Oscillating Solitons of the Parametrically Driven Damped Nonlinear Schrödinger Equation

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Parametrically driven damped NLS

Equation under study:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi.$$

$\gamma>0$ is the damping coefficient,

h is the amplitude of the parametric driver.

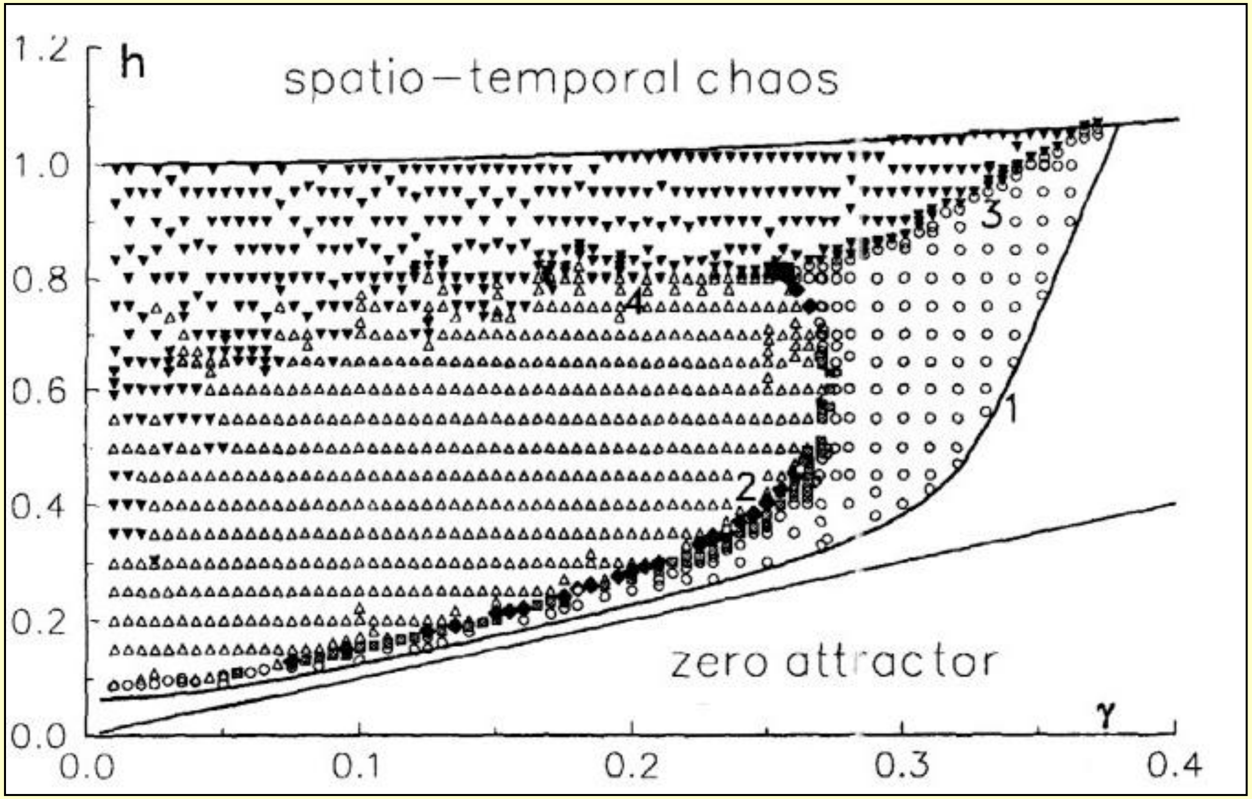
We are looking for periodic solutions by solving the NLS equation as a **boundary-value problem** on a two-dimensional domain $(-\infty, \infty) \times (0, T).$

Boundary conditions:

$$\psi(x,t) = 0 \quad \text{as } x \to \pm\infty,$$

$$\psi(x, t+T) = \psi(x,t).$$

Results of direct numerical simulation



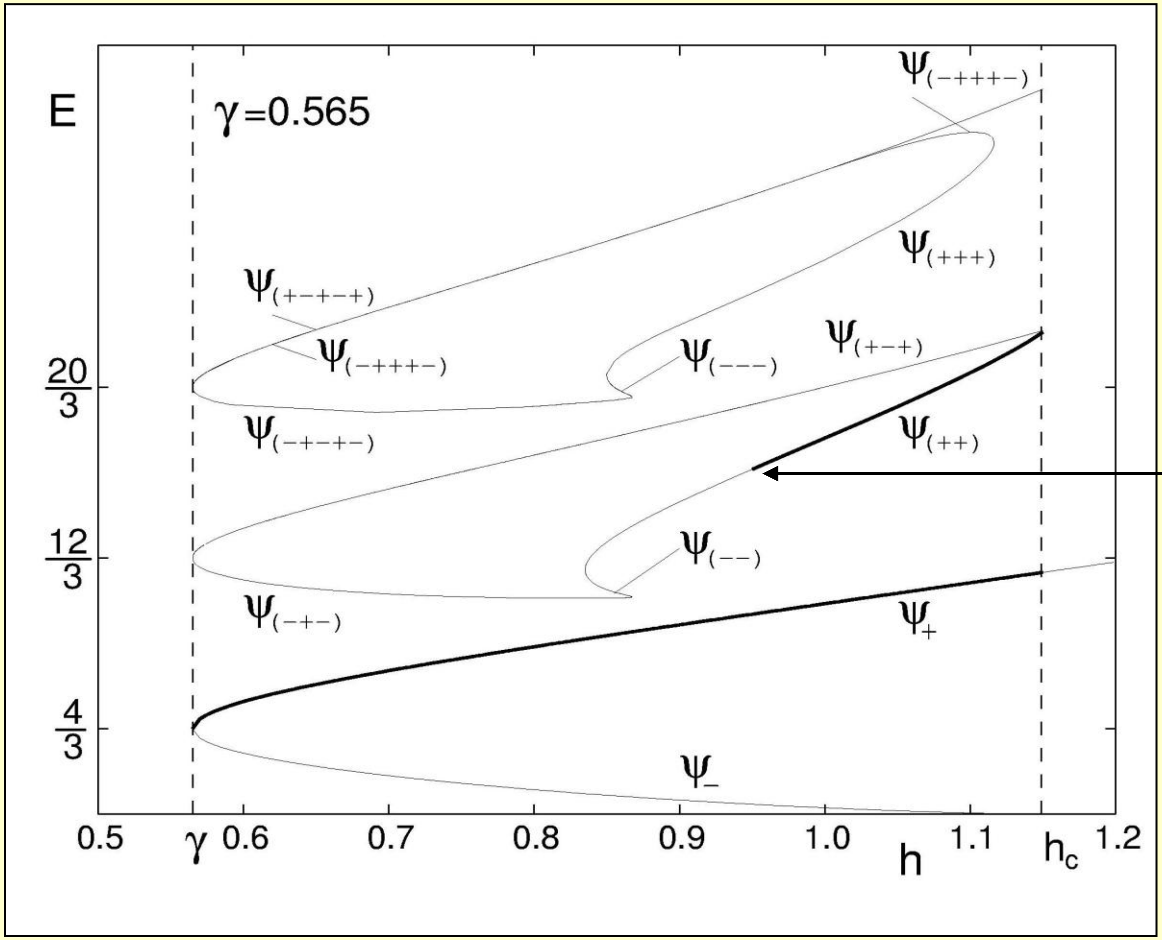
$$\psi(x) = Ae^{-i\theta}\mathrm{sech}(Ax),$$

$$A = \sqrt{1+\sqrt{h^2-\gamma^2}},$$

$$\theta = [\arcsin(\gamma/h)]/2$$

- M.Bondila, I.Barashenkov, M.Bogdan, Physica **D 87** 314 (1995)
- I.Barashenkov, M.Bogdan, V.Korobov, Europhys. Lett. **15** 113 (1991)

