Dense QCD Phases in Heavy-Ion Collisions, Dubna, 30. 8. 10

Clusters and Liquid-Gas Transition in Nuclear Matter

Gerd Röpke, Rostock

Outline

- Nuclear matter - a strongly interacting quantum liquid

  where it occurs, what do we know: nuclei, stars, HIC

- Many-particle theory:

  Equation of state, transport coefficients

  QCD? Effective interactions, Green functions, spectral functions

- Low-density limit: cluster formation

  Mass action law, nuclear statistical equilibrium, virial expansion

- Near saturation: medium effects

  mean-field and quasiparticles, dissolution of bound states

- Quantum condensates:

  transition from BEC to BCS, Hoyle states

Problems

1. Solution of the two-nucleon problem in nuclear matter. Separable interaction.

$$k \cot \quad = \# \frac{1}{a} + r_0 \frac{k^2}{2}$$

singlet (nn, pp): a = -23.678 fm, r = 1.726 fm

triplet (pn): a = 5.396 fm, r = 2.729 fm, E = -2.225 MeV

2. Thermodynamics with Skyrme I (Lattimer/Swesty)

Binding energy 16 MeV, saturation density 0.155 fm^{-3}

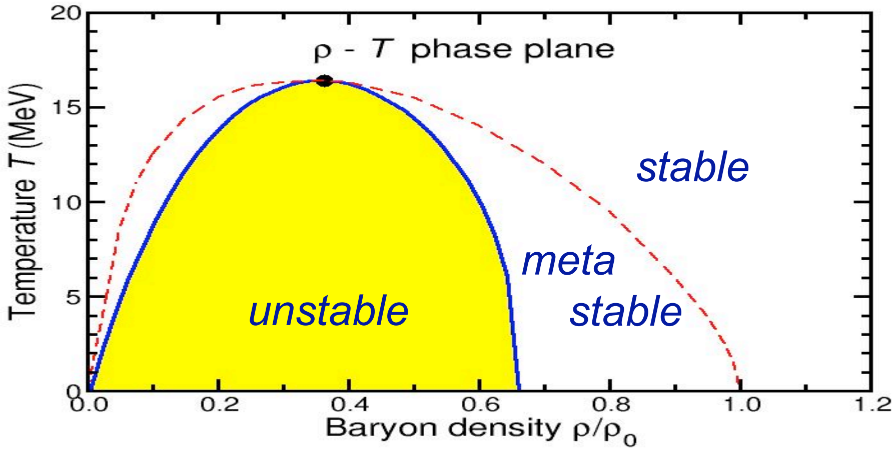
pot. energy density, n=n_p+n_n, y=n_p/n

w(n,y) = -[285.1 + 428.4 y(1-y)] MeV fm^3 n^2 +986.0 MeV fm^6 n^3

3. Bogoliubov transform. Transition from BCS to BEC

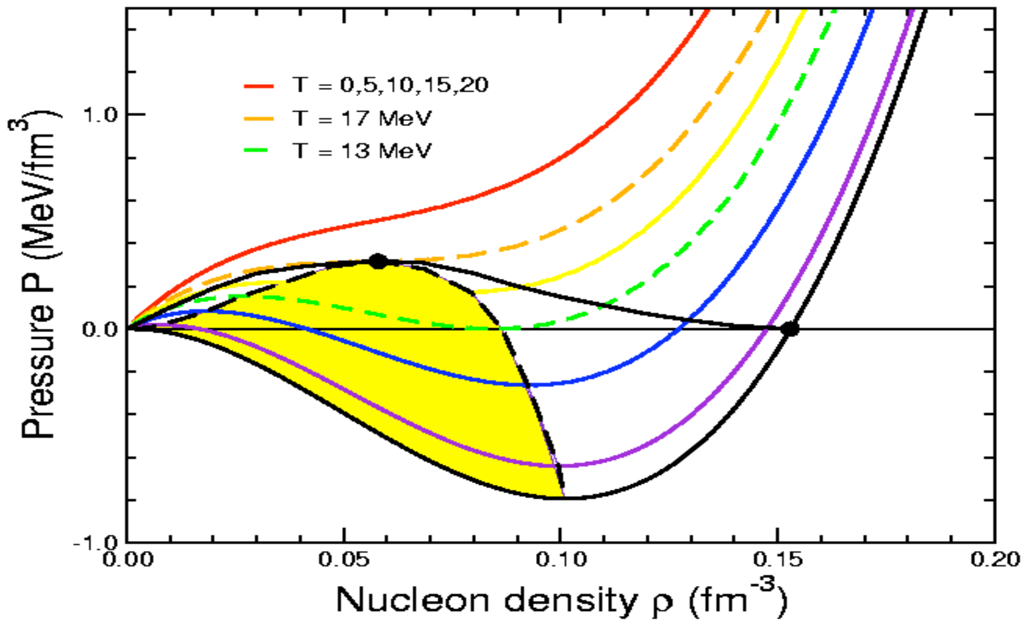
see G.R., D. Zablocki, Phys.Elem.Part.At.Nucl. **39**, 369 (2008)

Nuclear matter



Phase diagram

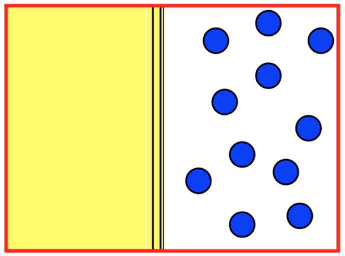
$$\varepsilon(T;\rho) = \varepsilon_{\rm FG}(T;\rho) + w(\rho)$$

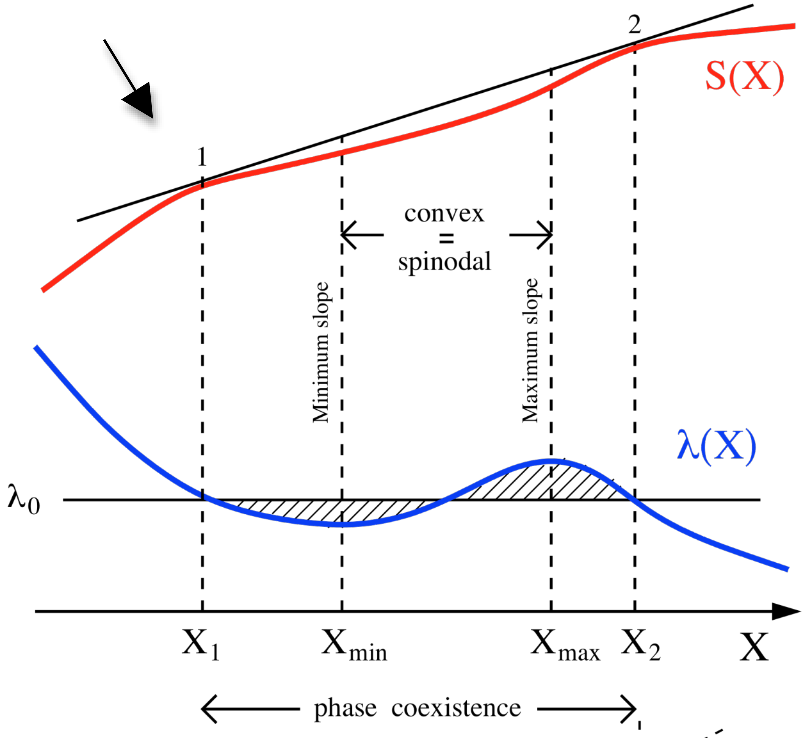


Equation of state:
$p_T(r)$

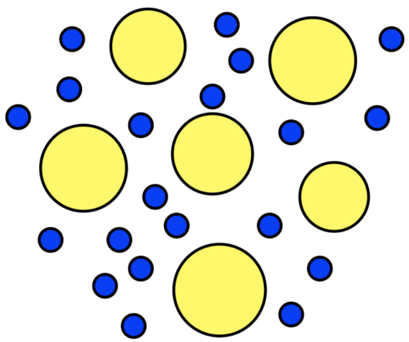
JRandrup: Dubna School, 2010

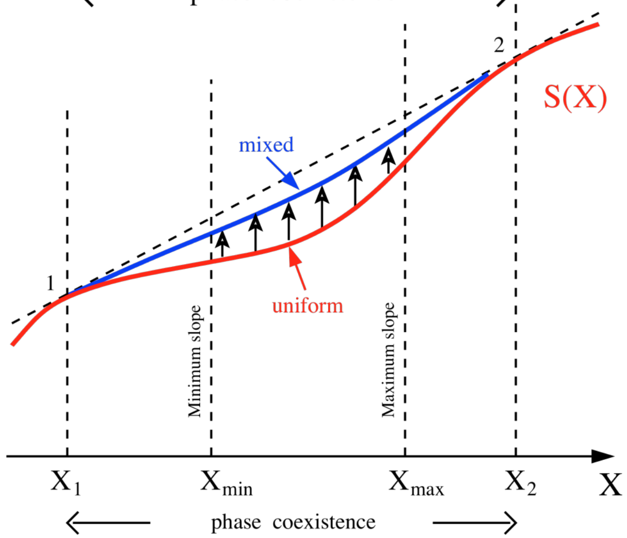
Separated phases:

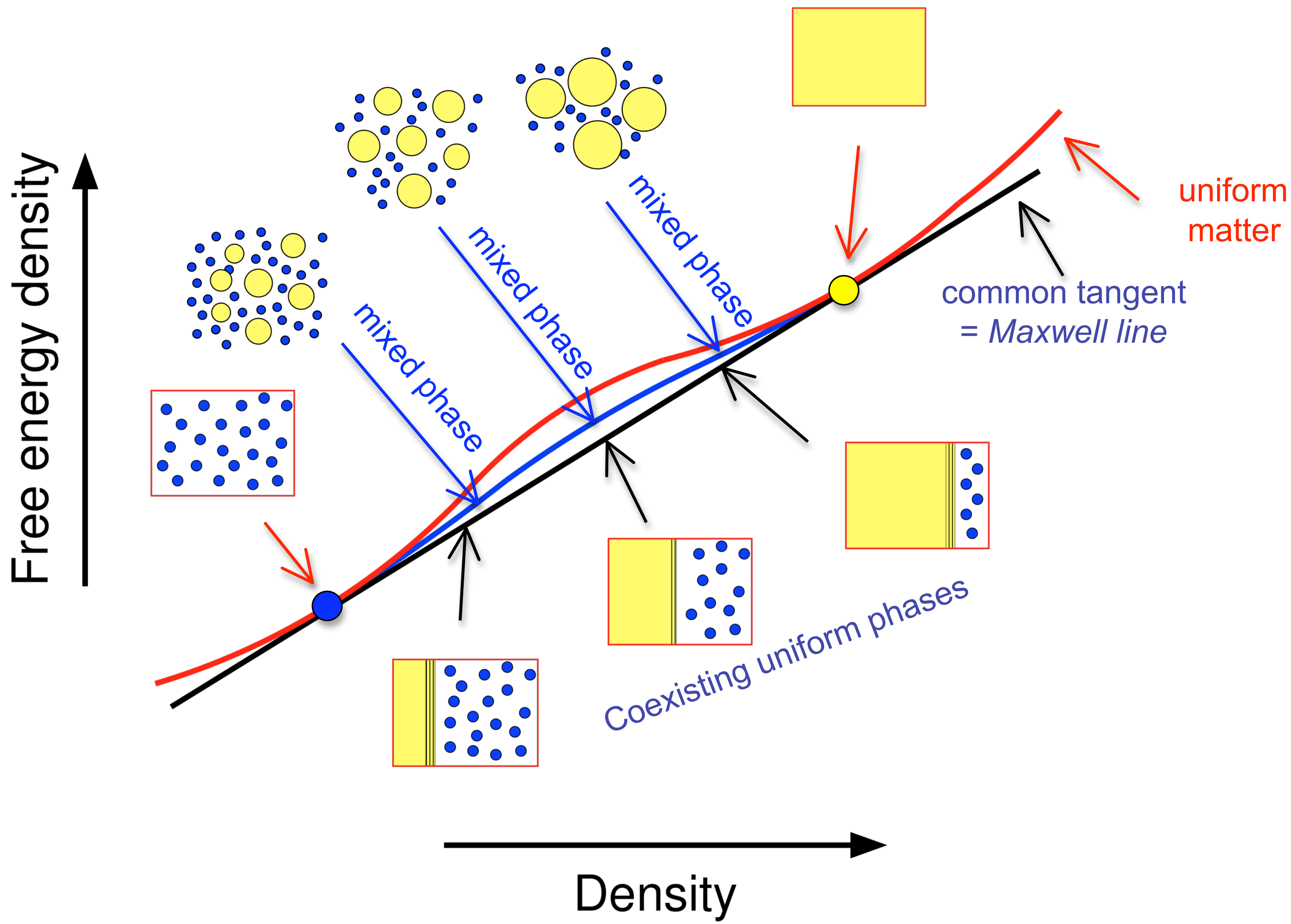




Mixed phase:







JRandrup: Dubna School, 2010

Hadron Gas *versus* *Quark-Gluon Plasma*

$$p^H = p_\pi + p_N + p_{\bar N} + p_w$$

$$p_\pi(T) = -g_\pi \int_{m_\pi}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 - \mathrm{e}^{-\beta\epsilon}]$$

$$p_N(T,\mu_0) = g_N \int_{m_N}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\epsilon-\mu_0)}]$$

$$p_{\bar N}(T,\mu_0) = g_N \int_{m_N}^{\infty} \frac{p\epsilon d\epsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\epsilon+\mu_0)}]$$

$$p_w(\rho) = \rho\partial_\rho w(\rho) - w(\rho)$$

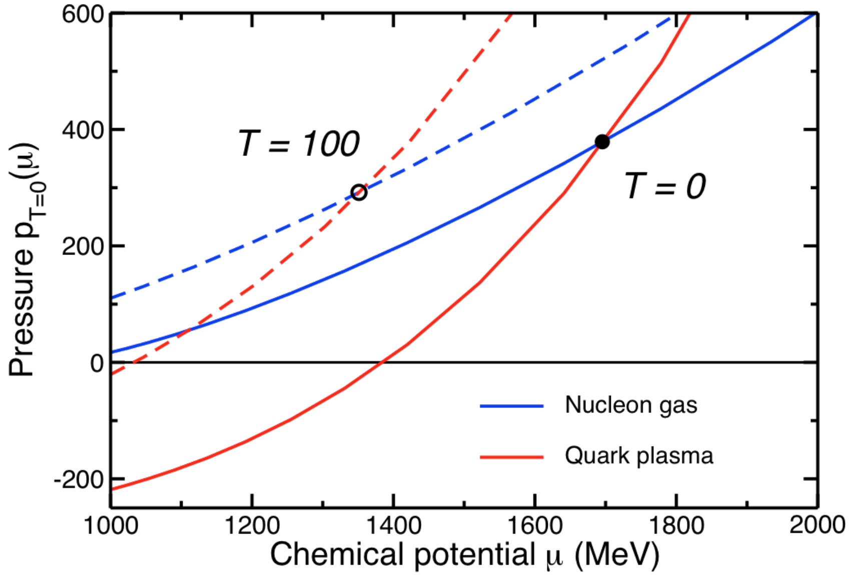
$$w(\rho) = \left[-A\left(\frac{\rho}{\rho_s}\right)^{\alpha} + B\left(\frac{\rho}{\rho_s}\right)^{\beta}\right]\rho$$

$$\mu = \mu_0 + \partial_\rho w = 3\mu_q$$

$$p^Q = p_g + p_q + p_{\bar q} - B$$

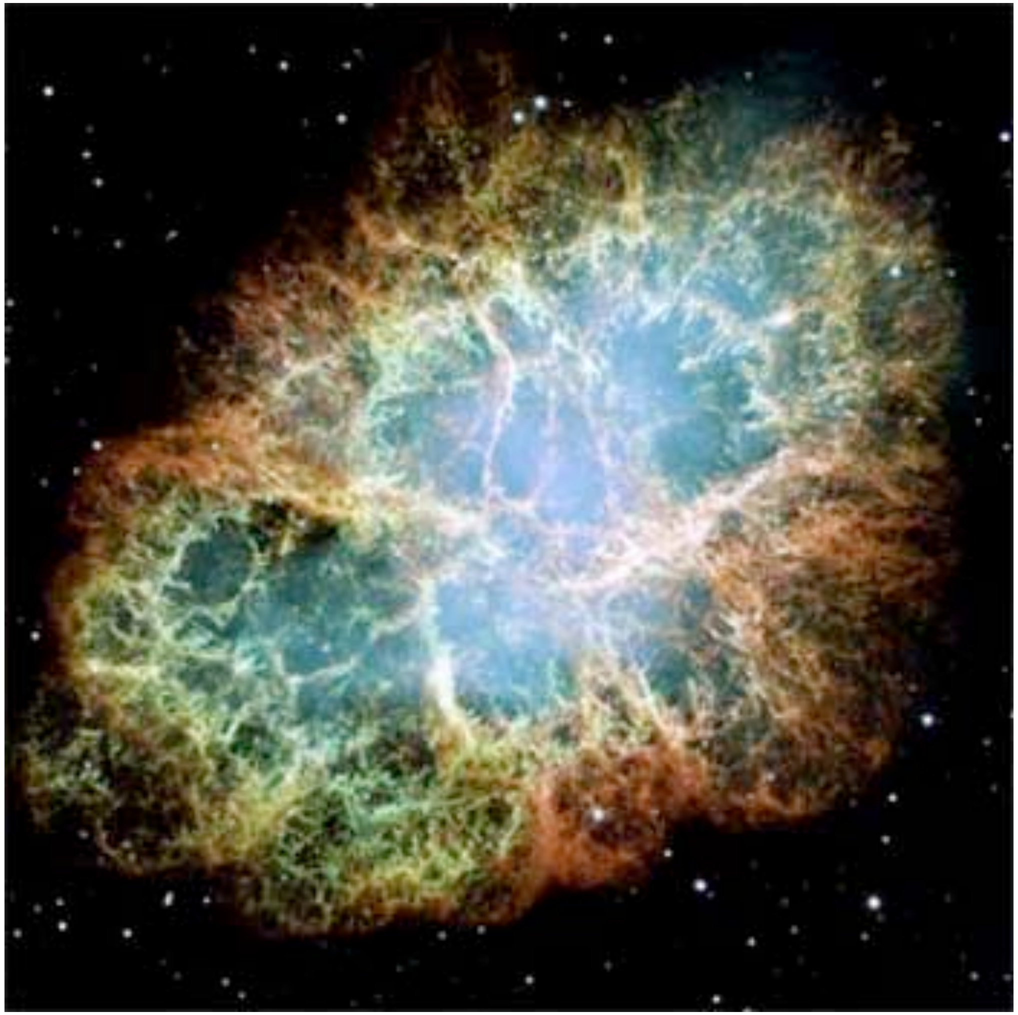
$$p_g = g_g \frac{\pi^2}{90} T^4$$

$$p_q + p_{\bar q} = g_q \left[\frac{7\pi^2}{360}T^4 + \frac{1}{12}\mu_q^2 T^2 + \frac{1}{24\pi^2}\mu_q^4\right]$$



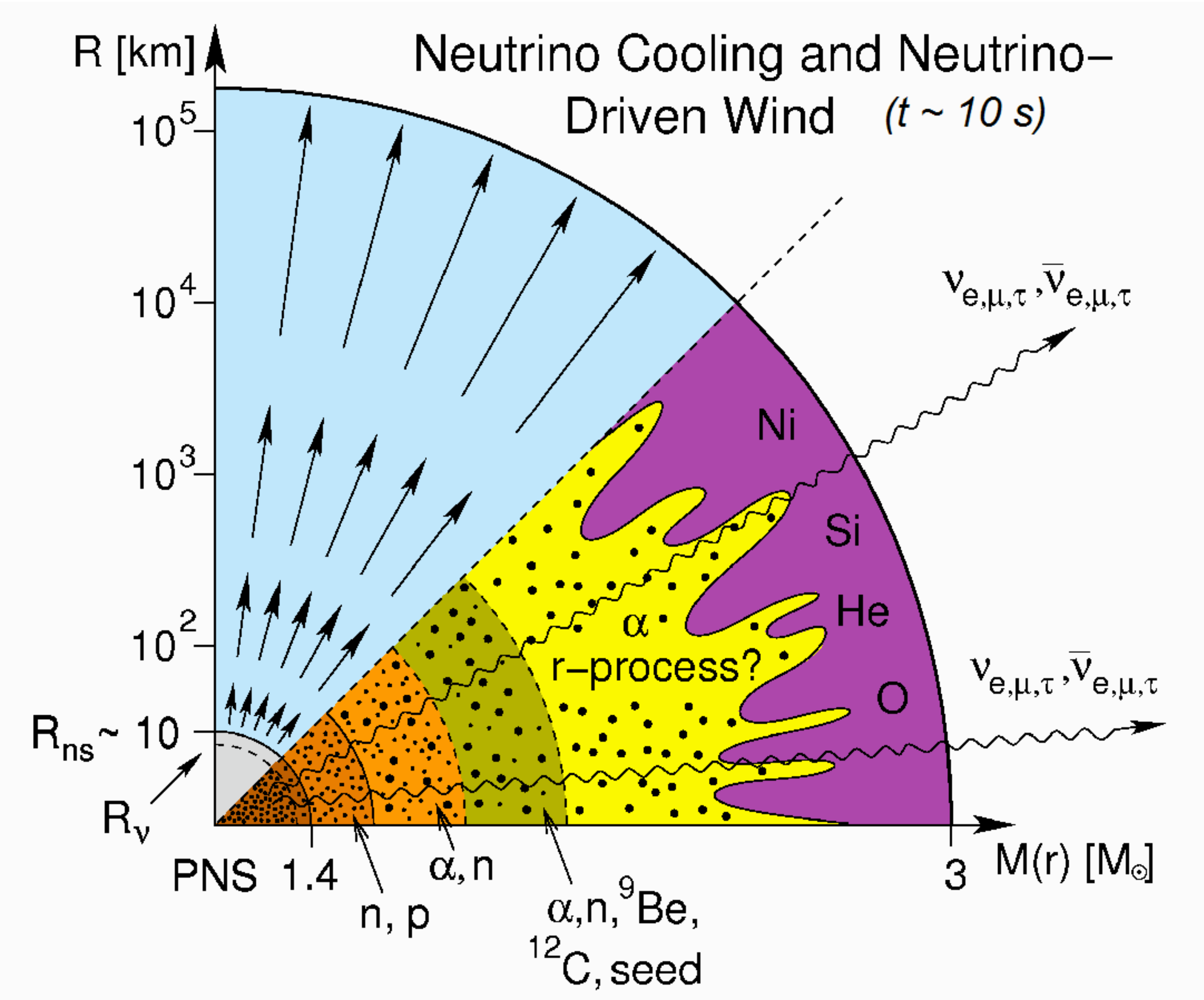
Supernova

Crab nebula, 1054 China, PSR 0531+21



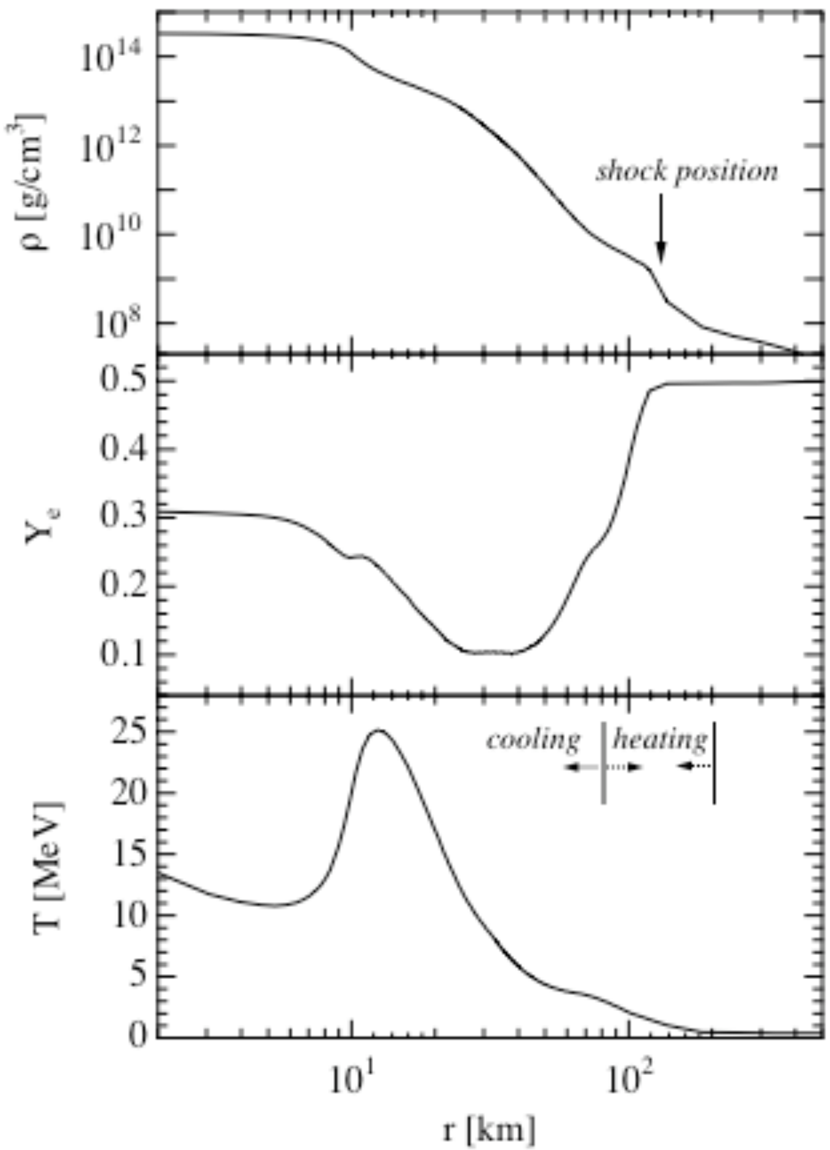
M1, the Crab Nebula. Courtesy of NASA/ESA

Supernova explosion



T.Janka

Core-collapse supernovae



Density.

electron fraction, and

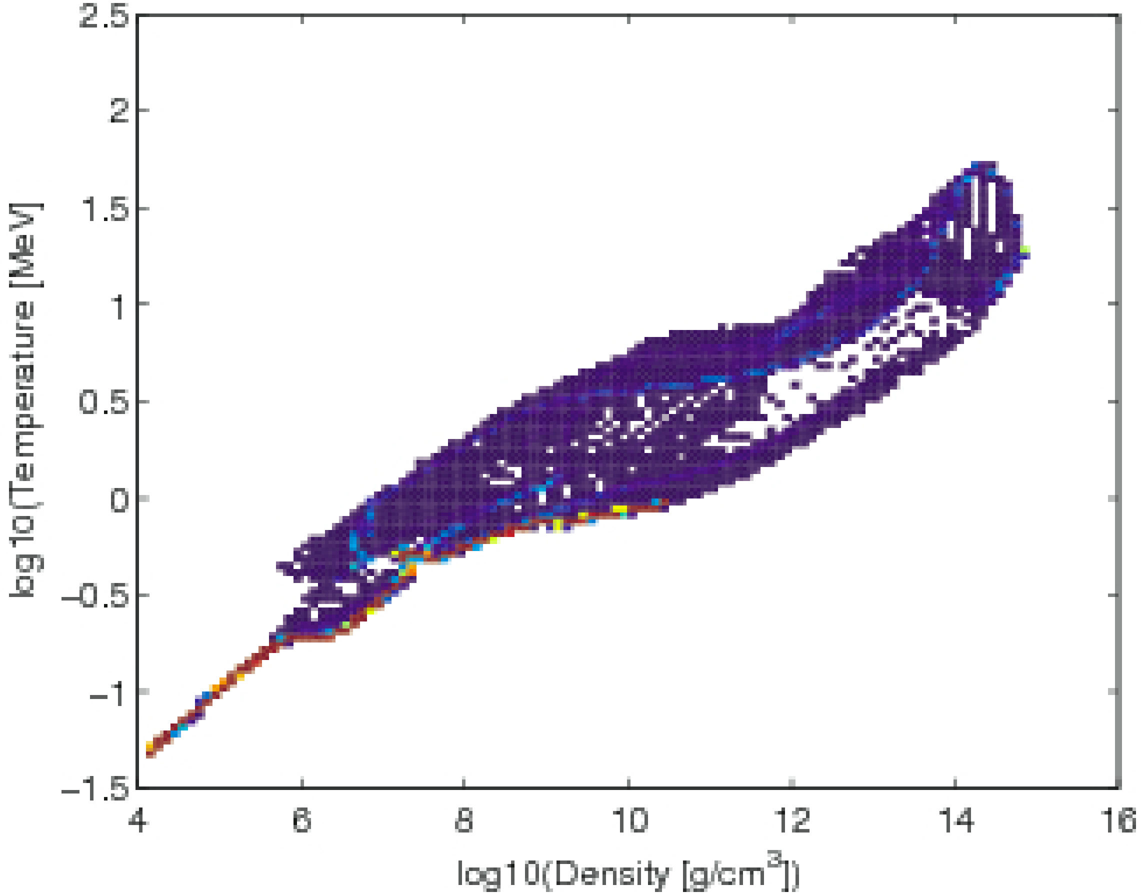
temperature profile

of a 15 solar mass supernova
at 150 ms after core bounce
as function of the radius.

