order in $\delta\varepsilon$:

$$\begin{cases} \partial_t \delta\varepsilon(x,t) \;\approx\; (\varepsilon_0 + p_0)\partial_x v_x(x,t) \\ (\varepsilon_0 + p_0)\partial_t v_x(x,t) \;\approx\; \partial_x p(x,t) \;\approx\; \dfrac{\partial p_0}{\partial \varepsilon_0}\,\partial_x \delta\varepsilon(x,t) \end{cases}$$

Sound equation:

$$\partial_t^2 \delta\varepsilon(x,t) \;=\; \frac{\partial p_0}{\partial \varepsilon_0}\,\partial_x^2 \delta\varepsilon(x,t)$$

$$v_s^2 \;=\; \frac{\partial p}{\partial \epsilon}$$

Harmonic disturbance: $\delta\varepsilon_k(x,t) \;\sim\; \mathrm{e}^{ikx - i\omega_k t}$

Dispersion relation:

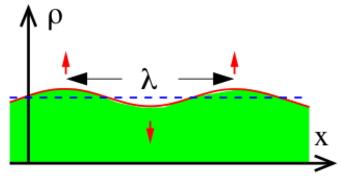
$$\omega_k^2 \;=\; v_s^2\, k^2$$

$$\begin{cases} v_s^2 > 0 : \quad \omega_k = \pm v_s k \\ v_s^2 < 0 : \quad \omega_k = \pm i\gamma_k = \pm i|v_s|k \end{cases}$$

Diverges for large k!

Ideal fluid dynamics with one conserved charge

$\eta, \zeta, \kappa = 0$



$$\left.\begin{array}{l}\varepsilon(x) = \varepsilon_0 + \delta\varepsilon(x)\\ \rho(x) = \rho_0 + \delta\rho(x)\end{array}\right\} \quad |v(x)| \ll 1$$

$T^{\mu\nu}(x)$: $\quad T^{00} \approx \varepsilon \qquad T^{i0} = T^{0i} \approx (\varepsilon + p)v^i \qquad T^{ij} = T^{ji} \approx \delta_{ij} p$

E $\quad 0 \doteq \partial_\mu T^{\mu 0} = \partial_t T^{00} + \partial_i T^{i0} = \partial_t \varepsilon + (\varepsilon + p)\partial_i v^i \quad \Rightarrow \quad \omega \varepsilon_k \doteq (\varepsilon_0 + p_0) k v_k$

M $\quad 0 \doteq \partial_\mu T^{\mu i} = \partial_t T^{0i} + \partial_j T^{ji} = (\varepsilon + p)\partial_t v^i + \partial_j T^{ji}$

C $\quad \partial_t \rho \doteq -\rho \partial_i v^i \quad \Rightarrow \quad \omega \rho_k \doteq \rho_0 k v_k$ *Continuity equation*

E & C => $\quad (\varepsilon_0 + p_0)\rho_k = \rho_0 \varepsilon_k$ *ρ tracks ε when κ=0*

$\partial_t E - \partial_i M \quad \Rightarrow \quad \partial_t^2 \varepsilon = \partial_i \partial_j T^{ji} = \partial_i \partial^i p$ => $\omega^2 \varepsilon_k = k^2 p_k$ *Sound equation*

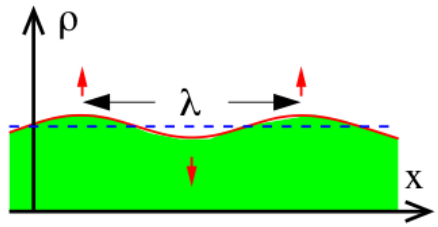
$$p(\varepsilon, \rho) \;\Rightarrow\; p_k = \frac{\partial p}{\partial \varepsilon}\varepsilon_k + \frac{\partial p}{\partial \rho}\rho_k = [\frac{\partial p}{\partial \varepsilon} + \frac{\rho_0}{\varepsilon_0 + p_0}\frac{\partial p}{\partial \rho}]\varepsilon_k = v_s^2 \varepsilon_k \qquad v_s^2 \equiv \frac{\rho}{\varepsilon + p}\left(\frac{\partial p}{\partial \rho}\right)_s$$

=> $\quad \omega_k^2 = v_s^2\, k^2$ => $\boxed{\gamma_k = |v_s| k}$ *Dispersion relation*

Diverges for large k!