Numerical continuation of stationary multi-soliton complexes



The Hopf bifurcation point: γ=0.565; h=0.9435

I.Barashenkov and E.Zemlyanaya, Phys Rev Lett **83** (1999) 2568

Method of numerical study. New variables

New variables τ and Ψ:

$$\tau T = t; \quad 0 \le \tau \le 1; \quad \Psi(x,\tau) = \psi(x,t).$$

Modified equation with respect of unknown Ψ and T:

$$i\Psi_\tau + T \cdot \Phi(\Psi(x,\tau),h,\gamma) = 0 \qquad \text{where}$$

$$\Phi \equiv \Psi_{xx} + 2|\Psi|^2 \cdot \Psi - \Psi - h\Psi^* + i\gamma\Psi.$$

Boundary conditions:

$$\Psi(-L,\tau) = \Psi(+L,\tau) = 0; \qquad \Psi(x,0) = \Psi(x,1);$$

Additional equation (phase condition)

$$R \equiv \mathrm{Re}[\Phi(\Psi(x^*,t^*),h,\gamma)] = 0; \qquad x^* = t^* = 0.$$

Method of numerical study. Newtonian scheme (1)

$$\Psi_{k+1} = \Psi_k + \xi_k v_k; \qquad T_{k+1} = T_k + \xi_k \mu_k;$$

k – number of iteration;

$0<\xi_k\leq 1$ parameter of the Newtonian scheme;

$$v_k = v^{(1)} + v^{(2)} \mu_k;$$

(1) $\quad iv^{(1)}_\tau + T_k v^{(1)}_{xx} + A_k v^{(1)} + B_k v^{(1)*} = -\Phi_k$

(2) $\quad iv^{(2)}_\tau + T_k v^{(2)}_{xx} + A_k v^{(2)} + B_k v^{(2)*} = -C_k$

BCs: $v^{(1)}(\pm L,\tau) = -\Psi_k(\pm L,\tau); \quad v^{(2)}(\pm L,\tau) = 0;$

$$v^{(1,2)}(x,0) - v^{(1,2)}(x,1) = -[\Psi^{(1,2)}(x,0) - \Psi^{(1,2)}(x,1)]$$

$$A_k = 4T_k\Psi_k(\Psi_k)^* - T_k - i\gamma T_k; \qquad B_k = 2T_k(\Psi_k)^2 - hT_k;$$

$$C_k = \Psi_{xx} + 2\Psi_k^*(\Psi_k)^2 - \Psi_k - h(\Psi_k)^* - i\gamma\Psi_k;$$

Method of numerical study. Newtonian scheme (2)

μ_k is calculated at each iteration as follows

$$\mu_k = \frac{-G - R}{F}$$

$$F = [V_R^{(2)}[_{xx} + 6\Psi_R^2 V_R^{(2)} + 4\Psi_I \Phi_R V_I^{(2)} + 2\Psi_I^2 V_R^{(2)} - V_R^{(2)} - hV_R^{(2)} - \gamma V_I^{(2)}$$

$$G = [V_R^{(1)}]_{xx} + 6\Psi_R^2 V_R^{(1)} + 4\Psi_I \Psi_R V_I^{(1)} + 2\Psi_I^2 V_R^{(1)} - V_R^{(1)} - hV_R^{(1)} - \gamma V_I^{(1)}$$

$$R = [\Psi_R]_{xx} + 2\Psi_R^3 + 2\Psi_I^2 \Psi_R - \Psi_R - h\Psi_R - \gamma \Psi_I$$

$$\Psi_R = \mathrm{Re}\,\Psi(x^*, 0); \quad \Psi_I = \mathrm{Im}\,\Psi(x^*, 0)$$

$$V_R^{(1,2)} = \mathrm{Re} V^{(1,2)}(x^*, 0); \quad V_I^{(1,2)} = \mathrm{Im} V^{(1,2)}(x^*, 0)$$

Spatial stepsize 0.05; stepsize in time 0.01; interval [-50,50]

Stability analysis

E.Zemlyanaya, I.Barashenkov, N.Alexeeva. Springer Lecture Notes in Computer Sciences **5434** (2009) 139

Periodic solution is linearized in small perturbation *u+iv:*

$$\psi(x,t) = \psi_0(x,t) + u(x,t) + iv(x,t)$$

$$\psi_0 = \begin{pmatrix} \mathcal{R}(x,t) \\ \mathcal{I}(x,t) \end{pmatrix}$$

After expansion *u* and *v* in the Fourier series on the interval (-L,L), according to the Floquet theory we obtain:

$$J\dot{\mathbf{w}}_m = \sum_{n=-N}^{N} H_{mn}(t)\mathbf{w}_n$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $q_n = \pi n/L$, $\mathbf{w}_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$

$$H_{mn}(t) = \frac{1}{2L}\int_{-L}^{L} e^{i(q_m-q_n)x} \begin{pmatrix} q_n^2+1+h-6\mathcal{R}^2-2\mathcal{I}^2 & -4\mathcal{R}\mathcal{I}+\gamma \\ -4\mathcal{R}\mathcal{I}-\gamma & q_n^2+1-h-2\mathcal{R}^2-6\mathcal{I}^2 \end{pmatrix} dx.$$

The system is solved numerically with initial condition

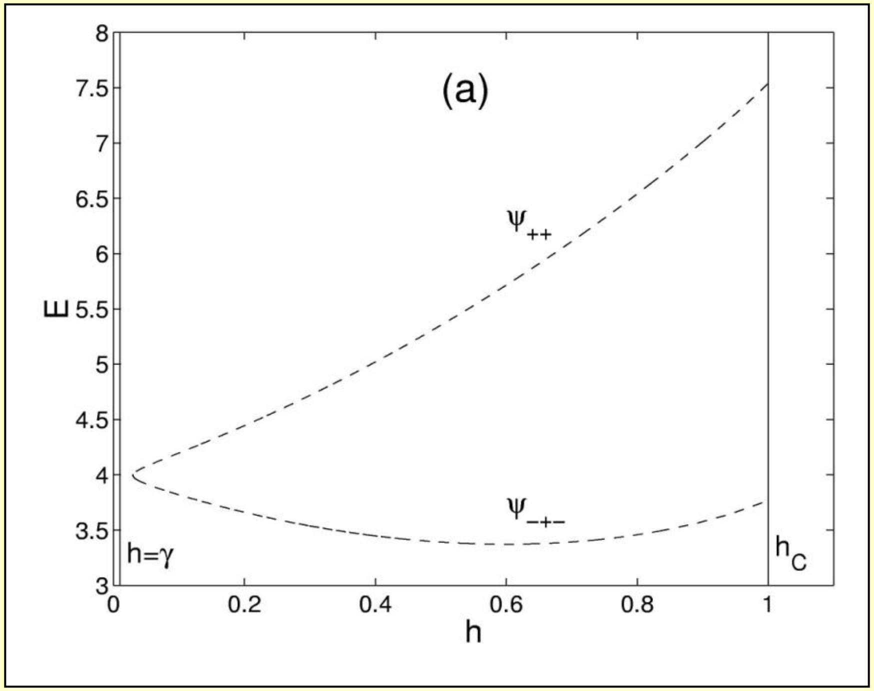
$$u_n(0) = \delta_{n\alpha}, \quad v_n(0) = 0 \quad (n = -N, ..., N),$$

