Lattice QCD at $\mu \neq 0$

Part II

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Outline :

1. Introduction
2. A Look inside Lattice Gauge Theory
3. Introducing Chemical Potentials
4. Fighting the Complex Measure Problem in Standard LGT
5. Results for Standard LGT
6. LGT-Motivated Effective Models on the Lattice
7. Complex Langevin Simulation
8. Summary and Conclusions

6. LGT-Motivated Effective Models on the Lattice (N colors)

- **Lattice, but not Lattice Gauge Theory: The Flux Tube Model**
- **$\mathbb{Z}(N)$ and $SU(N)$ Spin Models in 3D as "derived" from LGT**
- **Mean Field Solution of the Spin Models**
- **Flux Representation for the Spin Models**
- **A Model with Explicitly Chiral Quarks**

B. Svetitsky and L. Yaffe,
"Critical Behavior at Finite Temperature Confinement Transitions",
Nucl.Phys. B210 (1982) 423
postulated universality classes of the pure Yang Mills deconfinement transitions (sharing the second vs. first order of the transitions)

$$SU(2) \text{ deconfinement} \longleftrightarrow \text{3-dim } Z(2) \text{ spin model (Ising model)}$$
$$SU(3) \text{ deconfinement} \longleftrightarrow \text{3-dim } Z(3) \text{ spin model (Potts model)}$$
$$SU(N) \text{ deconfinement} \longleftrightarrow \text{3-dim } Z(N) \text{ spin model (as Potts models)}$$

This paradigm has led to build 3D spin models directly in terms of $\mathbb{Z}(N)$ spin variables ("Center-Valued Polyakov Spin Models") to be used as "starter kit" to add features representing quarks, including $\mu \neq 0$.

6. 1. Lattice, but not Lattice Gauge Theory: The Flux Tube Model

For the case of finite baryonic density, "Flux Representations" have been proposed for the effective 3D Spin Models on a 3D lattice in order to circumvent the sign problem inherited from the 4D Lattice Gauge Theory.

They should not be confused with the flux tube models independently proposed for the same problem to be discussed now.

- These are static models on a 3-dimensional lattice, without Euclidean "time".
- Therefore they are resembling conventional Statistical Mechanics, even more simplified by the neglect of kinetic energy (!)
- They are sub-hadronic in the sense that they are formulated in terms of static quarks and antiquarks (with mass).

Other energy-carrying degrees of freedom (part of the gluon field) are :

- **string bits (connect quark with antiquark or both with a vertex)**
- **vertices (connect N strings with each other, only for $N \geq 3$).**

Few papers

1. **"More on the Flux Tube Model of the Deconfining Transition", A. Patel, Phys. Lett. B139 (1984) 394**
2. **"Potts flux tube model at nonzero chemical potential", J. Condella and C. E. DeTar, Phys. Rev. D61 (2000) 074023, e-print: hep-lat/9910028**
3. **"Flux Tube Model Signals for Baryon Correlations in Heavy Ion Collisions", A. Patel, Phys. Rev. D85 (2012) 114019, e-print: arXiv:1111.0177**

Only these features/results of Lattice Gauge Theory are used :

- existence of (electric) flux tubes (strings, known from extremely elongated mesonic states)
- existence of baryonic vertices (a. k. a. "junctions", evidence from lattice studies telling the Y-shape from the Δ-shape of baryons)

The model is formulated on a (coarse) 3-dimensional lattice to provide a basic length scale (the lattice spacing a), a configuration entropy for bending of the strings and the possibility to reformulate it as a spin model.

"Vacuum structure" is considered as a network of strings, that changes its global structure with temperature while keeping the basic elements unchanged.

Unrealistic : before the string breaks, its tension and width change !

An eventual temperature dependence of string tension and vertex energy was not considered. It could be adapted, however, either

- **"microscopically" by LGT calculations (would show the problem !)**
- **"macroscopically" by fitting the thermodynamics (no predictive power)**

Even without such adaptation, the emerging XY or $\mathbb{Z}(N)$ models have a T-dependence already built in through coupling constants behaving like

$$J \propto \exp(-\sigma a/T)$$
$$h \propto \exp(-m/T)$$
$$w \propto \exp(-v/T)$$

which exactly reflects the contributions to the energy balance.

A system of links and sites is randomly occupied :

Links : occupation numbers on links (x, ν) :

- $l^{\pm}_{x,\nu}$ **in positive or negative direction**

Sites : occupation numbers at sites x :

- $p^{\pm}_{x}$ **of quarks/antiquarks**
- $q^{\pm}_{x}$ **of vertices/antivertices**

Energy balance : basic energy units :

- **energy of a "string bit"** σa
- **the mass of a quark** m
- **the "mass"** v **of a vertex (string junction)**

The energy of a "flux tube configuration" can be written :

$$E = \sigma a \sum_{x,\nu} \left(l^+_{x,\nu} + l^-_{x,\nu} \right) + m \sum_x \left(p^+_x + p^-_x \right) + v \sum_x \left(q^+_x + q^-_x \right)$$

The Gauss law results in a constraint :

$$\sum_{\nu=1}^{3} \left(l^+_{x,\nu} - l^+_{x-\hat{\nu},\nu} - l^-_{x,\nu} + l^-_{x-\hat{\nu},\nu} \right) - p^+_x + p^-_x + N \left(q^+_x - q^-_x \right) = \alpha_x = 0$$

The partition function now reads as a constrained sum :

$$Z_{fluxtube} = \sum_{\{n,p,q\}} \exp\left(-\frac{E}{T} \right) \prod_x \delta_{\alpha_x,0}$$

Writing the constraint as

$$\delta_{\alpha_x,0} = \int_{-\pi}^{+\pi} \frac{d\vartheta_x}{2\pi} \mathrm{e}^{i\alpha_x \vartheta_x}$$

the sums over the n, p and q can be performed.

A new version of the model emerges, an extended 3-dim XY-model :

$$Z_{XY} = \int_{-\pi}^{+\pi} \prod_x d\vartheta_x \exp\left[J \sum_{x,\nu} \cos(\vartheta_{x+\hat{\nu}} - \vartheta_x) + h \sum_x \cos(\vartheta_x) + w \sum_x \cos(N\vartheta_x) \right]$$

with two "magnetic fields" h ($\mathbb{Z}(N)$ breaking) and w ($\mathbb{Z}(N)$ symmetric) and one spin coupling J with a temperature dependence

$$\begin{aligned} J &= 2\exp\left(-\frac{\sigma a}{T}\right) \\ h &= 2\exp\left(-\frac{m}{T}\right) \\ w &= 2\exp\left(-\frac{v}{T}\right) \end{aligned}$$

The "action" is still real-valued !

If vertices do not cost energy ($v = 0$), the corresponding occupation numbers q do not occur in the energy balance.

Then, the Gauss law without vertices can be simplified to a constraint valid only modulo N :

$$\sum_{\nu=1}^{3}\left(l^{+}_{x,\nu}-l^{+}_{x-\hat{\nu},\nu}-l^{-}_{x,\nu}+l^{-}_{x-\hat{\nu},\nu}\right)-p^{+}_{x}+p^{-}_{x}=\beta_x=0 \ \textbf{mod}\ N$$

Writing the new constraint now as a "modulo-Kronecker symbol"

$$\delta_{\alpha_x,0}\ \textbf{mod}\ {}_N=\frac{1}{N}\sum_{k_x}\left(z_{k_x}\right)^{\beta_x}=\frac{1}{N}\sum_{k_x}\mathrm{e}^{i\beta_x\phi_x}$$

we get an $\mathbb{Z}(N)$ spin model with the partition function :

$$Z_{center}=\prod_{x}\left(\frac{1}{N}\sum_{k_x=0}^{N-1}\right)\exp\left(J\sum_{x,\mu}\cos(\phi_{x+\hat{\nu}}-\phi_x)+h\sum_{x}\cos(\phi_x)\right)$$

with angles ϕ_k that are restricted to center values $\phi_x=2\pi k_x/N$.

This is a $\mathbb{Z}(N)$ spin model, however, in the clock-model version, only one $\mathbb{Z}(N)$ breaking magnetic field h left (growing with quark mass).

A few remarks :

- **The clock model is the wrong (clock) $\mathbb{Z}(N)$ model for $N > 3$.**
- **Here, the $Z(N)$ spins are *not understood* as Polyakov loops.**
- **Polyakov loops have no place here (no "time", no gluon field).**
- **Still, the heavy quark free energy fulfills $\exp\left(-F_q/T\right) = \left\langle \exp\left(-i\phi_x\right)\right\rangle$.**
- **There is no relation to chiral symmetry breaking or restoration.**
- **Phase transitions can be studied through $\exp\left(-F_q/T\right)$ and via signals showing up in the specific heat, quark number susceptibility etc.**
- **This model was intended to describe the parameter range where a first order, resp. second order finite temperature deconfinement transition could be realized.**

- From hadron mass estimates within the XY-model, non-zero vertex (junction) energy $v = 150$ MeV seems to be more realistic.
- From this point of view, the XY-model is eventually more realistic.
- A description of hadronization has been attempted in Ref. [3] as break-up of the flux-tube network into pieces.

We are interested in the next step [2] : the extension to finite μ !

The model has no "time", therefore the Euclidean way of introducing a chemical potential μ is not possible.

But direct counting the baryon charge of a configuration is possible :

quark occupation number $n_x = -3, -2, -1, 0, +1, +2, +3$

color electric flux $l_{x,\nu} = -1, 0, +1$

Gauss' law holds modulo 3 : triality

$$\sum_{\nu=1}^{3} (l_{x,\nu} - l_{x,-\nu}) = n_x \ \textbf{mod}\ 3$$

energy of the flux tube network

$$E = \sum_{\{x,\nu\}} \sigma a \, |l_{x,\nu}| + \sum_{x} m \, |n_x|$$

total baryon number :

$$B = \sum_x n_x$$

partition function with triality constraint :

$$Z_{fluxtube} = \sum_{\{l,n\}'} \exp\left(-\beta(E - \mu B)\right) \delta_{n_x,0 \ \mathbf{mod}\ 3}$$

with the Gauss constraint restricting the sum over $\{l, n\}'$

lowest states of the model :

- **meson : quark and antiquark separated by one lattice spacing**
- **(anti)baryon : three quarks or antiquarks sitting on one site**

excited states of the model :

- **excited mesons : separated by longer strings**
- **exotic mesons : diquark and antidiquark separated by a short string**
- **excited (anti)baryons : quark and diquark (or antiquark and antidiquark) separated by short string**

others :

- **glueball : a plaquette of flux links (fixes the lattice spacing !)**

Represented by a Potts model with complex magnetic field, like that later derived from LGT, but here derived from the flux tube model :

The original flux tube model : (without complex weight if $\mu = 0$)

$$Z_{fluxtube} = \sum_{z_x} \prod_{\{x,\nu\}} \left(\sum_{l_{x,\nu}} \exp(-\beta\sigma|l_{x,\nu}|)(z_x \cdot z^{\star}_{x+\hat{\nu}})^{l_{x,\nu}} \right) \prod_x \left(\sum_{n_x} \exp[-\beta(m|n_x| - \mu n_x)] z_x^{-n_x} \right)$$

The Potts spin model for $N = 3$: (now with complex action !)

$$Z_{Potts} = \sum_{z_x} \exp\left(\beta' J \sum_{\{x,\nu\}} \mathrm{Re}(z_x z^{\star}_{x+\hat{\nu}}) + \beta' h \sum_x \mathrm{Re} z_x + i\beta' h' \sum_x \mathrm{Im} z_x \right)$$

- **For $\mu = 0$, the imaginary part of the magnetic field, h', vanishes, the action becomes real-valued.**
- **Then, if moreover $m = \infty$, the real part of the magnetic field, h, vanishes, too. Resembles pure gauge theory.**
- **Then the Potts model has a first order transition at $\beta' J = 0.36703$ corresponding to $\beta\sigma = 1.6265$ (mimicks pure gluodynamics)**
- **The first order transition persists at moderately heavy quarks.**
- **The critical endpoint corresponds to $\beta' J = 0.365$ and $\beta' h \in [0.002, 0.01]$ corresponding to $\beta\sigma = 1.632$ and a quark mass range $\beta m \in [3.3, 4.2]$**

At $\mu \neq 0$ simulations have been performed at $m/\sigma = 5$, (where at $\mu = 0$ there is a first order transition), in order to compare vanishing baryonic density, $\mu/\sigma = 0$, with finite density $\mu/\sigma = 1.75$.

- **Standard Monte Carlo simulations in the occupation number basis of $Z_{fluxtube}$.**
- **realized as standard Metropolis algorithm with local moves : adding or subtracting one of the following color-singlet hadrons**
  - **q-link-$\bar{q}$,**
  - **qq-link-$\bar{q}\bar{q}$**
  - **q-link-qq and $\bar{q}$-link-$\bar{q}\bar{q}$**
  - **plaquette**

  Metropolis-checked with respect to constraints and energy at given T.