Influence of cluster formation
on neutrino emission
in the cooling region and
on neutrino absorption
in the heating region ?

K.Sumiyoshi et al.,
Astrophys.J. **629**, 922 (2005)

Parameter range: Explosion



T. Fischer, On the possible fate of massive progenitor stars, Talk 25.2.08 Ladek

Nucleon-nucleon interaction

QCD? Effective Lagrangians, interaction potentials

- general form:

$$V_\alpha(p,p') = \sum_{i,j=1}^{N} w_{\alpha i}(p)\lambda_{\alpha ij} w_{\alpha j}(p') \qquad \text{uncoupled}$$

and

$$V_\alpha^{LL'}(p,p') = \sum_{i,j=1}^{N} w_{\alpha i}^{L}(p)\lambda_{\alpha ij} w_{\alpha j}^{L'}(p') \qquad \text{coupled}$$

p, p'	in- and outgoing relative momentum
α ...	channel
N ...	rank
$\lambda_{\alpha ij}$.	coupling parameter
L, L'	orbital angular momentum

Weak interaction - beta equilibrium? Coulomb interaction?

Many-particle theory

- equilibrium correlation functions

  e.g. equation of state $n(\beta,\mu) = \frac{1}{\Omega_0}\sum_1 \langle a_1^\dagger a_1 \rangle$

  density matrix $\langle a_1^\dagger a_{1'} \rangle = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} f_1(\omega) A(1,1',\omega)$

- Spectral function

$$A(1,1',\omega) = \mathrm{Im}\left[G(1,1',\omega+i\eta) - G(1,1',\omega-i\eta)\right]$$

- Matsubara Green function

$$G(1,1',iz_\nu), \qquad z_\nu = \frac{\pi\nu}{\beta} + \mu, \quad \nu = \pm 1, \pm 3, \cdots$$

$$1 \equiv \{\mathbf{p}_1, \sigma_1, c_1\}, \quad f_1(\omega) = \frac{1}{e^{\beta(\omega-\mu)}+1}, \quad \Omega_0 - \text{volume}$$

Many-particle theory

- Dyson equation and self energy (homogeneous system)

$$G(1, iz_\nu) = \frac{1}{iz_\nu - E(1) - \Sigma(1, iz_\nu)}$$

- Evaluation of $\Sigma(1, iz_\nu)$:
  perturbation expansion, diagram representation

$$A(1,\omega) = \frac{2\mathrm{Im}\,\Sigma(1,\omega + i0)}{[\omega - E(1) - \mathrm{Re}\,\Sigma(1,\omega)]^2 + [\mathrm{Im}\,\Sigma(1,\omega + i0)]^2}$$

approximation for self energy $\longrightarrow$ approximation for equilibrium correlation functions

alternatively: simulations, path integral methods

Simple approximations

1. Ideal Fermi gas
2. Hartree - Fock approximation
3. Quasiparticle approximation

Different approximations

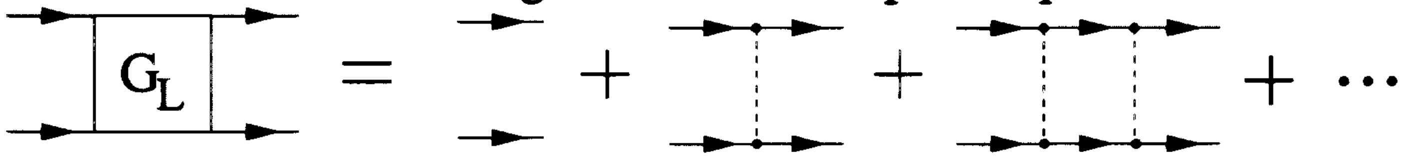
- Expansion for small Im $\Sigma(1, \omega + i\eta)$

$$A(1,\omega) \approx \frac{2\pi\delta(\omega - E^{\text{quasi}}(1))}{1 - \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Re}\ \Sigma(1,z)|_{z=E^{\text{quasi}}-\mu_1}} - 2\mathrm{Im}\ \Sigma(1,\omega+i\eta)\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{P}{\omega + \mu_1 - E^{\text{quasi}}(1)}$$

quasiparticle energy $E^{\text{quasi}}(1) = E(1) + \mathrm{Re}\ \Sigma(1,\omega)|_{\omega=E^{\text{quasi}}}$

- chemical picture: bound states $\hat{=}$ new species

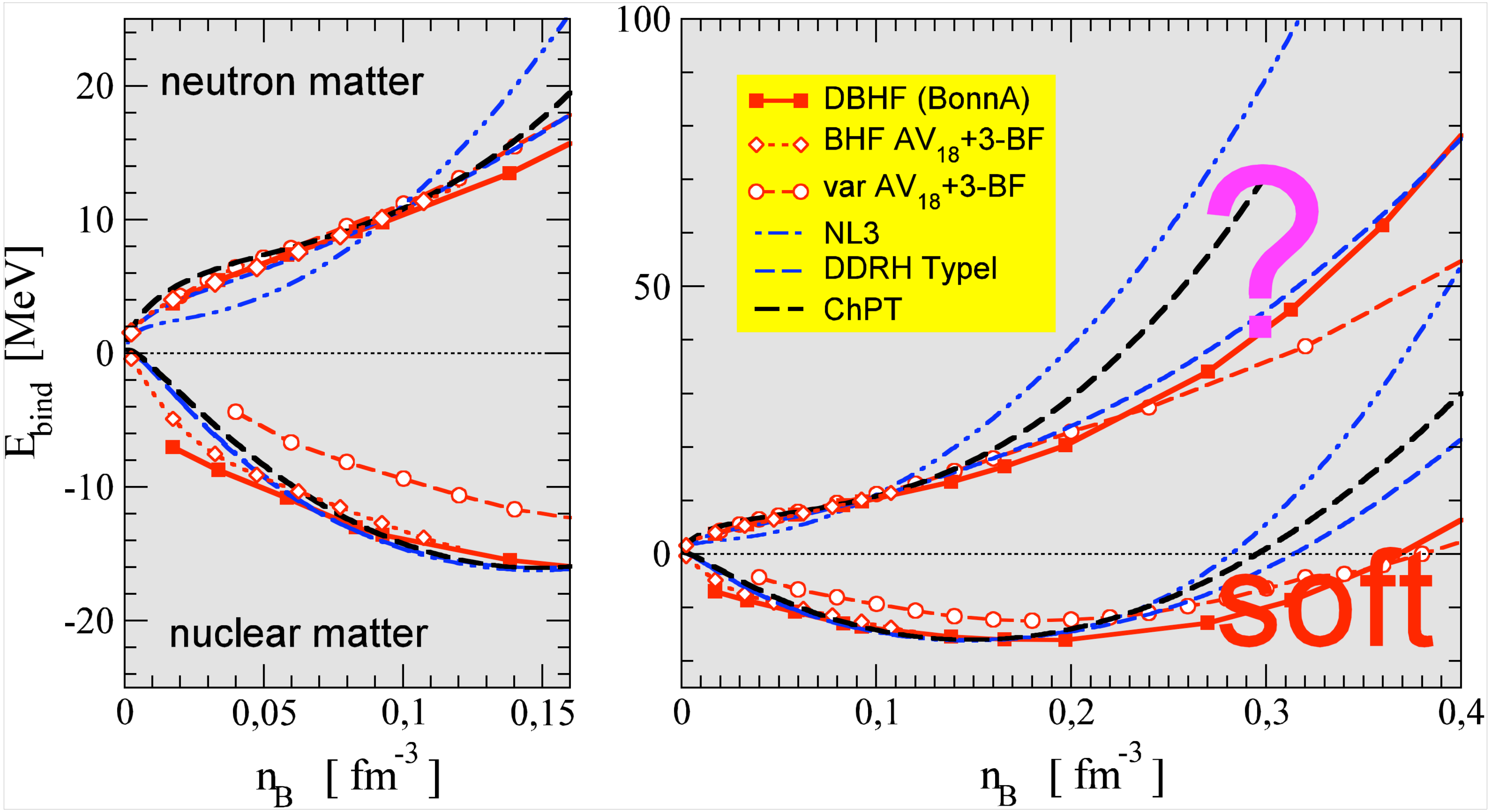
summation of ladder diagrams, Bethe-Salpeter equation



Medium effects:
Quasiparticle approximation

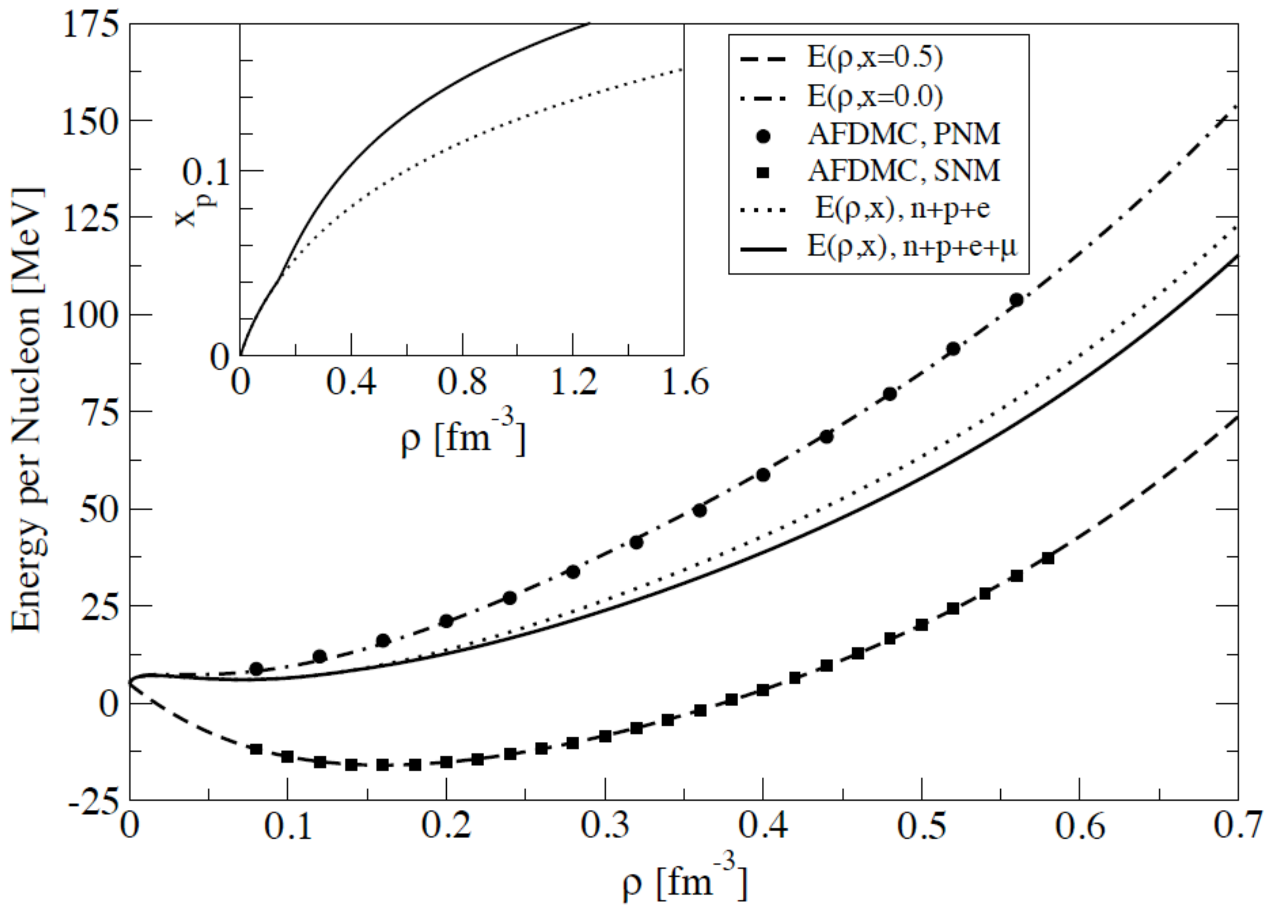
- Skyrme
- relativistic mean field (RMF)
  Lagrangian: non-linear sigma, TM1 parameters,
  single particle modifications, energy shift, effective mass
- DD-RMF [S.Typel, Phys. Rev. C **71**, 064301 (2007)]:
  expansion of the scalar field and the vector fields
  in powers of proton/neutron densities
- Dirac-Brueckner Hartree Fock (DBHF)
- Diffusion Monte Carlo EOS calculation (Talk Illarionov)

Quasiparticle picture: RMF and DBHF



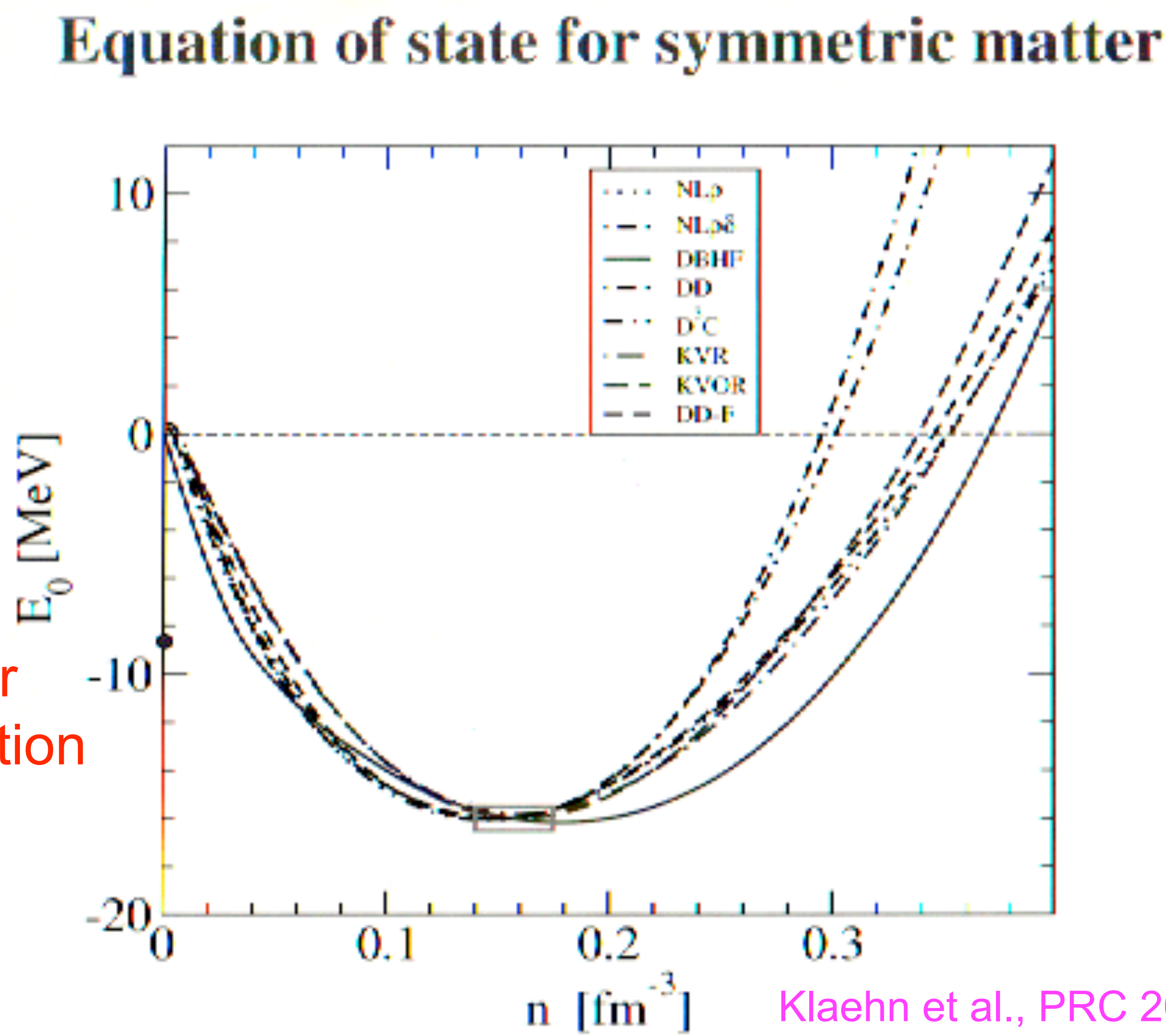
J.Margueron et al., Phys.Rev.C **76**,034309 (2007)

Diffusion Monte Carlo EOS calculation



S. Gandolfi, A. Yu. Illarionov, et al., Mon.Not.R.Astron.Soc., 2010

Quasiparticle approximation for nuclear matter



But: cluster formation

Incorrect low-density limit

Klaehn et al., PRC 2006

Simple approximations

1. Ideal Fermi gas
2. Hartree - Fock approximation
3. Quasiparticle approximation

Where are the light clusters?
(d,t,^{3}He,^{4}He)

Different approximations

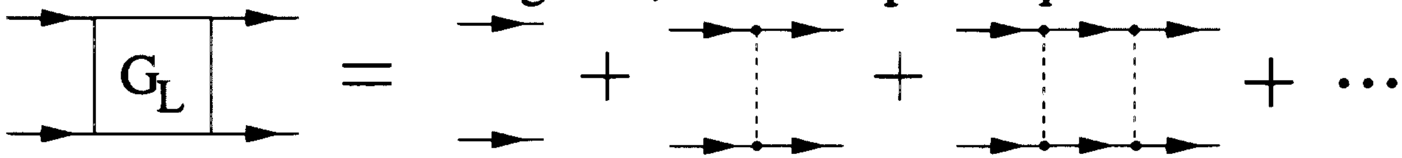
- Expansion for small $\mathrm{Im}\ \Sigma(1, \omega + i\eta)$

$$A(1,\omega) \approx \frac{2\pi\delta(\omega - E^{\mathrm{quasi}}(1))}{1 - \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Re}\ \Sigma(1,z)|_{z=E^{\mathrm{quasi}}-\mu_1}} - 2\mathrm{Im}\ \Sigma(1,\omega+i\eta)\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{P}{\omega+\mu_1-E^{\mathrm{quasi}}(1)}$$

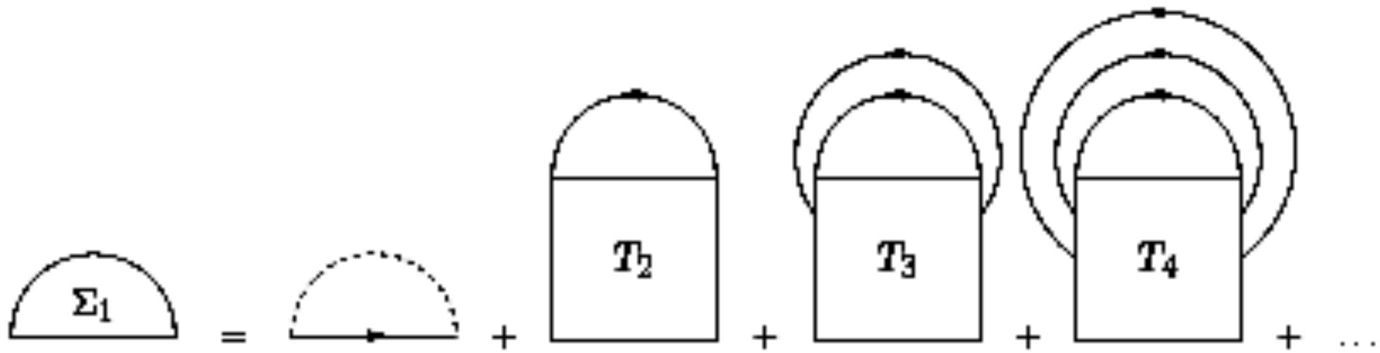
quasiparticle energy $E^{\mathrm{quasi}}(1) = E(1) + \mathrm{Re}\ \Sigma(1,\omega)|_{\omega=E^{\mathrm{quasi}}}$

- chemical picture: bound states $\hat{=}$ new species

summation of ladder diagrams, Bethe-Salpeter equation



Cluster decomposition of the self-energy



Ideal mixture of reacting nuclides

$$n_p(T, \mu_p, \mu_n) = \frac{1}{V} \sum_{A,\nu,K} Z_A f_A\{E_{A,\nu K} - Z_A \mu_p - (A - Z_A)\mu_n\}$$

$$n_n(T, \mu_p, \mu_n) = \frac{1}{V} \sum_{A,\nu,K} (A - Z_A) f_A\{E_{A,\nu K} - Z_A \mu_p - (A - Z_A)\mu_n\}$$

$$f_{A(z)} = \frac{1}{\exp(z/T) - (-1)^A}$$

mass number A,
charge Z_A,
energy $E_{A,\nu,K}$,
ν internal quantum number,
$\sim K$ center of mass momentum

Beth-Uhlenbeck formula

rigorous results at low density: virial expansion

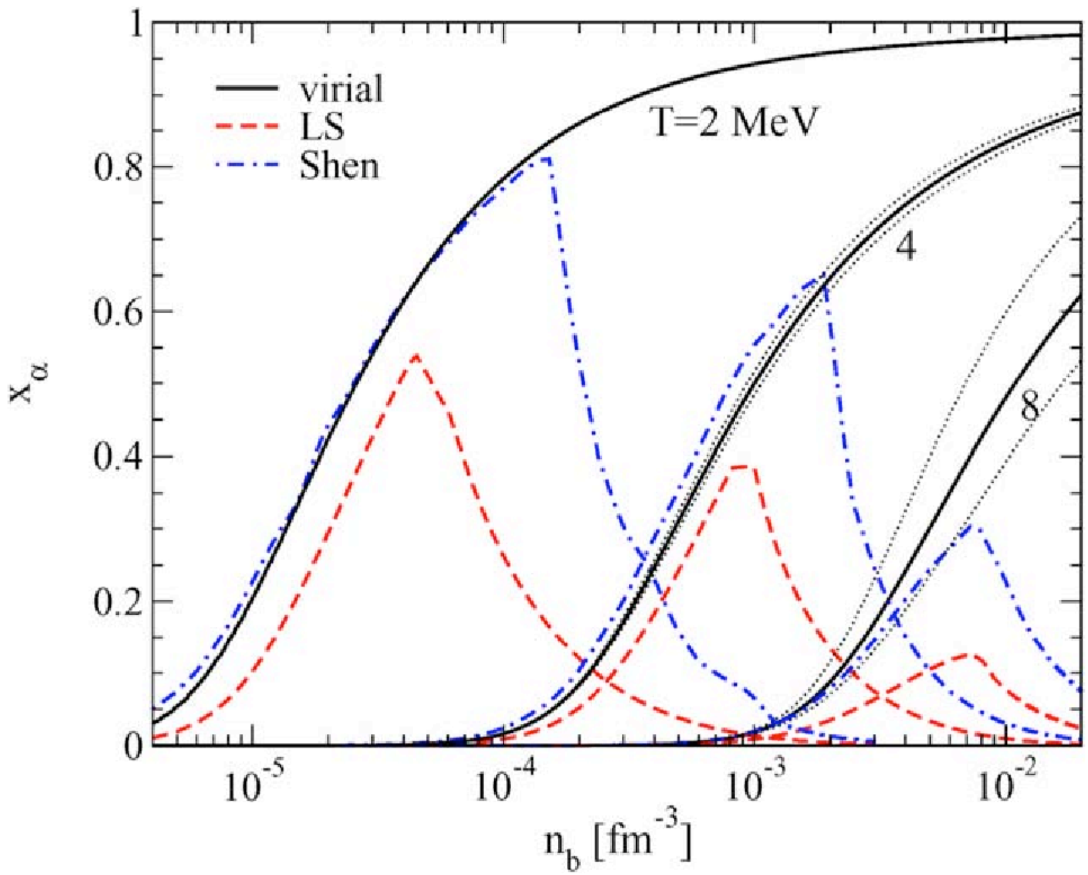
Beth-Uhlenbeck formula

$$
\begin{aligned}
n(T,\mu) &= \frac{1}{V}\sum_p e^{-(E(p)-\mu)/k_BT} \\
&+ \frac{1}{V}\sum_{nP} e^{-(E_{nP}-2\mu)/k_BT} \\
&+ \frac{1}{V}\sum_{\alpha P}\int_0^\infty \frac{dE}{2\pi} e^{-(E+P^2/4m-2\mu)/k_BT}\frac{d}{dE}\delta_\alpha(E) \\
&+ \dots
\end{aligned}
$$

$\delta_\alpha(E)$: scattering phase shifts, channel α

Alpha-particle fraction in the low-density limit

symmetric matter, T=2, 4, 8 MeV



C.J.Horowitz, A.Schwenk, Nucl. Phys. A **776**, 55 (2006)