Inclusion of gradient correction

$$\tilde{f}_T(\boldsymbol{r}) = f_T(\tilde{\rho}(\boldsymbol{r})) + \tfrac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2$$



=> $$p(\boldsymbol{r}) \approx p_0(\varepsilon(\boldsymbol{r}),\rho(\boldsymbol{r})) - C\rho_0\nabla^2\rho(\boldsymbol{r})$$

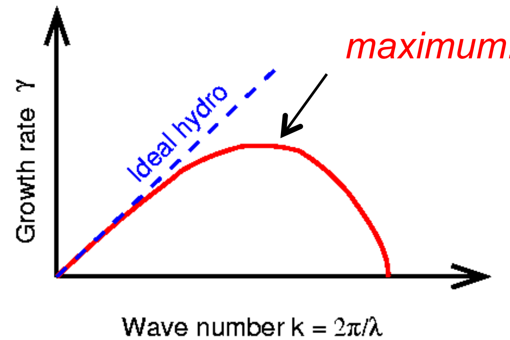
$$\rho(r,t) = \rho_0 + \delta\rho(x,t) \doteq \rho_0 + \rho_k \mathrm{e}^{ikx-i\omega t}$$

=> $$p_k \;\longrightarrow\; p_k + C\rho_0 k^2\rho_k \qquad = [v_s^2 + C\frac{\rho_0^2}{\varepsilon_0+p_0}k^2]\varepsilon_k$$

resists wriggling

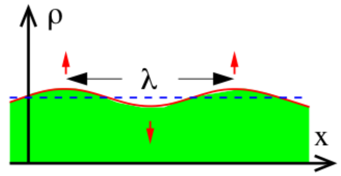
=> $$\omega_k^2 = v_s^2 k^2 + C\frac{\rho_0^2}{\varepsilon_0+p_0}k^4$$

=> $$\gamma_k^2 = |v_s^2|k^2 - C\frac{\rho_0^2}{\varepsilon_0+p_0}k^4$$



Viscous fluid dynamics with one conserved charge

$\eta, \zeta > 0$
$\kappa = 0$



$$\left.\begin{array}{l}\varepsilon(x) = \varepsilon_0 + \delta\varepsilon(x) \\ \rho(x) = \rho_0 + \delta\rho(x)\end{array}\right\} \; |v(x)| \ll 1$$

$T^{\mu\nu}(x)$:

$$\left\{\begin{array}{l} T^{00} \approx \varepsilon \qquad T^{i0} = T^{0i} \approx (\varepsilon + p)v^i \\ T^{ij} = T^{ji} \approx \delta_{ij}p - \eta[\partial_i v^j + \partial_j v^i - \tfrac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}] - \zeta\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}\end{array}\right.$$

=> $\boldsymbol{\nabla}\cdot\boldsymbol{T} \approx \boldsymbol{\nabla}p - \eta\boldsymbol{\Delta}v - [\tfrac{1}{3}\eta + \zeta]\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})$

E $\quad 0 \doteq \partial_\mu T^{\mu 0} = \partial_t T^{00} + \partial_i T^{i0} = \partial_t\varepsilon + (\varepsilon + p)\partial_i v^i \quad \Rightarrow \quad \omega\varepsilon_k \doteq (\varepsilon_0 + p_0)kv_k$

M $\quad 0 \doteq \partial_\mu T^{\mu i} = \partial_t T^{0i} + \partial_j T^{ji} = (\varepsilon + p)\partial_t v^i + \partial_j T^{ji}$