Search for transition as function of $\beta\sigma$ looking for signals E and B, i.e. in the specific heat

$$C_V/\beta^2 = \left(\langle E^2\rangle - \langle E\rangle^2\right)/L^3$$

and quark number susceptibility

$$\chi_q = \frac{\partial\langle B\rangle}{\partial\mu} = 9\left(\langle B^2\rangle - \langle B\rangle^2\right)/L^3$$

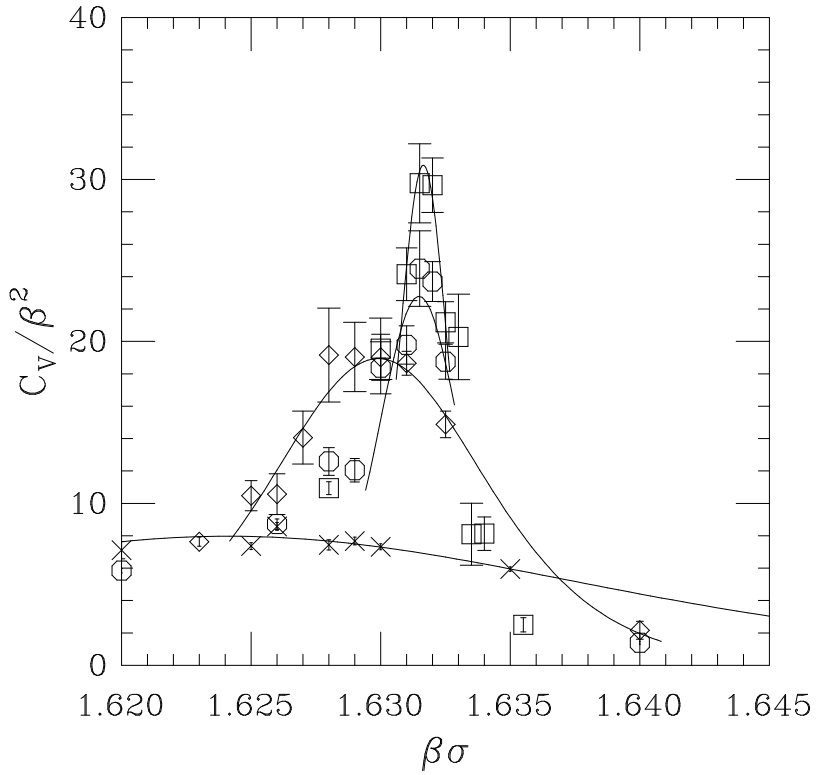
Typical result :

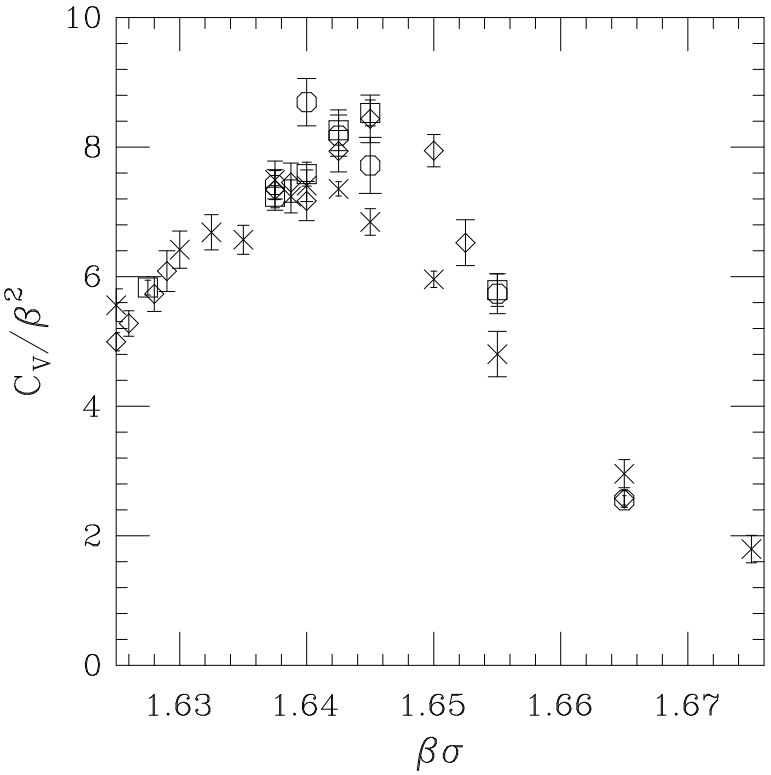
- **transition (which is first order at $\mu = 0$) turns into crossover,**
- **peak of specific heat is not rising anymore with volume,**
- **a weaker signal also in the quark number susceptibilty.**

Not studied intensively, because of the absence of chiral features.

The specific heat at the phase transition in Flux Tube Model, left at $\mu = 0$, right at $\mu/T = 1.75$

from hep-lat/9910028 Condella and DeTar





6. 2. $\mathbb{Z}(N)$ and $SU(N)$ Spin Models in 3D as "derived" from LGT

The spin variables are now explicitly representing Polyakov loops. This is a remnant from the gluon dynamics in Lattice Gauge Theory.

1. **Case of $SU(2)$**

   Ising spins, replacing $SU(2)$ pure gauge theory, located at sites $\vec{x}$.

   Action $S_G = H[s]$ ("energy of the spin system")

$$S_G[s] = -J \sum_{<\vec{x}\vec{y}>} z_{\vec{x}} \cdot z_{\vec{y}} \quad \textbf{for} \quad s_{\vec{x}} \in \{0, 1\}$$

   The $\mathbb{Z}(2)$ spins take values of the square roots of unity, expressed by the integers $s_{\vec{x}} \in (0, 1)$: $z_{\vec{x}} = (-1)^{s_{\vec{x}}} \in \mathbb{Z}(2)$.

   The coupling J is rising with temperature, say $J \propto \exp\left(-\frac{\sigma a}{T}\right)$, $\sigma =$ string tension, $a =$ lattice spacing.

2. Case of $SU(3)$

3-states Potts spins, replacing $SU(3)$ pure gauge theory, located at sites $\vec{x}$. For $N = 3$, Potts and clock model are the same :

Action $S_G = H[s]$ ("energy of the spin system")

$$S_G[s] = -J \sum_{<\vec{x}\vec{y}>} \delta_{s_{\vec{x}}, s_{\vec{y}}} \qquad \text{for} \quad s_{\vec{x}} \in \{-1, 0, 1\}$$

$$= -J' \sum_{<\vec{x}\vec{y}>} \left\{ 1 - \mathrm{Re} \ \left(z_{\vec{x}} \ z^{\star}_{\vec{y}} \right) \right\}$$

The $\mathbb{Z}(3)$ spins take values of the 3-rd roots of unity, expressed by the integers $s_{\vec{x}}$: $z_{\vec{x}} = \mathrm{e}^{i \frac{2\pi}{3} s_{\vec{x}}} \in \mathbb{Z}(3)$.

The coupling J is expressed through σ, a and T as for $SU(2)$.

Similarly for higher N : $\mathbb{Z}(N)$ resp. $SU(N)$ models.

However, there is a "clock vs. Potts model ambiguity" for $N > 3$!

"Does energy reflect the non-equality of neighbor spins or the angle ?"

The expectation values of $\mathbb{Z}(N)$-spins are then interpreted as Polyakov loop expectation values, and related observables, for example Polyakov loop correlators etc., can be evaluated, too :

$$\begin{aligned} \langle\ (1/3)\ \mathrm{tr}\ \ P_{\vec{x}} \rangle_{QCD} &\longleftrightarrow \langle z_{\vec{x}} \rangle_{\mathbb{Z}(3)\ \ spin\ \ model} \\ \left\langle\ (1/3)\ \mathrm{tr}\ \ P^{\dagger}_{\vec{x}} \right\rangle_{QCD} &\longleftrightarrow \langle z^{\star}_{\vec{x}} \rangle_{\mathbb{Z}(3)\ \ spin\ \ model} \end{aligned}$$

Before quarks are added to the model, the partition function of the Center-Valued Polyakov Spin Models reads :

$$Z_{\mathbb{Z}(N)} = \prod_{\vec{x}} \left(\sum_{s_{\vec{x}}} \right) \mathrm{e}^{-S_G^{eff}[s]}$$

$SU(N)$-Valued Polyakov Spin Models : what is the difference ?

The basic degrees of freedom are not $z_{\vec{x}} \in \mathbb{Z}(N)$, but the holonomies $P_{\vec{x}} \in SU(N)$, the action is expressed in terms of $P_{\vec{x}}$ (now via $L_{\vec{x}} = \frac{1}{N}$ tr $P_{\vec{x}}$).

Before quarks are added to the model, the partition function reads now (with Haar measure integration dP over $SU(N)$) :

$$Z_{SU(N)} = \prod_{\vec{x}} (dP_{\vec{x}})\, \mathrm{e}^{-S_G^{eff}[P]}$$

The inclusion of dynamical quarks, including the effect of non-vanishing baryonic density, is described by adding a fermionic part to S_G^{eff} :

$$S^{eff}[P] = -\frac{\beta'}{2} \sum_{<\vec{x},\vec{y}>} \left(\mathrm{tr}\ P_{\vec{x}} \cdot \mathrm{tr}\ P_{\vec{y}}^{\dagger} \right) + S_F^{eff}[P]$$

The effective models differ further in the way how quarks are included.

Wilson fermions : in lowest order of hopping parameter expansion, they influence the gluonic action by a shift in β :

$$\beta \to \tilde{\beta} = \beta + 48 \ N_f \ \kappa^4$$

multiplying the plaquette term $\left(1 - \frac{1}{3}\mathrm{Re\ tr}\ U_p\right)$ (leading to changed β').

Next orders of the κ expansion are $\mathcal{O}(\kappa^6), \ldots, \mathcal{O}(\kappa^{N_t}), \ldots)$

In $\mathcal{O}(N_t)$, in the hopping parameter expansion appear also loops encircling the periodic lattice in positive or negative time direction

$$S_F^{eff}[P] = 4 \ (2\kappa)^{N_t} \sum_{\vec{x}} \ \left(\mathrm{tr}\ P_{\vec{x}} + \mathrm{tr}\ P_{\vec{x}}^\dagger\right)$$

or at $\mu \neq 0$

$$S_F^{eff}[P] = 4 \ (2\kappa)^{N_t} \sum_{\vec{x}} \ \left(\mathrm{e}^{\mu/T}\ \mathrm{tr}\ P_{\vec{x}} + \mathrm{e}^{-\mu/T}\ \mathrm{tr}\ P_{\vec{x}}^\dagger\right)$$

Reduced back to the $\mathbb{Z}(N)$ variables, with $\mu \neq 0$, this reads

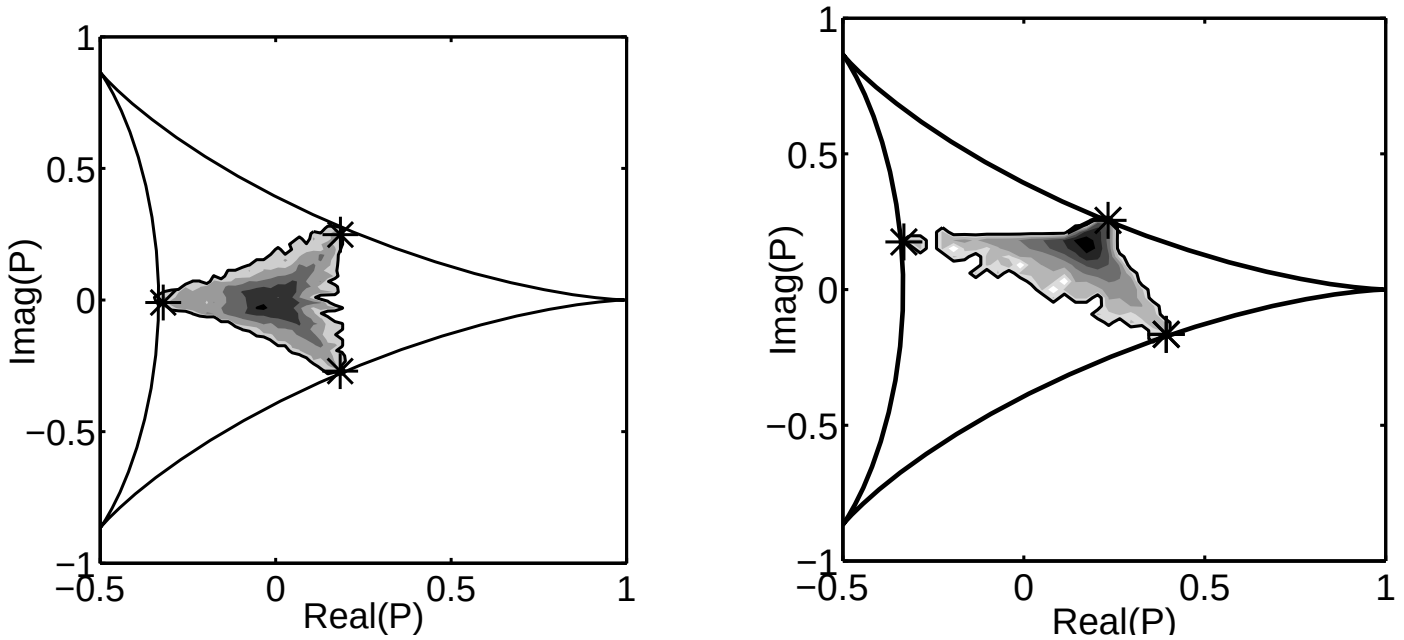
$$S_F^{eff}[z] = 4\ N\ (2\kappa)^{N_t} \sum_{\vec{x}}\ \left(\mathrm{e}^{\frac{\mu}{T}} z_{\vec{x}} + \mathrm{e}^{-\frac{\mu}{T}} z_{\vec{x}}^{\star} \right)$$

"The action of the $\mathbb{Z}(N)$ and the $SU(N)$ spin models is formally identical under the replacements $\frac{1}{N}\ \mathrm{tr}\ P_{\vec{x}} \to z_{\vec{x}}$ and $\frac{1}{N}\ \mathrm{tr}\ P_{\vec{x}}^{\dagger} \to z_{\vec{x}}^{\star}$. However, the range and the measure of the spins are different !"

Have a look at the fundamental domain for the Polyakov loop ($\frac{1}{3}\ \times$ trace of an arbitrary unitary matrix) in the complex plane :

- **in $SU(3)$ spin models the trace varies over the full triangle**
- **in $\mathbb{Z}(N)$ spin models the trace is locked to the three corners**
- **the next figure shows also specific semiclassical configurations**

The fundamental domain of the local Polyakov loop : shown for finite temperature instantons (calorons) in the confinement phase. The Polyakov loop in $SU(3)$ spin models varies over the full triangle, while $\mathbb{Z}(3)$ spins are locked in the corners. **from hep-lat/0309106 Regensburg/Berlin collaboration (Gattringer et al.)**



6. 3. Mean Field Solution of the Spin Models

From the hopping parameter expansion one gets (for $N_f = 4$)

$$\exp\left(-S_F^{eff}[P]\right) = \prod_{\vec{x}} \left[\det\left(1 + \epsilon P_{\vec{x}}\right)\right]^{\frac{N_f}{4}}$$
$$= \prod_{\vec{x}} \left[1 + \epsilon^3 + 3\epsilon L_{\vec{x}} + 3\epsilon^2 L^{\star}_{\vec{x}}\right]$$

Here the limit $m \to \infty$ and $\mu \to \infty$ has been taken, where only quarks (not antiquarks !) survive.

In this limit $\epsilon = \left(\frac{e^{\mu a}}{2m_q a}\right)^{N_t}$ ("density parameter") keeps a fixed value (it is measuring the final occupation of lattice points by quarks) :

- **for $\epsilon = 1$ one has half-filling $\langle n_q \rangle = 0.5$**
- **for $\epsilon \to \infty$ saturation $\langle n_q \rangle \to 1$.**

Such models schematically describe the ability of $\mu \neq 0$ to induce a loss of confinement, by gradually non-vanishing expectation values

$$\langle \mathrm{tr}\ \ P_{\vec{x}} \rangle \ \neq \ 0 \quad \textbf{at } T > T_{dec}(\mu)$$
$$\langle \mathrm{tr}\ \ P^{\dagger}_{\vec{x}} \rangle \ \neq \ 0 \quad \textbf{at } T > T_{dec}(\mu)$$

with transition temperatures $T_{dec}(\mu)$ considerably (?) lower than the deconfining temperature $T_{dec}(\mu = 0)$, if μ is large enough.

The effect of net-baryon density is described by different free energies of quarks and antiquarks in the medium :

$$\exp\left(-\frac{F_Q}{T}\right) \propto \left\langle \frac{1}{N}\ \mathrm{tr}\ \ P_{\vec{x}} \right\rangle \neq \left\langle \frac{1}{N}\ \mathrm{tr}\ \ P^{\dagger}_{\vec{x}} \right\rangle \propto \exp\left(-\frac{F_{\bar{Q}}}{T}\right)$$

while both average Polyakov loops are real-valued.

The chiral symmetry restoration aspect of the transition to the high density phase can hardly be exposed/understood in this lowest-order hopping-parameter approximation.

This model has been studied in the mean field approximation, with the fermion action split into modulus and phase part

$$S_F^{eff}[P] = S_F^{mag}[P] + i\Theta[P]$$

$$S^{mag}[P] = -\sum_{\vec{x}} \ln\left|1 + \epsilon^3 + 3\epsilon L_{\vec{x}} + 3\epsilon^2 L^\star_{\vec{x}}\right|$$

$$\Theta[P] = -\sum_{\vec{x}} \arg\left[1 + \epsilon^3 + 3\epsilon L_{\vec{x}} + 3\epsilon^2 L^\star_{\vec{x}}\right]$$

Variational ansatz for the "mean field action" with two parameters

$$S_{mf} = -\frac{x}{2}\sum_{\vec{x}}\left[L_{\vec{x}} + L^\star_{\vec{x}}\right] - \frac{y}{2}\sum_{\vec{x}}\left[L_{\vec{x}} - L^\star_{\vec{x}}\right]$$

One minimizes the "mean field free energy" with respect to x and y :

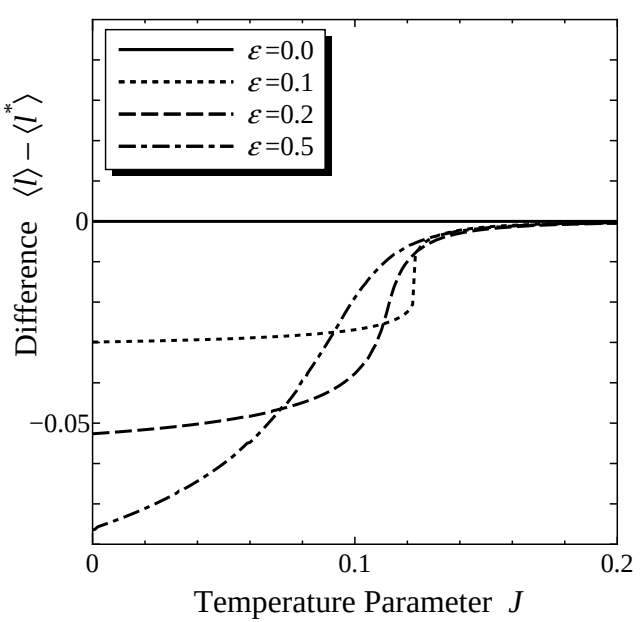
$$F_{mf}(x,y) = \langle S_G[P] + S_F^{eff}[P] - S_{mf}\rangle_{mf} - \ln\int dP\ \mathrm{e}^{-S_{mf}[P]} \geq F_{exact}$$

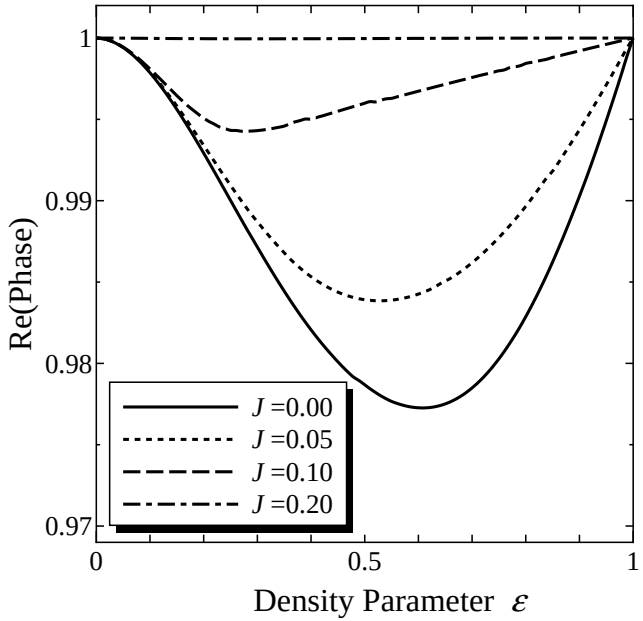
as "best approximation". The integral $\int dP$ is over the Haar measure !

Left: Difference between $\langle l \rangle$ and $\langle l^\star \rangle$ as function of temperature.

Right: Phase of the fermion determinant $\mathrm{Re}\langle \mathrm{e}^{-i\Theta} \rangle$ as function of density.

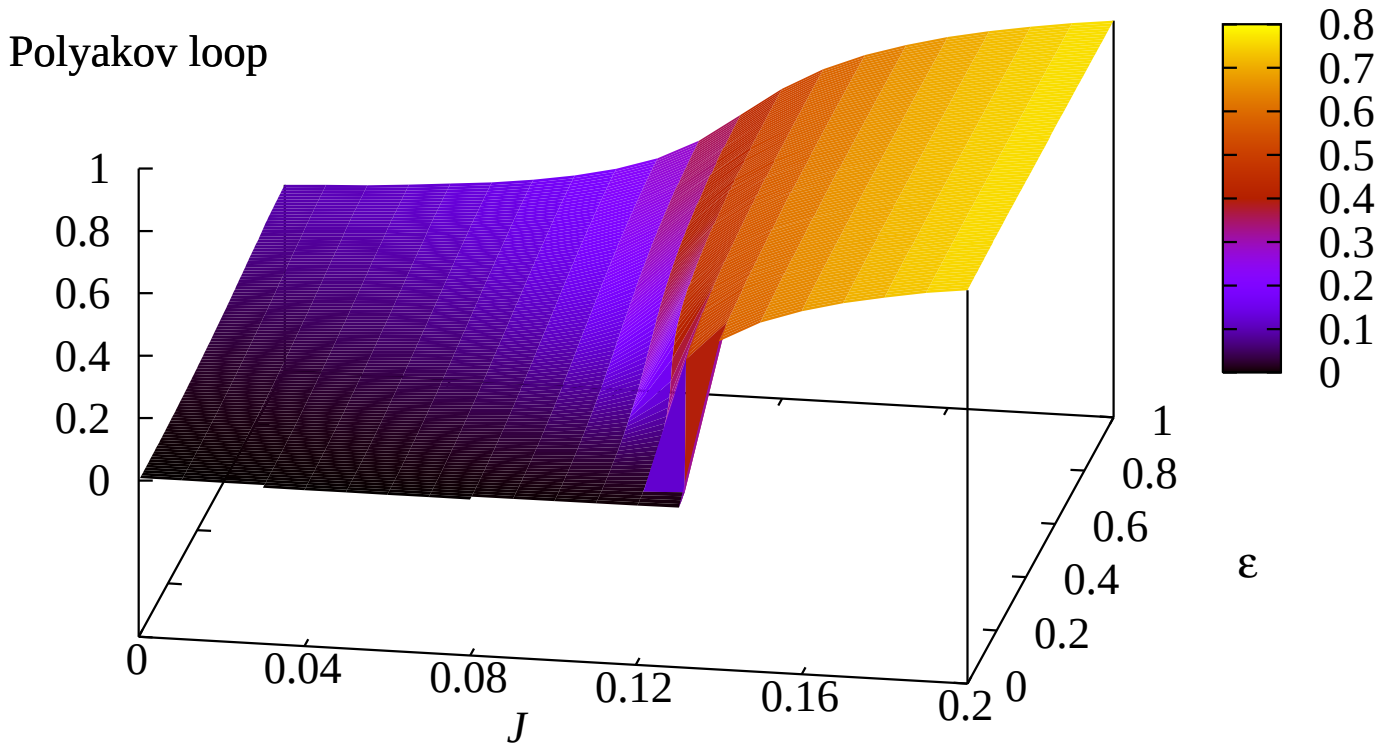
from hep-ph/0610323 Fukushima and Hidaka





The fundamental Polyakov loop $\langle l \rangle$ as function of the temperature parameter J and the density parameter ϵ.

from hep-ph/0610323 Fukushima and Hidaka



6. 4. Flux Representation for the Spin Models

The Flux Representation intends to solve the effective models exactly !

Simplified generic notation :

$$S[z] = -\sum_x \left(\tau \sum_{\nu=1}^{3} [z_x \ z^\star_{x+\nu} + c.c.] + \kappa \left[\mathrm{e}^{\mu} z_x + \mathrm{e}^{-\mu} z^\star_x \right] \right)$$

- $x = \vec{x}$ **in 3D space**
- κ **decreasing with increasing mass**
- τ **increasing with increasing temperature**
- μ **is the chemical potential in units of temperature** (μ/T)
  $\eta = \kappa \mathrm{e}^{\mu}$ **and** $\bar{\eta} = \kappa \mathrm{e}^{-\mu}$
- z_x **is either center-valued,** $z_x \in \mathbb{Z}(N)$**, or** $z_x = \frac{1}{N} \ \mathrm{tr} \ \ P_x$

Remark :

The Fermi statistics of quarks can be built in only by including higher order terms of the hopping parameter expansion.

The following universal picture of the spin models emerges :

The following steps towards finite baryonic density are always qualitatively realized :

- **For $\kappa = 0 \rightarrow$ standard Potts model (without quarks, no μ) : first order transition at $\tau = 0.183522$ (like $SU(3)$ Yang-Mills)**
- **First order weakens with growing κ (smaller quark mass) : critical endpoint at $\tau = 0.183127$ and $\kappa = 0.00026$ as long as $\mu = 0$**
- **Small non-zero μ weakens the transition further and shifts the critical endpoint to smaller κ.**

In advance, a critical general remark is appropriate :

Having finally exact solutions available for the $\mathbb{Z}(3)$ and $SU3)$ spin models, the following shortcomings cannot be ignored :

- **These models cannot reproduce that the crossover line eventually turns into a first order transition at large μ (with low mass) !**
- **This is what genuine LGT simulations in 4D show (or not) !**
- **Low mass (almost chiral) quarks cannot be described.**
- **Fermi statistics of localized quarks can only be accounted for by multiply winding loops included (with alternating sign) !**
- **The spin models in 3D will be considered mainly because of the interesting lessons they give about alternative simulation algorithms.**

Derivation of the Flux Representation for the $\mathbb{Z}(3)$ spin model :

For the nearest neighbor terms one can prove another closed form :

$$e^{\,\tau[P(x)P(x+\hat{\nu})^{*}+c.c.]} \;=\; C\sum_{b_{x,\nu}=-1}^{+1} B^{|b_{x,\nu}|}\,\Big(\,P(x)P(x+\hat{\nu})^{*}\,\Big)^{b_{x,\nu}}$$

with a dimer variable taking values $b_{x,\nu}\in\{-1,0,+1\}$

$$C \;=\; \frac{e^{2\tau}+2e^{-\tau}}{3} \quad , \qquad B \;=\; \frac{e^{2\tau}-e^{-\tau}}{e^{2\tau}+2e^{-\tau}}$$

The terms with the external field (acting on each $\mathbb{Z}(3)$ spin) :

$$e^{\,\eta\,P(x)\,+\,\overline{\eta}\,P(x)^{*}} \;=\; \sum_{s_x=-1}^{+1} M_{s_x}\,P(x)^{s_x}$$

can be expressed, with a monomer variable taking values $s_x\in\{-1,0,+1\}$
and the monomer weight function, as follows :

$$M_s=\frac{1}{3}\left[e^{\,\eta+\overline{\eta}}\;+\;2\,e^{-(\eta+\overline{\eta})/2}\,\cos\left((\eta-\overline{\eta})\frac{\sqrt{3}}{2}-s\frac{2\pi}{3}\right)\right]$$

Then the partition function is :

$$Z \;=\; C^{dV} \sum_{\{b\}} \sum_{\{s\}} W[b,s] \left(\prod_{x} \sum_{P(x)} P(x)^{\sum_\nu [b_{x,\nu} - b_{x-\hat{\nu},\nu}] + s_x} \right)$$

The outer sums are over dimer and monomer variables, b and s, subject to simulation.

The inner sums can be performed at once due to the properties of $\mathbb{Z}(3)$:

$$\sum_{P} P^n \;=\; 3\,T(n) \quad \textbf{with} \quad T(n) \;=\; \delta_{n \bmod 3\,,\,0} i$$

This is finally the Flux Representation of the $\mathbb{Z}(3)$ Potts model :

$$Z = \left(3C^3\right)^V \sum_{\{b,s\}} \left(\prod_{x,\nu} B^{|b_{x,\nu}|} \right) \left(\prod_{x} M_{s_x} \right) \prod_{x} T \left(\sum_{\nu} (b_{x,nu} - b_{x-\nu,\nu}) + s_x \right)$$

The variables characterizing a configuration are :

- **a single monomer variable** $s_x \in \{-1, 0, +1\}$
- **a single dimer variable** $b_{x,\nu} \in \{-1, 0, +1\}$

with the building blocks (expressions) B **and** M **defined above.**

Important for 3 colors: the triality function (constraining the divergence of flux at each site to vanish up to multiples of 3) :

$$T(n) = \delta_{0,n} \bmod 3$$

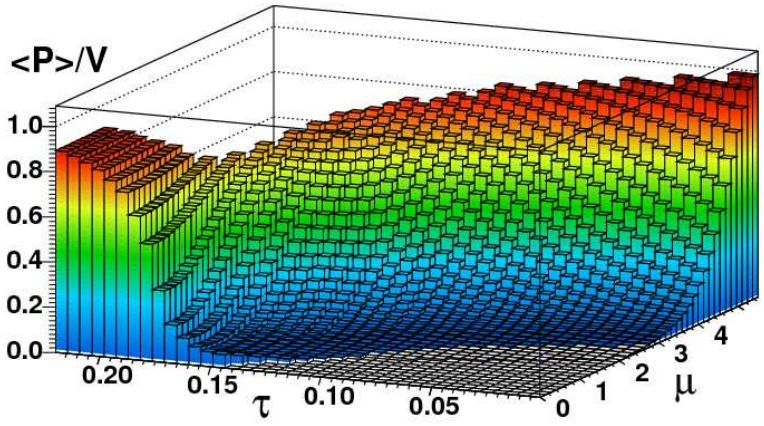
This model has become a testbed for intense studies of the worm algorithms.

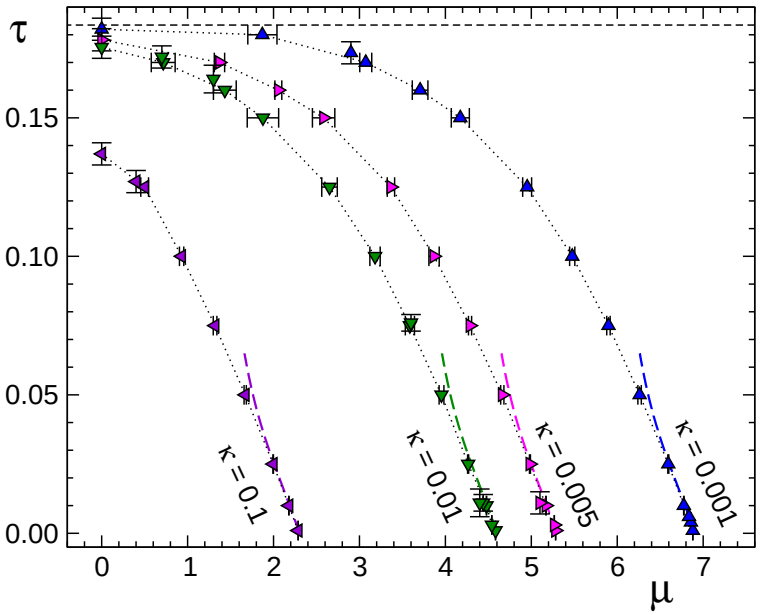
arXiv:1202.4293 Y. Delgado, H. G. Evertz, Ch. Gattringer

Results :

- **average Polyakov loop vs. T and μ**
- **transition line obtained from peak in Polyakov loop susceptibility**
- **transition line obtained from peak in the specific heat**
- **the respective lines are not on top of each other**
  (this suggests a crossover rather than a transition !)

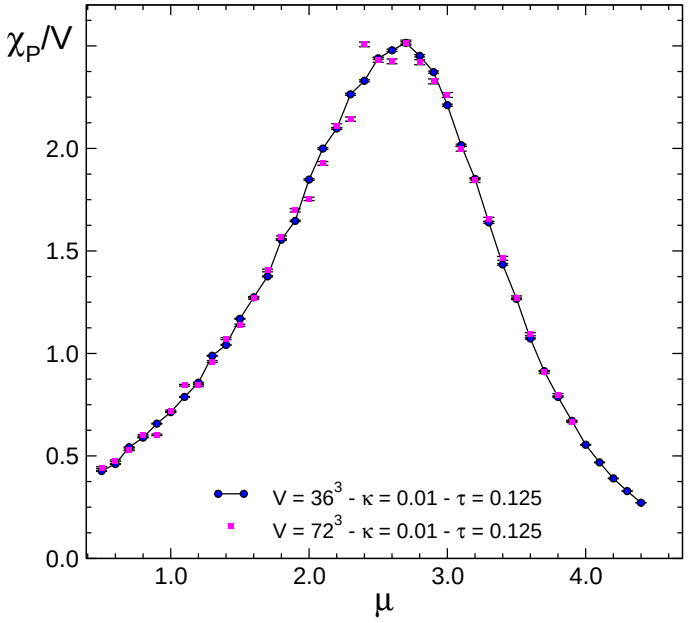
Left: phase structure in the T-μ plane of the $\mathbb{Z}(3)$ Potts model at $\kappa = 0.01$.
Right: transition lines for 4 values of κ.
from arXiv:1110.6862 Delgado, Evertz and Gattringer

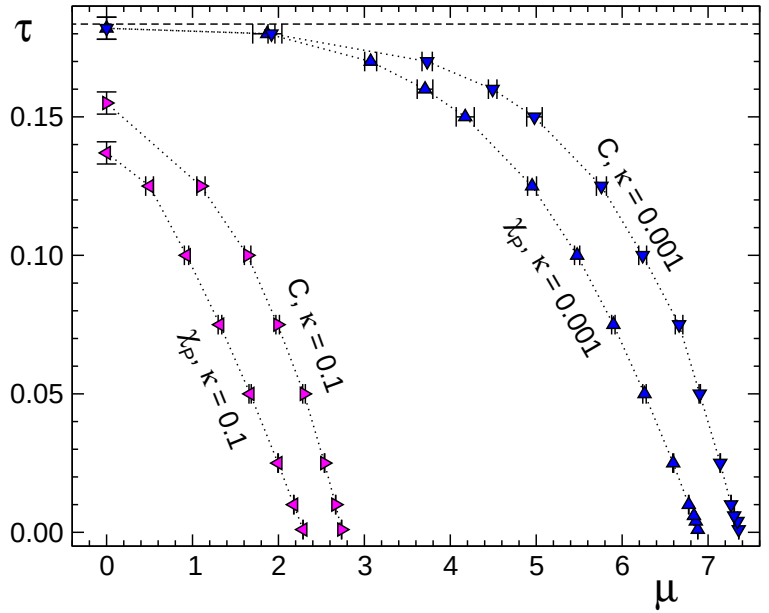




Left: example of Polyakov loop susceptibility χ_P as seen in a μ-scan.
Right: Comparing phase boundaries according to χ_P and heat capacity C.

from arXiv:1110.6862 Delgado, Evertz and Gattringer





Derivation for the $SU(3)$ spin model :

Similar steps, but now $\frac{1}{3}\,\mathrm{tr}\;\; P_{\vec{x}} \notin \mathbb{Z}(3)$ **(construction more complicated).**

Therefore, the useful identities of the $\mathbb{Z}(3)$ case are not applicable.

$$Z \;=\; \sum_{\{l,\bar{l}\}} \sum_{\{s,\overline{s}\}} \left(\prod_{\overline{x},\nu} \frac{\tau^{l_{x,\nu}+\bar{l}_{x,\nu}}}{l_{x,\nu}!\;\bar{l}_{x,\nu}!} \right) \left(\prod_{x} \frac{\eta^{s_x}\;\overline{\eta}^{\overline{s}_x}}{s_x!\;\overline{s}_x!} \right) \left(\prod_{x} I(f_x, \overline{f}_x) \right) \;\; .$$

The price is a sum over two sets of dimer and monomer variables :

- **two dimer variables** $l_{x,\nu}, \bar{l}_{x,\nu} \in [0, +\infty)$**, living on the links** (x, ν)
- **two monomer variables** $s_x, \overline{s}_x \in [0, +\infty)$**, living on the sites** x

The symbol $I(f_x, \overline{f}_x)$ summarizes the pointwise integral over the $SU(3)$ Haar measure :

$$I(n, \overline{n}) \;=\; \int_{SU(3)} dP\, (\mathbf{Tr} P)^n\, (\mathbf{Tr} P^\dagger)^{\overline{n}}\;,$$

with n and $\overline{n}$ being non-negative integers.

It vanishes when the triality condition

$$(n - \overline{n}) = 0 \;\mathbf{mod}\; 3$$

is violated.

Otherwise it is real, non-negative, but not independent of n and $\overline{n}$!

The actual arguments are :

$$f_x \;=\; \sum_{\nu=1}^{3} [\, l_{x,\nu} + \overline{l}_{x-\hat{\nu},\nu}\,] + s_x \quad , \qquad \overline{f}_x \;=\; \sum_{\nu=1}^{3} [\, \overline{l}_{x,\nu} + l_{x-\hat{\nu},\nu}\,] + \overline{s}_x \;.$$

A closed expression is available (Uhlmann, Meinel, Wipf: hep-th/0611170, also Gattringer arXiv:1104.1503). $I(n, \overline{n})$ is not a simple function !

Despite the more complicated set of variables and constraints, the model can be simulated by a variant of the Prokof'ev and Svistunov worm algorithm provided the dimer and monomer variables have been suitably reparametrized.

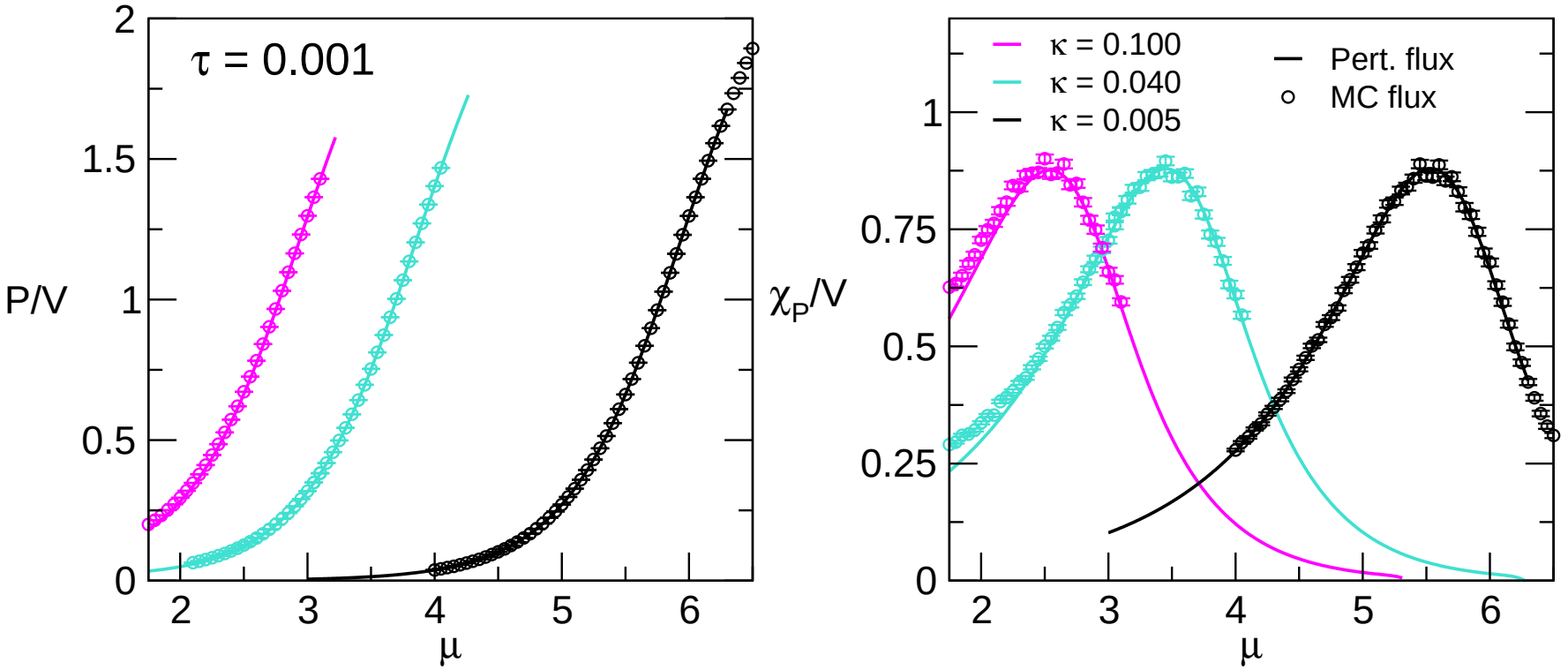
For details see : Delgado and Gattringer,

arXiv:1204.6074

arXiv:1208.1169

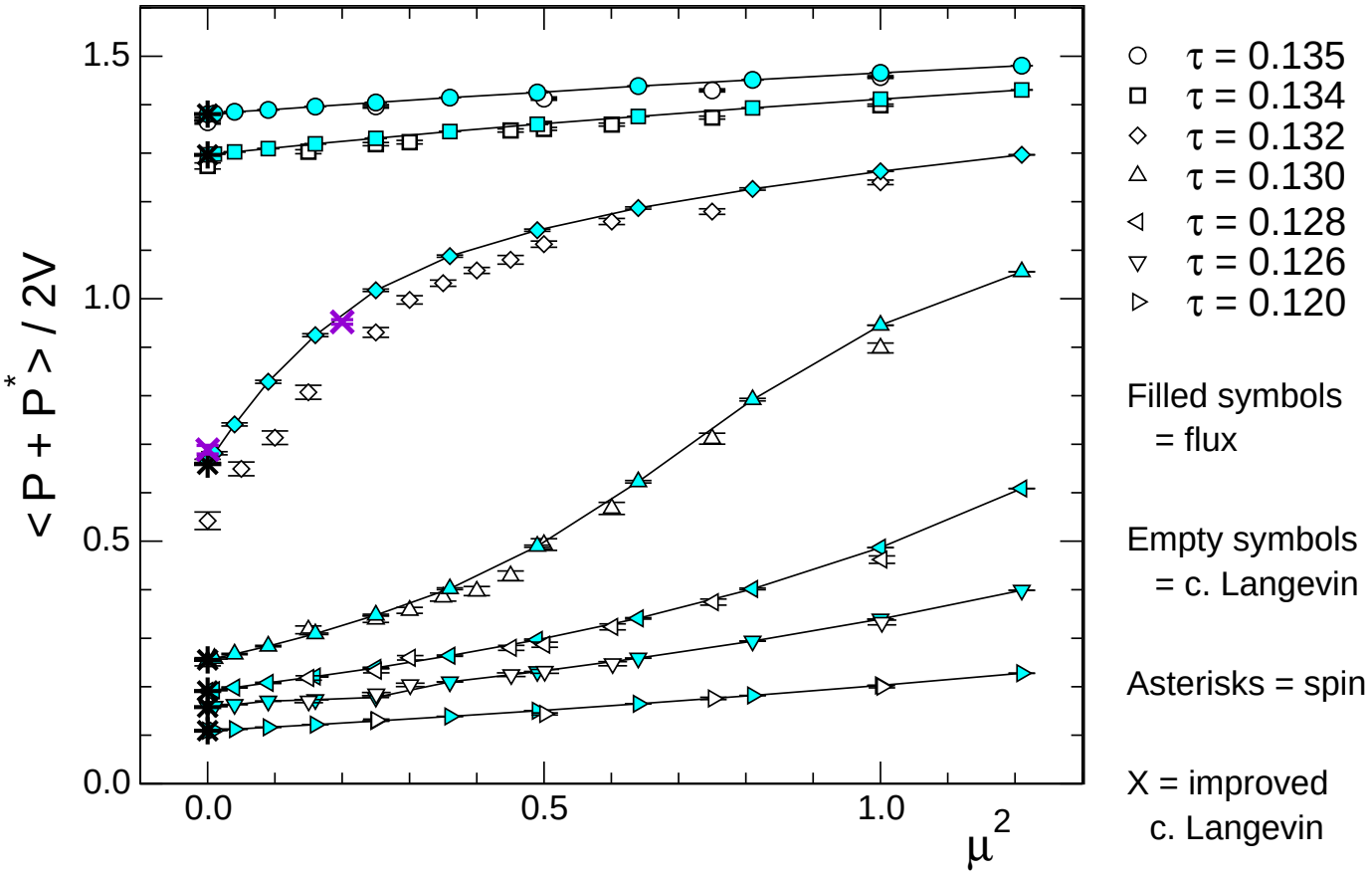
The critical lines according to different susceptiblities agree even less !

Polyakov loop P (left) and its susceptibility χ_P (right) for $\kappa = 0.1, 0.04$ and 0.005 and $\tau = 0.001$. Simulation results on 10^3 lattices are compared with an expansion in τ. **from arXiv:1208.1169 Delgado and Gattringer**

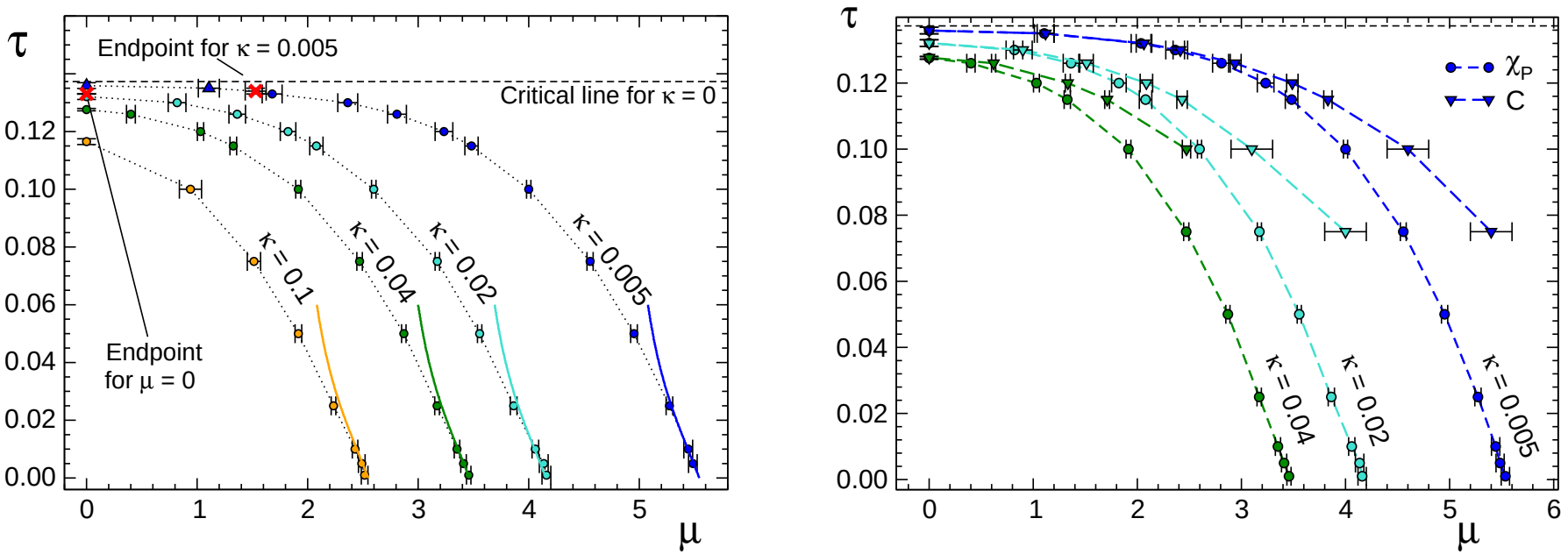


Comparison of $\langle P + P^\star \rangle / 2V$ from the flux simulation (filled symbols) with results from the Complex Langevin simulation (empty symbols) for various temperatures τ as function of μ^2 for $\kappa = 0.02$ on 10^3 lattices. For $\mu = 0$ data from conventional Monte Carlo simulations are added.

from arXiv:1208.1169 Delgado and Gattringer



Left: Phase diagram from maxima of χ_P for 4 values of κ with critical endpoint for $\mu = 0$, the critical line for $\kappa = 0$ and the critical endpoint for $\kappa = 0.005$. Right: Comparing phase boundaries according to χ_P and heat capacity C. from arXiv:1208.1169 Delgado and Gattringer



6. 5. A Model with Explicitly Chiral Quarks

Kogut-Susskind fermions : have a remnant $U(1) \times U(1)$ global chiral symmetry in the limit $m \to 0$. KS fermions are chosen in order to describe light quarks :

$$S_F = \sum_x \left[m\bar{\chi}(x)\chi(x) + \sum_\mu \frac{1}{2}\eta_\mu(x) \left(U_\mu(x)\chi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\chi(x-\hat{\mu}) \right) \right]$$

describes four "tastes" (roughly $= 4$ flavors) if the determinant is not "rooted", $\det(D_F) \to (\det(D_F))^{1/4}$ (as is done in most Monte Carlo studies).

Introduce baryonic chemical potential as usual

$$\begin{aligned} U_4(\vec{x},\tau) &\to U_4(\vec{x},\tau)\ \mathrm{e}^{\mu a} \\ U_4^\dagger(\vec{x},\tau) &\to U_4^\dagger(\vec{x},\tau)\ \mathrm{e}^{-\mu a} \end{aligned}$$

Take an highly anisotropic lattice $a_t << a_s$, $r = a_t/a_s << 1$, with many (N_t) timeslices, and neglect spatial plaquettes.

Integrating over spatial links, leads to an effective action for gluons

$$S_G = -\frac{\beta'}{2} \sum_{<\vec{x},\vec{y}>} \left(\text{tr } P_{\vec{x}} \cdot \text{tr } P^\dagger_{\vec{y}} + c.c. \right)$$

as before, as function of the (untraced) Polyakov loops P and $P^\dagger$.

Their traces are the Polyakov loops $L_{\vec{x}}$ and its conjugate operator $L^\star_{\vec{x}}$

$$L_{\vec{x}} = \frac{1}{N} \text{tr } P_{\vec{x}} \quad \textbf{and} \quad L^\star_{\vec{x}} = \frac{1}{N} \text{tr } P^\dagger_{\vec{x}}$$

A single expectation value

$$L = \frac{1}{N_1 N_2 N_3} \left\langle \sum_{\vec{x}} L_{\vec{x}} \right\rangle$$

is the order parameter of deconfinement only as long as $\mu = 0$.

The (already well-known) "coupling constant" of the spin part is :

$$\beta' = 2 \, \mathrm{e}^{-\frac{\sigma a_s}{T}}$$

Integration over the spatial links gives an four-fermion effective action

$$
\begin{aligned}
S'_F \;=\; & \sum_{\vec{x},\tau} m\bar{\chi}(\vec{x},\tau)\chi(\vec{x},\tau) \\
& + \frac{1}{2}\sum_{\vec{x},\tau} \eta_\mu(\vec{x},\tau)\eta_4(\vec{x}) \Big(U_4(\vec{x},\tau)\chi(\vec{x},\tau+1) - U_4^\dagger(\vec{x},\tau-1)\chi(\vec{x},\tau-1) \Big) \\
& + \frac{r^2}{4N}\sum_{\vec{x},\tau}\sum_i \bar{\chi}(\vec{x},\tau)\chi(\vec{x},\tau)\bar{\chi}(\vec{x}+\hat{i},\tau)\chi(\vec{x}+\hat{i},\tau)
\end{aligned}
$$

representing an antiferromagnetic coupling between the color singlet operators $\bar{\chi}^a(\vec{x},\tau)\chi^a(\vec{x},\tau)$ at spatially neighboring sites.

Anticipating a staggered order parameter, we replace this by a field $M(\vec{x},\tau) = \eta_4(\vec{x})\bar{\chi}^a(\vec{x},\tau)\chi^a(\vec{x},\tau)$ with a uniform sign of M. ($\eta_4 = (-1)^{x+y+z}$)

The neighboring site interaction is linearized by means of an auxiliary field integration (over λ)

$$\mathrm{e}^{\frac{1}{2}\sum_\tau \sum_{\vec{x}\vec{y}} M(\vec{x},\tau) K^{-1}(\vec{x}\vec{y}) M(\vec{y},\tau)} \propto \int \prod_\tau \prod_{\vec{x}} d\lambda(\vec{x},\tau) \mathrm{e}^{-\frac{1}{2}\sum_\tau \sum_{\vec{x}\vec{y}} \lambda(\vec{x},\tau) K(\vec{x},\vec{y}) \lambda(\vec{y},\tau)} \times \mathrm{e}^{-\sum_\tau \sum_{\vec{x}} \lambda(\vec{x},\tau) M(\vec{x},\tau)}$$

with

$$K^{-1}(\vec{x},\vec{y}) = \frac{1}{4N} \sum_i \left(\delta_{\vec{x}+\hat{i},\vec{y}} + \delta_{\vec{x},\vec{y}+\hat{i}} \right)$$

Leaving the integration over λ (a dynamical quark mass) and over the Polyakov loops undone, the spatial sites now are decoupled with respect to the fermion action, and the fermion field can be integrated separately on all spatial sites $\vec{x}$.

The result is a product of $N_t \times N_t$ determinants,
with $N \times N$ subblocks, at each spatial site :

$$\prod_{\vec{x}} \det W(\vec{x})$$

with $W(\vec{x}) =$

$$\begin{bmatrix}
m(1) & +U(1) & 0 & \cdots & +U^\dagger(1) \\
-U^\dagger(1) & m(2) & +U(2) & \cdots & 0 \\
0 & -U^\dagger(2) & m(3) & \cdots & 0 \\
0 & 0 & -U^\dagger(3) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & +U(N_t-2) & 0 \\
0 & 0 & \cdots & m(N_t-1) & +U(N_t) \\
-U(N_t) & 0 & \cdots & -U^\dagger(N_t-1) & m(N_t)
\end{bmatrix}$$

with the abbreviations :

$$m(k) = (m + \eta_4 \lambda(\tau = k)) \mathbb{1}_{N\times N}$$

and U and $U^\dagger$ which stand for the $N \times N$ gauge links

$$U(k) \;=\; \frac{\eta_4}{2}\, U_4(\vec{x}, \tau = k)\; \mathrm{e}^{+\mu a}$$
$$U^\dagger(k) \;=\; \frac{\eta_4}{2}\, U_4^\dagger(\vec{x}, \tau = k)\; \mathrm{e}^{-\mu a}$$

Within the static Polyakov gauge

$$U_4(\vec{x}, \tau) = \mathrm{diag}\left(\mathrm{e}^{i\phi_\alpha(\vec{x})/N_t}\right)$$

the determinants (so-called "circulants") are known in closed form (in the $\lambda =$ constant approximation):

$$\det W(\vec{x}) = \prod_{\alpha=1}^{N} \left[\cosh\left(N_t s\right) + \cosh\left(i\phi_\alpha + \frac{\mu}{T}\right)\right]$$

with

$$r\ \lambda = \sinh(s)$$

The effective fermionic action is :

$$S_F^{eff} = -\sum_{\vec{x}} \frac{N_f}{4} \sum_{\alpha=1}^{N} \ln\left[\cosh\left(N_t s\right) + \cosh\left(i\phi_\alpha(\vec{x}) + \frac{\mu}{T}\right)\right]$$

For large λ (assumed homogeneous, a staggered "magnetic field" creating the antiferromagnetic ordering of the quark condensate $\bar{\chi}\chi$), the effective action behaves like :

$$S_F^{eff} = -\sum_{\vec{x}} \frac{N_f}{4} \left\{ N N_t \ln\lambda + \left(\frac{1}{2r\lambda}\right)^{N_t} \left[\mathrm{e}^{+\frac{\mu}{T}}\ \mathrm{tr}\ \ P_{\vec{x}} + \mathrm{e}^{-\frac{\mu}{T}}\ \mathrm{tr}\ \ P_{\vec{x}}^{\dagger}\right] \right\}$$

similar to the form that would be obtained from Wilson fermions.

However, λ is a variational parameter. It must be found simultaneously with the minimization of the mean-field free energy for the $SU(N)$ spin model.

The sum rule for $K(\vec{x},\vec{y})$,

$$\sum_{\vec{x}} K(\vec{x},\vec{y}) = \frac{2N}{d-1}$$

introduces the dimensionality. The dimensionality is important for the relative correctness of the mean-field approximation.

Now the model is ready for a mean-field treament.

In our 1984 paper (wrongly) considered only one Polyakov loop $L = \mathrm{tr}\ P$.

The mean-field free energy per site had been taken in the form

$$F = (d-1)\beta' L^2 + NN_t\lambda^2/(d-1) - \ln \int dP\,(1+\beta' L\ \mathrm{Re\ tr}\ P)^{2*(d-1)} \times \det W$$

to be minimized w.r.t. L and λ (at fixed β' and μ).

The result

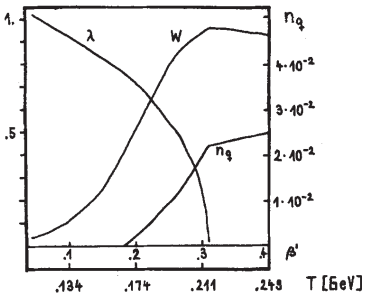


Fig. 1. Chiral order parameter λ, Wilson line W and quark density n_q as function of β', for zero chemical potential. The temperature in physical units is obtained as explained at the end of Sect. 2

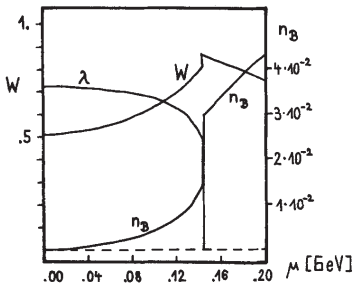


Fig. 2. λ, W, and baryon density n_B as function of the chemical potential at $\beta' = 0.2$ ($T \sim 174$ MeV)

At lower temperature the picture is different. For $\beta' = 0.05$ (corresponding to $T = 110$ MeV, shown in Fig. 3b) there is a clear local minimum at $\lambda = 0$ also at μ below $\mu_c \sim 215$ MeV. It remains separated from the minimum at $\lambda \cong 1.0$ by a potential threshold within some interval in μ around μ_c.

Our results are reassuring for those interested in high energy heavy ion experiments. Transition phenomena should be even more violent in the fragmentation region at heavy ion collisions. A significant increase in the density of heavy quarks (characterized by the behaviour of the Wilson line, (16)) should however be expected in the central region only.

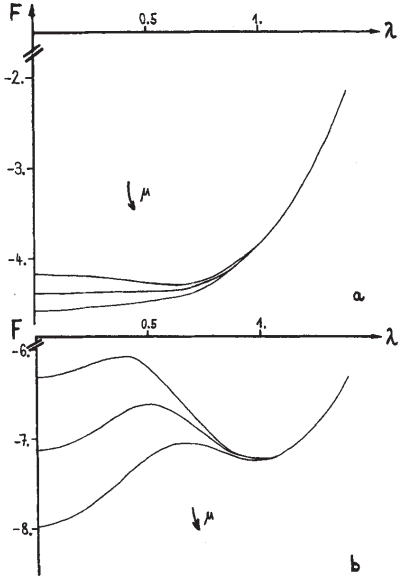


Fig. 3a and b. Mean free energy per site as function of λ for two values of β': **a)** $\beta' = 0.2$, $W = 0.9$, $\mu = 0.12, 0.14, 0.16$ GeV; **b)** $\beta' = 0.05$, $W = 0.2$, $\mu = 0.18, 0.21, 0.24$ GeV

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Our Result of 1984 :

- **a β-scan at $\mu = 0$, shows a second order transition at $\beta'_c = 0.38$,**
- **a μ-scan for $\beta' < \beta'_c$, shows a first order transition.**

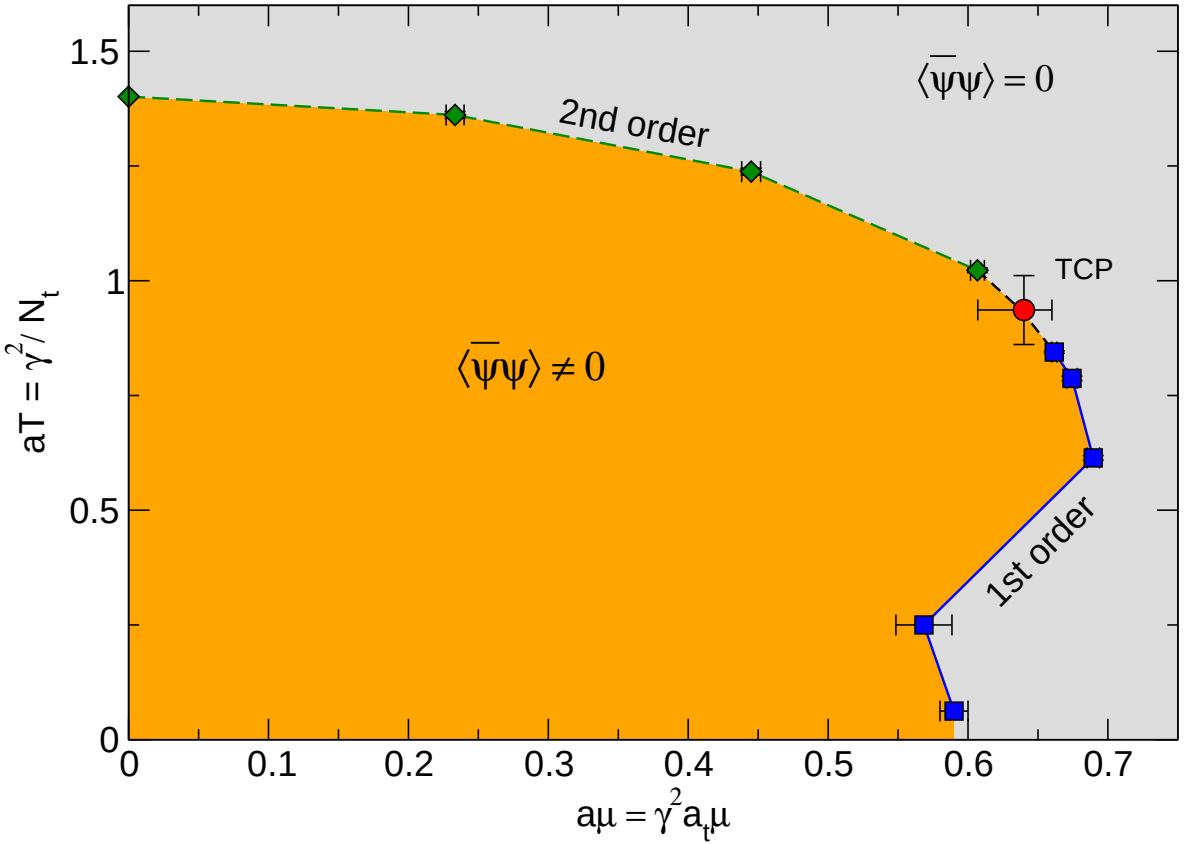
Since then, the model has been improved in many respects, including a correct mean field ansatz allowing for two Polyakov loops L and $L^\star$.

$$S_{mf} = -\frac{x}{2}\sum_{\vec{x}}[L_{\vec{x}} + L^\star_{\vec{x}}] - \frac{y}{2}\sum_{\vec{x}}[L_{\vec{x}} - L^\star_{\vec{x}}]$$

This class of effective models is under discussion until now, describing various scenarios for the phase diagram with light quarks in the strong coupling approximation for the gluons (gradually including higher corrections). See for example :
Ph. de Forcrand, M. Fromm, J. Langelage, K. Miura, O. Philipsen, W. Unger arXiv:1111.4677 (talk by O. Philipsen at LATTICE 2011)

The phase diagram for $N_f = 1$ QCD in the strong coupling and chiral limit on an $N_t = 4$ lattice contains 2nd and 1st order transitions and a tricritical point !

from arXiv:1111.4677 (talk by O. Philipsen at LATTICE 2011)



Mainly Japanese authors (Kawamoto et al.) have developed this over many years :

T. Z. Nakano, K. Miura, A. Ohnishi, arXiv:1009.1518

K. Miura, T. Z. Nakano, A. Ohnishi, N. Kawamoto, arXiv:1012.1509

K. Miura, T. Z. Nakano, A. Ohnishi, N. Kawamoto, PoS(Lattice 2011) 318

A. Ohnishi, K. Miura, T. Z. Nakano, arXiv:1104.1029

T. Misumi, T. Kimura, A. Ohnishi, arXiv:1206.1977

7. Complex Langevin Simulation

- General Remarks
- Unsuccessful Attempts
- Successful Attempts
- Complex Langevin for an Effective Action in Higher β Order

7. 1. General Remarks

Stochastic quantization : In normal cases (with **real-valued action,** bounded below), the simulation is done with one (or more, parallel) **Langevin process(es) in a (discrete) ficticious time τ.**

They evolve according to the Langevin equation

$$\frac{\partial \phi_x}{\partial \tau} = -\frac{\delta S[\phi]}{\partial \phi_x}_{|\phi(\tau)} + \eta_x(\tau)$$

This is **equivalent to a Monte Carlo simulation,** provided the step size is extrapolated $\epsilon \to 0$.

This simulation is **driven by one (or more, parallel) realization(s)** of a white noise process $\eta(\tau)$ per degree of freedom,

with $\sqrt{2}$ variance :

$$\langle \eta_x(\tau) \rangle = 0$$

and

$$\langle \eta_x(\tau) \eta_y(\tau') \rangle = 2\delta_{xy}\delta(\tau - \tau')$$

This leads, in the limit $\tau \to \infty$, to a distribution over subsequent Langevin configurations (or over an ensemble in case of parallel processes), representing the weight

$$W[\phi] \propto \mathrm{e}^{-S[\phi]}$$

This is an alternative to the Monte Carlo method for equilibrium simulations in order to generate ensembles of configurations.

For complex action, however, importance sampling by Monte Carlo simulation fails.

It was hoped, that the stochastic (Langevin) method might still work.

There are the following possibilities :

The Complex Langevin Simulation method

- works and leads to the correct result,
- works and leads to a fake result,
- does not work.

In the latter case little improvements (e.g. dynamically adapted step size) may be of help.

A systematic investigation of the actual status of the Complex Langevin Simulation method for various models has started not long ago. Stamatescu, Seiler, Aarts et al.

In general, the field ϕ_x (for $x \in \Lambda$, the lattice system) must be complexified

$$\phi \to \phi^R + i\phi^I$$

Then the Langevin equation becomes a series of simultaneous updates for ϕ^R and ϕ^I (with a suitable time step $\epsilon = \delta\tau$) :

$$\frac{\partial \phi_x^R}{\partial \tau} = F_x^R + \eta_x \qquad F_x^R = -\mathrm{Re}\left(\frac{\delta S}{\delta \phi_{x\,|\phi\to\phi^R+i\phi^I}}\right)$$

$$\frac{\partial \phi_x^I}{\partial \tau} = F_x^I \qquad F_x^I = -\mathrm{Im}\left(\frac{\delta S}{\delta \phi_{x\,|\phi\to\phi^R+i\phi^I}}\right)$$

7. 2. Unsuccessful Attempts

(2+1)-dimensional XY-model at finite chemical potential

G. Aarts and F. A. James (Lattice 2010) arXiv:1009.5838

Without chemical potential the action is :

$$S_{XY} = -\beta \sum_{x} \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}}\right)$$

the model apparently is similar to the Bose gas.

For the 4D Bose gas Complex Langevin works well also if $\mu \neq 0$.

With chemical potential the action reads :

$$S_{XY} = -\beta \sum_{x} \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

The chemical potential is coupled to the Noether charge related to the global symmetry $\phi_x \to \phi_x + \alpha$. Property $S^{\star}(\mu) = S(-\mu^{\star})$

At vanishing chemical potential the model has a phase transition at $\beta_c = 0.45421$ separating

- **a discordered phase for $\beta < \beta_c$,**
- **an ordered phase for $\beta > \beta_c$.**

Observations :

- **correct results of complex Langevin at large β (ordered phase) : cold start and hot start (for the imaginary part of the fields) lead to the same distribution.**
- **wrong results at small β (disordered phase); here the start matters !**
- **The reason for failure is not the sign problem, but an insufficient exploration of the configuration space.**

7. 3. Successful Attempts

Example 1

Bose gas by Complex Langevin simulation : G. Aarts arXiv:0810.2089

The problem is again the ϕ^4 theory in 4D (with or without interaction term $\lambda|\phi|^4$), that later (2012) has been successfully dealt with in the Flux Representation (by Gattringer and and Kloiber).

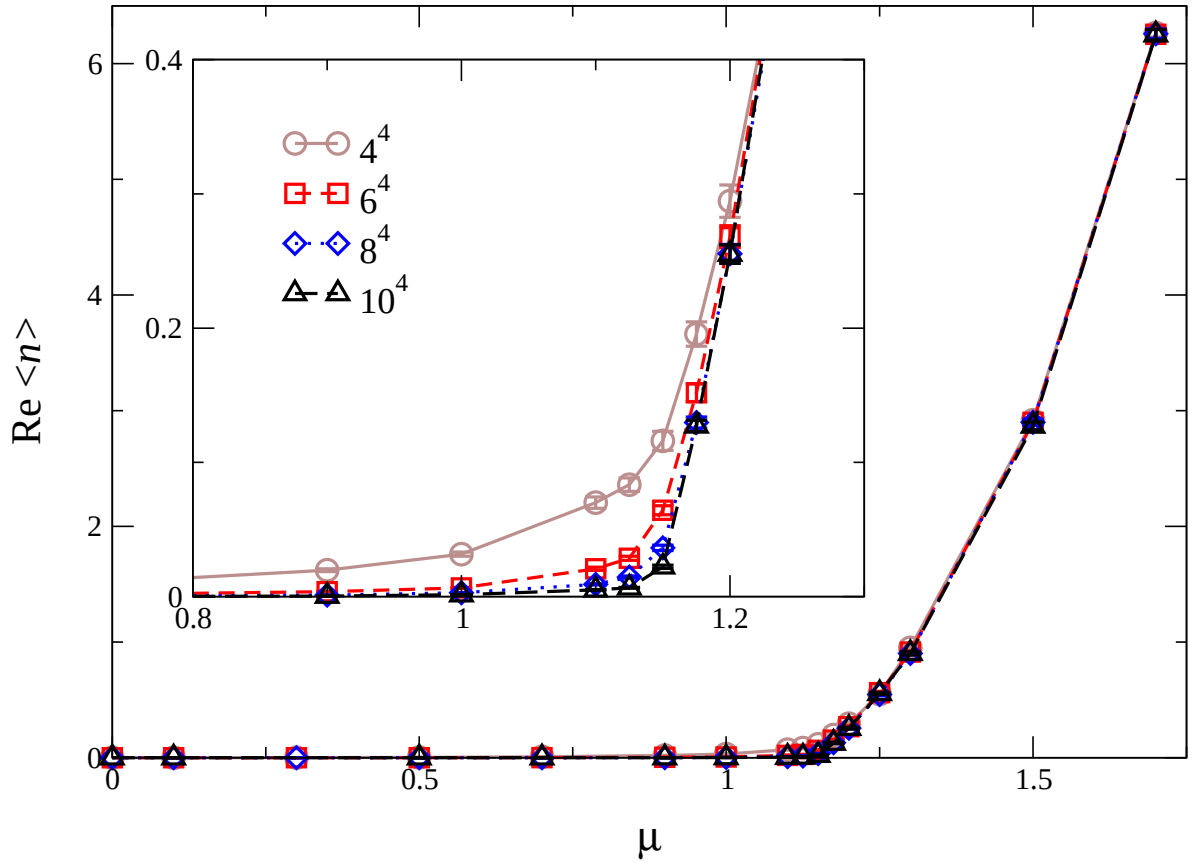
The first non-perturbative analysis of the model was performed using the Complex Langevin method, not the Flux Representation !

The results show the silver blaze effect.

Side remark : the imaginary parts of the observables (although not exactly zero) average to zero in the course of the Langevin process.

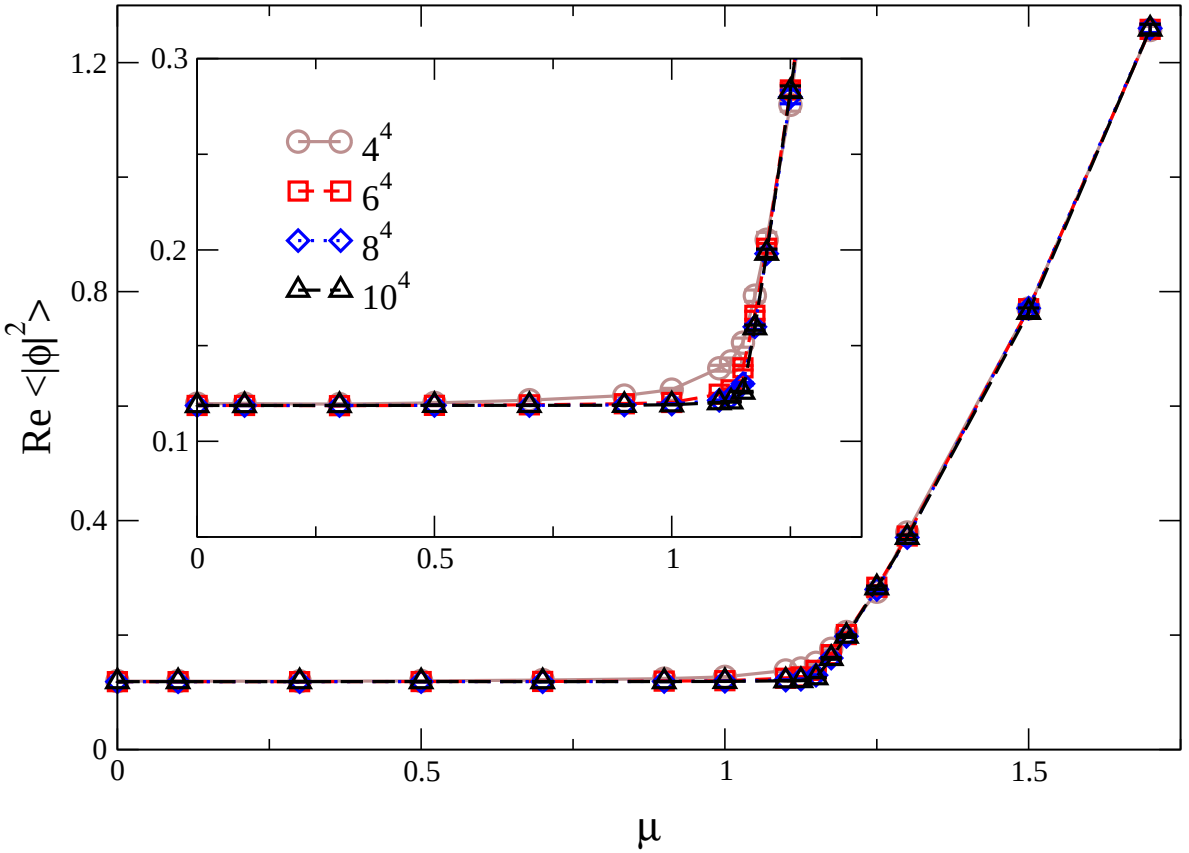
Real part of the density $\langle n \rangle$ as function of μ on lattices N^4 with $N = 4$, 6, 8, and 10. $m = \lambda = 1$, step size $\epsilon = 5 \times 10^{-5}$.

from G. Aarts arXiv:0810.2089



Real part of $\langle|\phi|^2\rangle$ as function of μ.

from G. Aarts arXiv:0810.2089



These results are not reproduced in the phase-quenched theory where for any observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} \mathrm{e}^{i\Theta} \rangle_{|pq}}{\langle \mathrm{e}^{i\Theta} \rangle_{|pq}}$$

with the phase

$$\Theta = \text{ imaginary part of the action}$$

while sampling is done with a weight

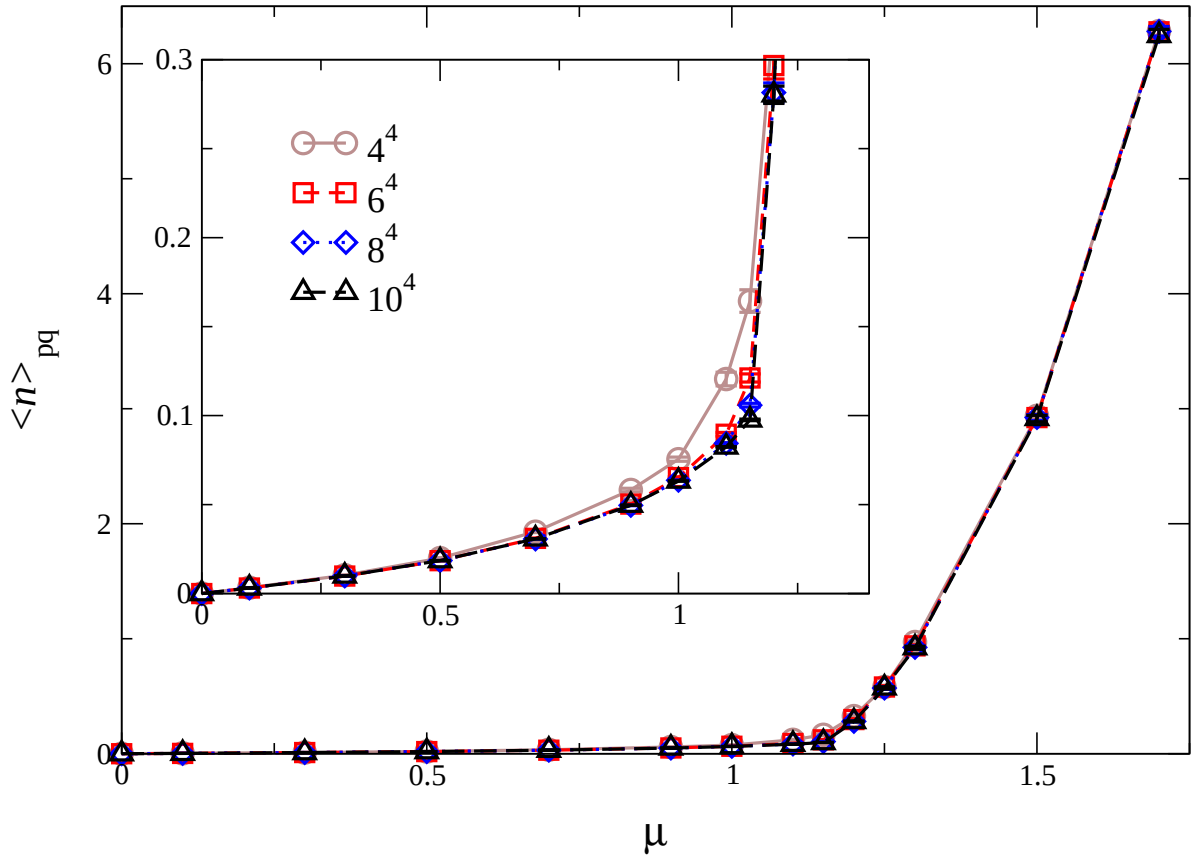
$$P \propto \exp\left(- \text{ real part of the action}\right)$$

An analysis of the average phase factor shows that the phase-quenched reweighting technique is rapidly deteriorating with increasing N.

The Complex Langevin method is free of this problem, as well as the simulation in the Flux Representation.

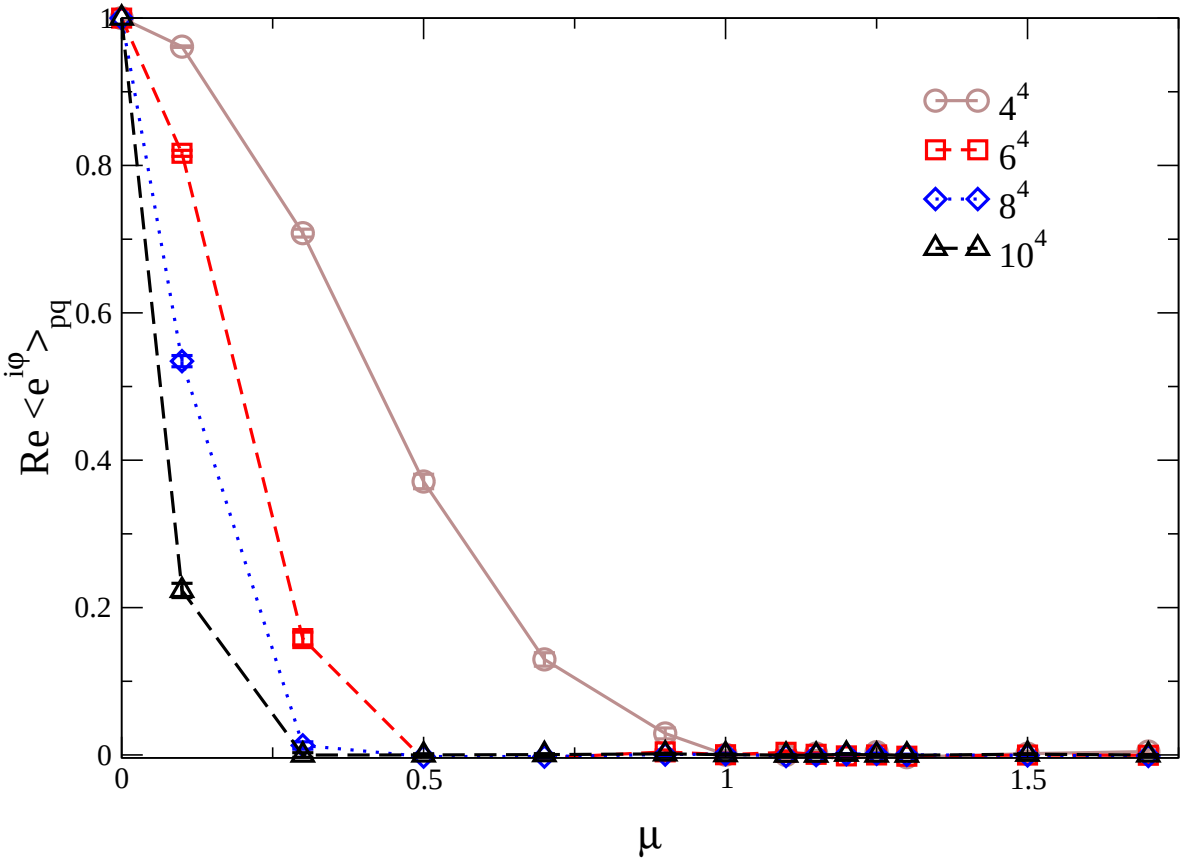
Density, calculated in the phase-quenched theory, as function of μ.

from G. Aarts arXiv:0810.2089



Average phase factor corresponding to the phase-quenched theory, $\mathrm{Re}\langle \mathrm{e}^{i\Theta}\rangle_{|pq}$, as function of μ. The drop with increasing N shows the increasing severeness of the sign problem in the thermodynamic limit.

from G. Aarts arXiv:0810.2089



Bose gas analytically : G. Aarts arXiv:0902.4686

With $\lambda = 0$ it is easy to follow (analytically) what is going on.

The probability distribution for the complex Langevin process $\rho[\phi^R, \phi^I, \tau]$, is defined by

$$\langle O[\phi^R + i\phi^I, \tau] \rangle_{|\eta} = \int D\phi^R D\phi^I \rho[\phi^R, \phi^I, \tau] \; O[\phi^R + i\phi^I]$$

and satisfies an extended Fokker-Planck equation

$$\frac{\partial \rho[\phi^R, \phi^I, \tau]}{\partial \tau} = \sum_x \left[\frac{\delta}{\delta \phi^R_{a,x}(\tau)} \left(\frac{\delta}{\delta \phi^R_{a,x}(\tau)} - K^R_{a,x}(\tau) \right) - \frac{\delta}{\delta \phi^I_{a,x}(\tau)} K^I_{a,x}(\tau) \right] \rho[\phi^R, \phi^I, \tau]$$

The stationary solution for the non-interacting case can be inferred analytically and can be shown to correspond to the $\tau \to \infty$ solution.

This distribution is seen to be real, nonlocal and becoming singular (!) in the limits $\mu \to 0$, $p_4 \to 0$.

Treating the interaction on a mean-field level reproduces well the previous numerical results in the silver blaze region.

Example 2

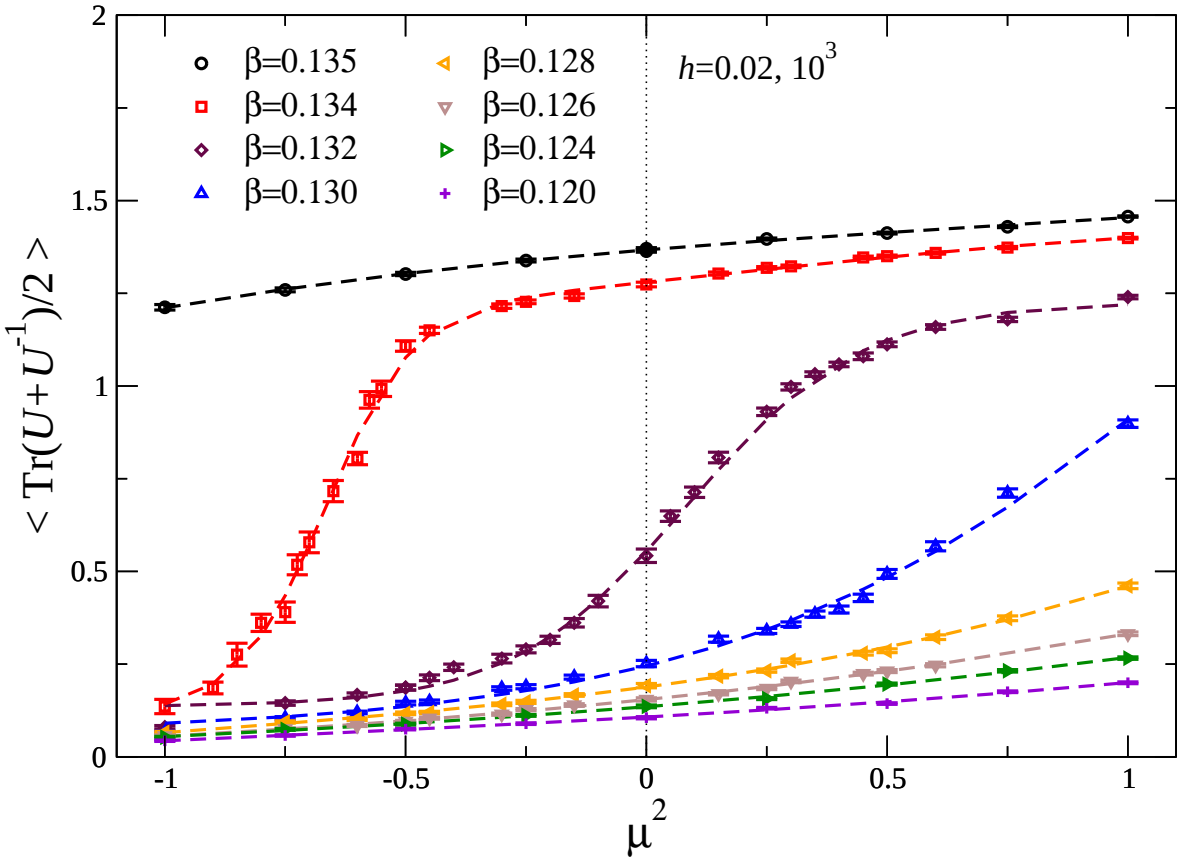
$SU(3)$ spin model in 3D by Complex Langevin simulation :
G. Aarts and F. A. James, arXiv:1112.4655

This has been already compared with the Flux Representation of the $SU(3)$ Potts model earlier, but only for $\mu^2 > 0$.

Here data for $\mu^2 > 0$ are obtained by Complex Langevin simulation, whereas data for $\mu^2 < 0$ are obtained by Real Langevin simulation.

The average of the two Polyakov loops, $\langle P + P^\star \rangle / 2V$ from Complex and Real Langevin simulation for various temperatures β as function of μ^2 for $h = 0.02$ on 10^3 lattices.

from G. Aarts and F. A. James, arXiv:1112.4655



7. 4. Complex Langevin for an Effective Action in Higher β Order

The lattice QCD partition function with Wilson gauge action $S_g[U]$ and flavors $f = 1, \ldots, N_f$ with Wilson fermion matrix $Q(\kappa_f, \mu_f)$ can be written in higher order strong coupling approximation :

$$
\begin{aligned}
Z &= \int D[U] \prod_f \det[Q] e^{-S_G[U]} = \int D[P] e^{-S_{\mathbf{eff}}[P]} \\
S_{\mathbf{eff}} &= S^s_{\mathbf{eff}} + S^a_{\mathbf{eff}} \\
S^s_{\mathbf{eff}}[P] &= -\sum_{i=1}^{\infty} \lambda_i S^s_i[P] \\
S^a_{\mathbf{eff}}[P] &= 2 \sum_{f=1}^{N_f} \sum_{i=1}^{\infty} \left[h_{if} S^a_i[P] + \bar{h}_{if} S^{a,\dagger}_i[P] \right]
\end{aligned}
$$

3D effective action is obtained after integration over the spatial links.

All $S^{s,a}_i[P]$ depend on untraced Polyakov loops $P(\vec{x}) = \prod_{\tau=1}^{N_t} U_0(\vec{x}, \tau)$.

Terms S_i^s are $\mathbb{Z}(N)$-symmetric, whereas S_i^a break $\mathbb{Z}(N)$ symmetry.

The couplings of the effective theory depend on

- **temporal extent N_t of the 4D lattice,**
- **the fundamental representation character coefficient $u(\beta) = \beta/18 + O(\beta^2)$ ($\beta = 2N_c/g^2$ is the 4D lattice coupling)**
- **the hopping parameters κ_f; for heavy quarks related to the quark masses by $\kappa_f = \exp(-am_f)/2$, entering via certain ....**
- **"magnetic fields" $\bar{h}_{if}(\mu_f) = h_{if}(-\mu_f)$.**

The gauge sector is described by the nearest-neighbor interaction of $L_{\vec{x}} = \frac{1}{N}\ \mathrm{tr}\ \ P_{\vec{x}}$ with $L^{\star}_{\vec{y}} = \frac{1}{N}\ \mathrm{tr}\ \ P^{\dagger}\vec{y}$,

$$e^{-S^{s}_{\mathbf{eff}}[P]} \ = \ \prod_{<\vec{z}\vec{y}>} [1 + 2\lambda \mathbf{Re} L_{\vec{x}} L^{*}_{\vec{y}}]$$

$$\lambda(u, N_t \geq 5) \ = \ u^{N_t} \exp\left[N_t \left(4u^4 + 12u^5 - 14u^6 - 36u^7 + \frac{295}{2}u^8 + \frac{1851}{10}u^9 + \frac{1055797}{5120}u^{10} + \ldots \right) \right]$$

Well described for $\beta \leq 6.5$! Higher couplings are negligible for $N_t \geq 6$.

When fermions are present, β is shifted by $O(\kappa^4)$ corrections.

The $\mathbb{Z}(N)$-breaking terms are

$$e^{-S^a_{\rm eff}[P]} = \prod_n \Delta_n[P] \; .$$

a product of single-site terms Δ_1 and of nearest neighbor site terms Δ_2

$$\Delta_1 \; = \; \prod_{f,\vec{x}} \det[1 + h_{1f} P_{\vec{x}}]^2 [1 + \bar{h}_{1f} P^\dagger_{\vec{x}}]^2$$

$$\Delta_2 \; = \; \prod_{f,<\vec{x}\vec{y}>} \left[1 - h_{2f} N_t \; {\rm tr} \; \frac{P_{\vec{x}}}{1 + C_f P_{\vec{x}}} \; {\rm tr} \; \frac{P_{\vec{y}}}{1 + C_f P_{\vec{y}}} \right]^2$$

The latter contains quark-quark ($P_{\vec{x}} P_{\vec{y}}$) interactions, with summations over multiple windings !

$$h_{1f} \; = \; C_f \left[1 + 6 \kappa_f^2 N_t \frac{u - u^{N_t}}{1 - u} + \ldots \right] \; ;$$

$$h_{2f} \; = \; C_f^2 \frac{\kappa_f^2}{N} \left[1 + 2 \frac{u - u^{N_t}}{1 - u} + \ldots \right] \; ,$$

with $C_f = (2\kappa_f e^{a\mu_f})^{N_t} = e^{(\mu_f - m_f)/T}$ and $\bar{C}_f(\mu_f) = C_f(-\mu_f)$.

The static and (extremely) strong coupling limit ($\beta = \lambda = 0$) has reasonable properties :

The partition function factorizes into single site partition functions Z_1 in the volume $V = N_s^3$:

$$Z(\beta = 0) = Z_1^V = \left[\int dP \left(1 + CL + C^2 L^* + C^3\right)^2 \left(1 + \bar{C} L^\star + \bar{C}^2 L + \bar{C}^3\right)^2 \right]^V$$

Only singlet terms survive the remaining group integration over dP.

$$\begin{aligned} Z_1 \;=\; & \left[1 + 4C^3 + C^6\right] + 2C \left[2 + 3C^3\right] \bar{C} \\ & + \; 2C^2 \;\; \left[5 + 3C^3\right] \;\; \bar{C}^2 + 2 \left[2 + 10C^3 + 2C^6\right] \;\; \bar{C}^3 \\ & + \; 2C \;\; \left[3 + 5C^3\right] \;\; \bar{C}^4 + 2C^2 \;\; \left[3 + 2C^3\right] \;\; \bar{C}^5 \\ & + \; \left[1 + 4C^3 + C^6\right]^3 \;\; \bar{C}^6 \;\; \stackrel{T \to 0}{\longrightarrow} \;\; \left[1 + 4C^N + C^{2N}\right] \end{aligned}$$

Here the terms denote

- **baryons** $\sim C^3$
- **mesons** $\sim \bar{C}C$
- **and other composites.**

For $\mu > 0$ at $T = 0$:

$\bar{C} \to 0$ **(only baryons survive).**

Quark density :

$$n = \frac{T}{V}\frac{\partial}{\partial \mu} \ln Z = \frac{1}{a^3}\frac{4N\ C^N + 2N\ C^{2N}}{1 + 4C^N + C^{2N}};$$

reduces in the high density limit to :

$$\lim_{\mu \to \infty} (a^3 n) = 2N = N\ (a^3 n_{B,\mathrm{sat}})\ .$$

Quark density in lattice units saturates when all available states per lattice site labeled by spin, color and flavor are occupied.

The geometric series over all windings of the Wilson lines is necessary in order to obtain saturation !

For $\mu \neq 0$ at $T = 0$:

$$\lim_{T\to 0} a^4 f = \begin{cases} 0, & \mu < m \\ 2N(a\mu - am), & \mu > m \end{cases} ;$$

$$\lim_{T\to 0} a^3 n = \begin{cases} 0, & \mu < m \\ 2N, & \mu > m \end{cases} .$$

The static strong coupling limit shows the silver blaze property, with $n = 0$ for $\mu < m$ and a jump to saturation density for $\mu > m$ (first order phase transition at $\mu_c = m$).

Baryon and meson masses :

$$am_M = -2\ln(2\kappa) - 6\kappa^2 - 24\kappa^2 \frac{u}{1-u} + \dots ,$$

$$am_B = -3\ln(2\kappa) - 18\kappa^2 \frac{u}{1-u} + \dots .$$

In the static strong coupling limit reduces to $am_B = -3\ln(2\kappa) = 3am$, i.e. the onset happens at $\mu_B = m_B$ (Cohen).

The effective theory has a mild sign problem.

This can treated by reweighting methods using a standard Metropolis algorithm.

As long as $h_{2f} = 0$, the effective theory can alternatively be treated in the Flux Representation and simulated with the worm algorithm, i. e. without any sign problem.

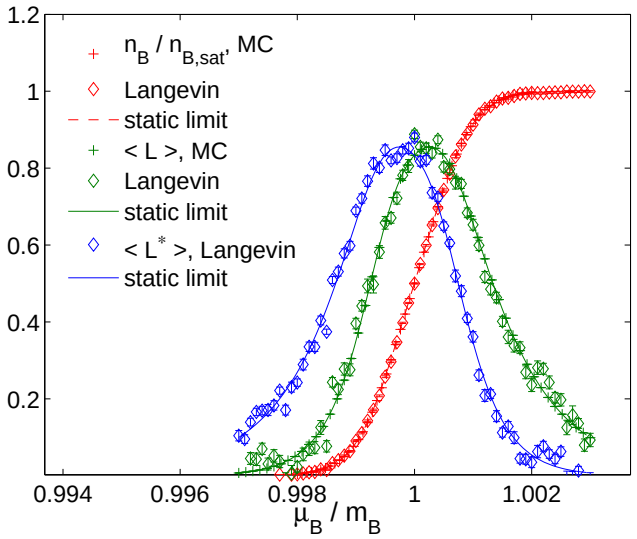
If $h_2 \neq 0$, a first quark-quark coupling $\sim L(\vec{x})L(\vec{y})$ enters.

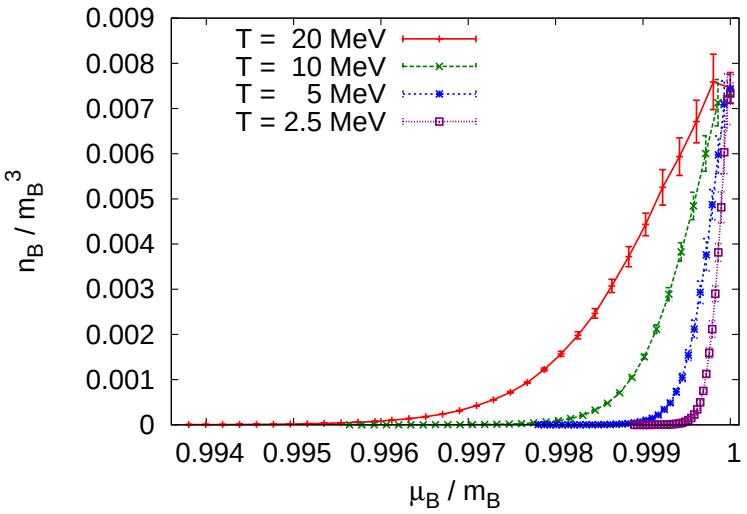
Then the Flux Representation does not work !

Instead, Complex Langevin simulation is used.

Left: Baryon density $n_B/N_{B,sat}$, Polyakov loop $\langle L \rangle$ and conjugate Polyakov loop $\langle L^\star \rangle$ as functions of μ_B/m_B. Right: Baryon density showing the silver blaze effect for temperature approaching 0.

From Philipsen et al. arXiv:1207.3005





8. Summary and Conclusions

Lattice gauge theory at baryonic $\mu \neq 0$ is still an exceptionally difficult field.

- **Direct simulations are impossible because of the complex weight.**
- **Only part of the phase diagram is indirectly accessible for simulations, due to**
  - **the sign problem,**
  - **the overlap problem.**
- **This hampers the reweighting technique, with limitations that are partly under control by the following diagnostic tools**
  - **the average phase factor,**
  - **the overlap parameter.**

- **What causes the overlap problem is not understood in detail.**
- **One would like to inspect configurations of the simulating ensemble (together with their reweighting factor) and confront them with configurations of the genuine target ensemble for inspection.**
- **This is only possible with imaginary chemical potential where the overlap problemi, after all, still exists.**
- **Standard methods of extrapolation (diagnostic quantities) are**
  - **reweighting (average phase factor, overlap parameter),**
  - **analytical continuation (requires series expansion at $\mu^2 < 0$), for the transition temperature, for screening lengths etc.**
  - **Taylor expansion (limited by radius of convergence in μ/T).**

- **The reconstruction of the Canonical Ensemble is very costly for realistic baryon numbers and volumes because of the Fourier transformation involved.**
- **Effective field theories obtained by strong coupling methods (applied to the gauge part) are useful, because the sign problem is partly ameliorated :**
  - **they may be studied on 3D lattices by mean field methods,**
  - **in special cases, their sign problem can be solved in the Flux Representation,**
  - **they may be eventually studied by Complex Langevin simulation.**
- **Such models are useful for obtaining qualitative results (say, connecting chiral vs. confinement aspects), but cannot replace a solution for the real 4D lattice gauge theory.**

- **They might be useful as source of inspiration for further search for a principal solution.**

An ambitious program has been outlined for LGT in the paper by M. Cristoforetti, F. Di Renzo, and L. Scorzato,
"High density QCD on a Lefschetz thimble ?" arXiv:1205.3996

combining

- **searching for stationary points of action,**
- **methods of integration along contours in complex plane,**
- **identifying an optimal integration domain associated with each stationary point (the "thimble"),**
- **evaluating the path integral as sum over integrals over thimbles.**
- **This program was motivated by E. Witten and Morse theory.**