Effective wave equation for the deuteron in matter

$$\left[\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}\right]\Psi_{n,P}(p_1,p_2)+\sum_{p_1',p_2'}(1-f_{p_1}-f_{p_2})V(p_1,p_2;p_1',p_2')\Psi_{n,P}(p_1',p_2')$$

$$=E_{n,P}\,\Psi_{n,P}(p_1,p_2)$$

Pauli-blocking

Fermi distribution function

$$f_p=\left[e^{(p^2/2m-\mu)/k_BT}+1\right]^{-1}$$

BEC-BCS crossover:
Alm et al.,1993

Effective wave equation for the deuteron in matter

$$\frac{p_1^2}{\#2m_1} + \frac{p_2^2}{2m_2} \overset{\%}{\underset{\&}{(}} \;_{n,P}(p_1,p_2) + \sum_{p_1\ ,p_2} (1)\ f_{p_1}\ )\ f_{p_2})V(p_1,p_2;p_1\ ,p_2\ )(\ _{n,P}(p_1\ ,p_2\ )$$

$$= E_{n,P}\quad {}_{n,P}(p_1,p_2)$$

Add self-energy

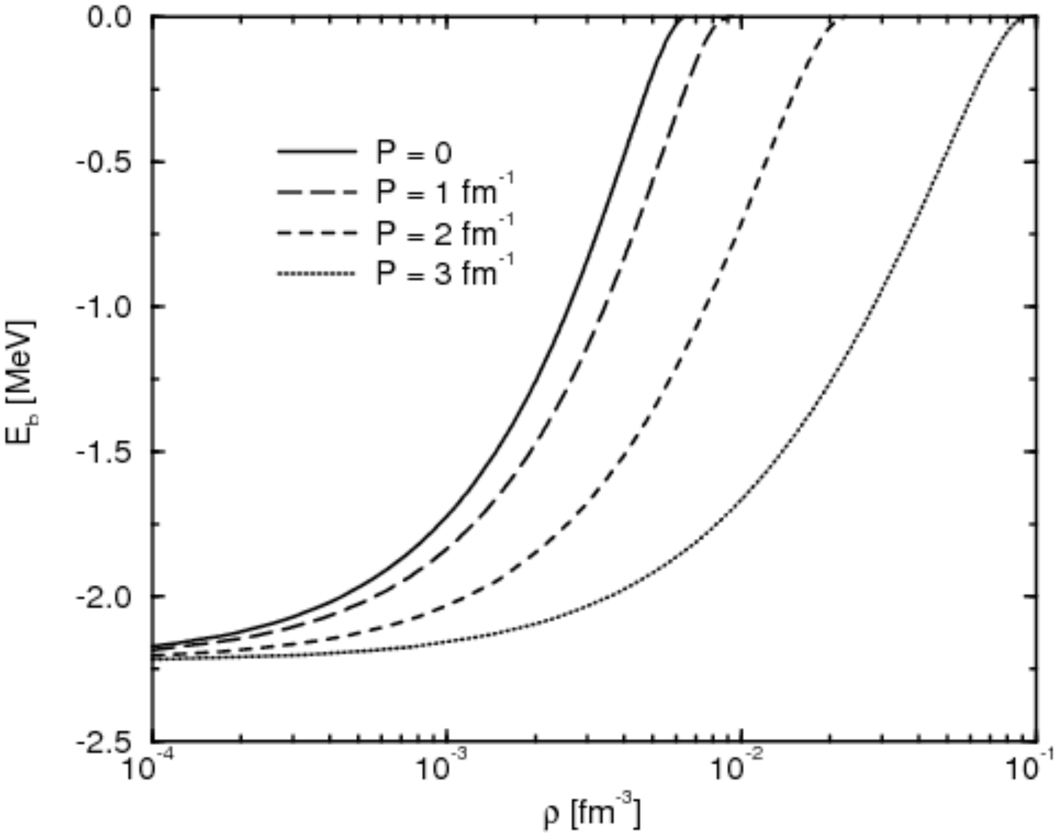
Pauli-blocking

Fermi distribution function

$$f_p = \left[e^{(p^2/2m-\mu)/k_B T} + 1 \right]^{-1}$$

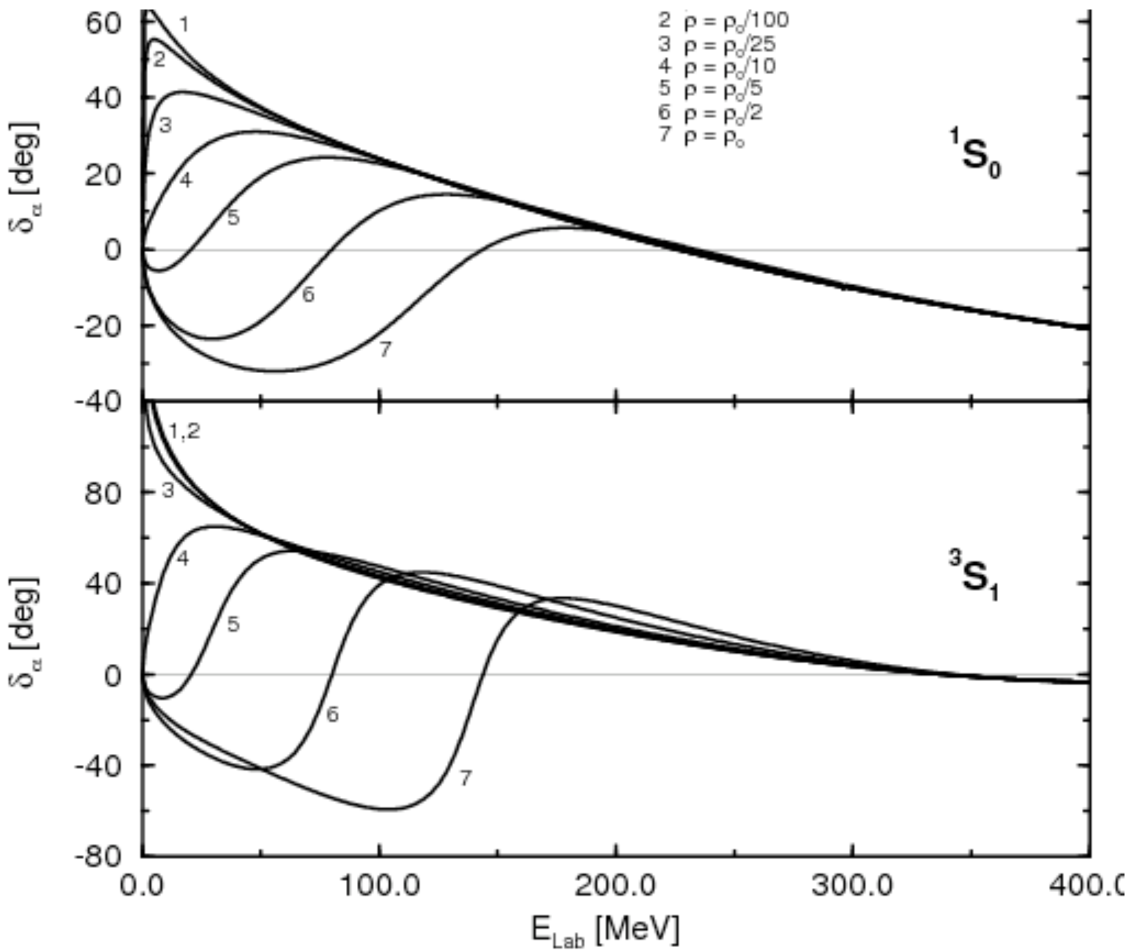
BEC-BCS crossover:
Alm et al.,1993

Deuterons in nuclear matter



T=10 MeV, P: center of mass momentum

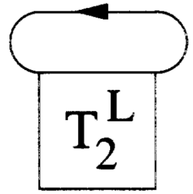
Scattering phase shifts in matter



Different approximations

low density limit:

$$G_2^L(12,1'2',i\lambda) = \sum_{n\mathbf{P}} \Psi_{n\mathbf{P}}(12) \frac{1}{i\omega_\lambda - E_{n\mathbf{P}}} \Psi^*_{n\mathbf{P}}(12)$$

$\Sigma =$ 

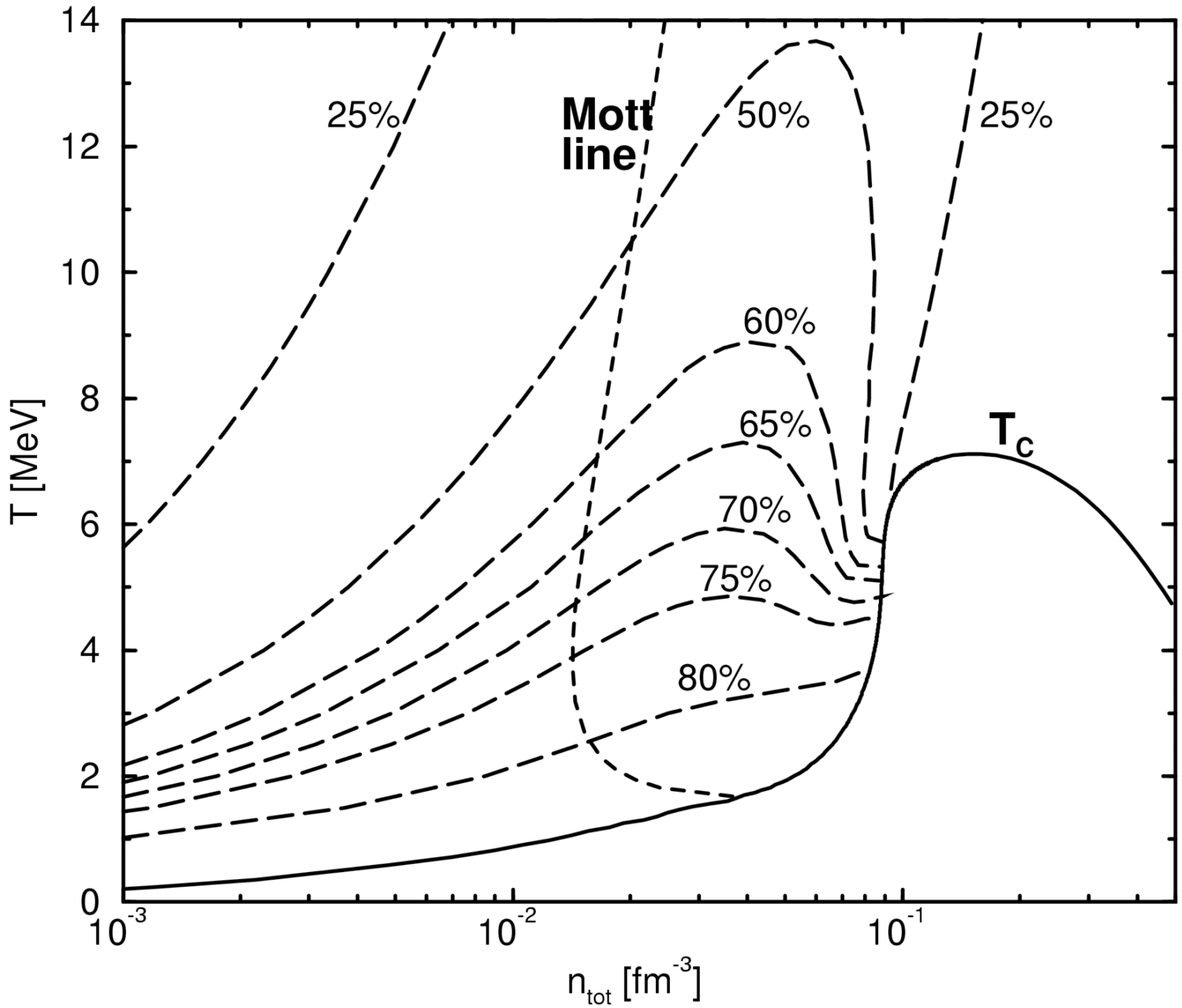
$$n(\beta,\mu) = \sum_1 f_1(E^{\text{quasi}}(1)) + \sum_{2,n\mathbf{P}}^{\text{bound}} g_{12}(E_{n\mathbf{P}})$$

$$+ \sum_{2,n\mathbf{P}} \int_0^\infty \mathrm{d}k\, \delta_{\mathbf{k},\mathbf{p}_1-\mathbf{p}_2} g_{12}(E^{\text{quasi}}(1) + E^{\text{quasi}}(2)) 2\sin^2\delta_n(k) \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}k} \delta_n(k)$$

- generalized Beth-Uhlenbeck formula
  correct low density/low temperature limit:
  mixture of free particles and bound clusters

Composition of symmetric nuclear matter

Fraction of correlated matter (virial expansion, Generalized Beth-Uhlenbeck approach, contribution of bound states, of scattering states, phase shifts)



H. Stein et al.,
Z. Phys. **A351**, 259 (1995)

Deuteron quasiparticle properties

$$E_d^{\text{qu}}(P) = E_d^{\text{free}} + \Delta E_d + \frac{\hbar^2}{2m_d^*}P^2 + O(P^4)$$

$$\Delta E_d^{\text{Pauli}}(T, n_B, \alpha) = \delta E_d^{(0)}(T, \alpha) n_B + O(n_B^2)$$

$$\frac{m_d^*}{m_d}(T, n_B, \alpha) = 1 + \delta m_d^{(0)}(T, \alpha) n_B + O(n_B^2)$$

T [MeV]	delta E [MeV fm³]	delta m* [fm³]
10	364.3	21.3
4	712.9	87.1

$$E_d^{\text{free}} = -2.225\text{MeV}$$

G.R.,PRC **79**, 014002 (2009)

Few-particle Schrödinger equation in a dense medium

4-particle Schrödinger equation with medium effects

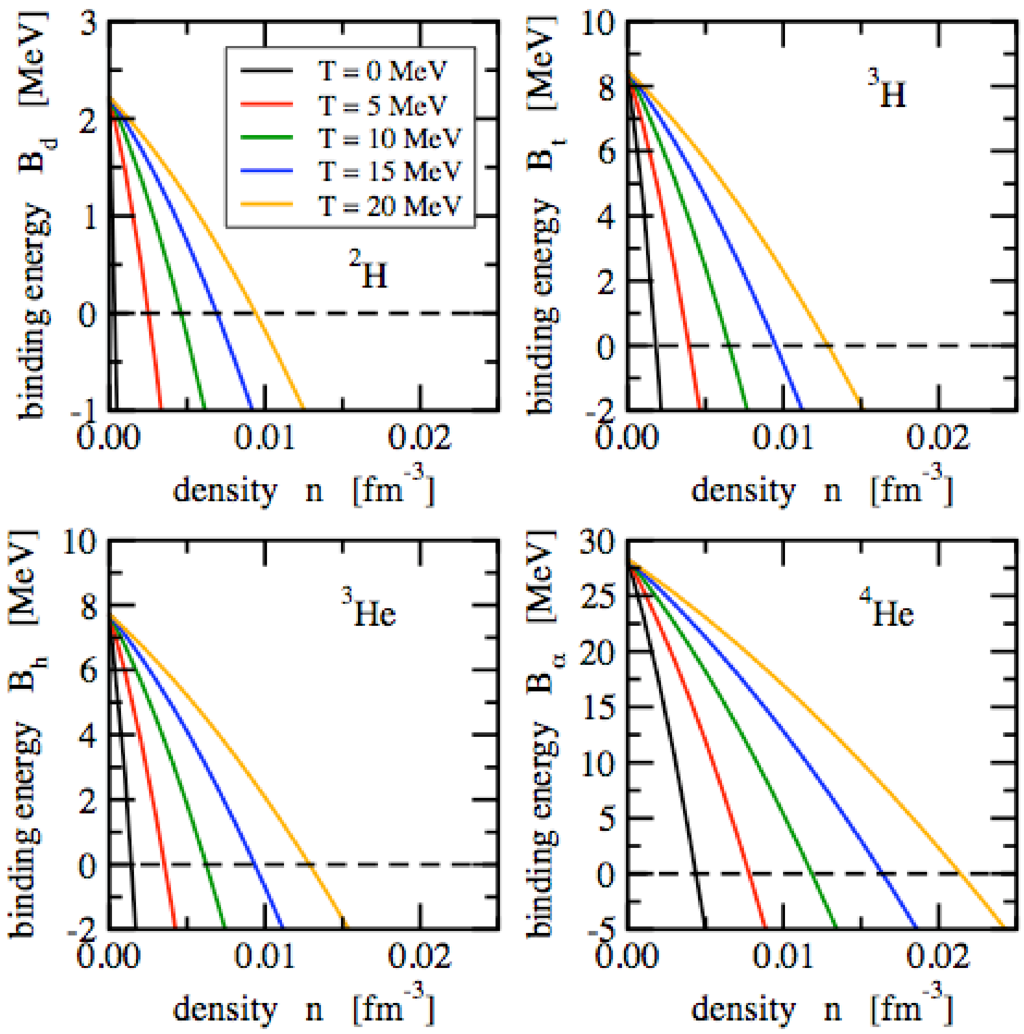
$$\left(\left[E^{HF}(p_1)+E^{HF}(p_2)+E^{HF}(p_3)+E^{HF}(p_4)\right]\right)\Psi_{n,P}(p_1,p_2,p_3,p_4)$$

$$+\sum_{p_1',p_2'}(1-f_{p_1}-f_{p_2})V(p_1,p_2;p_1',p_2')\Psi_{n,P}(p_1',p_2',p_3,p_4)$$

$$+\{permutations\}$$

$$=E_{n,P}\Psi_{n,P}(p_1,p_2,p_3,p_4)$$

Shift of Binding Energies of Light Clusters



Symmetric matter

G.R., PRC **79**, 014002 (2009)
S. Typel et al.,
PRC **81**, 015803 (2010)

Composition of dense nuclear matter

$$n_p(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} Z_A f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$
$$n_n(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} (A-Z_A) f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$

mass number A
charge Z_A
energy $E_{A,\nu,K}$
ν: internal quantum number

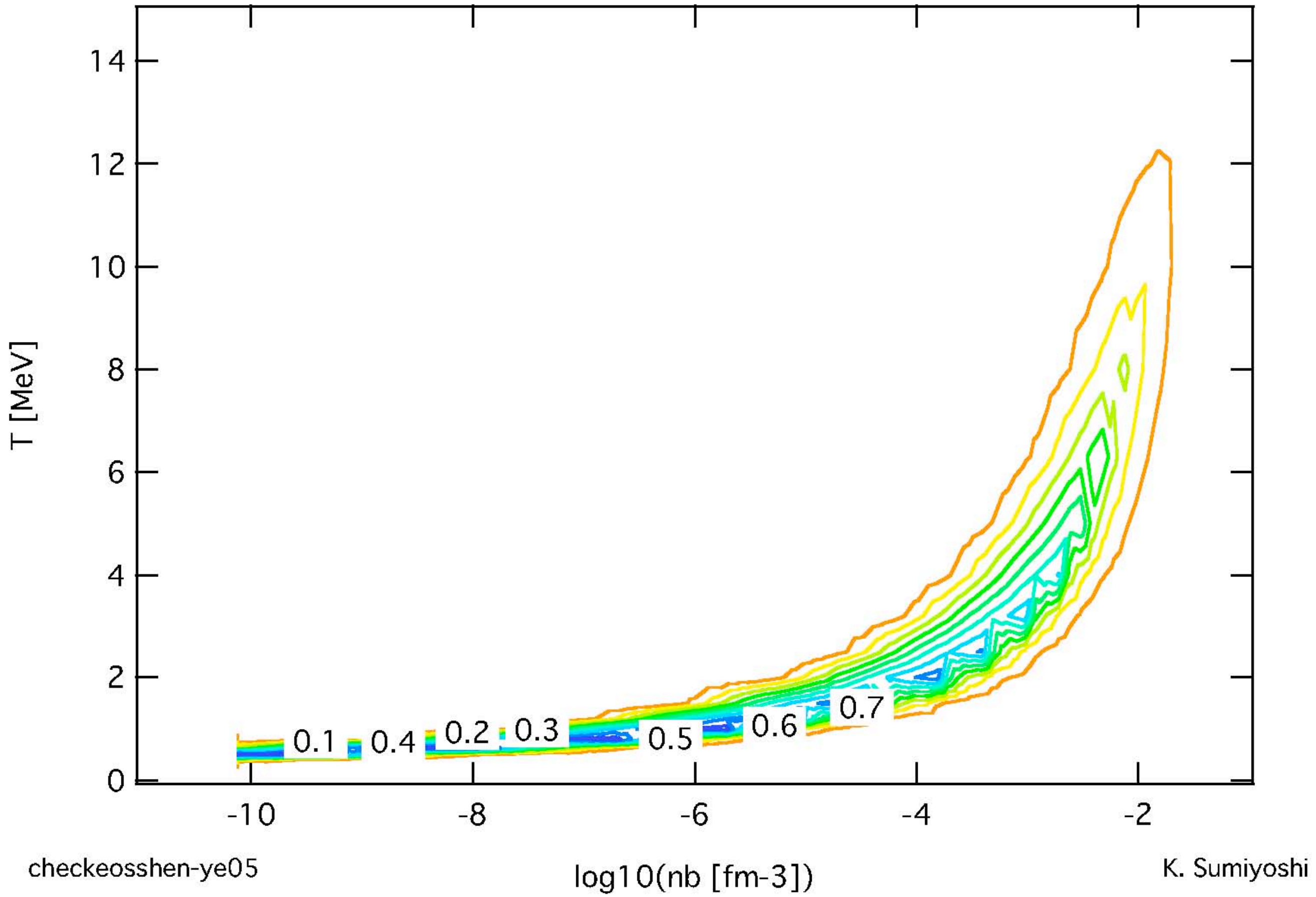
$$f_{A(z)} = \frac{1}{\exp(z/T) - (-1)^A}$$

- Inclusion of excited states and continuum correlations

- Medium effects:
  self-energy and Pauli blocking shifts of binding energies,
  Coulomb corrections due to screening (Wigner-Seitz,Debye)

- Bose-Einstein condensation

alpha-fraction in symmetric matter

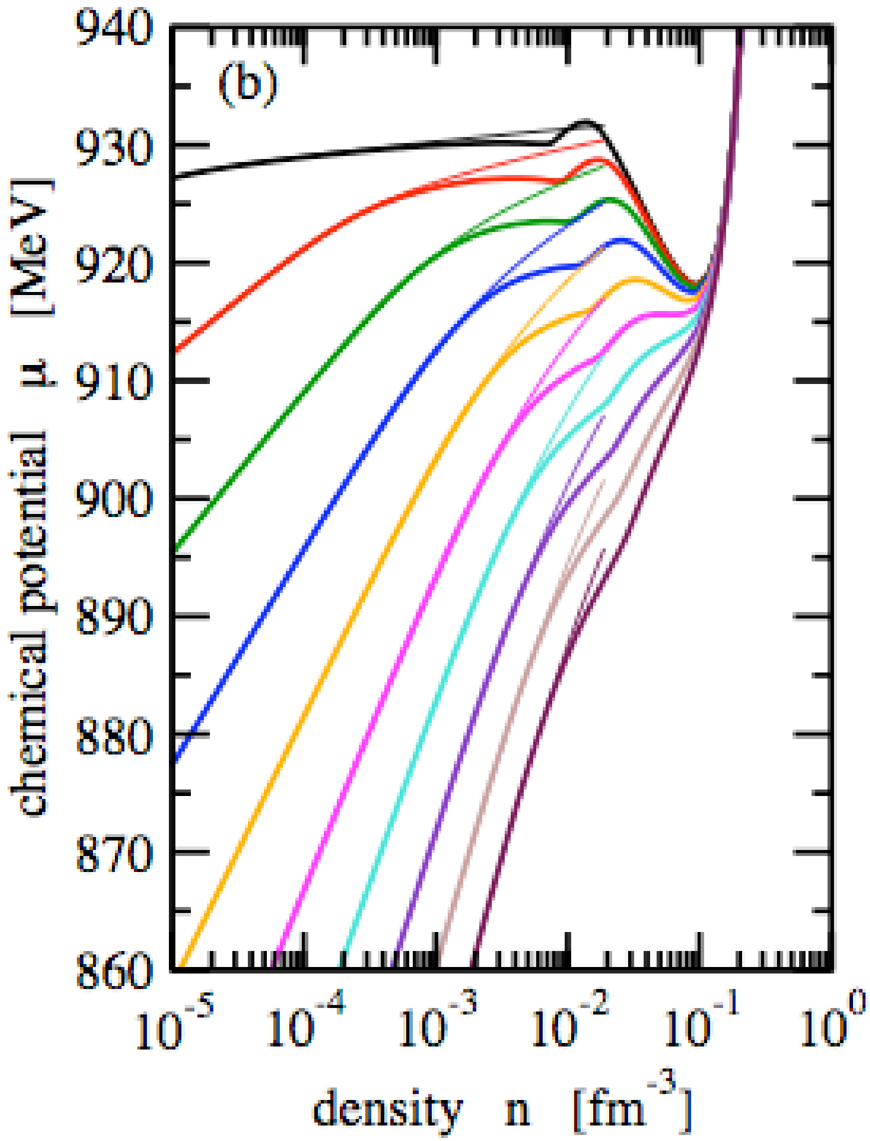


Chemical potential of symmetric matter

Isotherms

T[MeV]
2
4
6
8
10
12
14
16
18
20

thin lines: NSE

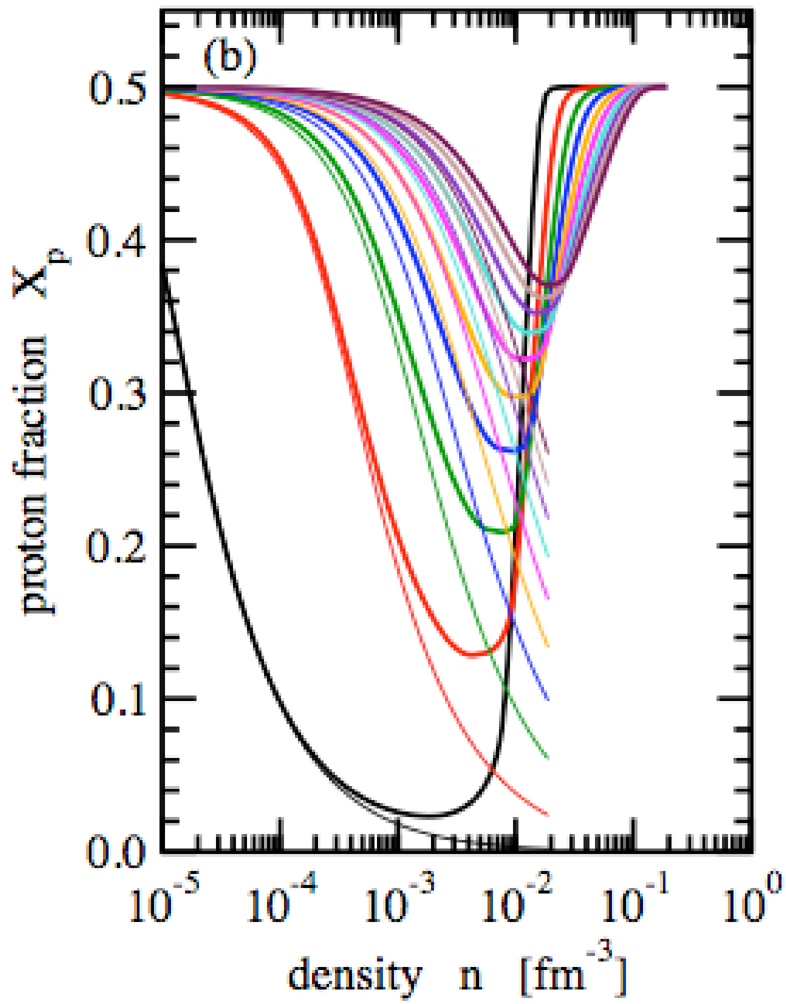


Proton fraction in symmetric matter

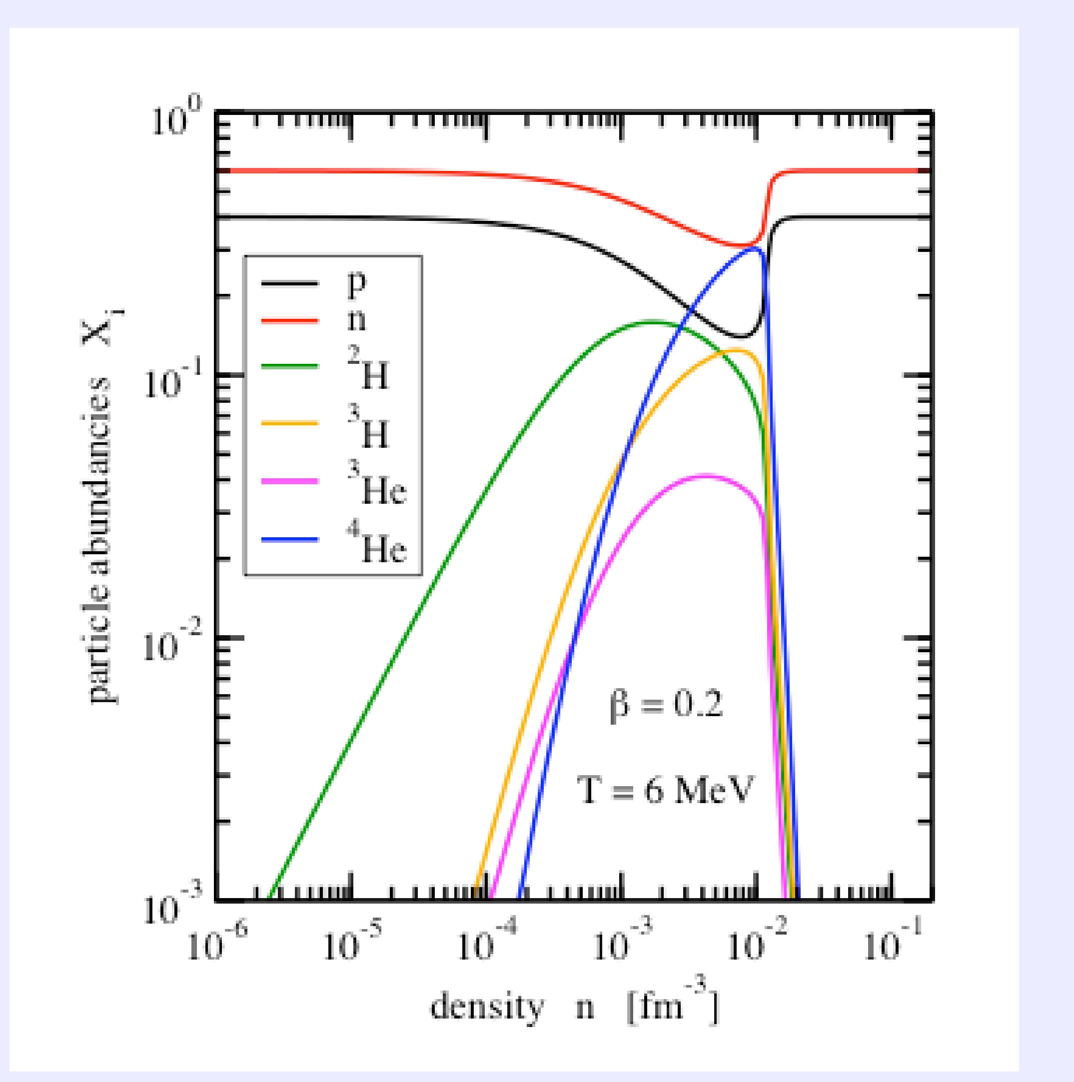
Isotherms

T[MeV]
20
18
16
14
12
10
8
6
4
2

thin lines: NSE



Light Cluster Abundances



S. Typel et al.,
PRC **81,** 015803 (2010)

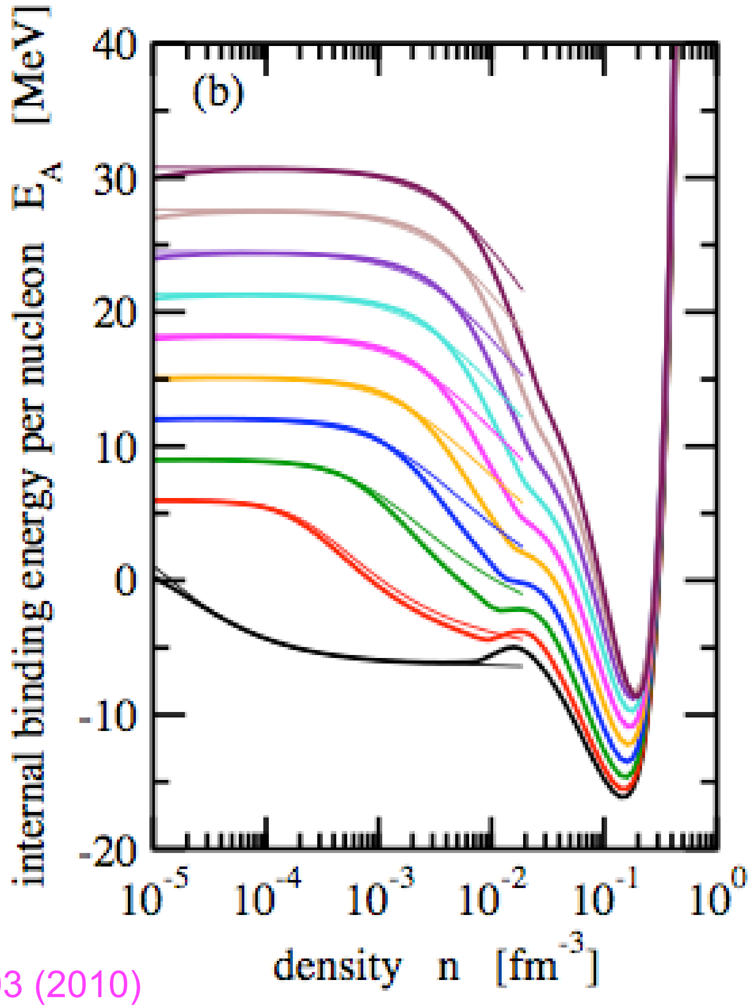
Internal energy per nucleon

Isotherms

T[MeV]

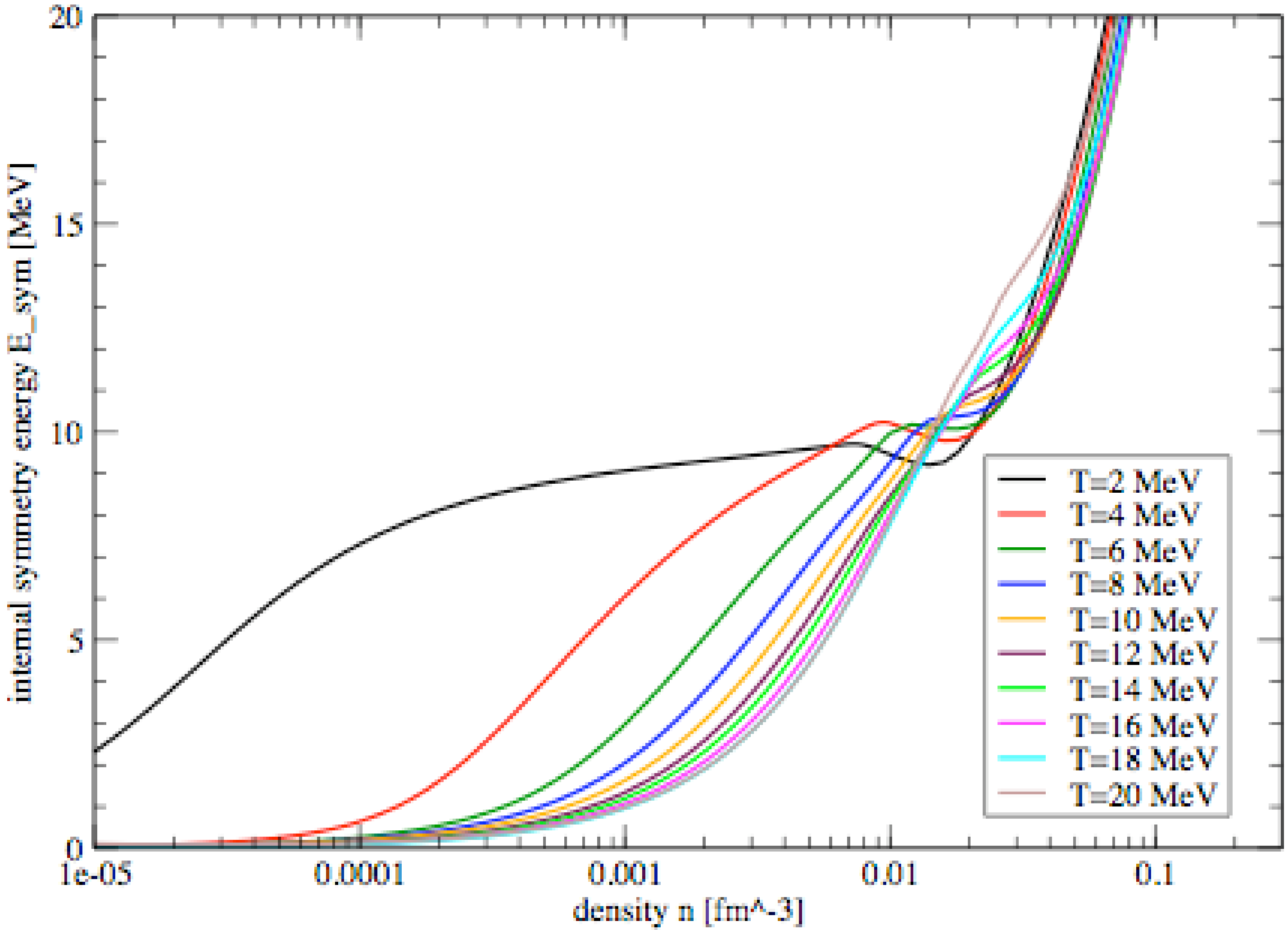
20
18
16
14
12
10
8
6
4
2

thin lines: NSE



S. Typel et al., PRC **81**, 015803 (2010)

Internal symmetry energy

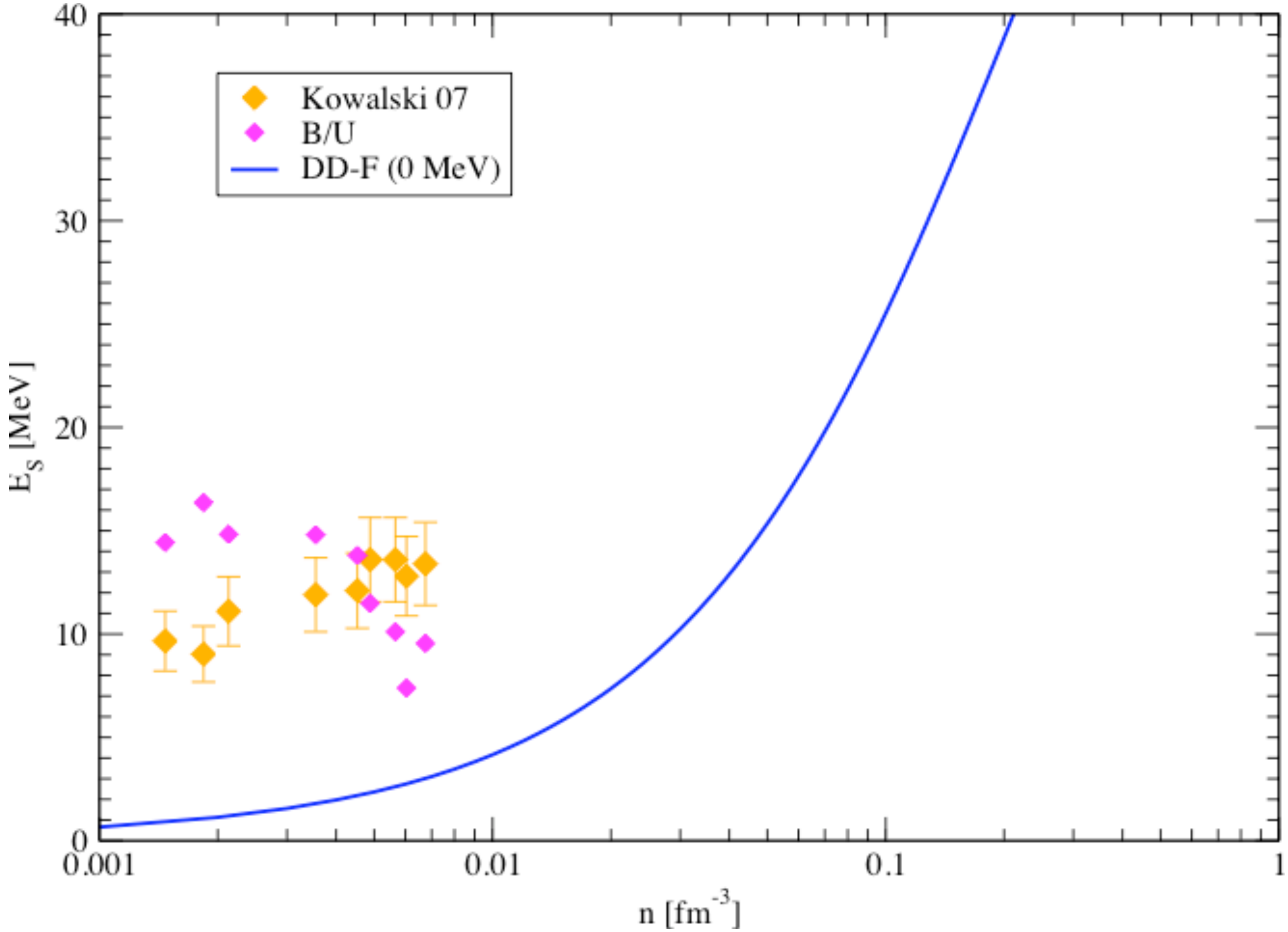


Application to Heavy Ion Reactions

- Test the EOS
  (NSE, virial,… at low densities,
  Skyrme, DBHF, RMF,… near saturation)
- Unifying quantum statistical approach
- Symmetry energy

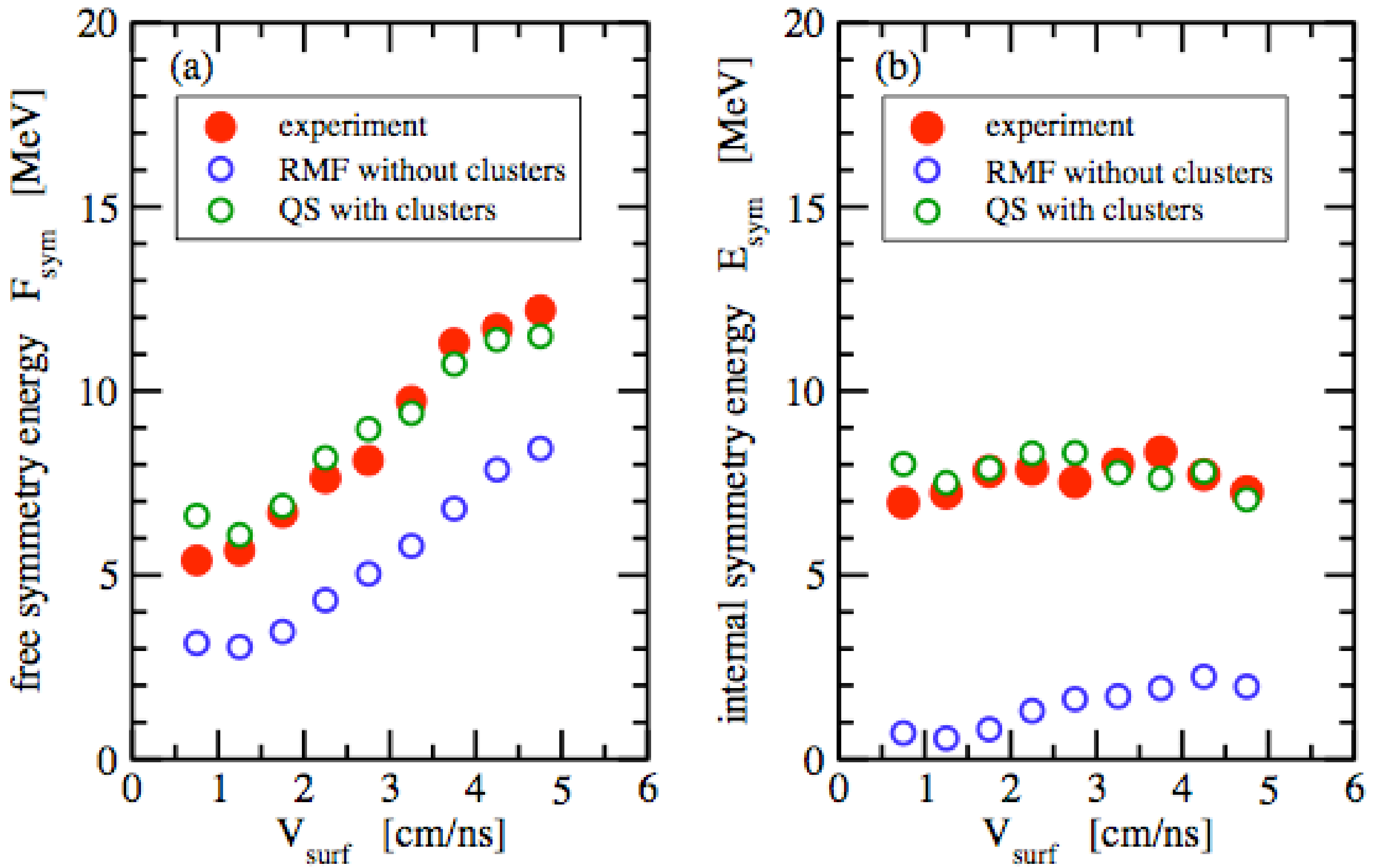
Symmetry energy

Heavy-ion collisions, spectra of emitted clusters, temperature (3 - 10 MeV), free energy



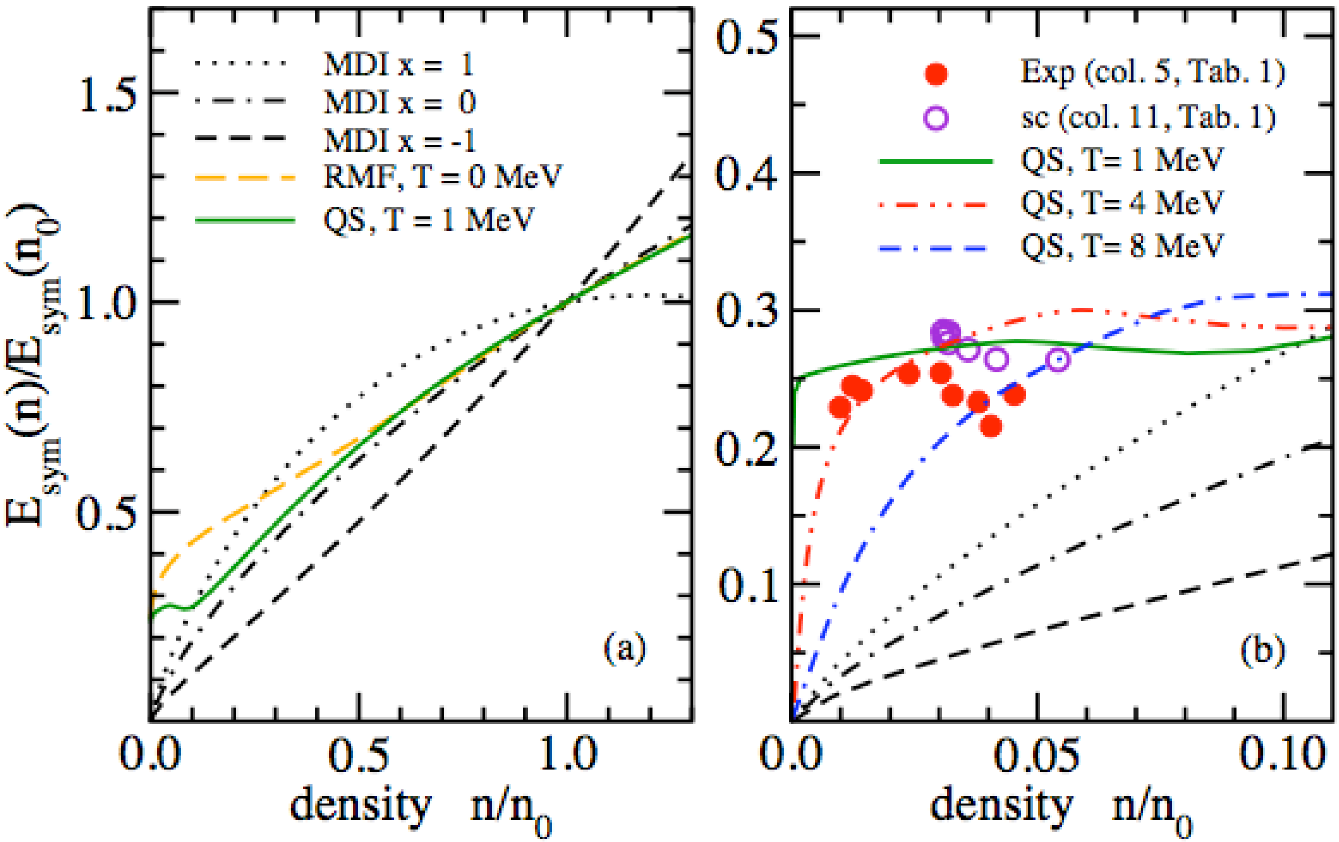
S. Kowalski et al., PRC **75**, 014601 (2007)

Symmetry energy, comparison experiment with theories



J.Natowitz et al., PRL 2010

Symmetry Energy



Scaled internal symmetry energy as a function of the scaled total density.
MDI: Chen et al., QS: quantum statistical, Exp: experiment at TAMU

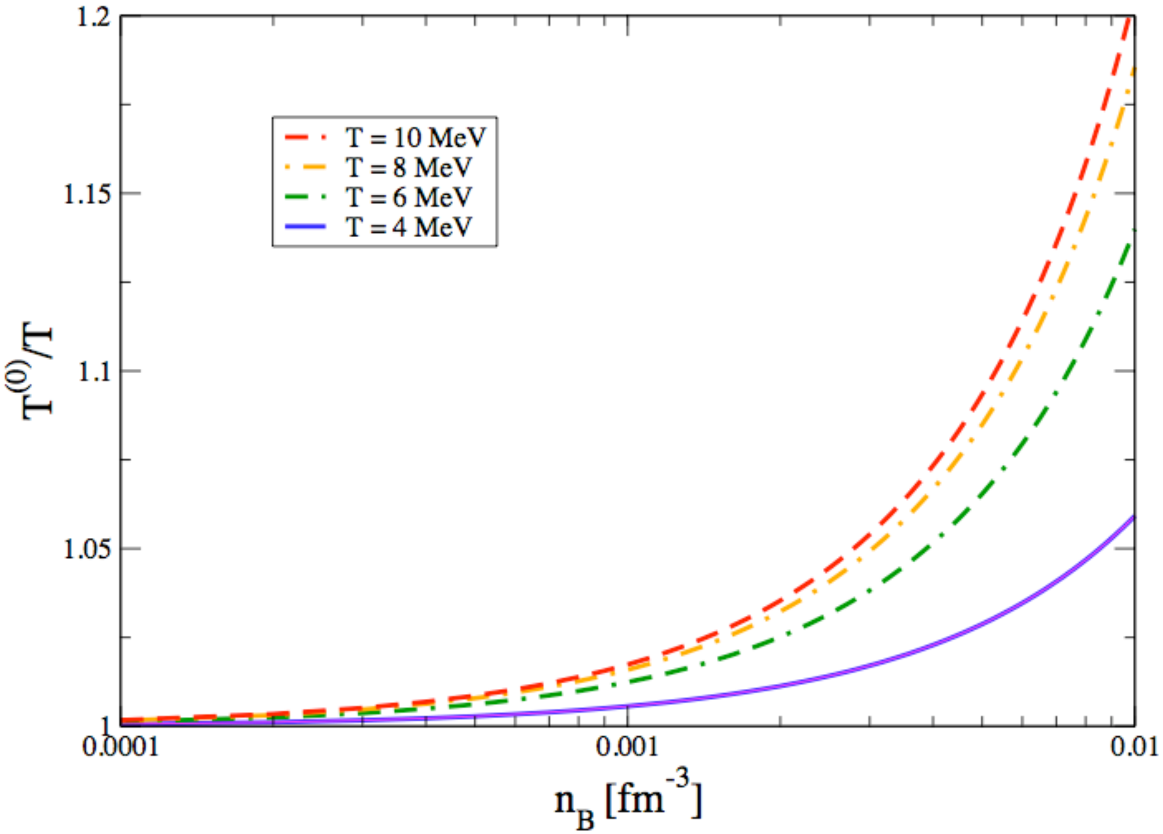
J.Natowitz et al. PRL, May 2010

Cluster yields in Heavy Ion Collisions

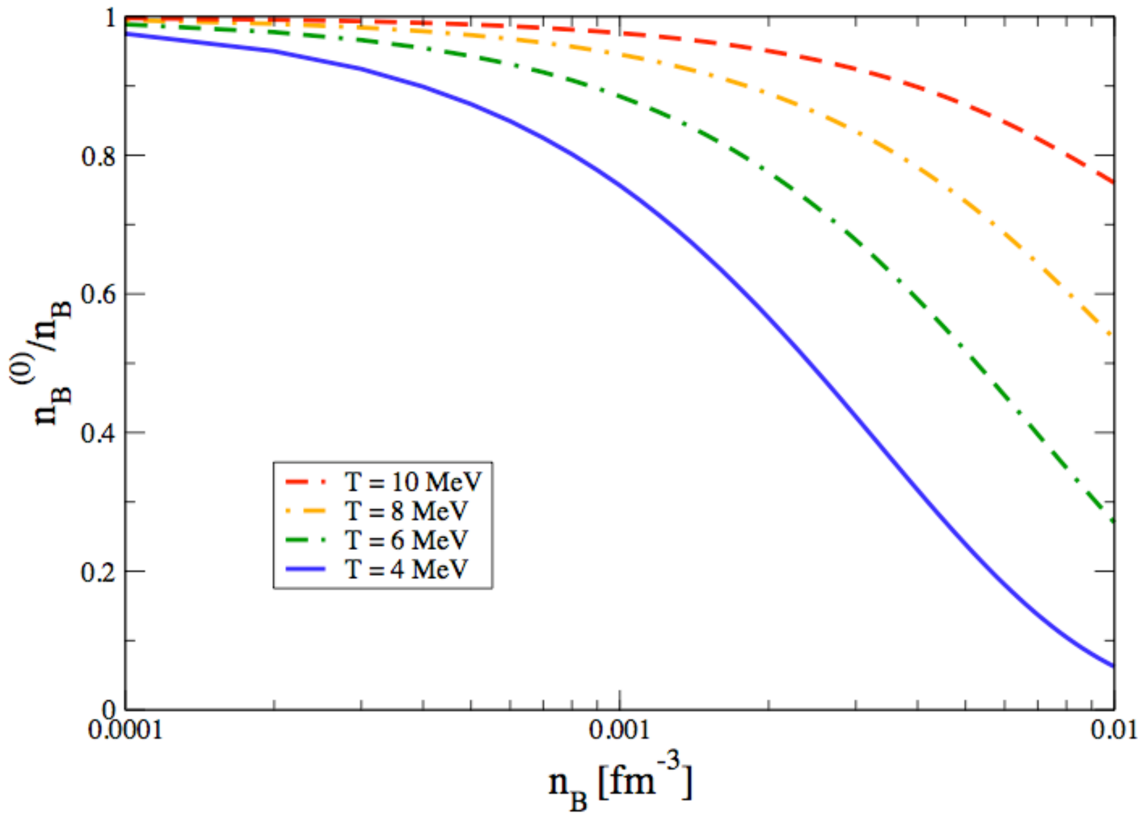
- Thermodynamic parameters (temperature and nucleon density) at freeze out from cluster yields
- Saha equation in plasma physics and Albergo thermometer (mass action law)
- Albergo temperature: double ratios of yields to eliminate the chemical potentials (e.g. 2H, 4He vs. 3H, 3He)
- Is nuclear statistical equilibrium (NSE) or statistical multifragmentation justified?
- Account of medium effects changes the inferred values for temperature and density.

S. Shlomo, G. R., J.B. Natowitz,
PRC **79**, 034604 (2009)

Albergo Temperature Misfit

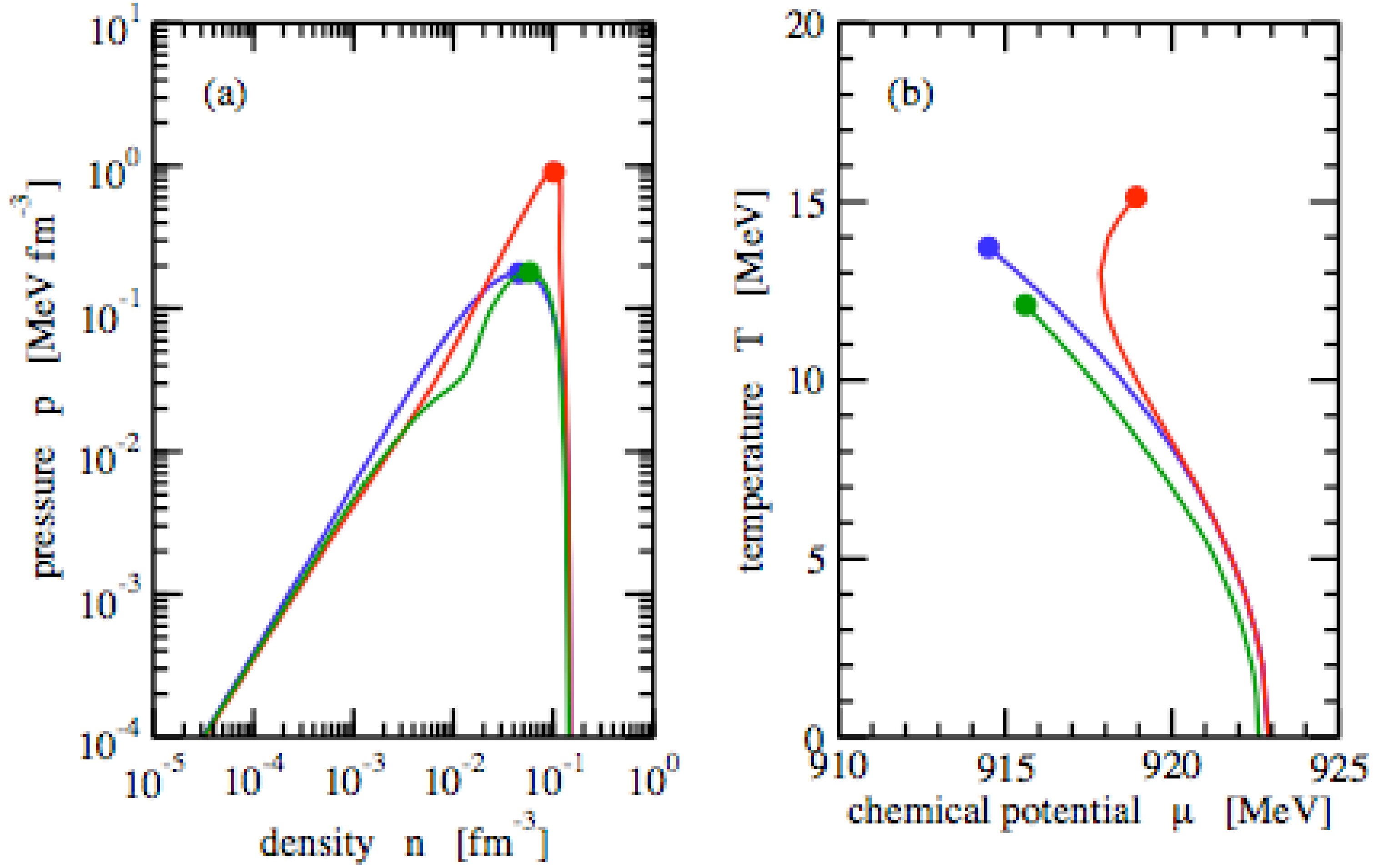


Albergo Density Misfit



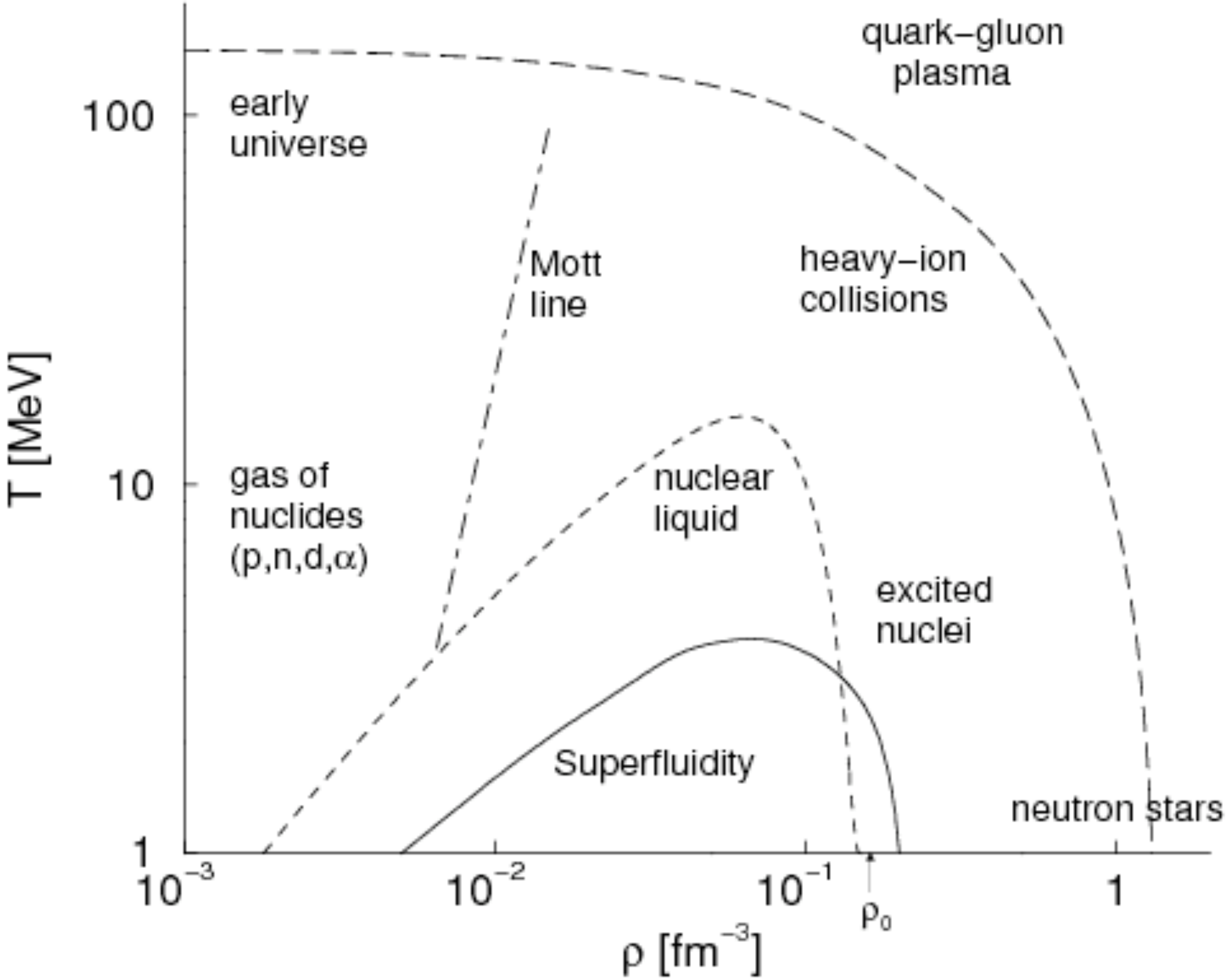
S. Shlomo, G. R., J.B. Natowitz,
PRC **79**, 034604 (2009)

Liquid-vapor phase transition



blue: no light cluster, green: with light clusters

Symmetric nuclear matter: Phase diagram



Conclusion I

• Due to the interaction, cluster are formed in nuclear matter that are of significance in the low-density limit. Here, the nuclear statistical equilibrium or cluster virial expansions can be used to describe the thermodynamic properties.

• Medium effects become of relevance for densities > 10^{-4} fm^{-3}. Single nucleon quasiparticle energies can be introduced. In addition, Pauli blocking modifies the cluster properties so that they are dissolved with increasing density.

• Properties of nuclear matter such as the symmetry energy are determined in the low-density region by the formation of bound states.

Outline

- Nuclear matter - a strongly interacting quantum liquid

  where it occurs, what do we know: nuclei, stars, HIC
- Many-particle theory:

  Equation of state, transport coefficients

  QCD? Effective interactions, Green functions, spectral functions
- Low-density limit: cluster formation

  Mass action law, nuclear statistical equilibrium, virial expansion
- Near saturation: medium effects

  mean-field and quasiparticles, dissolution of bound states
- Quantum condensates:

  transition from BEC to BCS, Hoyle states

Pauli blocking and Mott effect

Two different fermions (a,b: proton,neutron) form a bound state (c: deuteron).

$$c_q = \sum_p F(q,p) a_p b_{q-p}$$

Is the bound state a boson? Commutator relation

$$\left[c_q, c_{q'}^+\right]_- = \sum_{p,p'} F(q,p) F^*(q',p') \left[a_p b_{q-p}, b_{q'-p'}^+ a_{p'}^+\right]_-$$

$$\underline{a_p b_{q-p} b_{q'-p'}^+ a_{p'}^+} + a_p b_{q'-p'}^+ b_{q-p} a_{p'}^+ - a_p b_{q'-p'}^+ b_{q-p} a_{p'}^+ - b_{q'-p'}^+ a_p b_{q-p} a_{p'}^+$$

$$+ b_{q'-p'}^+ a_p b_{q-p} a_{p'}^+ + b_{q'-p'}^+ a_p a_{p'}^+ b_{q-p} - b_{q'-p'}^+ a_p a_{p'}^+ b_{q-p} \underline{- b_{q'-p'}^+ a_{p'}^+ a_p b_{q-p}}$$

$$= a_p a_{p'}^+ \delta_{q-p,q'-p'} - b_{q'-p'}^+ b_{q-p} \delta_{p,p'} = \left(\delta_{p,p'} - a_{p'}^+ a_p\right) \delta_{q-p,q'-p'} - b_{q'-p'}^+ b_{q-p} \delta_{p,p'}$$

$$\left[c_q, c_{q'}^+\right]_- = \sum_{p,p'} F(q,p) F^*(q',p') \left[\left(\delta_{p,p'} - a_{p'}^+ a_p\right) \delta_{q-p,q'-p'} - b_{q'-p'}^+ b_{q-p} \delta_{p,p'}\right]$$

$$= \sum_p F(q,p) F^*(q,p) \delta_{q,q'} - \sum_{p,p'} F(q,p) F^*(q',p') \left[\left(a_{p'}^+ a_p\right) \delta_{q-p,q'-p'} + \left(b_{q'-p'}^+ b_{q-p}\right) \delta_{p,p'}\right]$$

averaging

$$\left\langle \left[c_q, c_{q'}^+\right]_- \right\rangle = \delta_{q,q'} \left[1 - \sum_p F(q,p) F^*(q,p) \left(\left\langle a_p^+ a_p \right\rangle + \left\langle b_{q-p}^+ b_{q-p} \right\rangle\right)\right]$$

Fermionic substructure: phase space occupation, “excluded volume”

Quantum condensates: Outline

1. Many-particle system
   - Mean field approach: self-energy and Pauli blocking
   - Generalized Beth-Uhlenbeck equation
   - Mott effect and transition from BEC to BCS
   - Self-consistent solutions and pseudogap
2. Correlations: account of higher clusters
   - Cluster expansion of the self-energy
   - Cluster - mean field approximation
   - Quantum condensates: Pairing and quartetting
3. Finite systems: 4-n nuclei
   - Cluster formation in dilute nuclei
   - BEC states: Hoyle state and THSR wave function
   - Suppression of the condensate at increasing density

Bose - Einstein Condensation

Int. Workshop BEC 93
Levico Terme
Ed.: Griffin, Snoke, Stringari
Cambridge Univ. Press, 1995

D. W. Snoke and G. Baym

Table 1. *Bosons under study*

Particle	Composed of	In	Coherence seen in
Cooper pair	e^-, e^-	metals	superconductivity
Cooper pair	h^+, h^+	copper oxides	high-T_c superconductivity
exciton	e^-, h^+	semiconductors	luminescence and drag-free transport in Cu_2O
biexciton	$2(e^-, h^+)$	semiconductors	luminescence and optical phase coherence in CuCl
positronium	e^-, e^+	crystal vacancies	(proposed)
hydrogen	e^-, p^+	magnetic traps	(in progress)
^4He	$^4He^{2+}, 2e^-$	He-II	superfluidity
^3He pairs	$2(^3He^{2+}, 2e^-)$	^{3}He-A,B phases	superfluidity
cesium	$^{133}Cs^{55+}, 55e^-$	laser traps	(in progress)
interacting bosons	nn or pp	nuclei	excitations
nucleonic pairing	nn or pp	nuclei neutron stars	moments of inertia superfluidity and pulsar glitches
chiral condensates	$\langle \bar{q}q \rangle$	vacuum	elementary particle structure
meson condensates	pion condensate $= \langle \bar{u}d \rangle$, etc. kaon condensate $= \langle \bar{u}s \rangle$	neutron star matter	neutron stars, supernovae (proposed)
Higgs boson	$\langle \bar{t}t \rangle$ condensate (proposed)	vacuum	elementary particle masses

"This is the first book devoted to Bose - Einstein Condensation (BEC) as an interdisciplinary subject, covering atomic and molecular physics, laser physics, low temperature physics, nuclear physics and astrophysics."

Many-particle system

- Hamiltonian (non-relativistic)

$$H = \sum_{1} E(1) a_1^+ a_1 + \frac{1}{2} \sum_{12,1'2'} V(12,1'2') a_1^+ a_2^+ a_{2'} a_{1'}$$

- fermions in states $\{1\} = \{p, \sigma, \tau\}$
- interaction: Coulomb, nuclear, …
- bound states (bosons)
- quantum condensates
- homogeneous system in equilibrium: $\rho = \exp[-S / k_B]$
- variation in time and space

  (finite systems? non-equilibrium?)

Many-particle system

- Hamiltonian (non-relativistic)

$$H = \sum_{1} E(1) a_1^+ a_1 + \frac{1}{2} \sum_{12,1'2'} V(12,1'2') a_1^+ a_2^+ a_{2'} a_{1'}$$

- Entropy $\rho(t) = \exp[-S(t)/k_B]$
- cluster decomposition, non-equilibrium

$$S(t) = S_0(t) + S_1(t) + S_2(t)$$

$$S_1(t) = \sum_{1,2} \xi(12,t) a_2^+ a_1 + \frac{1}{2} \sum_{1,2} \psi(12,t) a_1^+ a_2^+ + c.c.$$

$$S_2(t) = \frac{1}{2} \sum_{12,1'2'} \omega(12,1'2',t) a_1^+ a_2^+ a_{2'} a_{1'}$$

Lagrange parameter ξ, ψ, ω are determined by $\left\langle a_2^+ a_1 \right\rangle^t, \left\langle a_2 a_1 \right\rangle^t, \ldots$

Many-particle system

- Hamiltonian (non-relativistic)

$$H = \sum_{1} E(1) a_1^+ a_1 + \frac{1}{2} \sum_{12,1'2'} V(12,1'2') a_1^+ a_2^+ a_{2'} a_{1'}$$

- Entropy $\rho(t) = \exp[-S(t)/k_B]$
- cluster decomposition, equilibrium

$$S = S_0 + S_1 + S_2$$

$$S_1 = \sum_{1} (E(1) - \mu)/T a_1^+ a_1$$

$$S_2 = \frac{1}{2} \sum_{12,1'2'} V(12,1'2')/T a_1^+ a_2^+ a_{2'} a_{1'}$$

Lagrange parameter T, μ are determined by $\langle H \rangle, \langle N \rangle$

BEC - BCS crossover
in mean-field approximation

only single-particle contributions to the entropy

$$S_1(t) = \sum_{1,2} \xi(12,t) a_2^+ a_1 + \frac{1}{2} \sum_{1,2} \psi(12,t) a_1^+ a_2^+ + c.c.$$

Lagrange multipliers are determined by the given mean values

$$\langle a_2^+ a_1 \rangle^t = \delta_{12} n(1,t) \qquad \langle a_2 a_1 \rangle^t = F(12,t) = \delta_{p_1+p_2,2q} \chi(12) e^{i\alpha_p(t)} F(p,t)$$

diagonalization by Bogoliubov-Valatin transformation

$$a_{p+q,\uparrow} = u_p b_{q+p,\uparrow} + v_p b^+_{q-p,\downarrow} \qquad 2\,|u_p|^2 = 1 + (1+\theta_p^2)^{-1/2}$$

$$\theta_p = \frac{\sqrt{2}\,|F(12)|}{1 - n(1) - n(2)}$$

the anomalous mean values $\langle b_2 b_1 \rangle^t$ vanish

Effective wave equation for the deuteron in matter

In-medium two-particle wave equation in mean-field approximation

$$\left(\frac{p_1^2}{2m_1}+\Delta_1+\frac{p_2^2}{2m_2}+\Delta_2\right)\Psi_{d,P}(p_1,p_2)+\sum_{p_1',p_2'}(1-f_{p_1}-f_{p_2})V(p_1,p_2;p_1',p_2')\Psi_{d,P}(p_1',p_2')$$

$$=E_{d,P}\Psi_{d,P}(p_1,p_2)$$

Add self-energy

Pauli-blocking

Thouless criterion

$$E_d(T,\mu)=2\mu$$

Fermi distribution function

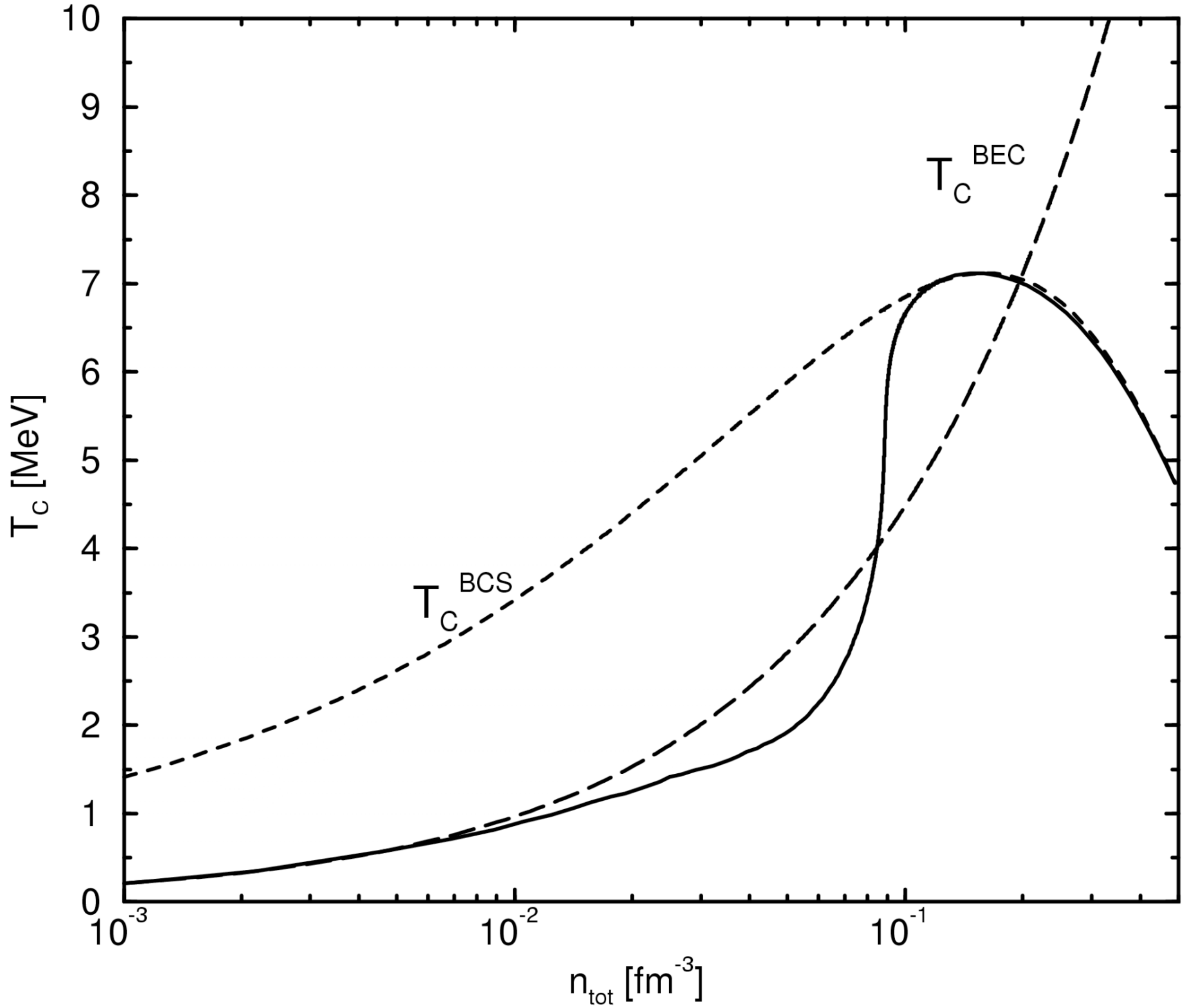
$$f_p=\left[e^{(p^2/2m-\mu)/k_BT}+1\right]^{-1}$$

BEC-BCS crossover:
Alm et al.,1993

Quantum condensate

Bose-Einstein-
Condensation
of deuterons
(BEC)

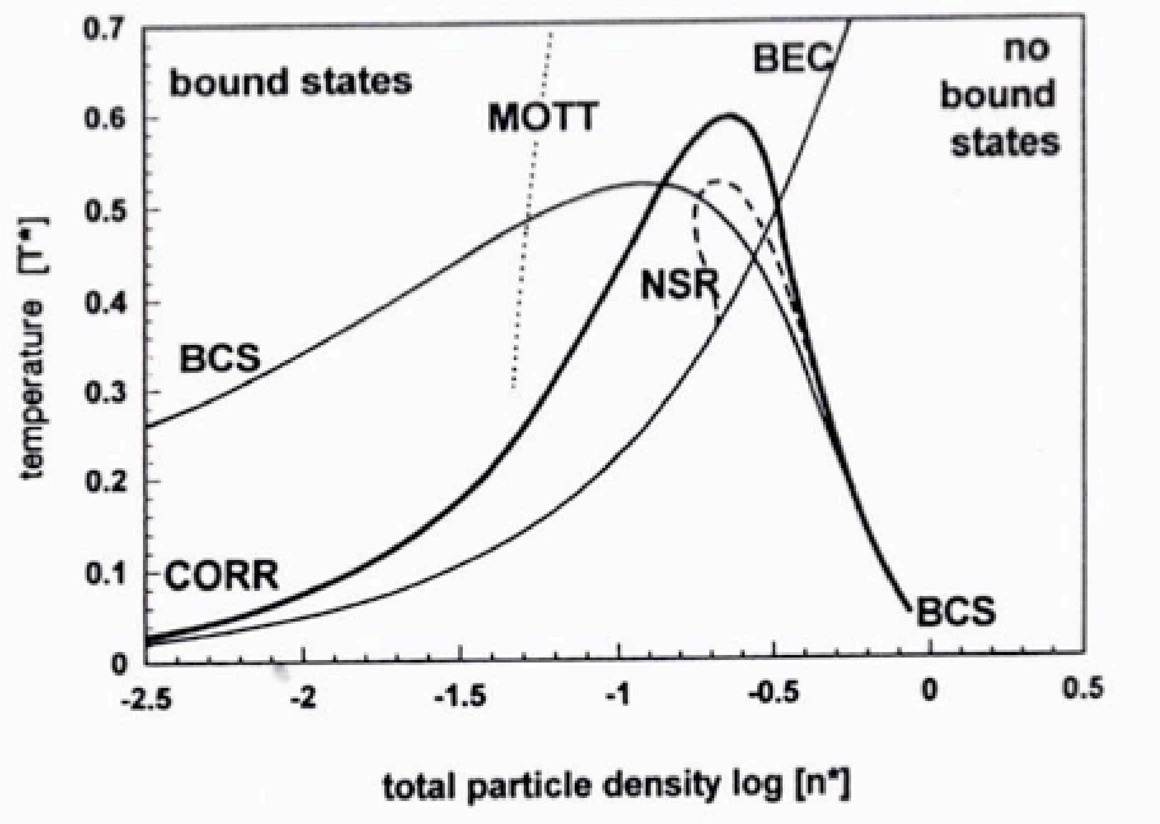
Bardeen-Cooper
Schrieffer
pairing
(BCS)



Crossover from BEC to BCS

Phase transition to the superfluid state

fermionic model system with separable interaction, $T^* = T/E_0, n^* = n(\hbar^2/mE_0)^{3/2}$



NSR[a]: blocking by single-particle distribution function

thick line[b]: including the interaction with the correlated component of the medium

a P. Nozieres, S. Schmitt-Rink, J. Low Temp. Phys. **59**, 159 (1985)

b G. Röpke, Ann. Phys. (Leipzig) **3**, 145 (1994)

Inclusion of correlations in the medium

- Self-consistent description of the medium
- Formation of higher clusters
- Correlated condensates (quartetting)

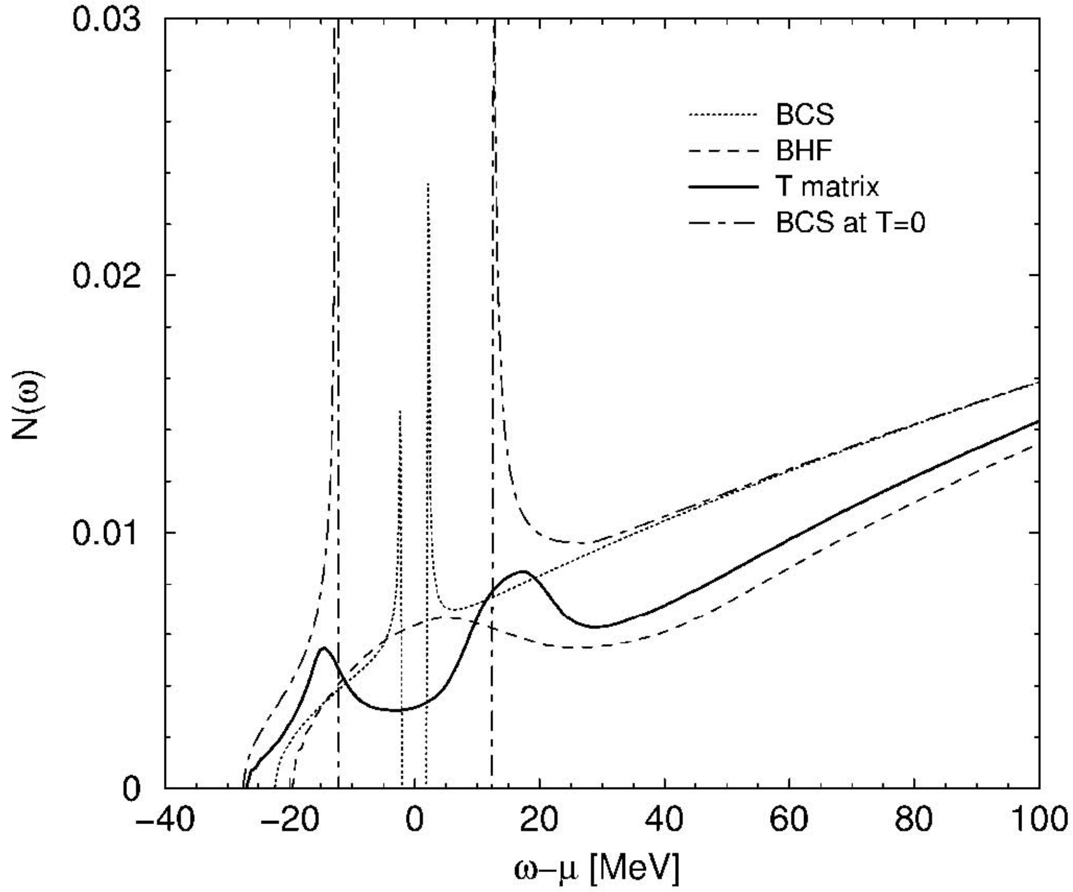
Pseudogap

Precritical Pair Fluctuations and Formation of a Pseudogap in Low-Density Nuclear Matter

A. Schnell, G. Roepke, P. Schuck, PRL **83,** 1926 (1999)

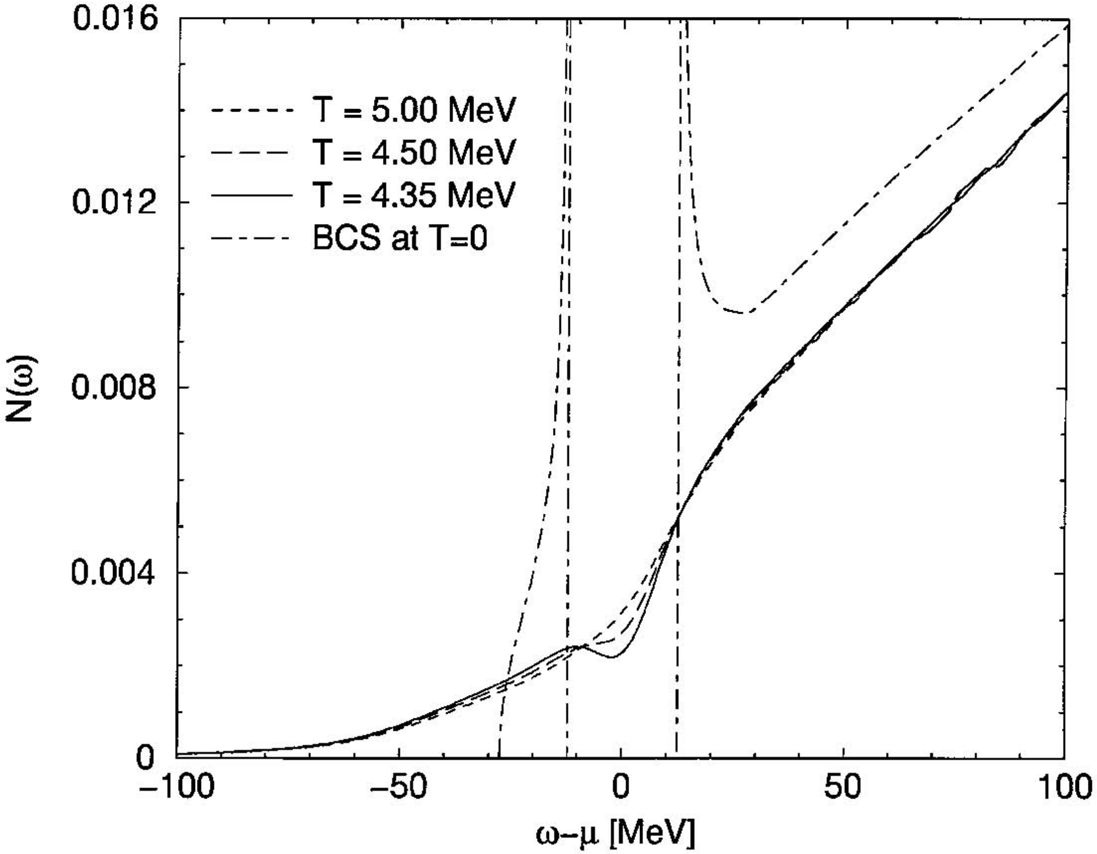
- Self-consistent solution of the two-nucleon Bethe Salpeter equation and evaluation of the density of states
- above the critical temperature: depletion near the chemical potential instead opening of the gap

Density of states near phase transition



T=5 MeV, rho=rho$_0$/3: T-matrix in quasiparticle approximation, compared with BCS and BHF. Also shown: BCS at T=0

Density of states near phase transition



rho=rho$_0$/3: T-matrix, self-consistent spectral function

Few-particle Schrödinger equation in a dense medium

4-particle Schrödinger equation with medium effects

$$\left(\left[E^{HF}(p_1)+E^{HF}(p_2)+E^{HF}(p_3)+E^{HF}(p_4)\right]\right)\Psi_{n,P}(p_1,p_2,p_3,p_4)$$

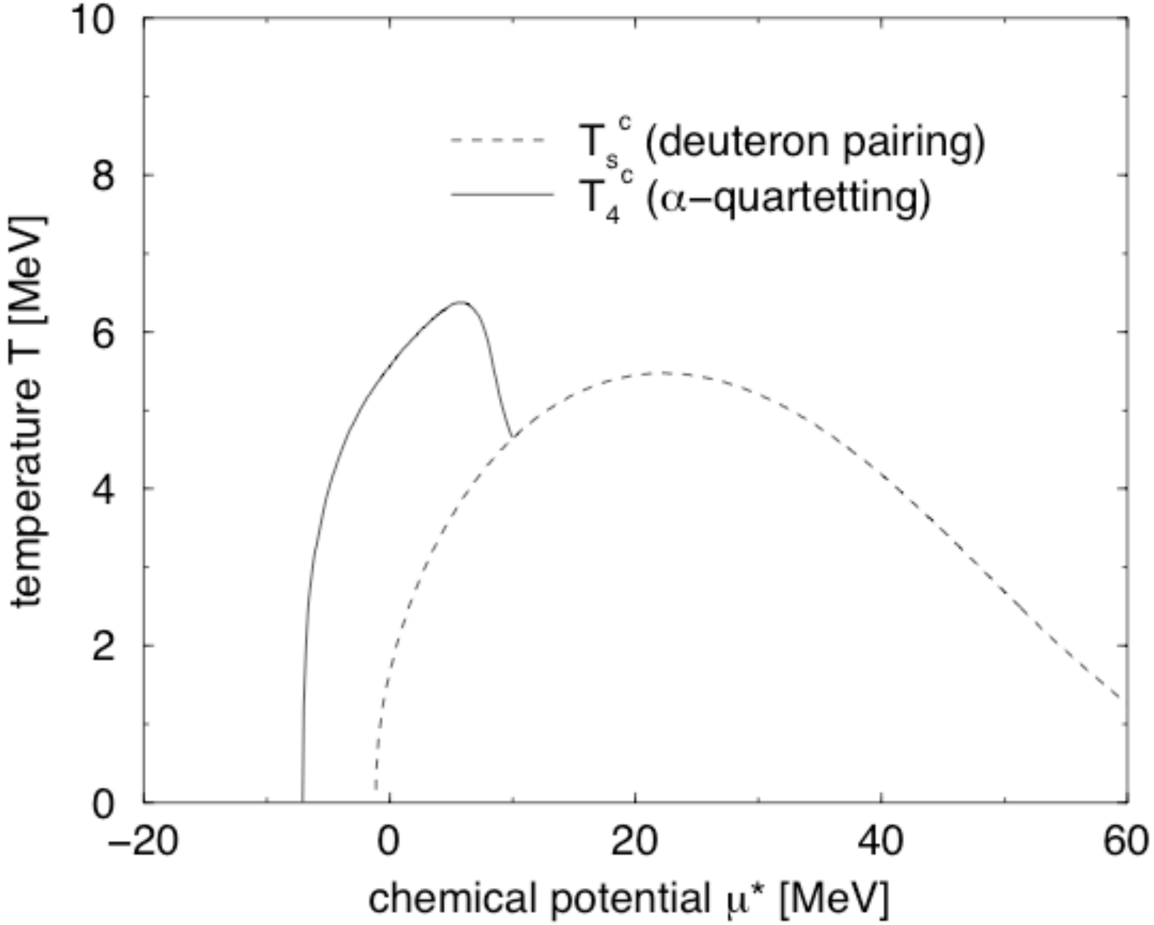
$$+\sum_{p_1',p_2'}(1-f_{p_1}-f_{p_2})V(p_1,p_2;p_1',p_2')\Psi_{n,P}(p_1',p_2',p_3,p_4)$$

$$+\{permutations\}$$

$$=E_{n,P}\Psi_{n,P}(p_1,p_2,p_3,p_4)$$

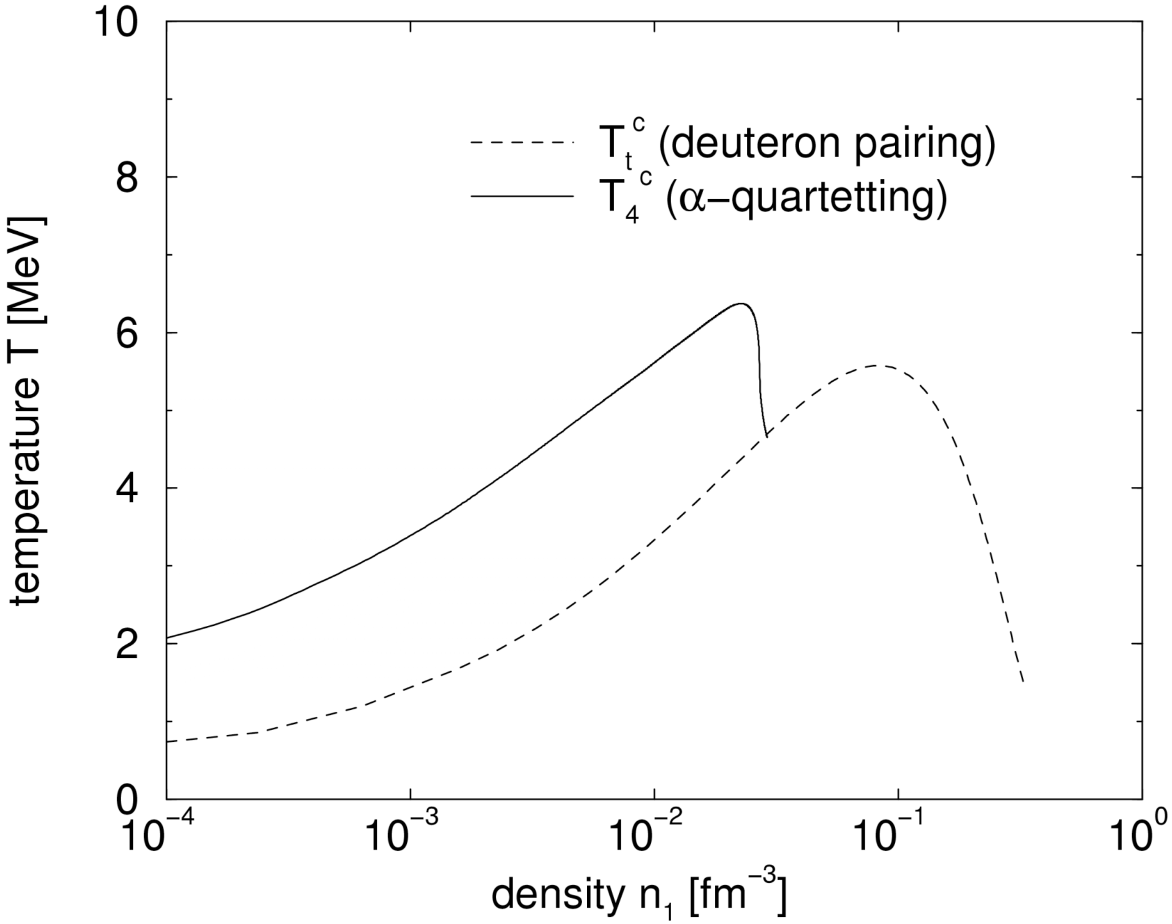
α-cluster-condensation

(quartetting)



α-cluster-condensation

(quartetting)



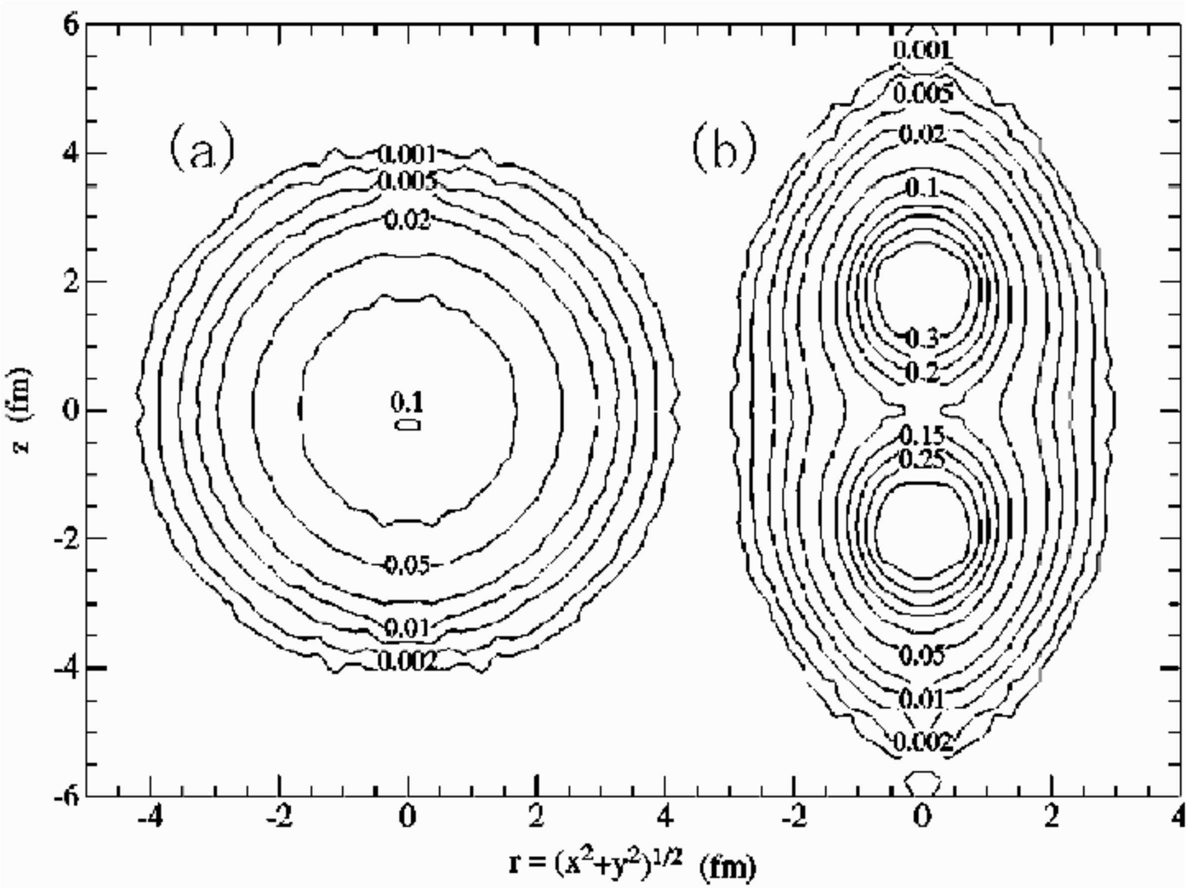
G.Röpke, A.Schnell, P.Schuck, and P.Nozieres, PRL 80, 3177 (1998)

Clusters in nuclei

- Low-density isomers
- Alpha matter at low densities
- Quartetting
- Condensate wave function

- Suppression of the condensate with increasing density
- Dissolution of clusters with increasing density

Alpha cluster structure of Be 8



R.B. Wiringa et al.,
PRC **63**, 034605 (01)

Contours of constant density, plotted in cylindrical coordinates, for 8Be(0+) .
The left side is in the laboratory frame while the right side is in the intrinsic frame.

Self-conjugate 4n nuclei

^{12}C:

0^+ state at 0.39 MeV above the 3α threshold energy:
α cluster interact predominantly in relative S waves,
gaslike structure

α-particle condensation in low-density nuclear matter
$(\rho \leq \rho_0/5)$

$n\alpha$ cluster condensed states
– a general feature in $N = Z$ nuclei?

Self-conjugate 4n nuclei

$n\alpha$ nuclei: ^{8}Be, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ...

Single-particle shell model, or

Cluster type structures

ground state, excited states

$n\alpha$ break up at the threshold energy $E_{n\alpha}^{\mathrm{thr}} = nE_{\alpha}$

Variational ansatz

$$|\Phi_{n\alpha}\rangle = \left(C_\alpha^\dagger\right)^n |\mathrm{vac}\rangle$$

α- particle creation operator

$$C_\alpha^\dagger = \int d^3R e^{-\vec{R}^2/R_0^2}$$

$$\times \int d^3r_1 \dots d^3r_4 \phi_{0s}(\vec{r}_1-\vec{R}) a^\dagger_{\sigma_1\tau_1}(\vec{r}_1) \dots \phi_{0s}(\vec{r}_4-\vec{R}) a^\dagger_{\sigma_4\tau_4}(\vec{r}_4)$$

with

$$\phi_{0s}(\vec{r}) = \frac{1}{(\pi b^2)^{3/4}} e^{-\vec{r}^2/(2b^2)}$$

Variational ansatz

total $n\alpha$ wave function

$$\langle \vec{r}_1 \sigma_1 \tau_1 \ldots \vec{r}_{4n} \sigma_{4n} \tau_{4n} | \Phi_{n\alpha} \rangle$$

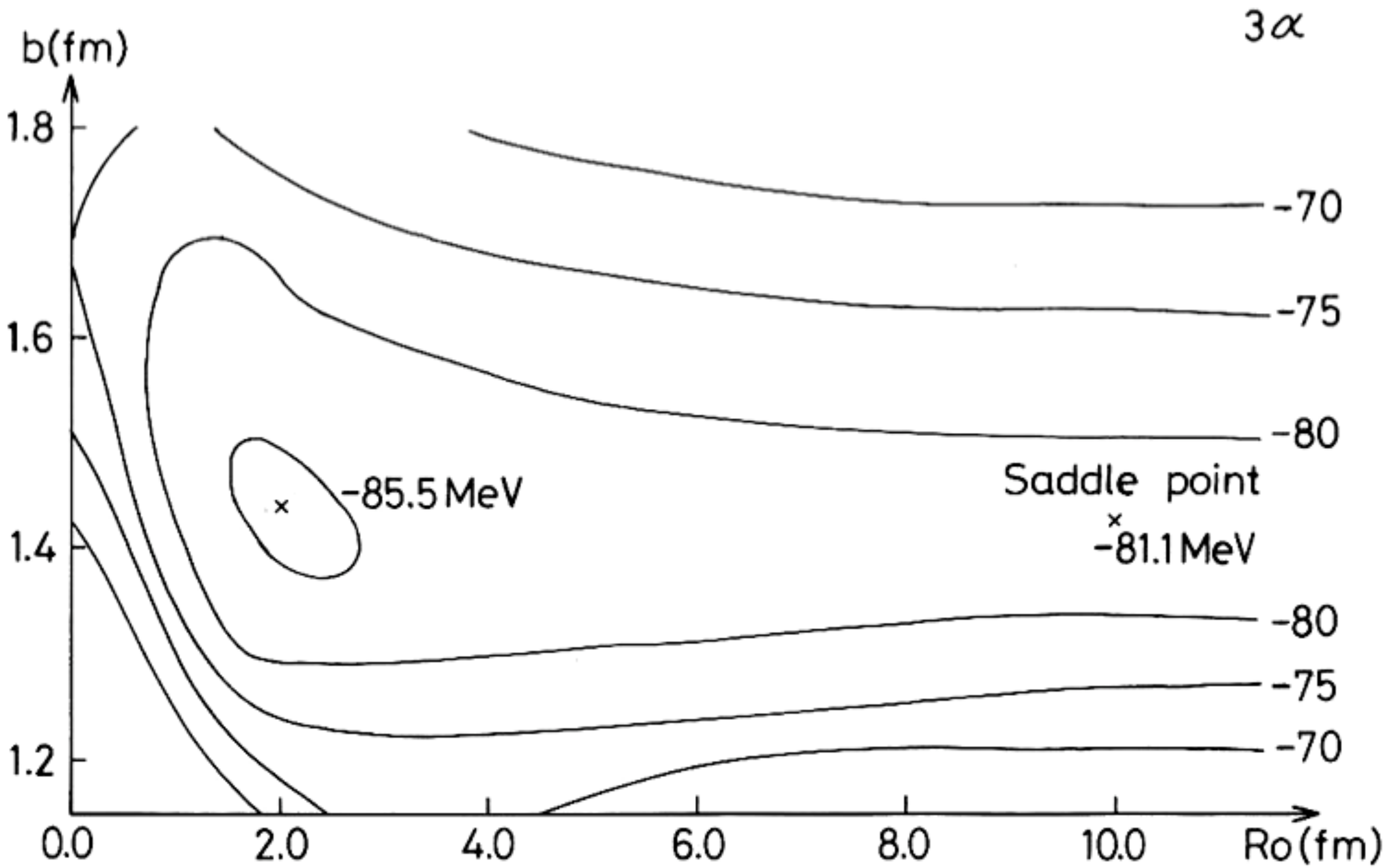
$$\propto \mathcal{A} \left\{ e^{-\frac{2}{B^2}(\vec{X}_1^2 + \ldots \vec{X}_n^2)} \phi(\alpha_1) \ldots \phi(\alpha_n) \right\}$$

where $B^2 = (b^2 + 2R_0^2)$, $\vec{X}_i = \frac{1}{4} \Sigma_n \vec{r}_{in}$,

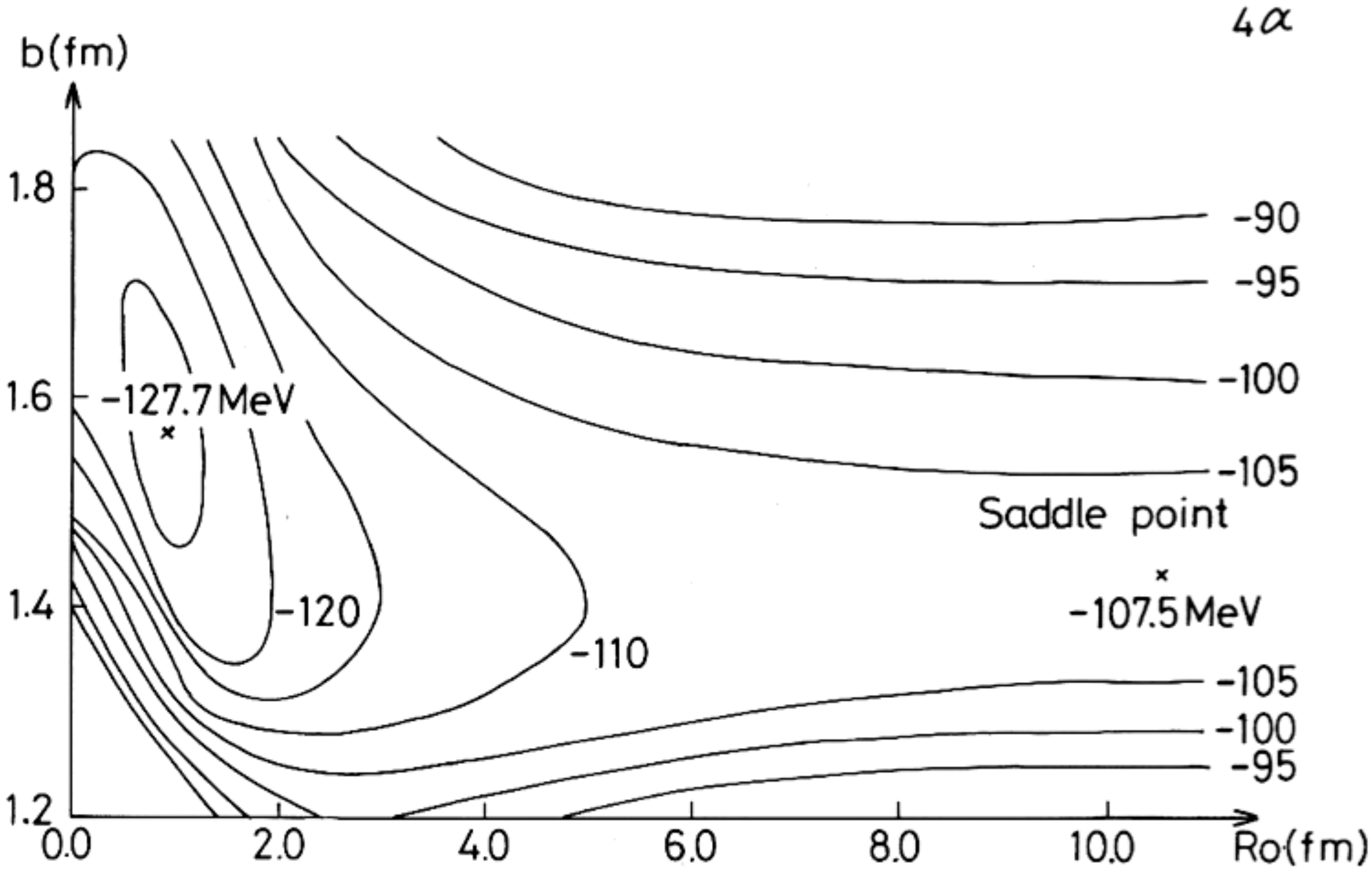
$\phi(\alpha_i) = e^{-\frac{1}{8b^2} \Sigma_{m>n}^4 (\vec{r}_{im} - \vec{r}_{in})^2}$ - internal α wave function

A. Tohsaki, H. Horiuchi, P. Schuck, G. Röpke, PRL **87**, 192501 (2001)

3 alpha variational energy



4 alpha variational energy



A. Tohsaki et al., PRL **87**, 192501 (2001)

Results

		E_k (MeV)	E_{exp} (MeV)	$E_k - E_{n\alpha}^{thr}$ (MeV)	$(E - E_{n\alpha}^{thr})_{exp}$ (MeV)	$\sqrt{\langle r^2 \rangle}$ (fm)	$\sqrt{\langle r^2 \rangle}_{exp}$ (fm)
^{12}C	$k = 1$	-85.9	-92.16 (0_1^+)	-3.4	-7.27	2.97	2.65
	$k = 2$	-82.0	-84.51 (0_2^+)	+0.5	0.38	4.29	
	$E_{3\alpha}^{thr}$	-82.5	-84.89				
^{16}O	$k = 1$	-124.8	-127.62 (0_1^+)	-14.8	-14.44	2.59	2.73
		$(-128.0)^*$		$(-18.0)^*$			
	$k = 2$	-116.0	-116.36 (0_3^+)	-6.0	-3.18	3.16	
	$k = 3$	-110.7	-113.62 (0_5^+)	-0.7	-0.44	3.97	
	$E_{4\alpha}^{thr}$	-110.0	-113.18				
^{8}Be				-0.17	+0.1		

Tabelle 1: Comparison of the generator coordinate method calculations with experimental values. $E_{n\alpha}^{thr} = nE_\alpha$ denotes the threshold energy for the decay into α-clusters, the values marked by * correspond to a refined mesh.