C $\quad \partial_t\rho \doteq -\rho\partial_i v^i \quad \Rightarrow \quad \omega\rho_k \doteq \rho_0 kv_k$ *Continuity equation*

E & C => $\quad (\varepsilon_0 + p_0)\rho_k = \rho_0\varepsilon_k$ *ρ tracks ε when κ=0*

$\partial_t E - \partial_i M \quad \Rightarrow \quad \partial_t^2\varepsilon = \partial_i\partial_j T^{ji} = \partial_i\partial^i[p - (\tfrac{\mathbf{4}}{\mathbf{3}}\eta + \zeta)\partial_j v^j]$ *Sound equation*

=> $\quad \omega^2\varepsilon_k = k^2 p_k - i\xi k^3 v_k = v_s^2 k^2\varepsilon_k - i\xi\dfrac{\omega}{\varepsilon_0 + p_0}k^2\varepsilon_k$ $\qquad \xi \equiv \tfrac{4}{3}\eta + \zeta$

=> $$\gamma_k^2 = |v_s^2|k^2 - C\frac{\rho_0^2}{\varepsilon_0 + \rho_0}k^4 - \xi\frac{k^2}{\varepsilon_0 + p_0}\gamma_k$$ *Dispersion relation*

JRandrup: Dubna School II, 2012

$\eta, \zeta, \kappa > 0 \longrightarrow$ *Dissipative fluid dynamics*

Energy-momentum tensor:

$$T^{00} \approx \varepsilon \quad \& \quad T^{0i} \approx (\varepsilon + p)v^i + q^i \quad \&$$

A Muronga, PRC 76, 014909 (2007)

$$T^{ij} \approx \delta_{ij} p - \eta[\partial_i v^j + \partial_j v^i - \tfrac{2}{3}\delta_{ij}\partial^k v^k] - \zeta \delta_{ij}\partial^k v^k$$

$$|\rho_k| \ll \rho_0 \;\Rightarrow\; |v| \ll 1$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} \;\approx\; \boldsymbol{\nabla} p - \eta \Delta \boldsymbol{v} - [\tfrac{1}{3}\eta + \zeta]\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \;\asymp\; \partial_x p - [\tfrac{4}{3}\eta + \zeta]\partial_x^2 v$$

Eckart frame

Equations of motion:

$$C: \; \partial_t \rho \;\doteq -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v} \;\Rightarrow\; \omega \rho_k \doteq \rho_0 k v_k$$ charge

$$\boldsymbol{M}: \; h_0 \partial_t \boldsymbol{v} \;\doteq -\boldsymbol{\nabla}[p - \zeta \boldsymbol{\nabla} \cdot \boldsymbol{v}] - \boldsymbol{\nabla} \cdot \boldsymbol{\pi} - \partial_t \boldsymbol{q}$$ momentum

$$E: \; \partial_t \varepsilon \;\doteq -h_0 \boldsymbol{\nabla} \cdot \boldsymbol{v} - \boldsymbol{\nabla} \cdot \boldsymbol{q}$$ energy

Sound equation:

$$\partial_t E - \boldsymbol{\nabla} \cdot \boldsymbol{M}: \; h_0 \partial_t^2 \varepsilon \;\doteq\; \Delta[p - \zeta \boldsymbol{\nabla} \cdot \boldsymbol{v}] + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\pi})$$

=> $$\omega^2 \varepsilon_k \;\doteq\; k^2 p_k - i[\tfrac{4}{3}\eta + \zeta]\frac{\omega}{\rho_0} k^2 \rho_k$$

$$\xi \equiv \tfrac{4}{3}\eta + \zeta$$

Heat flow:

$$\boldsymbol{q} \approx -\kappa[\boldsymbol{\nabla} T + T_0 \partial_t \boldsymbol{v}]: \; q_k = -i\kappa[k T_k - \frac{T_0}{\rho_0}\frac{\omega^2}{k}\rho_k]$$