$$\alpha = -N, ..., N$$

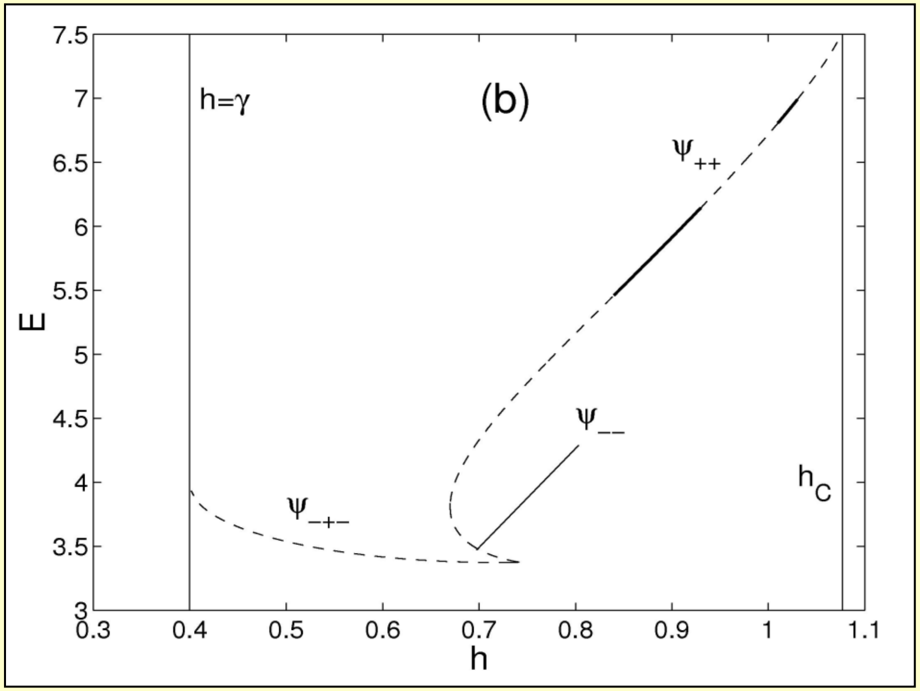
The monodromy matrix is constructed. Its eigenvalues allow us to make conclusions about stability properties of periodic solitons.

Numerical continuation of stationary two-soliton solutions

(a) γ=0.01



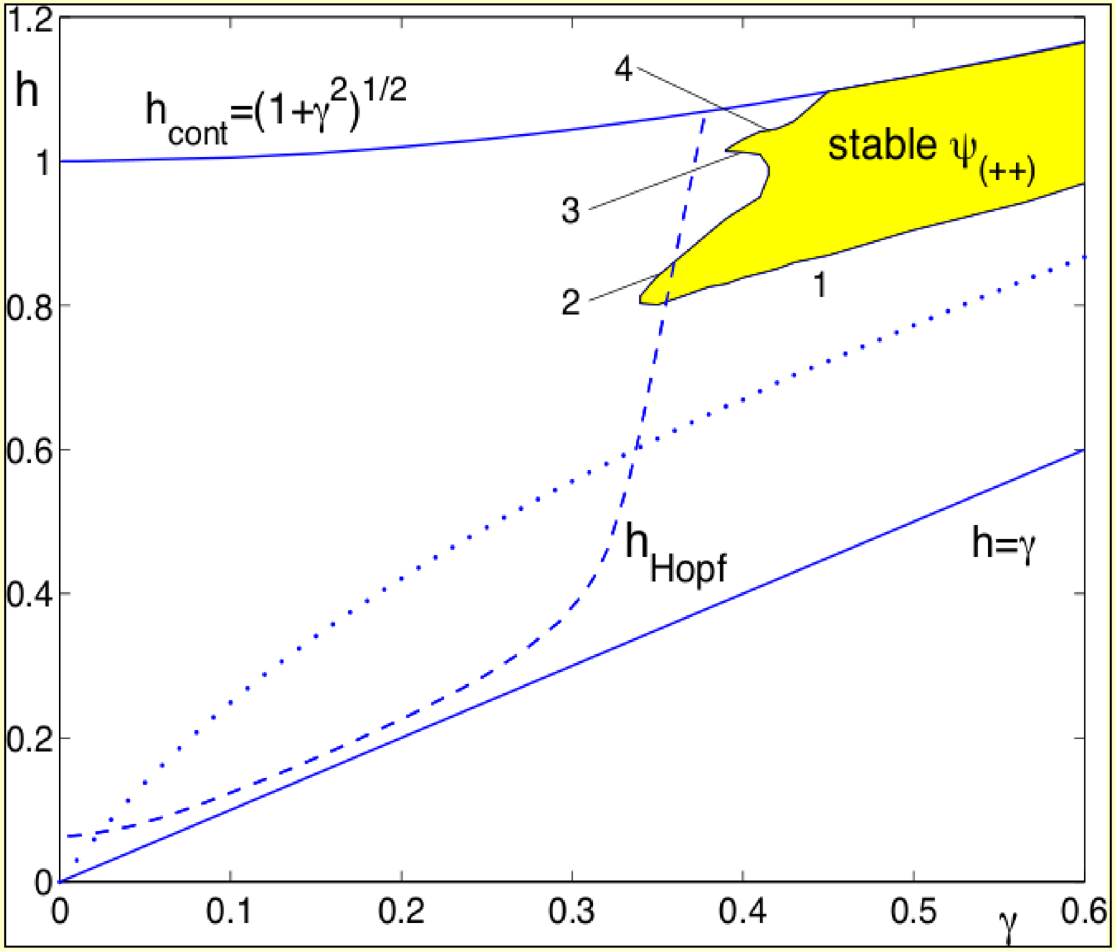
(b) γ=0.4



E.Zemlyanaya, I.Barashenkov, N.Alexeeva. Springer Lecture Notes in Computer Sciences **5434** (2009) 139
E.Zemlyanaya, A.Alexeeva. Theor. and Math. Phys. **159** No.3 (2009) 536-544

Numerical continuation of stationary two-soliton solutions

Hopf bifurcations points of stationary solitons at the (h,γ)-plane



Numerical continuation in h for the fixed γ.

Weak damping:

- γ=0.1
- γ=0.2
- γ=0.265

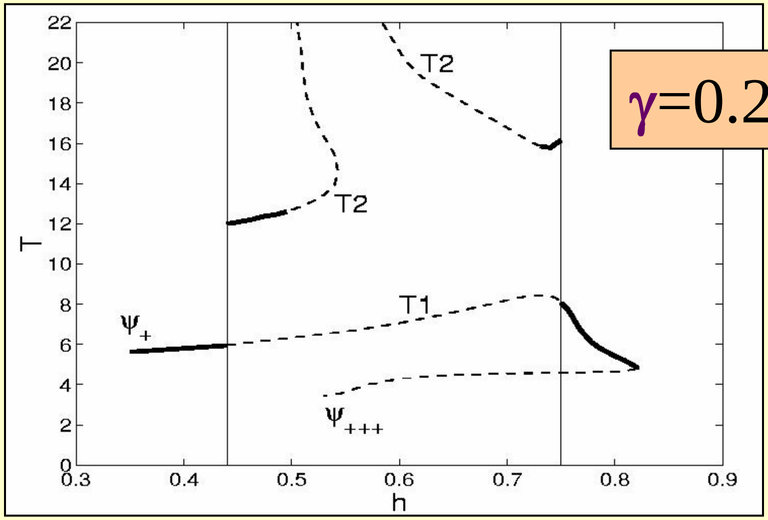
Moderate damping:

- γ=0.3
- γ=0.35
- γ=0.38
- γ=0.41

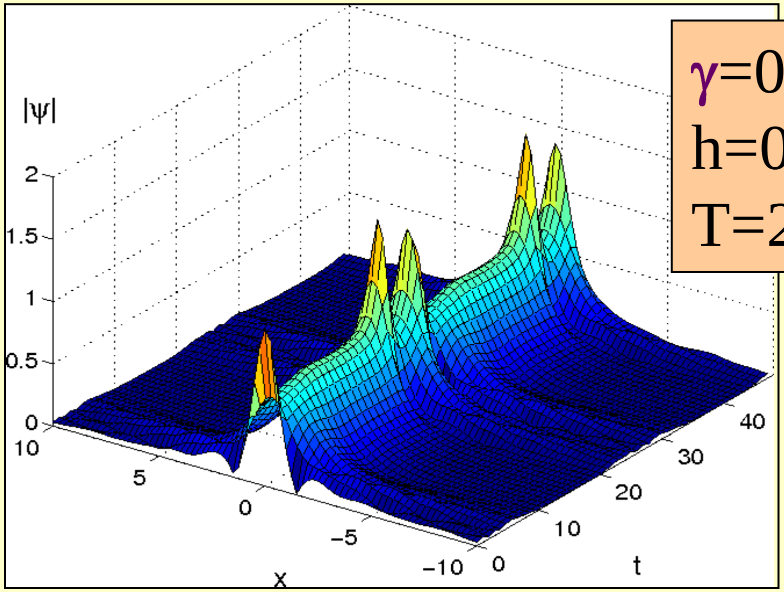
Strong damping:

- γ=0.565

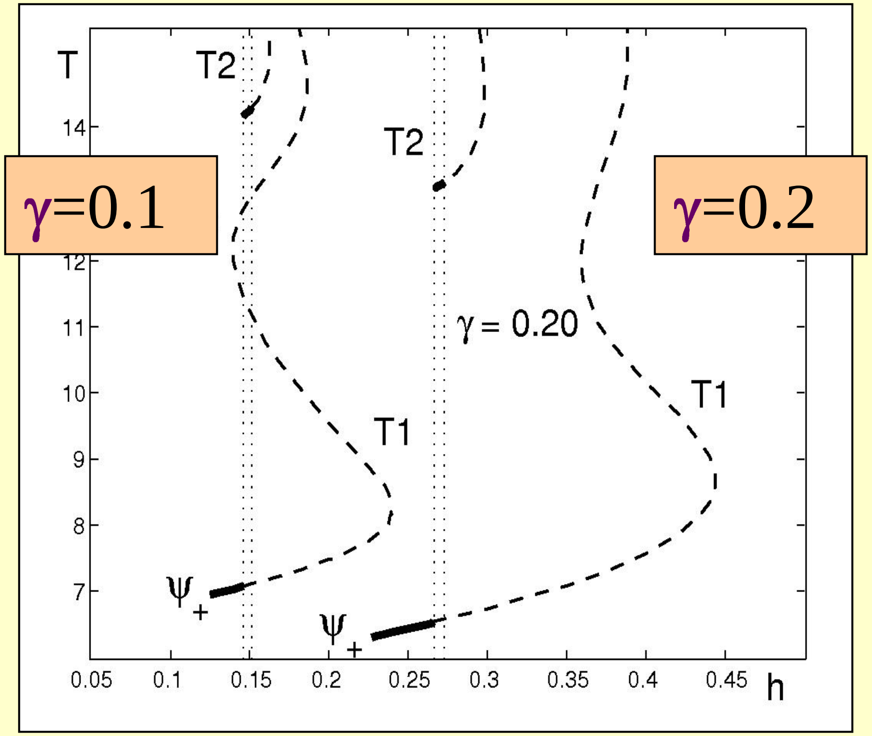
Numerical results (1)
Weak damping: γ=0.265; 0.2; 0.1



γ=0.265



γ=0.265
h=0.502
T=22.985

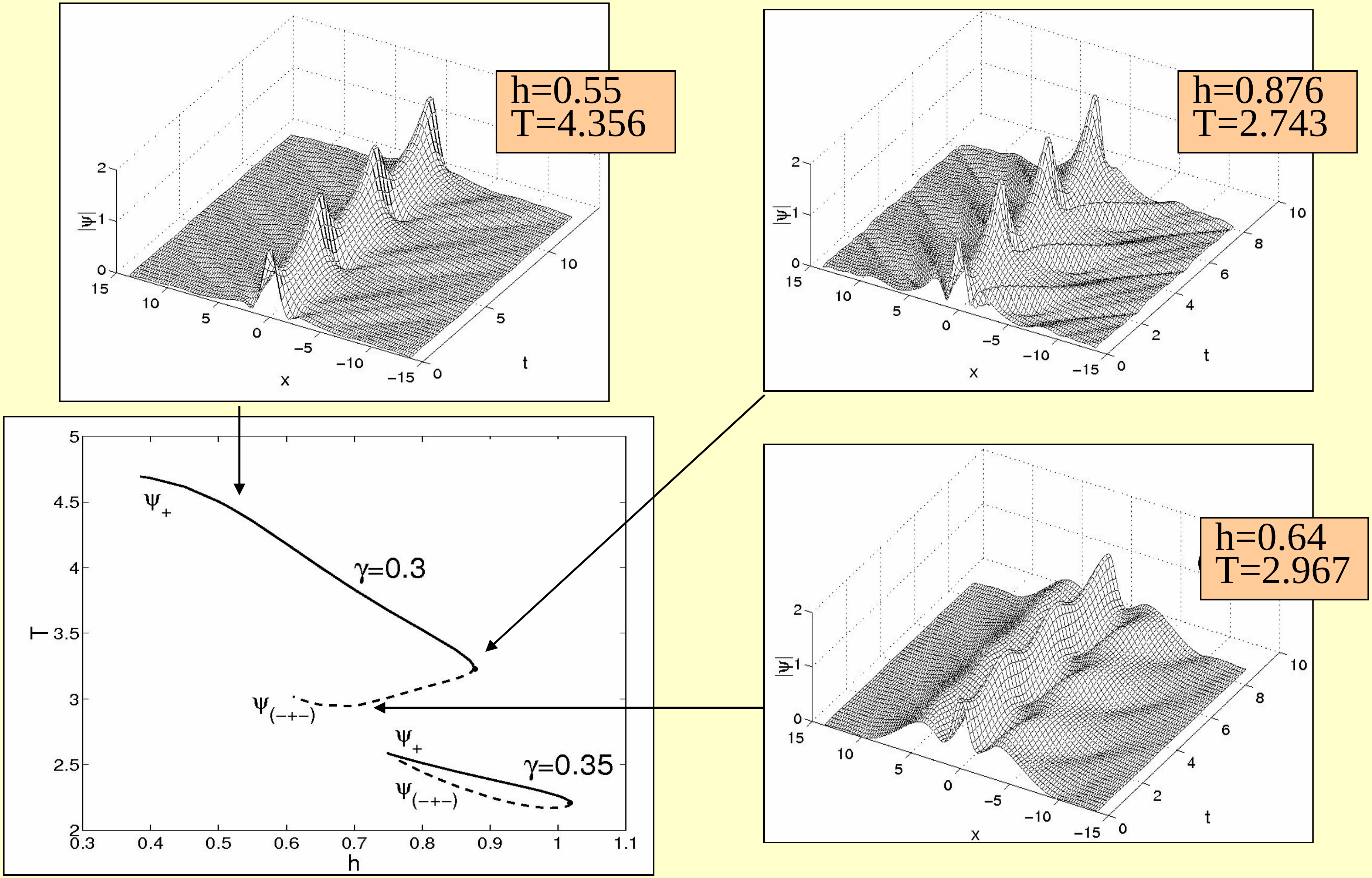


γ=0.1

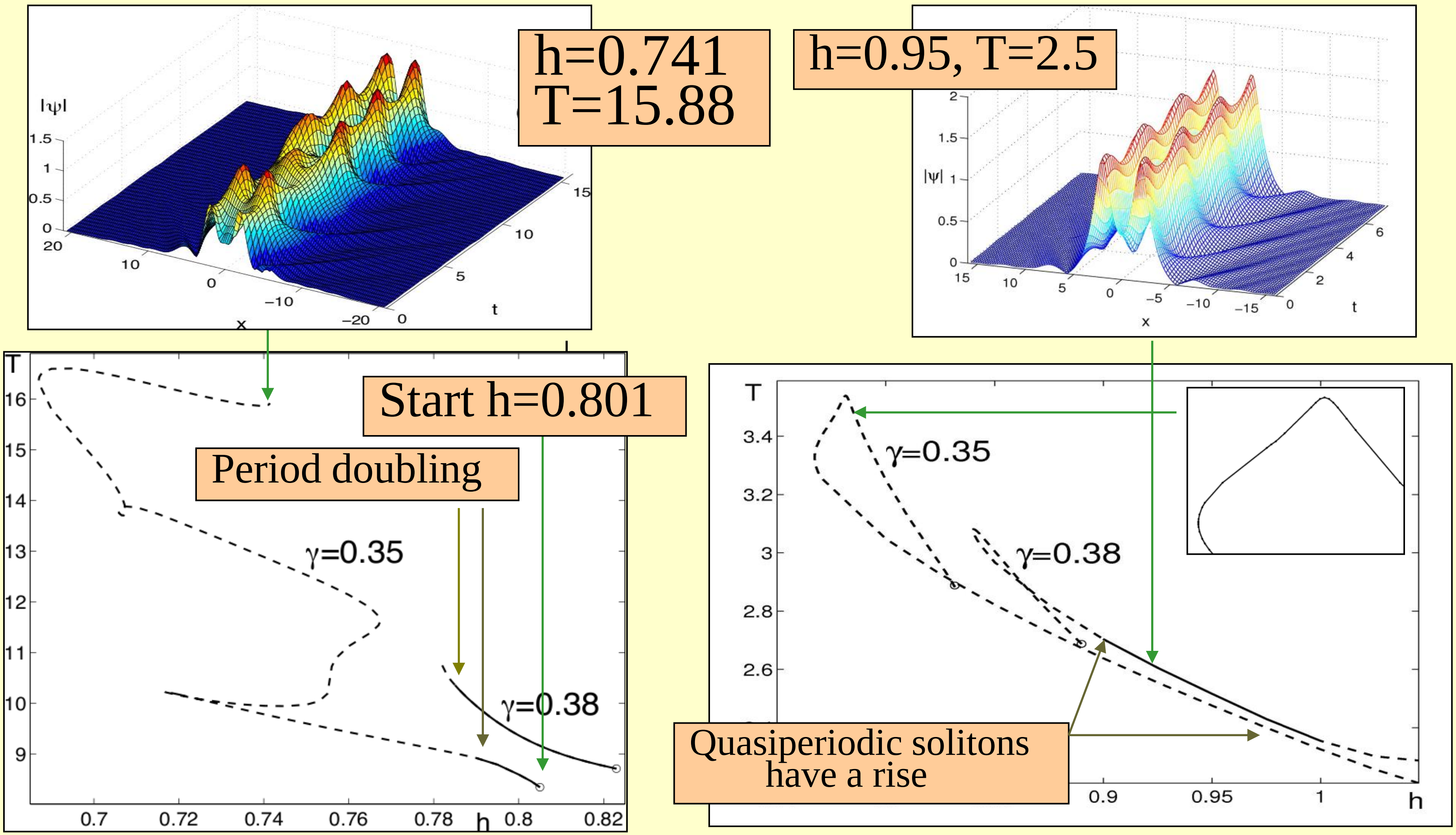
γ=0.2

I.V.Barashenkov, E.V.Zemlyanaya,
E.V.Zemlyanaya, Time-periodic solitons in a damped-driven nonlinear Schroedinger equation. Submitted to Phys. Rev. E

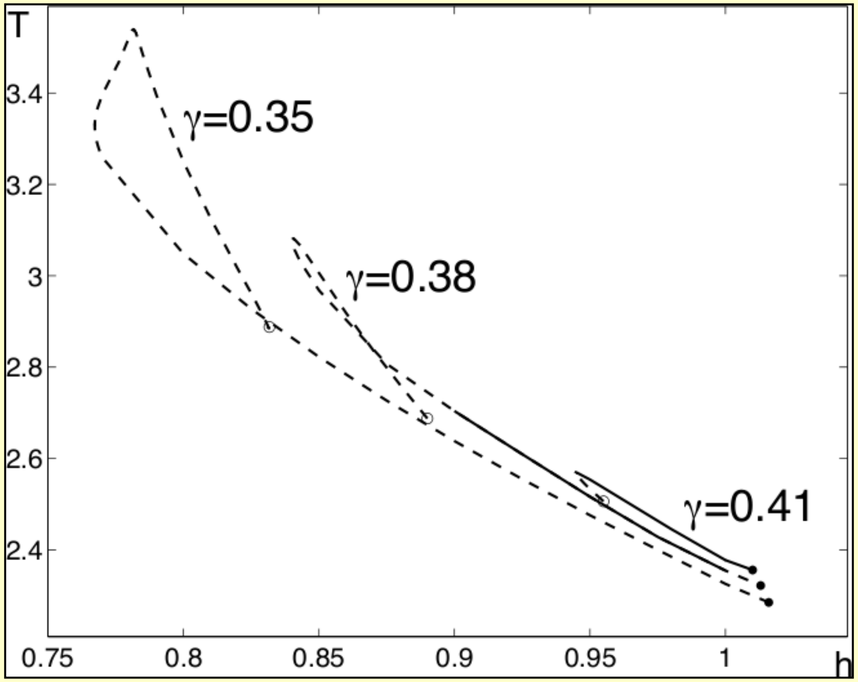
Numerical results (2)
Moderate damping: γ=0.3, 0.35

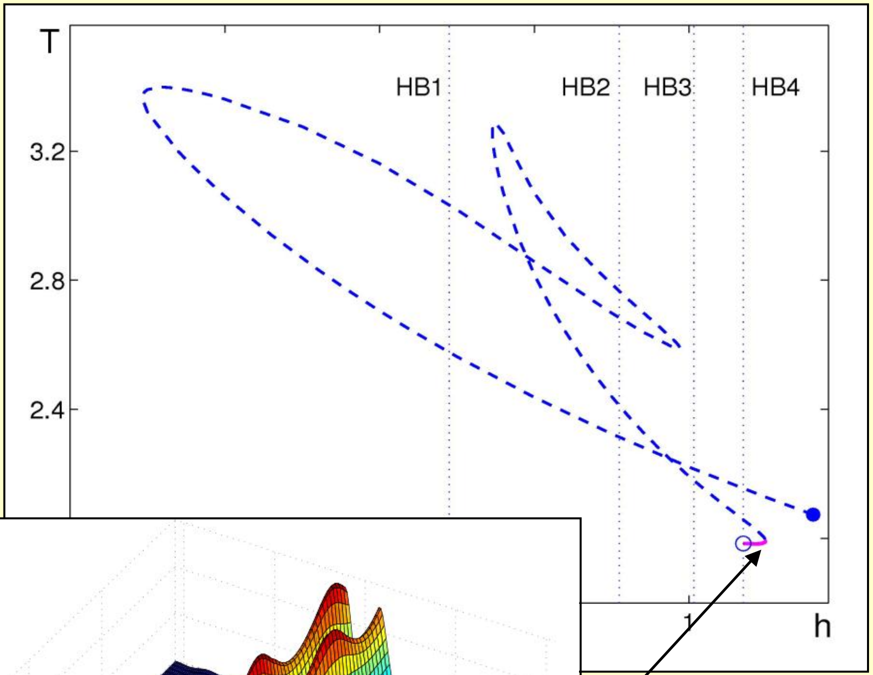


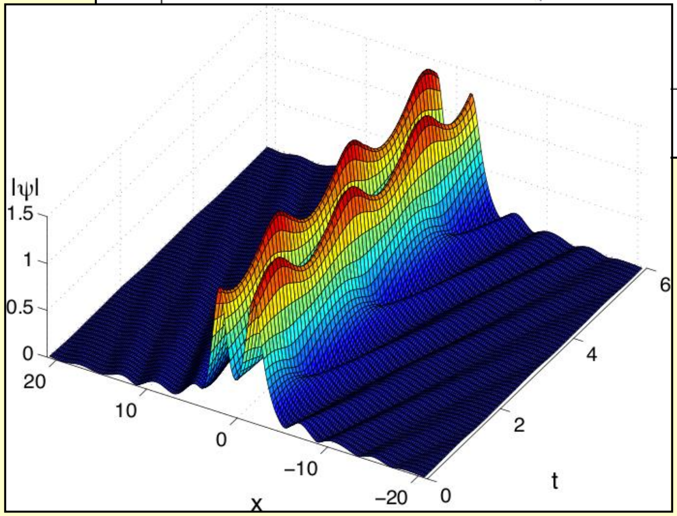
Numerical results (3)
One-periodic two-soliton solutions
Moderate damping: γ=0.35, 0.38.



Numerical results (4)
One-periodic two-soliton solutions
Moderate damping: γ=0.41 (the case of 4 HBs)

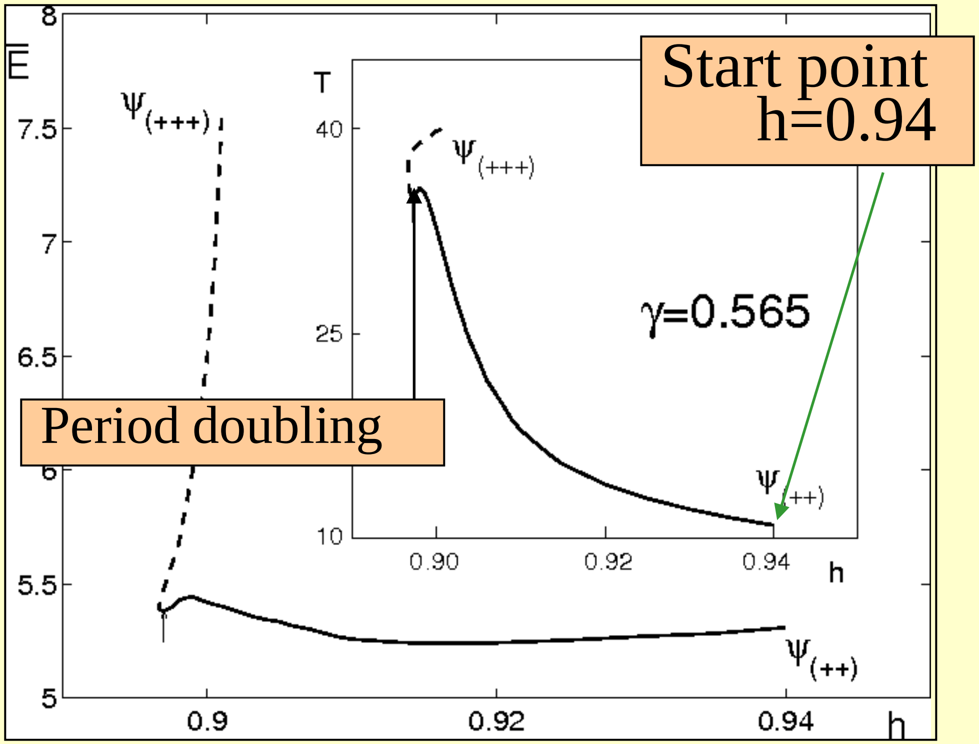


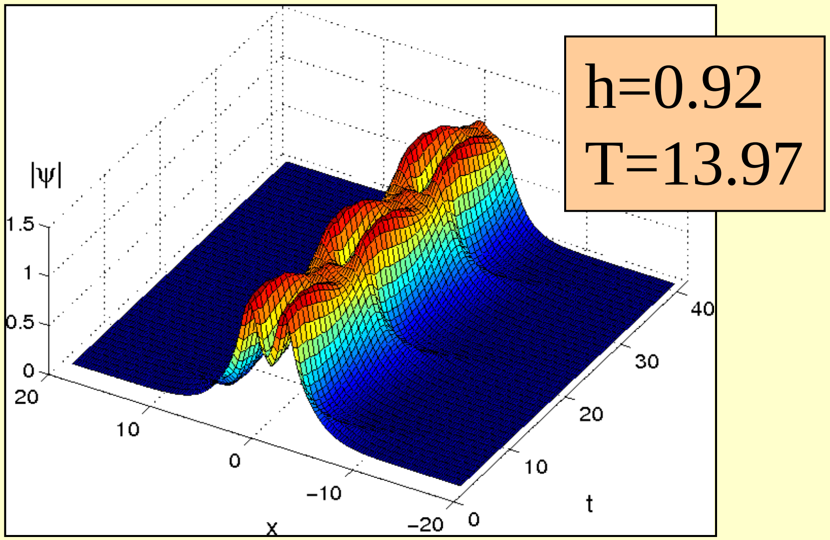


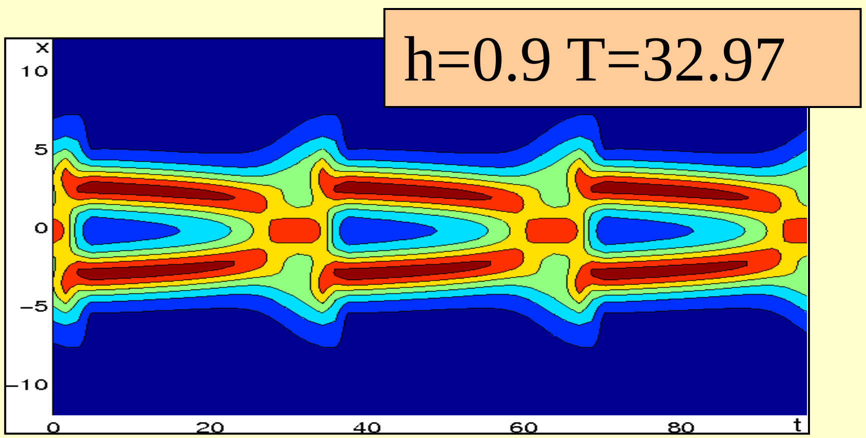


A time-periodic two-soliton complex oscillating out of phase with each other. $\gamma = 0.41$, $h = 1.049$, $T = 1.991$

Numerical results (5)
One-periodic two-soliton solutions
Strong damping: γ=0.565

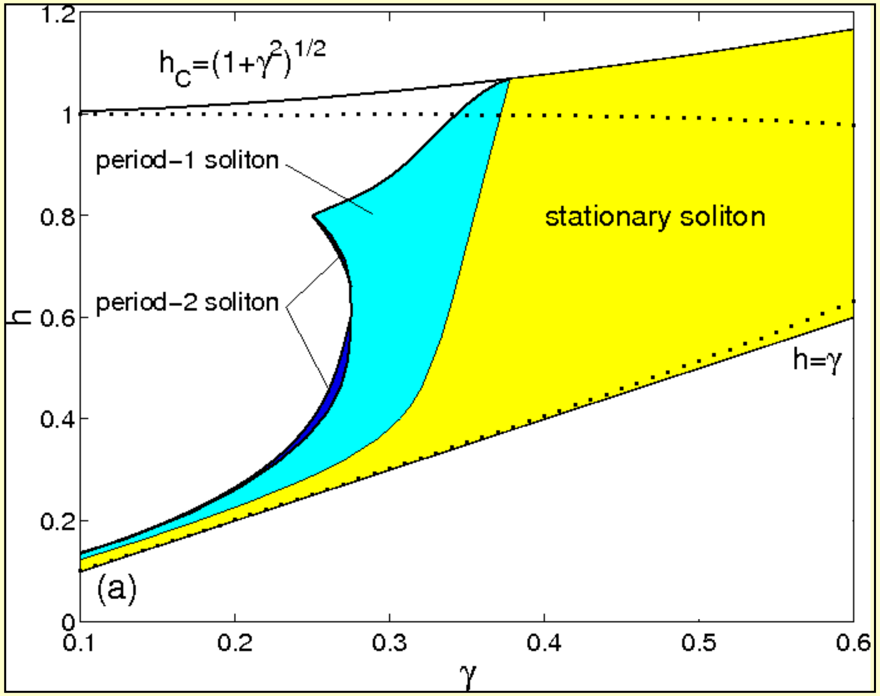


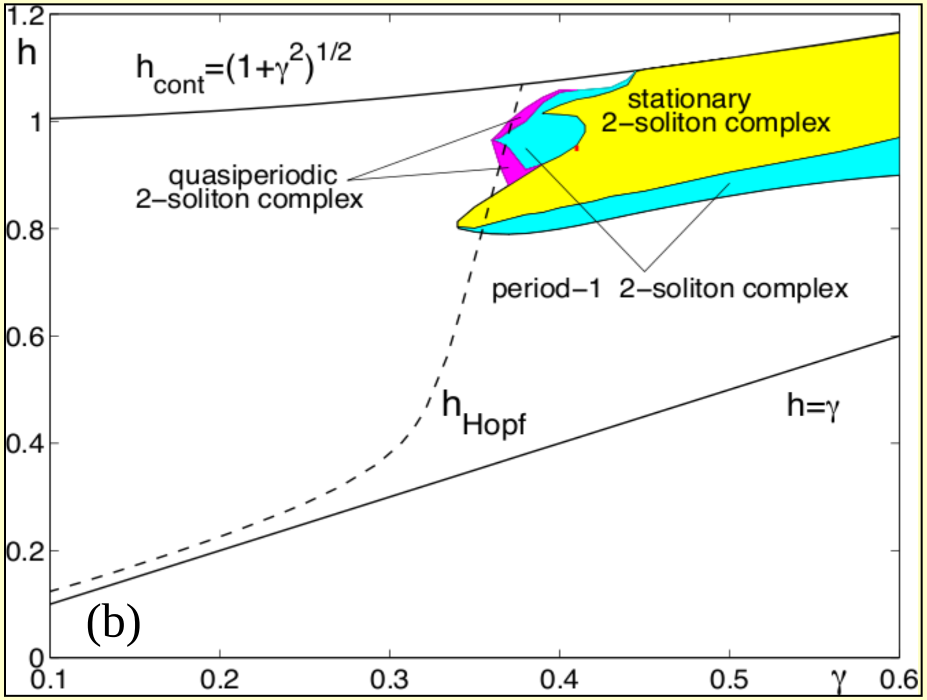




I.V.Barashenkov, E.V.Zemlyanaya. Soliton complexity in the damped-driven nonlinear Schroedinger equation: stationary, periodic, quasiperiodic complexes. Submitted to Phys.Rev.E

Numerical results (6)
Stability diagram of stationary and time-periodic solitons at the (h,γ)-plane





3- and 5-mode approximation (1)

We decompose ψ as

$$\psi = A_+[U(\bar{x},\bar{t}) + iV(\bar{x},\bar{t})]e^{-i\theta_+},$$

This casts the NLS in the form:

$$-V_t - 2\Gamma V = -U_{xx} + U - 2(U^2+V^2)U,$$
$$+U_t + 2HV = -V_{xx} + V - 2(U^2+V^2)V$$

We expand ψ (*x;t*) as:

u and *v* are real;

A and *B* are complex

$$U(x,t) = u(x) + \mathcal{A}(x)e^{i\Omega t} + \mathcal{A}^*(x)e^{-i\Omega t},$$
$$V(x,t) = v(x) + \mathcal{B}(x)e^{i\Omega t} + \mathcal{B}^*(x)e^{-i\Omega t},$$

Resulting system of equations:

$$u_{xx} - u + 2(u^2+v^2)u + 4(3|\mathcal{A}|^2 + |\mathcal{B}|^2)u + 4(\mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B})v - 2\Gamma v = 0$$
$$v_{xx} - v + 2(u^2+v^2)v + 4(|\mathcal{A}|^2 + 3|\mathcal{B}|^2)v + 4(\mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B})u + 2Hv = 0,$$
$$\mathcal{A}_{xx} - \mathcal{A} + 2(3u^2+v^2)\mathcal{A} + 2(3|\mathcal{A}|^2 + 2|\mathcal{B}|^2)\mathcal{A} + 2(2uv + \mathcal{A}^*\mathcal{B})\mathcal{B} - 2\Gamma\mathcal{B} - i\Omega\mathcal{B} = 0,$$
$$\mathcal{B}_{xx} - \mathcal{B} + 2(u^2+3v^2)\mathcal{B} + 2(2|A|^2 + 3|\mathcal{B}|^2)\mathcal{B} + 2(2uv + \mathcal{B}^*\mathcal{A})A + 2H\mathcal{B} + i\Omega\mathcal{A} = 0.$$

where $\Omega=2\pi/(A_+^2 \cdot T)$, $\Gamma = \gamma/A_+^2$ and $H = \sqrt{h^2-\gamma^2}/A_+^2$.

3- and 5-mode approximation (2)

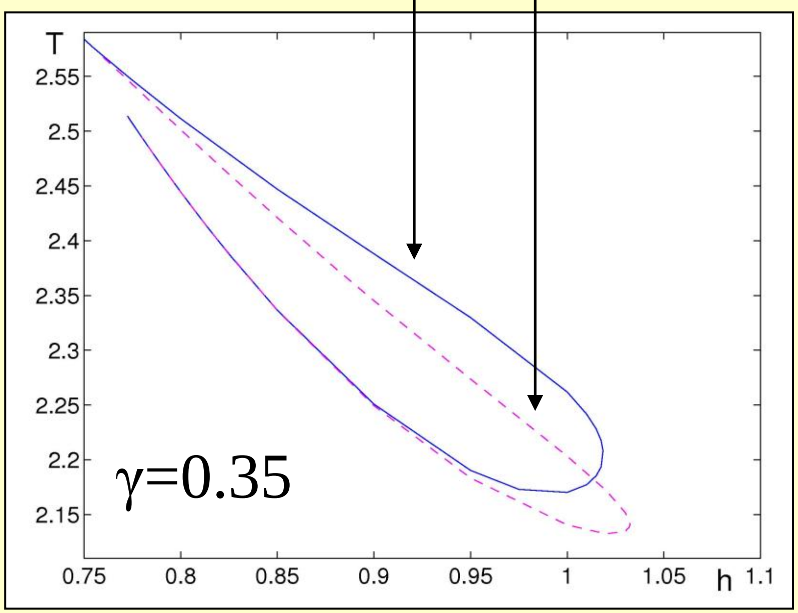
E.V.Zemlyanaya, A.N.Alexeeva. Breathers in a damped-driven nonlinear Schroedinger equation. Accepted to J. Theor. Math. Phys. (2011)

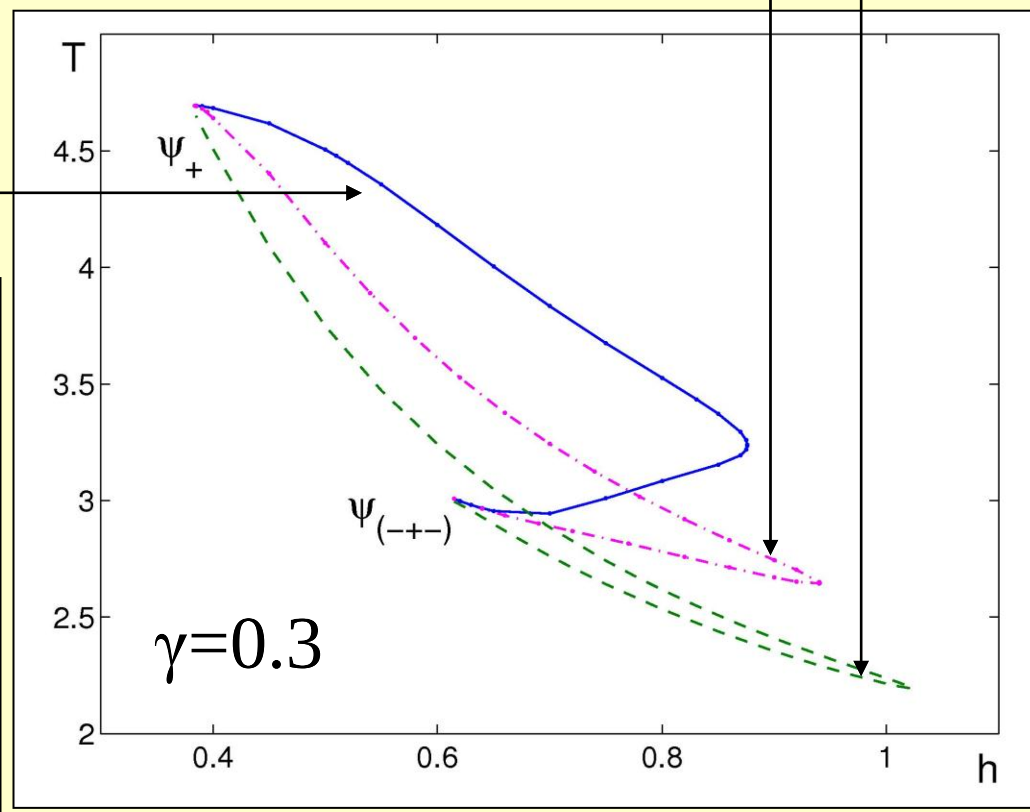
3-mode approximation

5-mode approximation

5-mode approximation

Full NLS





Summary

- A boundary-value problem and stability problem have been formulated for numerical investigation of temporally periodic solitons of parametricaly driven damped NLS.
- Transformations of temporally periodic solitons have been numerically studied; interconnection between coexisting branches of stable and unstable solutions has been analyzed.
- New temporally periodic solitons have been found.
- Stability diagram of stationary and oscillating two-soliton complexes has been constructed at the (h,γ)-plane.
- Shown that the bifurcation diagram can be reproduced a three- and five-mode approximation.

THANK YOU!