M. Chernykh et al., PRL **98**, 032501 (07); Y. Funaki et al., PRL **101**, 082502 (08)

Estimation of condensate fraction in zero temperature α-matter

α-cluster condensate in ^{12}C, ^{16}O:
resonating group method
occupation numbers of α-orbits in ^{12}C

	RMS radii	S-orbit	D-orbit	G-orbit
0_1^+ (g.s.)	2.44 fm	1.07	1.07	0.82
0_2^+	4.31 fm	2.38	0.29	0.16

80 % condensate at 1/8 nuclear matter density

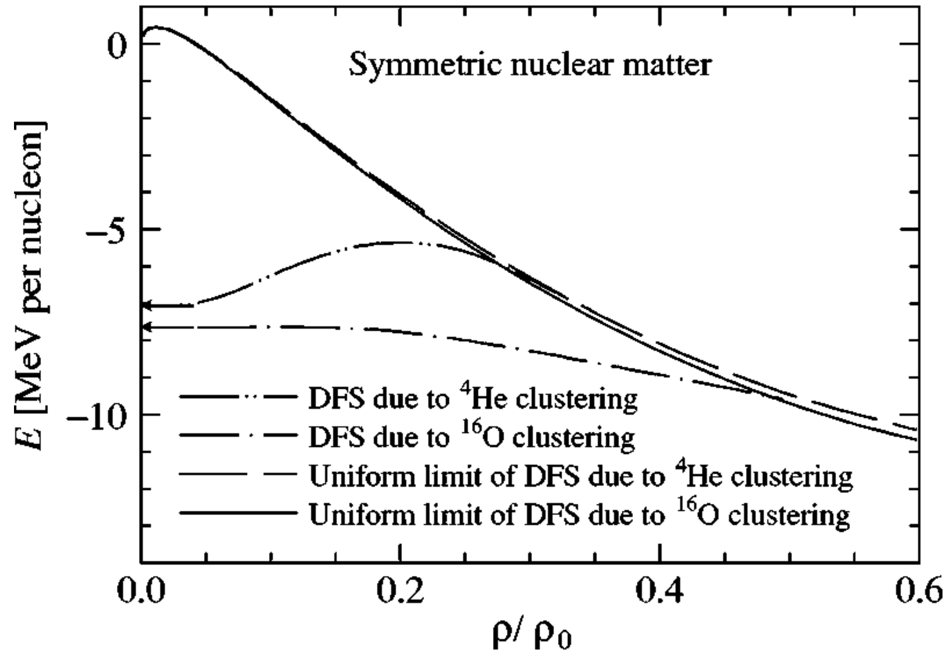
T. Yamada, P. Schuck: $(2.16 - \text{normal})/3 \simeq 60\%$

Open problems

Matter at low densities:

- wave function,
- correlations,
- cluster formation,
- quantum condensates

Clustering phenomena in nuclear matter below the saturation density

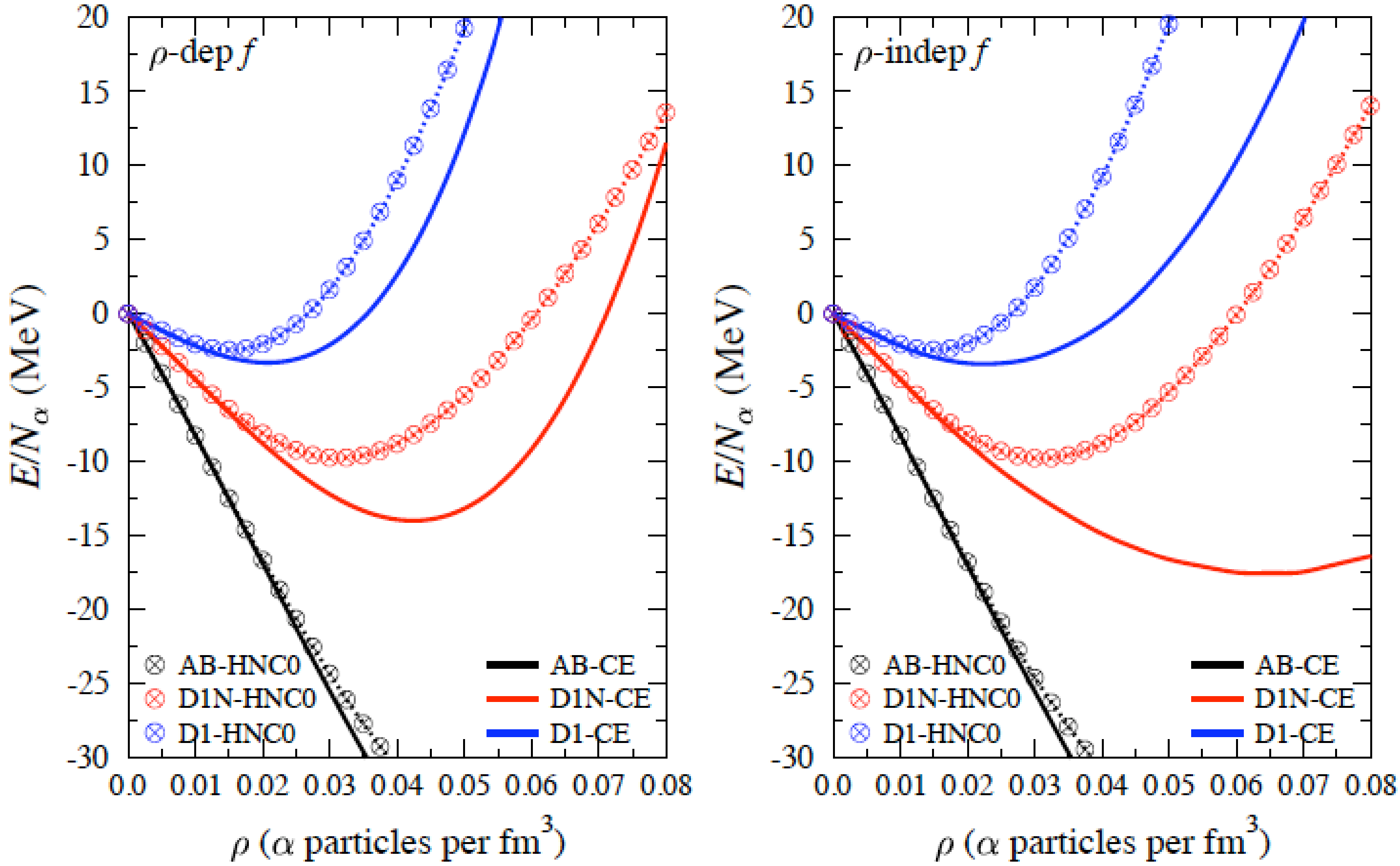


- FIG. 8. Energy curves of DFSs due to a and 16O clustering in
- the symmetric nuclear matter by the use of the BB s*B*4d force. The
- density of matter is normalized by the saturation density of the
- uniform matter with the Fermi sphere, r0=0.206 fm−3. The presentation
- of the curves is similar to that in Fig. 4.

Hiroki Takemoto et al.,

PR C **69**, 035802 (2004)

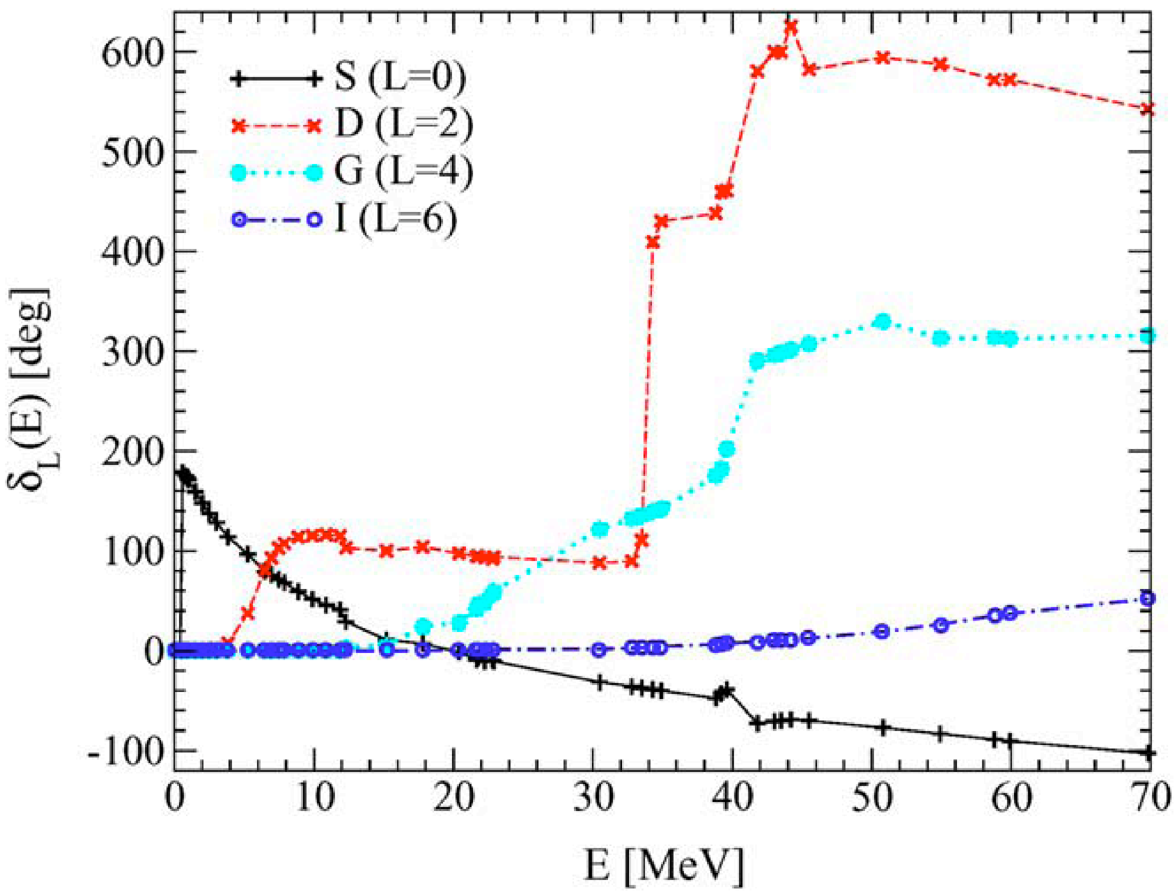
Energy of α-Matter at T=0



Total energy calculated with the cluster expansion
within the HNC/0 (circles) and HNC/4 (solid lines) approximation.
Different interaction potentials

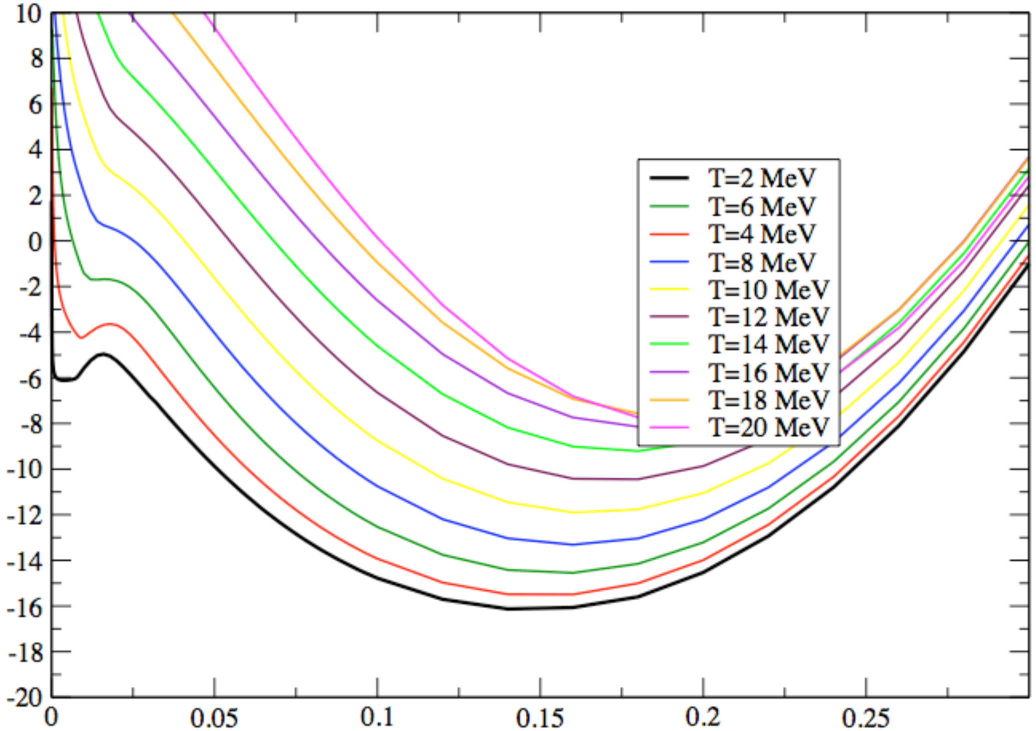
F.Carstoiu, S.Misicu, PLB, 2009

Alpha-alpha scattering phase shifts



Horowitz & Schwenk, NPA (2006)

Internal energy per nucleon



Quantum statistical approach:

Cluster ?

Condensate?

EOS for symmetric matter - low density region?

Summary

- The low-density limit of the nuclear matter EoS can be rigorously treated. The Beth-Uhlenbeck virial expansion is a benchmark.

- An extended quasiparticle approach can be given for single nucleon states and nuclei. In a first approximation, self- energy and Pauli blocking is included. An interpolation between low and high densities is possible.

- Compared with the standard quasiparticle approach, significant changes arise in the low-density limit due to clustering. Examples are Bose-Einstein condensation (quartetting), and the behavior of the symmetry energy.

- Problems: Instability of homogeneous matter at low temperatures, condensation energy for Bose condensate state (compare pairing)? „strict“ solution of the 4-nucleon equation including Pauli blocking?

Thanks

to D. Blaschke, C. Fuchs, Y. Funaki, H. Horiuchi, J. Natowitz, T. Klaehn, S. Shlomo, P. Schuck, A. Sedrakian, K. Sumiyoshi, A. Tohsaki, S. Typel, H. Wolter, T. Yamada

for collaboration

to you

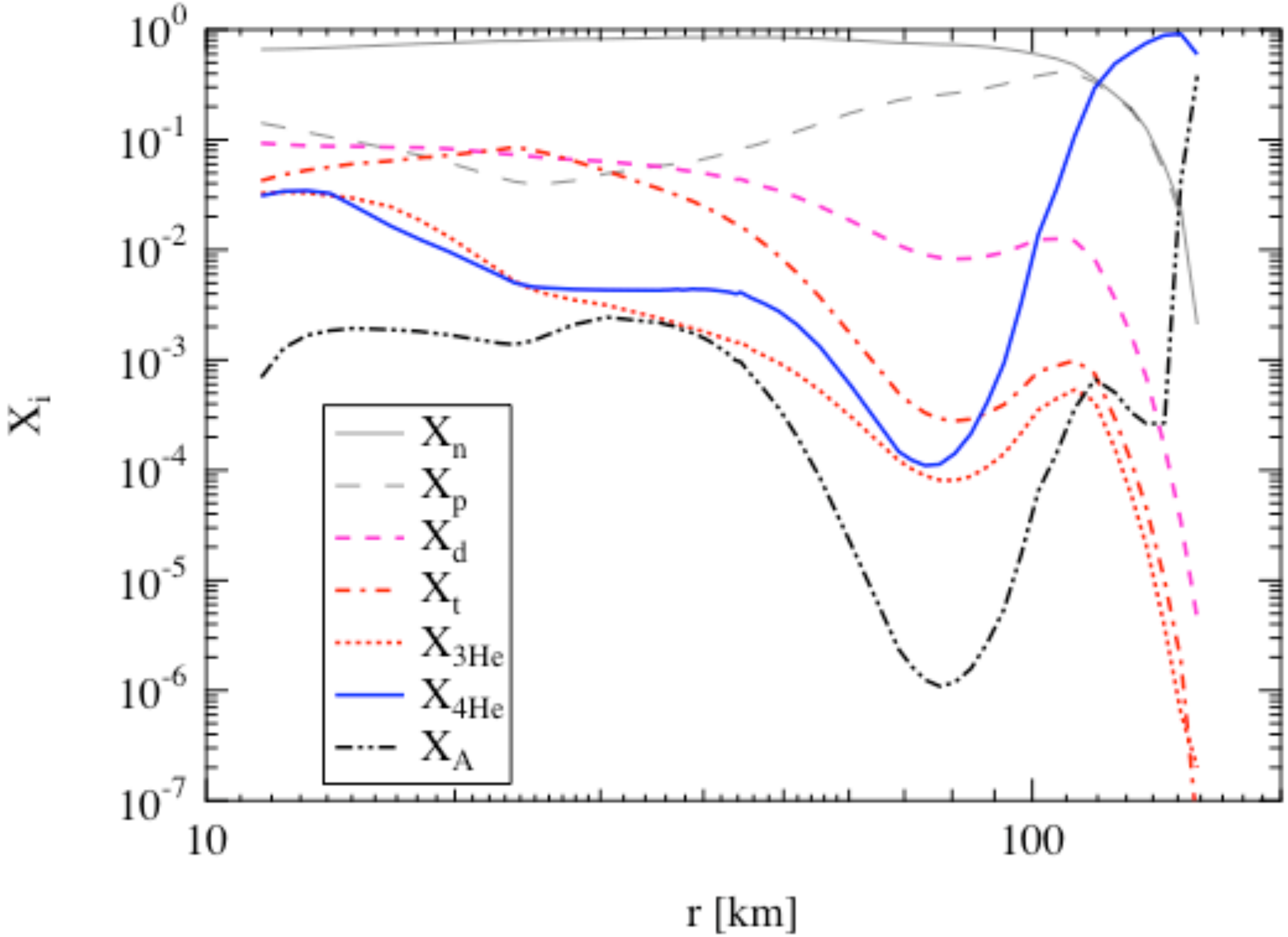
for attention

D.G.

Astrophysical Applications

- Supernova explosions
- Neutrino transport
- Neutron star structure
- Equation of state (EOS)
- Composition
- Transport properties (cross sections)

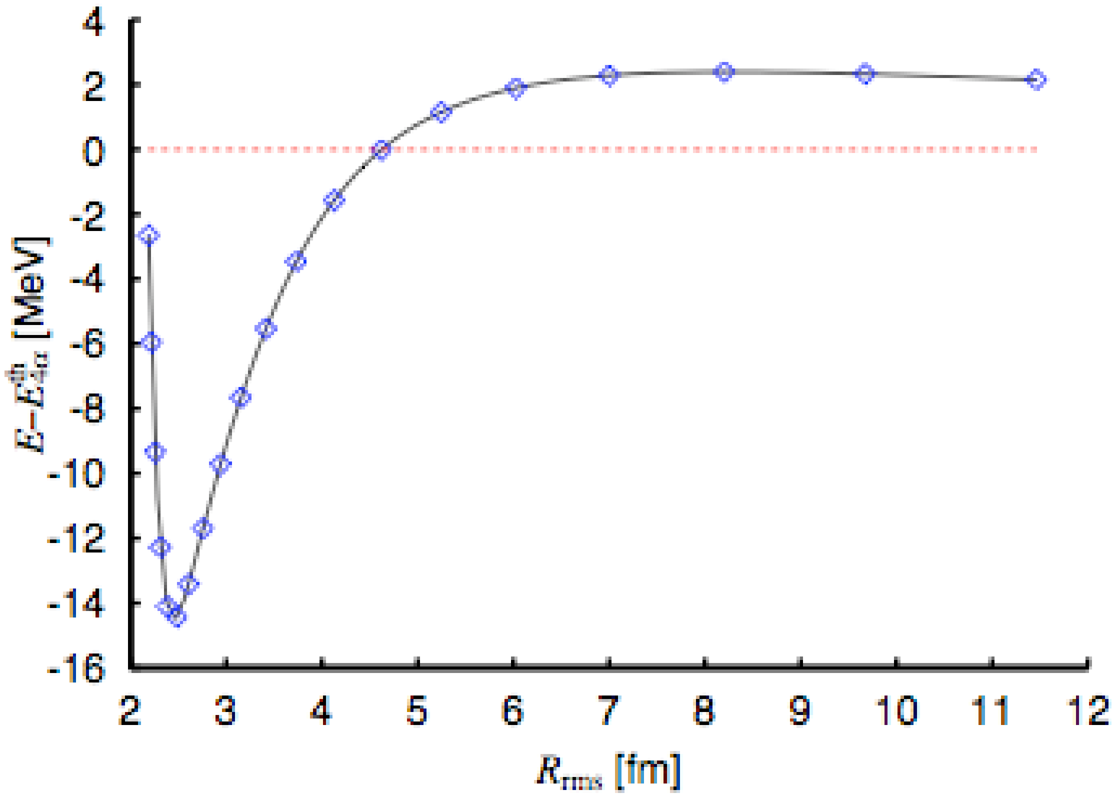
Composition of supernova core



Mass fraction X of light clusters for a post-bounce supernova core

K.Sumiyoshi,
G. R.,
PRC **77**,
055804 (2008)

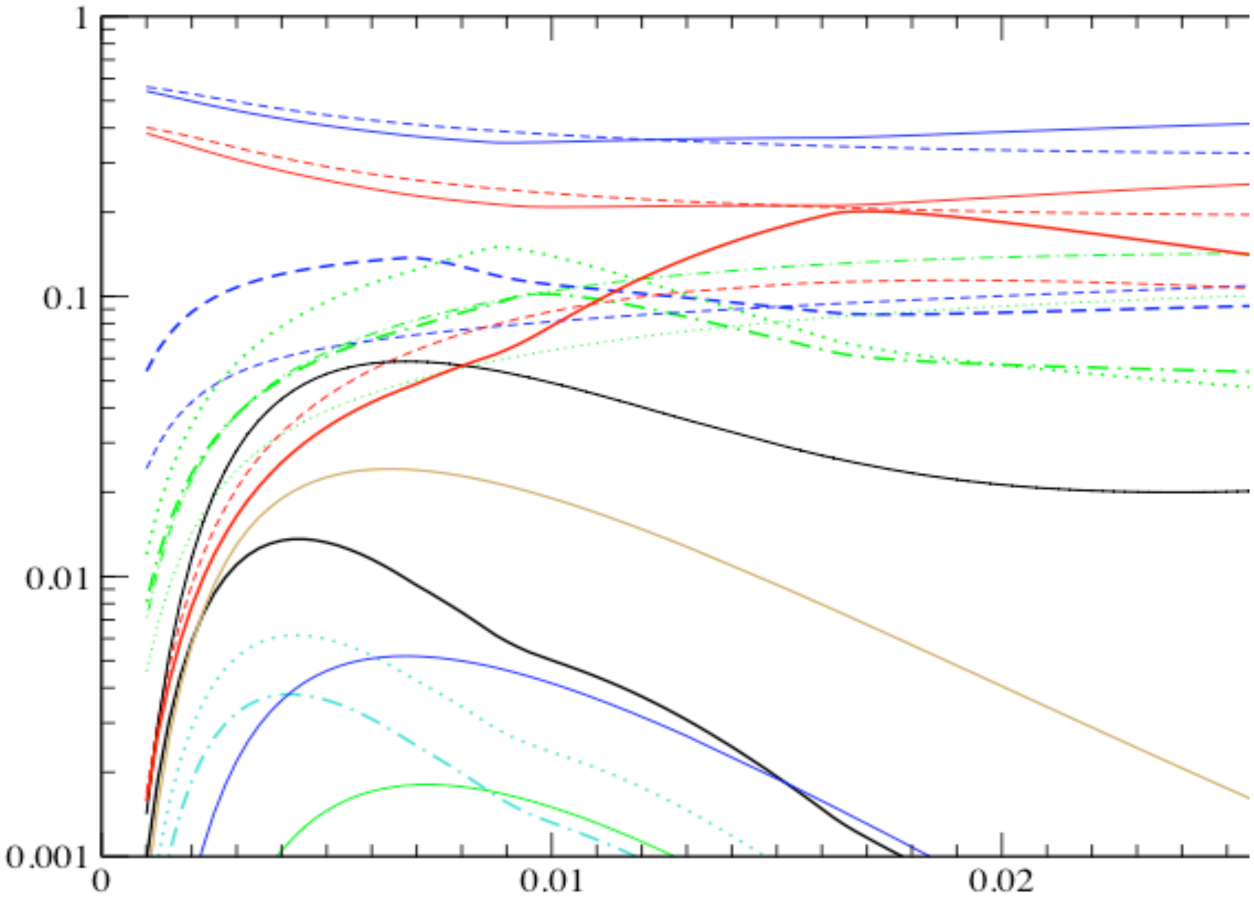
Rms-dependence of condensate energies



Variational energy for the Gaussian condensate of 4 alpha

Heavy nuclei abundances in nuclear matter

T=10 MeV, asymmetry 0.42, as function of baryon density



n, p, d, t, He3, He4, Li5,…

Problems:

- Warm Dilute Matter: Nuclear matter at subsaturation densities (T, n_p, n_n):

  Temperature $T \le 16$ MeV = E_s/A, baryon density $n_B \le 0.17$ fm^{-3} = n_s, asymmetry

- Formation of clusters (nuclei in matter):

  A = 1,2,3,4: free neutrons, free protons, deuterons (^2H), tritons (^3H), helions (^3He), alphas (^4He)

- Low-density, low-temperature limit:

  Virial expansion, non-interacting nuclides, quantum condensates

- Transition to higher densities:

  Medium effects, quasiparticles. Interpolation between Beth-Uhlenbeck and DBHF / RMF

- Cluster formation (correlations) *vs.* mean field:

  Consistent quantum-statistical approach

Ideal mixture of reacting nuclides

$$n_p(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} Z_A f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$
$$n_n(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} (A-Z_A) f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$

(statistical multifragmentation)

$$f_{A(z)} = \frac{1}{\exp(z/T) - (-1)^A}$$

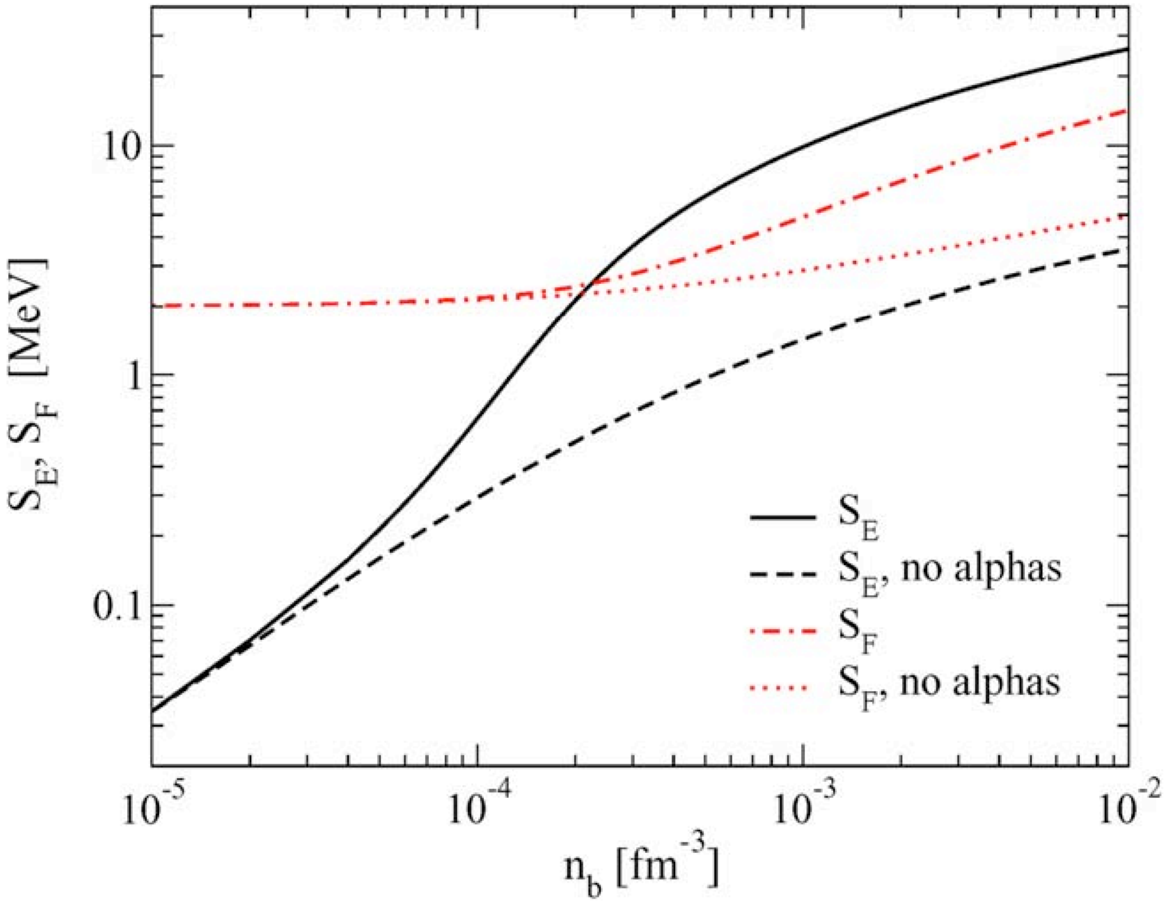
mass number A,
charge Z_A,
energy $E_{A,\nu,K}$,
ν: internal quantum number,
K: center of mass momentum

Virial expansion

- excited nuclei
- resonances
- scattering phase shifts (no double counting)
- virial expansions
- quantum statistical approach

Particle clustering and Mott transition in nuclear matter at finite temperatures,
G. Röpke, M. Schmidt, L. Münchow, H. Schulz: NPA **399**, 587-602 (1983).
Generalized Beth-Uhlenbeck Approach for Hot Nuclear Matter,
M. Schmidt, G. Röpke, H. Schulz: Annals of Physics **202**, 57 - 99 (1990).
Cluster formation and the virial equation of state of low-density nuclear matter
C. J. Horowitz and A. Schwenk, Nucl. Phys. **A 776**, 55 (2006).

Symmetry energy and symmetry free energy



T=4MeV

Horowitz & Schwenk,
NPA (2006)

Few-particle Schrödinger equation in a dense medium

Four-particle Schrödinger equation with medium effects

$$
\begin{aligned}
&\left[E^{\mathrm{HF}}(p_1) + E^{\mathrm{HF}}(p_2) + E^{\mathrm{HF}}(p_3) + E^{\mathrm{HF}}(p_4)\right] \psi_{nP}(p_1, p_2, p_3, p_4) \\
&+ \sum_{p_1' p_2' p_3' p_4'} \Big\{ \left[1 - f(p_1) - f(p_2)\right] V(p_1 p_2, p_1' p_2') \delta_{p_3 p_3'} \delta_{p_4 p_4'} \\
&\qquad + \left[1 - f(p_1) - f(p_3)\right] V(p_1 p_3, p_1' p_3') \delta_{p_2 p_2'} \delta_{p_4 p_4'} \\
&\qquad + \text{permutations} \Big\} \psi_{nP}(p_1', p_2', p_3', p_4') \\
&= E_{nP}\ \psi_{nP}(p_1, p_2, p_3, p_4)
\end{aligned}
$$

Estimation of condensate fraction in zero temperature α-matter

$$n_0 = \frac{\langle \Psi | a_0^\dagger a_0 | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

destruction of the BEC of the ideal Bose gas:
thermal excitation, but also correlations

"excluded" volume for α-particles $\approx 20\ \mathrm{fm}^3$
at nucleon density $\rho = 0.048\ \mathrm{fm}^{-3}$ filling factor $\approx 28\ \%$
(liquid ^4He: 8 % condensate),
destruction of the condensate at $\approx \rho_0/3$

Estimation of condensate fraction in zero temperature α-matter

α-cluster condensate in ^{12}C, ^{16}O:
resonating group method
occupation numbers of α-orbits in ^{12}C

	RMS radii	S-orbit	D-orbit	G-orbit
0_1^+ (g.s.)	2.44 fm	1.07	1.07	0.82
0_2^+	4.31 fm	2.38	0.29	0.16

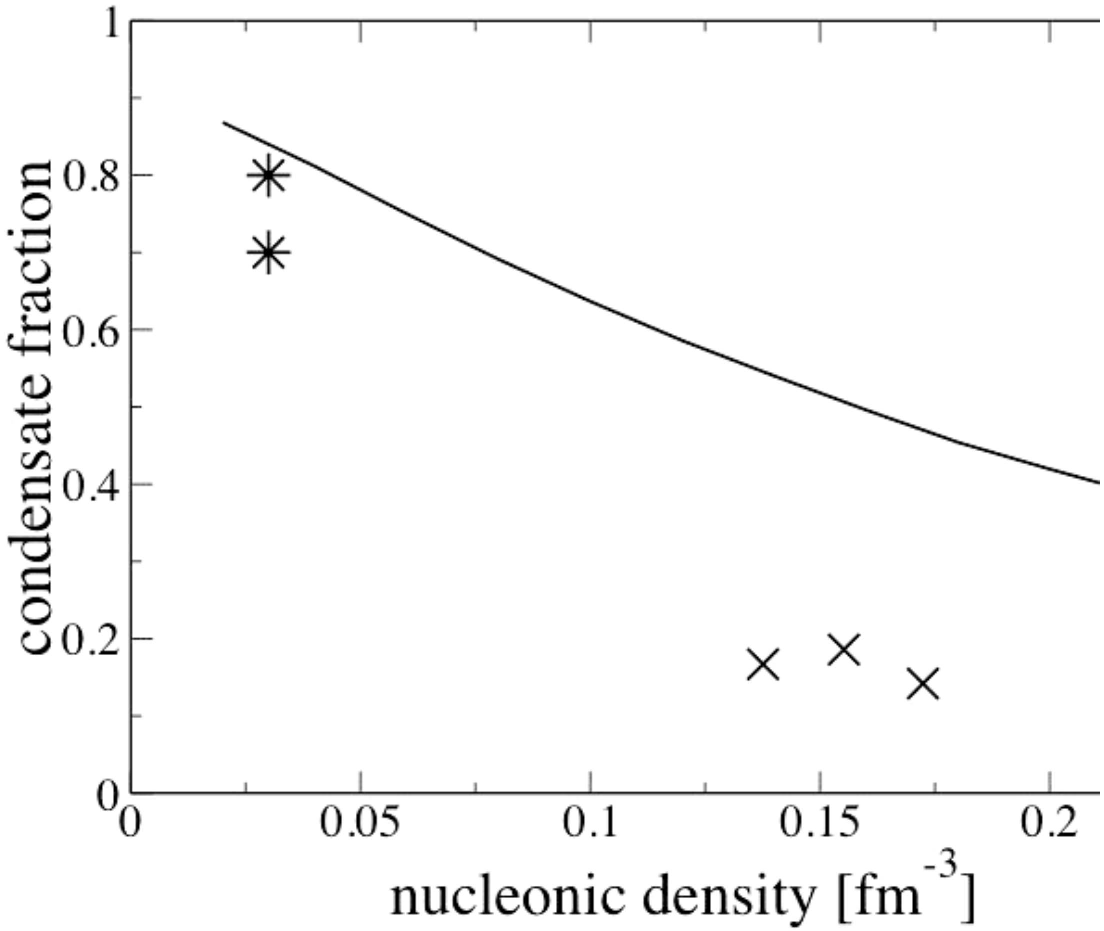
80 % condensate at 1/8 nuclear matter density

T. Yamada, P. Schuck: (2.16 − normal)/3 ≈ 60%

Suppresion of condensate fraction

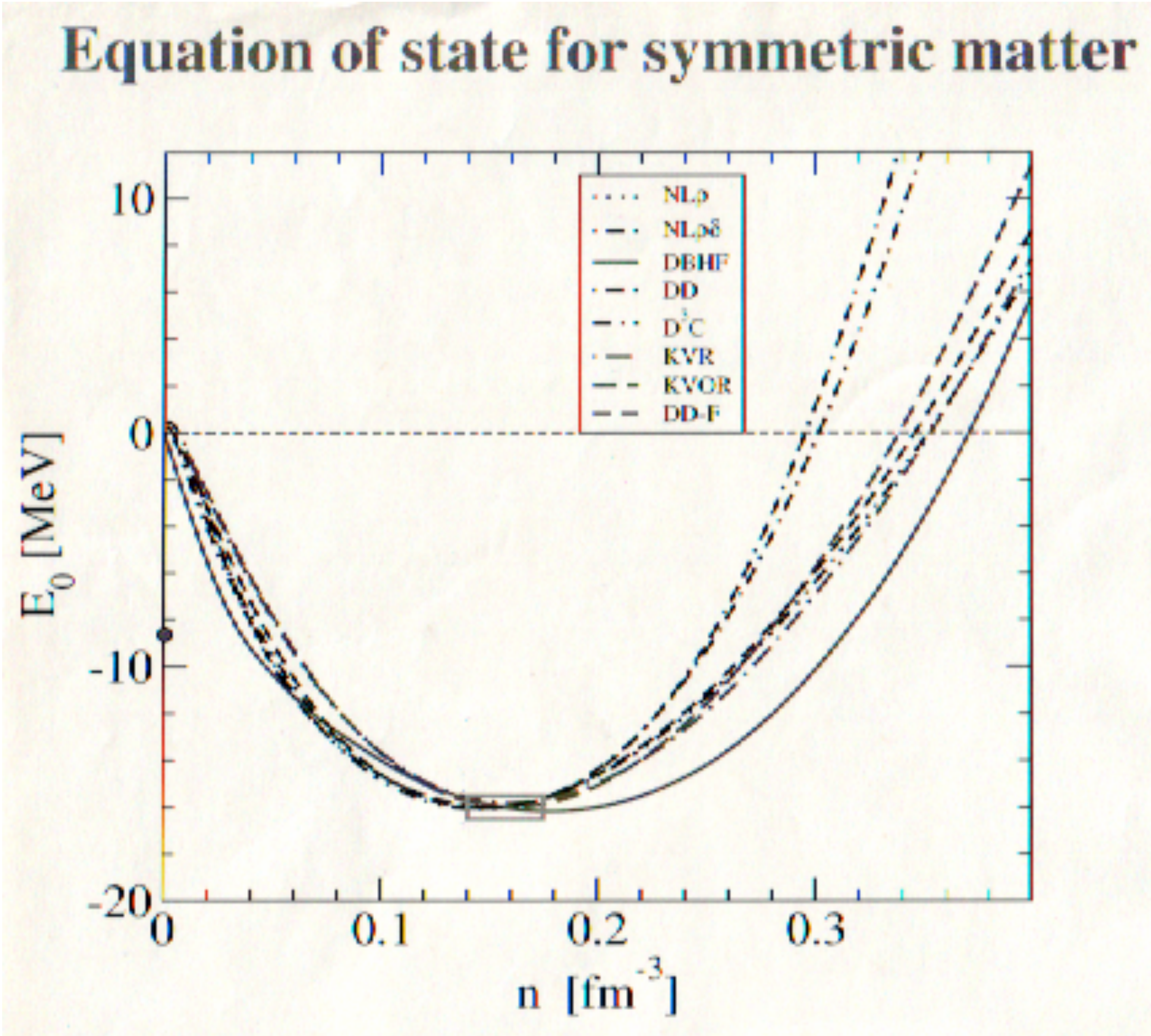
- Alpha-alpha interaction (Ali/Bodmer),
  no Pauli blocking:
- Variational calculation (Clark/Jastrow approach to the alpha-particle condensate amplitude)
  (crosses)
- First order approximation
  (full line)

- Yamada/Schuck's result for condensate in C12 - O2+
  (stars)

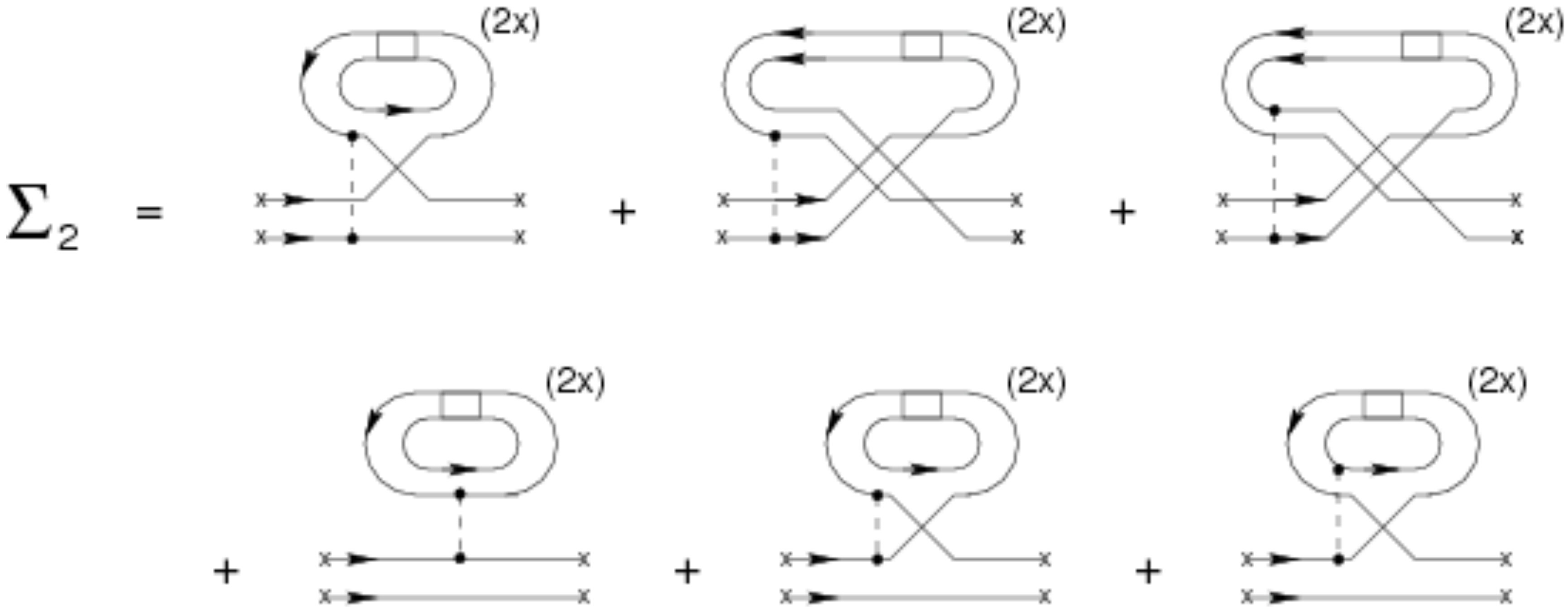


Y. Funaki et al., PRC **77**, 064312 (08)

Quasiparticle approximation for nuclear matter



Correlations in the medium



Cluster - mean field approximation

Cluster (A) interacting with a distribution of clusters (B) in the medium, fully antisymmetrized

$$\sum_{1'\ldots A'} \{H_A^0(1\ldots A,1'\ldots A') + \sum_i \Delta_i^{A,mf}\delta_{k,k'} + \frac{1}{2}\sum_{i,j}\Delta V_{ij}^{A,mf}\delta_{l,l'} - E_{A\nu P}\delta_{k,k'}\}\psi_{A\nu P}(1'\ldots A') = 0$$

self-energy

$$\Delta_1^{A,mf}(1) = \sum_2 V(12,12)_{ex} f^*(2) + \sum_{B\nu P}\sum_{2\ldots B'} f_B(E_{B\nu P})\sum_i V_{1i}(1i,1'i')\psi^*_{B\nu P}(1\ldots B)\psi_{B\nu P}(1'\ldots B')$$

effective interaction

$$\Delta V_{12}^{A,mf} = -\frac{1}{2}[f^*(1) + f^*(2)]V(12,1'2') - \sum_{B\nu P}\sum_{2^*\ldots B''} f_B(E_{B\nu P})\sum_i V_{1i}\psi^*_{B\nu P}(22^*\ldots B^*)\psi_{B\nu P}(2'2''\ldots B'')$$

phase space occupation $\quad f^*(1) = f_1(1) + \sum_{B\nu P}\sum_{2\ldots B} f_B(E_{B\nu P})\,|\psi_{B\nu P}(1\ldots B)|^2$

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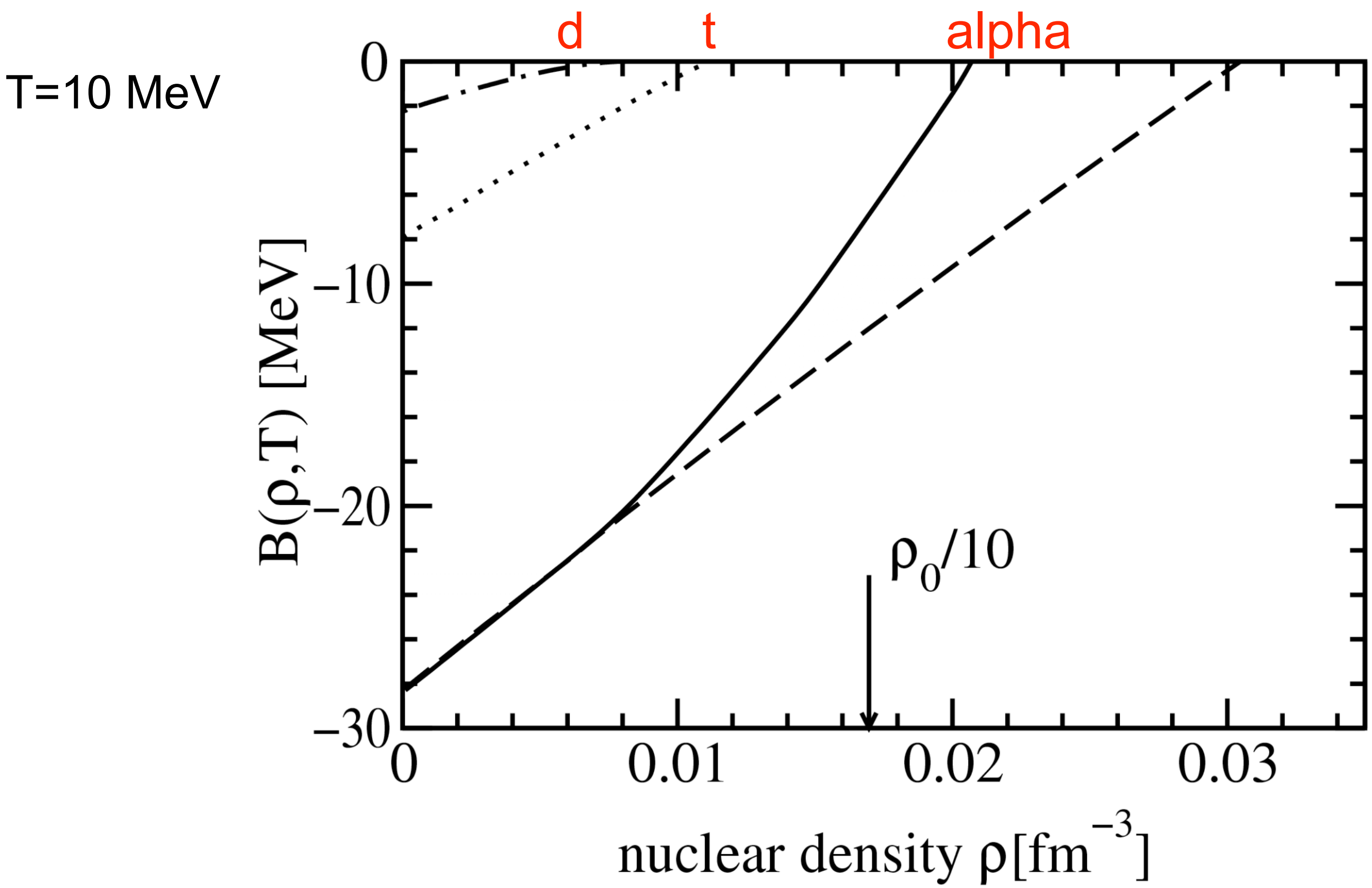
In-medium shift of binding energies of clusters

Solution of the Faddeev-Yakubovski equation with Pauli blocking



M. Beyer et al., PLB **488**, 247 (00), A. Sedrakian et al., Ann. Phys; PRC **73**, 035803 (06)

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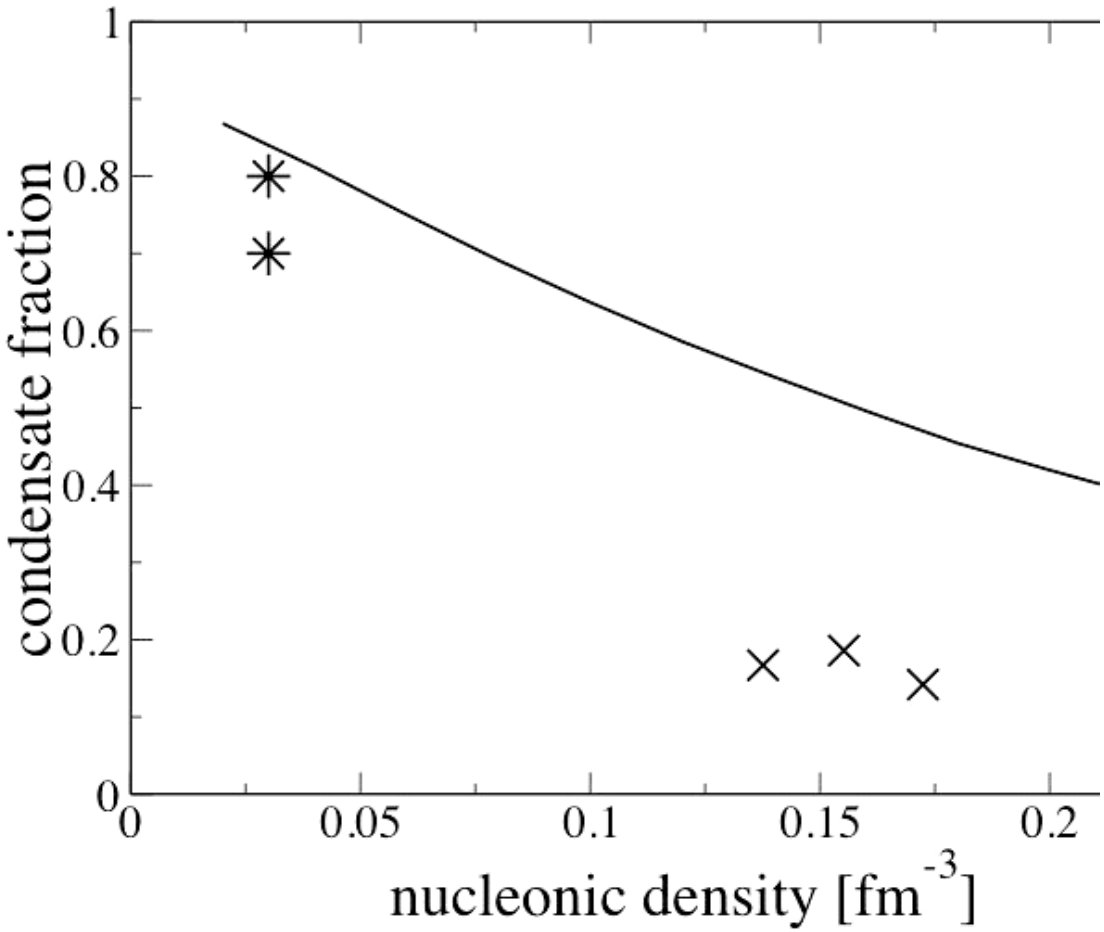

Y.Funaki et al., PRC 77, 064312 (2008),   F.Carstoiu, S.Misicu, PLB, 2009