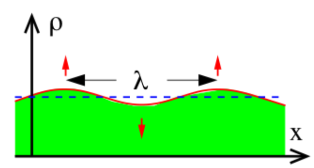
$$T_k \approx \frac{1}{1 + i\kappa k^2/\omega c_v} \frac{T_0}{\rho_0} \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \rho_k$$

Equation of state:

$$p_T(\rho) \;\Rightarrow\; p_k = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho c_v T_k + \frac{h_0}{\rho_0} v_T^2 \rho_k$$

Dispersion equation:

$$\omega^2 \;\doteq\; v_T^2 k^2 + C\frac{\rho_0^2}{h_0} k^4 - i\xi \frac{\omega}{h_0} k^2 + \frac{v_s^2 - v_T^2}{1 + i\kappa k^2/\omega c_v} k^2$$



Transport coefficients

$$\eta_0 \geq 1$$
$$\kappa_0 \geq 1$$

1) *Bulk viscosity ζ: Ignore* $\quad \zeta \ll \eta \Rightarrow \xi \equiv \frac{4}{3}\eta + \zeta \approx \frac{4}{3}\eta$

2) *Shear viscosity η:*

$$\rho = 0: \quad h \equiv p + \varepsilon = T\sigma$$

*) $$\rho = 0: \quad \eta \geq \frac{\hbar}{4\pi}\sigma = \frac{\hbar}{4\pi}\frac{h}{T}$$

$$\eta(\rho,T) = \eta_0 \frac{c_0}{c} d(\rho,T)\, h(\rho,T)$$

$$\rho > 0,\ T \ll mc^2: \ h \asymp mc^2 n \gg T\sigma$$

$$\rho = 0: \quad n \sim T^3 \Rightarrow \frac{\hbar c}{T} = 4\pi c_0\, d$$

$$d \equiv n^{1/3}$$

$$\lambda_{\rm visc} \equiv \frac{1}{c}\frac{\xi(\rho,T)}{h(\rho,T)/c^2} \approx \tfrac{4}{3}\eta_0\, c_0\, d(\rho,T)$$

3) *Heat conductivity κ:*

$$\eta \approx \tfrac{1}{3} n\bar{p}\ell \qquad \kappa \approx \tfrac{1}{3}\bar{v}\ell c_v$$

$$\frac{\kappa}{\eta} \approx \frac{c_v}{h/c^2}$$

$$\bar{p} = m\bar{v} \qquad c_v \equiv \partial_T \varepsilon_T(\rho)$$

$$h \asymp mc^2 n \qquad c_v \asymp \frac{3}{2}n$$

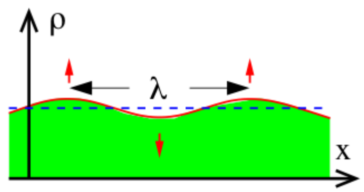
$$\kappa(\rho,T) = \kappa_0\, c_0\, c\, d(\rho,T)\, c_v(\rho,T)$$

$$\lambda_{\rm heat} \equiv \frac{1}{c}\frac{\kappa(\rho,T)}{c_v(\rho,T)} = \kappa_0\, c_0\, d(\rho,T)$$

$$c_0 = \frac{1}{4\pi}\left[(g_g + \tfrac{3}{3}g_q)\frac{\zeta(3)}{\pi^2}\right]^{\frac{1}{3}} \approx 0.12779$$

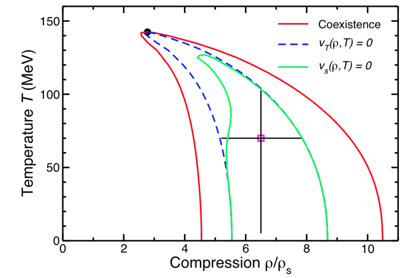
*) V Koch & J Liao, PRC81, 014902 (2010)

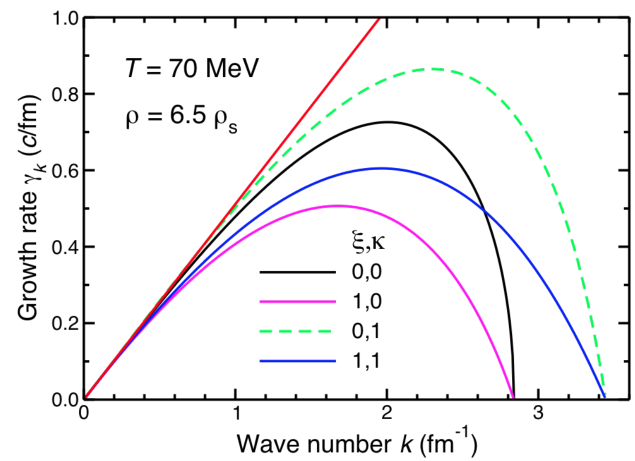
Evolution of density fluctuations with dissipative fluid dynamics



$A_k \sim \exp(-i\omega_k t)$

$$\omega_k \;=\; \epsilon_k + i\gamma_k$$





Dispersion equations:

Ideal fluid dynamics: $\omega^2 \doteq v_s^2 k^2$

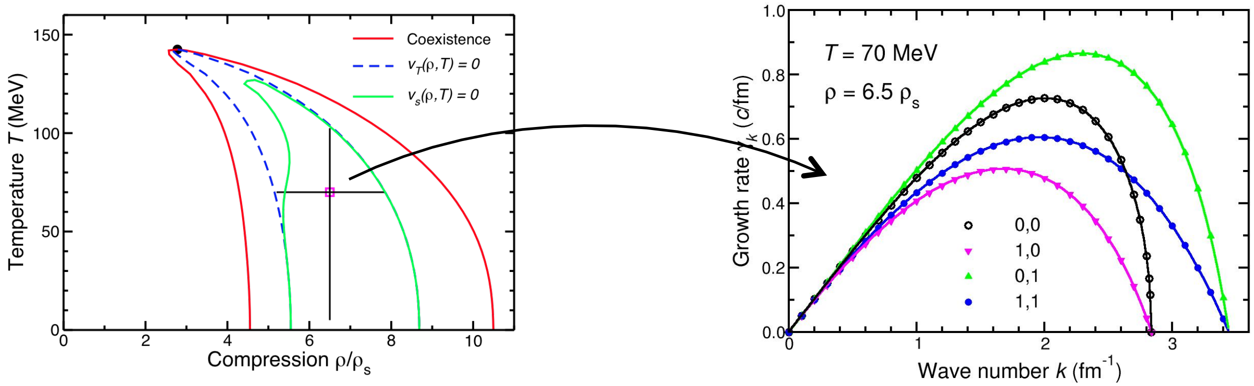
\+ gradient term: $\omega^2 \doteq v_s^2 k^2 + C\frac{\rho_0^2}{h_0}k^4$ <= $p_k \longrightarrow p_k + C\rho_0 k^2 \rho_k$

\+ shear & bulk viscosity: $\omega^2 \doteq v_s^2 k^2 + C\frac{\rho_0^2}{h_0}k^4 - i\xi\frac{\omega}{h_0}k^2$ <= $\xi \equiv \frac{4}{3}\eta + \zeta$

\+ heat conduction: $\omega^2 \doteq v_T^2 k^2 + C\frac{\rho_0^2}{h_0}k^4 - i\xi\frac{\omega}{h_0}k^2 + \frac{v_s^2 - v_T^2}{1 + i\kappa k^2/\omega c_v}k^2$ <= κ

Spinodal growth rates

(ρ,T): $\omega^2 \doteq v_T^2 k^2 + C\frac{\rho_0^2}{h_0}k^4 - i\xi\frac{\omega}{h_0}k^2 + \frac{v_s^2 - v_T^2}{1 + i\kappa k^2/\omega c_v}k^2$ => $\gamma_k(\rho,T)$



Fastest growth rates: