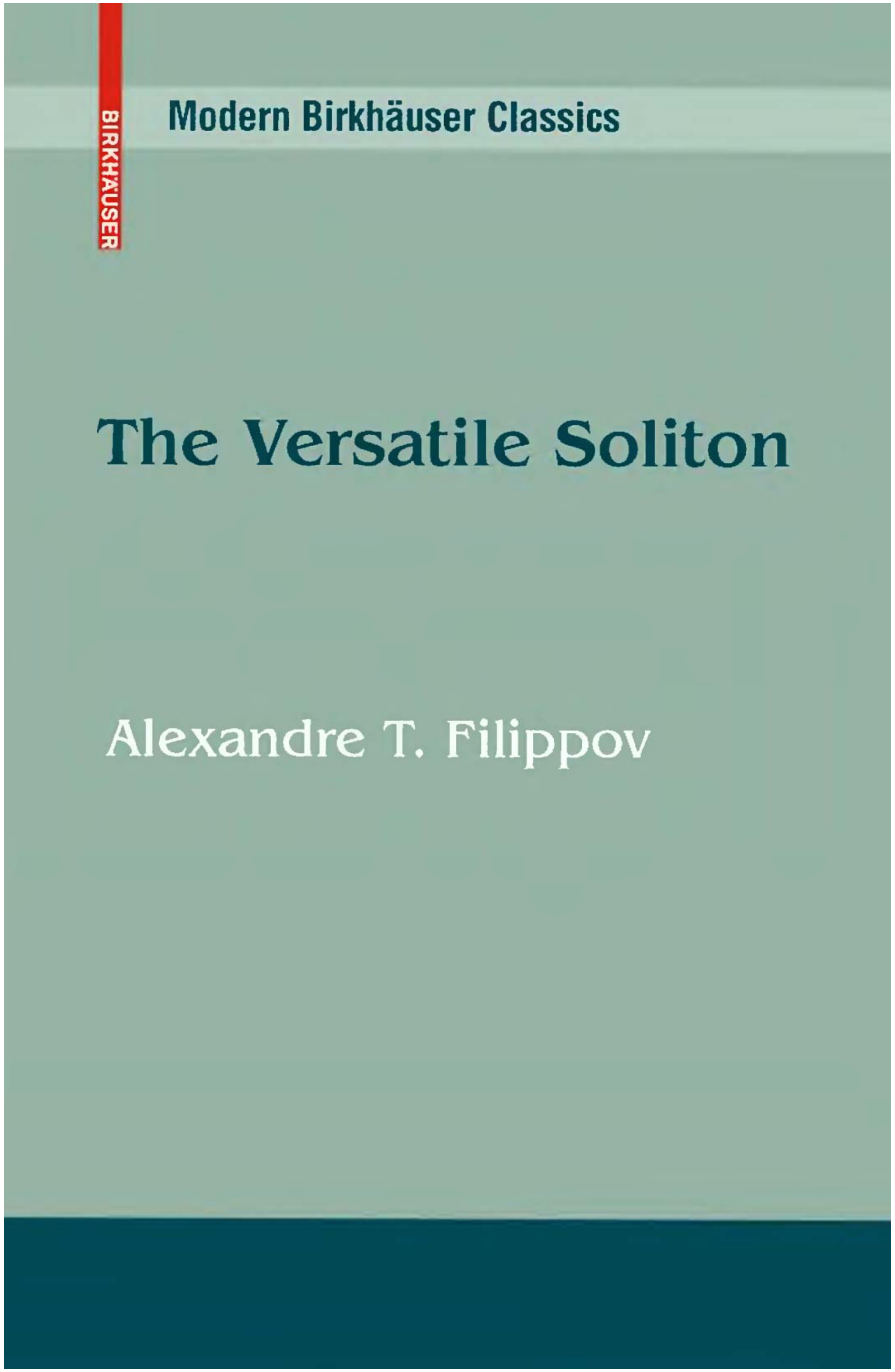


Modern Birkhäuser Classics

Many of the original research and survey monographs in pure and applied mathematics published by Birkhäuser in recent decades have been groundbreaking and have come to be regarded as foundational to the subject. Through the MBC Series, a select number of these modern classics, entirely uncorrected, are being re-released in paperback (and as eBooks) to ensure that these treasures remain accessible to new generations of students, scholars, and researchers.

1. The **KdV equation** was written in Chapter 7 (eqs.(7.1) and (7.2)). Usually one writes it for the dimensionless function $u \equiv 3y/4h$ depending on the dimensionless time and space variables $T \equiv \sqrt{6}v_0t/h$ and $X \equiv \sqrt{6}x/h$:

$$\dot{u} + (u + u^2 + u'')' = 0.$$

Here the dot denotes the derivative with respect to "time" T and the prime denotes the "space" derivative (with respect to X). The one-soliton solution of this equation is

$$u = 6k/\cosh^2[k(X - VT)],$$

where k is an arbitrary real number and $V = 1 + 4k^2$ (compare this to eqs. (7.1) and (7.2)). By replacing in the KdV equation u^2 with u^3, we get the so-called "modified" KdV equation. Its one-soliton solution is

$$u = \sqrt{2}k/\cosh[k(X - VT)], \quad V = 1 + k^2.$$

Finally, let us write a typical equation describing **nonlinear diffusion**

$$u'' - \dot{u} = u(u - 1)(u - a)$$

and its solitary wave solution

$$u = \left[1 + \exp[(X - VT)/\sqrt{2}]\right]^{-1}, \quad V = (1 - 2a)/\sqrt{2}.$$

The sine-**Gordon equation** was written in Chapter 6 (eq. (6.11)). One usually writes it for the function $u = \pi + \phi$ of the dimensionless variables $T = \omega_0 t$ and $X = \omega_0 x / v_0$:

$$u'' - \ddot{u} = \sin u.$$

Its one-soliton solution was also written in Chapter 6:

$$u = 4 \arctan \exp[\beta(X - VT)], \quad \beta = 1/\sqrt{1 - V^2}.$$

Two solitons are described by the solution

$$u = 4 \arctan[V \sinh(\beta x) / \tanh(\beta V T)].$$

The soliton-antisoliton solution is

$$u = 4 \arctan[V^{-1} \sinh(\beta V T) / \cosh(\beta x)],$$

and the breather solution is

$$u = 4 \arctan[a \sin(bT)/b \cosh(aX)], \quad a^2 + b^2 = 1.$$

Typical solitons in discrete lattices are the solitons in the nonlinear **Toda lattice.** The system of equations describing movements of "atoms" in the Toda lattice is the following:

$$\ddot{u}_n = \exp(u_{n+1} - u_n) - \exp(u_n - u_{n-1}),$$

where u_n are the discrete (dimensionless) coordinates of the atoms. The solitonic solution of the Toda system is given by the formulae

$$u_n = s_n - s_{n+1}\ , \quad s_n = \ln\{1 + \exp[2(an + T\sinh a)]\},$$

where a is an arbitrary real number. Note that the discretized KdV equation has the form

$$\dot{u}_n = \exp u_{n+1} - \exp u_{n-1},$$

while the continuum limit of the Toda equations is the Boussinesq equation

$$\ddot{u} = (u + u^2 + u'')'',$$

which sometimes is called the nonlinear string equation.

Solitons with the PENDULUM and other related physical systems

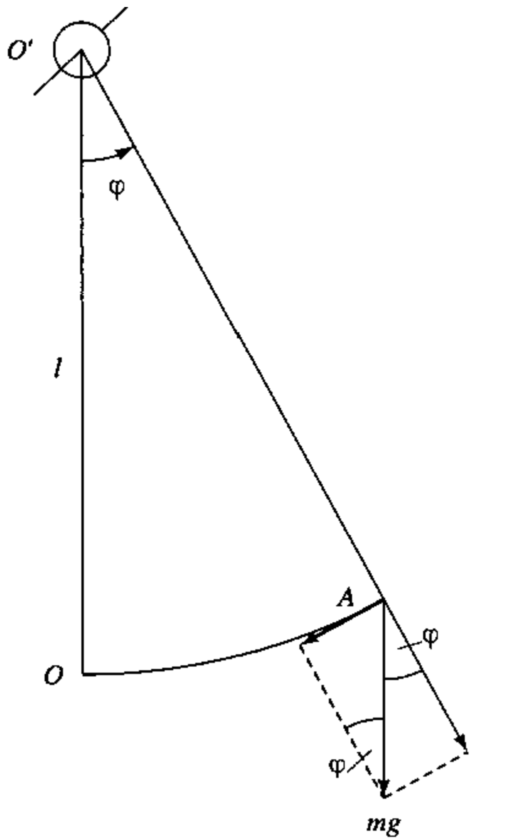
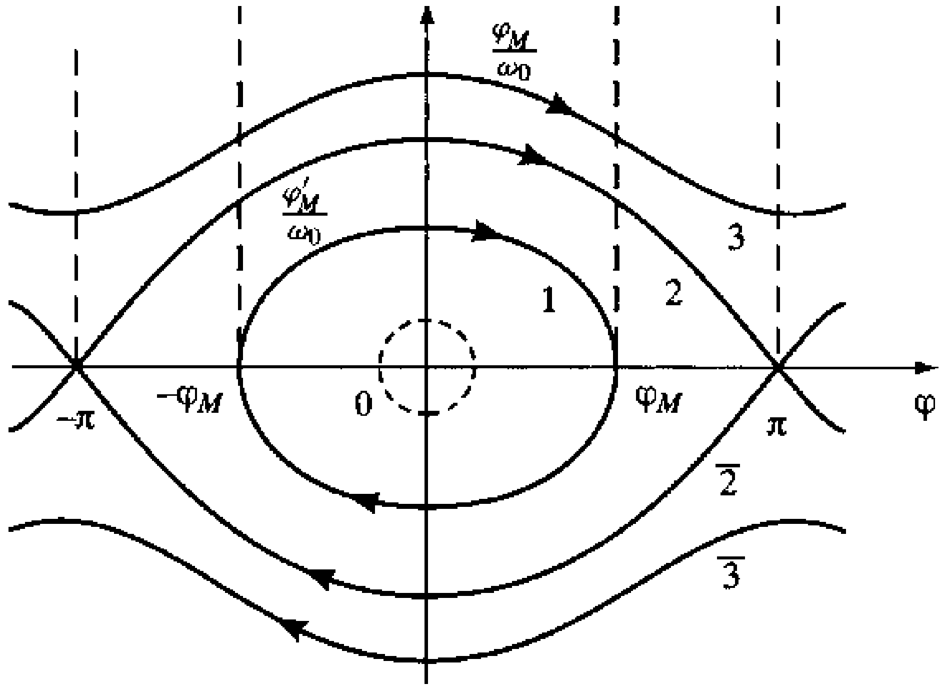


Figure 4.1 The mathematical pendulum



Phase portrait of the pendulum

The “Soliton” Solution of the Pendulum Equation

To better understand the behavior of the soliton solution, let us directly express ϕ in terms of t by using the inverse to the tangent function

$$\phi = \pi - 4\arctan(e^{-\omega_0 t}). \tag{4.9}$$

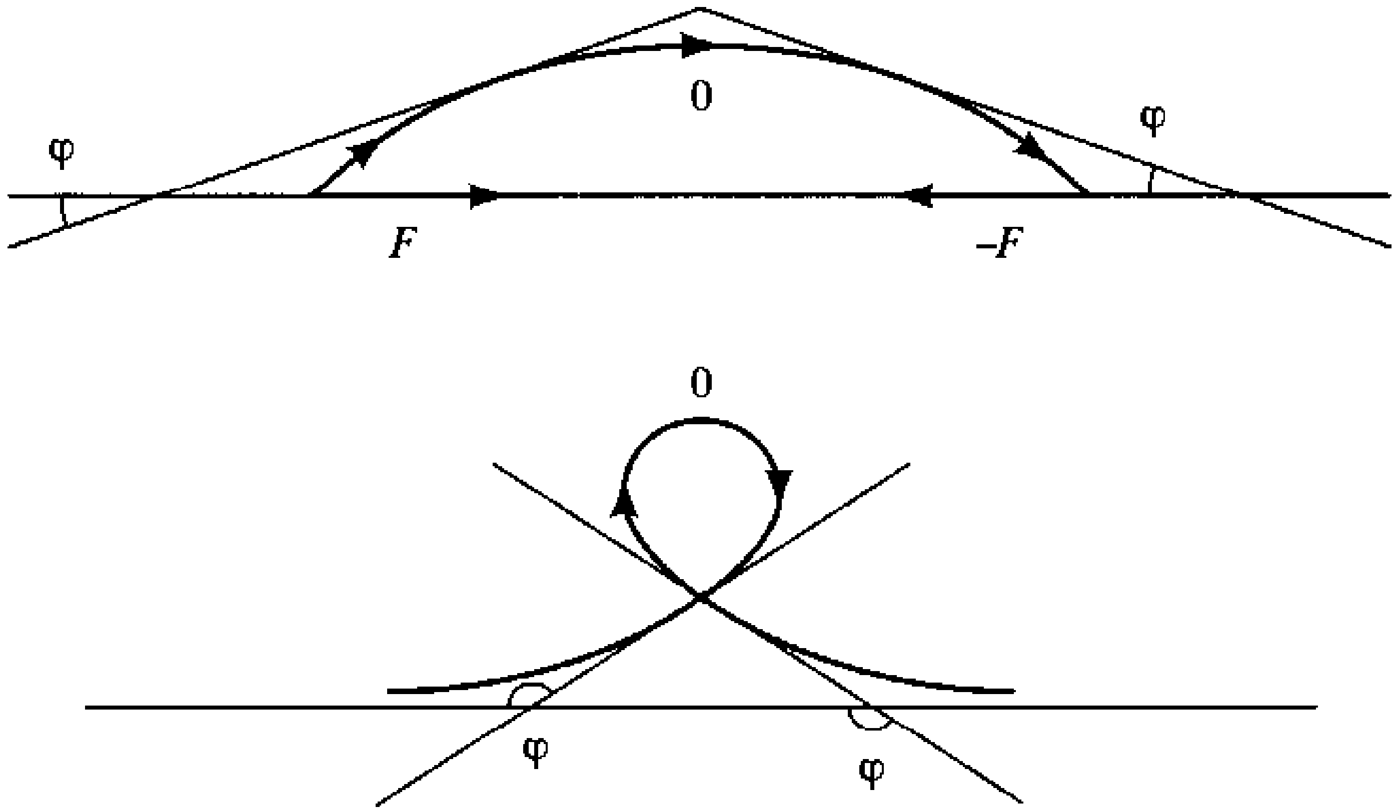


Figure 4.15 Bending shapes of a wire

Motions of the Pendulum and the "Tame" Soliton

Duality between the pendulum and wire shapes

Both are described by the same graphs

ϕ grows from $-\pi$ to $+\pi$. Its dependence on s is described by the formula (4.9) after substituting s for t. The parameter ω_0 depends on the applied force F. If the wire is infinitely long and ideally elastic, the loop can freely move along the wire. This loop is one of the simplest solitons. We may call it the "tame" soliton.

Note that all bending shapes of the ideal infinitely long wire $\phi(s)$ describe motions of the pendulum. This remarkable analogy between seemingly very different phenomena was discovered[57] by the German physicist Gustav Kirchhoff (1824–1887) and is called "Kirchhoff's analogy." In fact he discovered a much more general analogy between the states of deformed elastic bodies and some motions of rigid bodies. Unfortunately, this beautiful analogy is practically forgotten. We will say more about it when we turn to Frenkel's soliton.

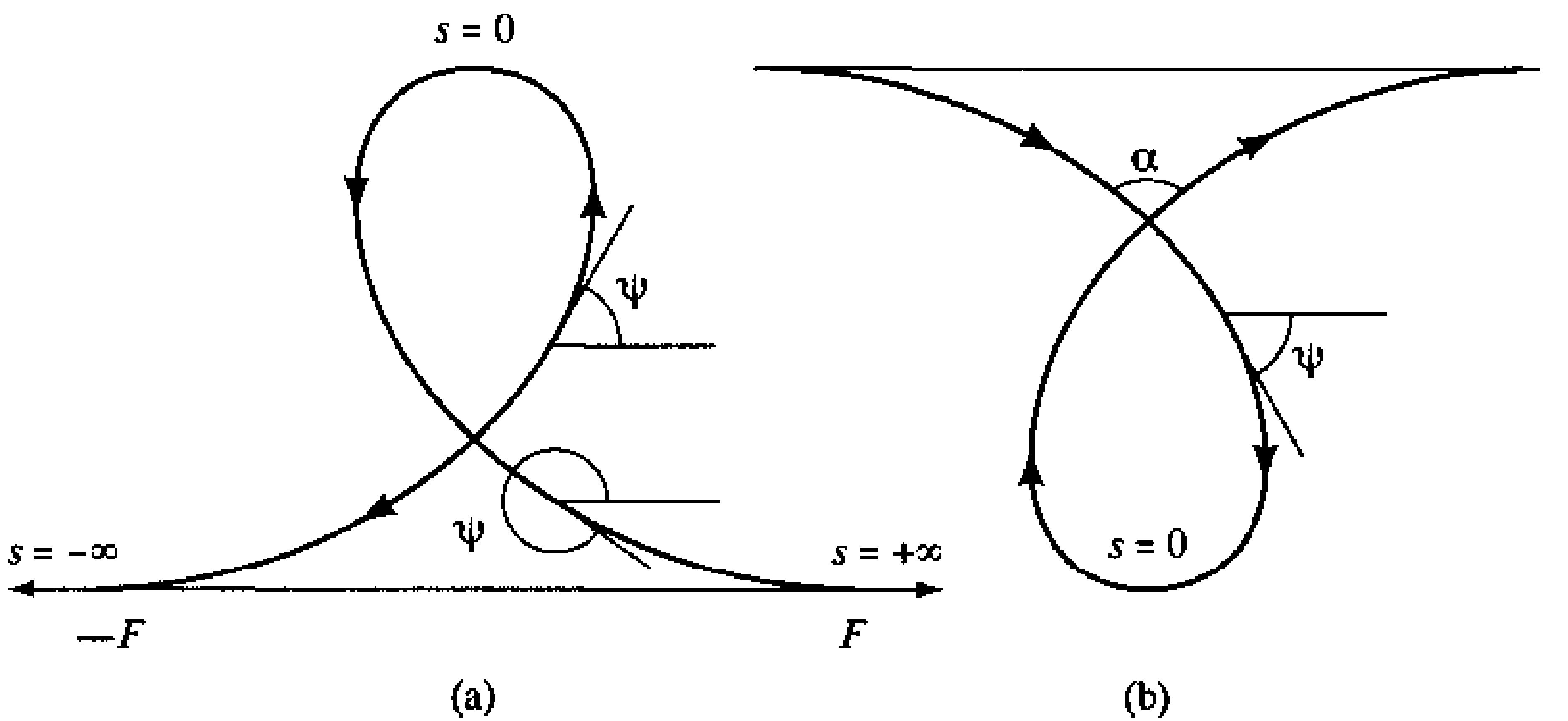
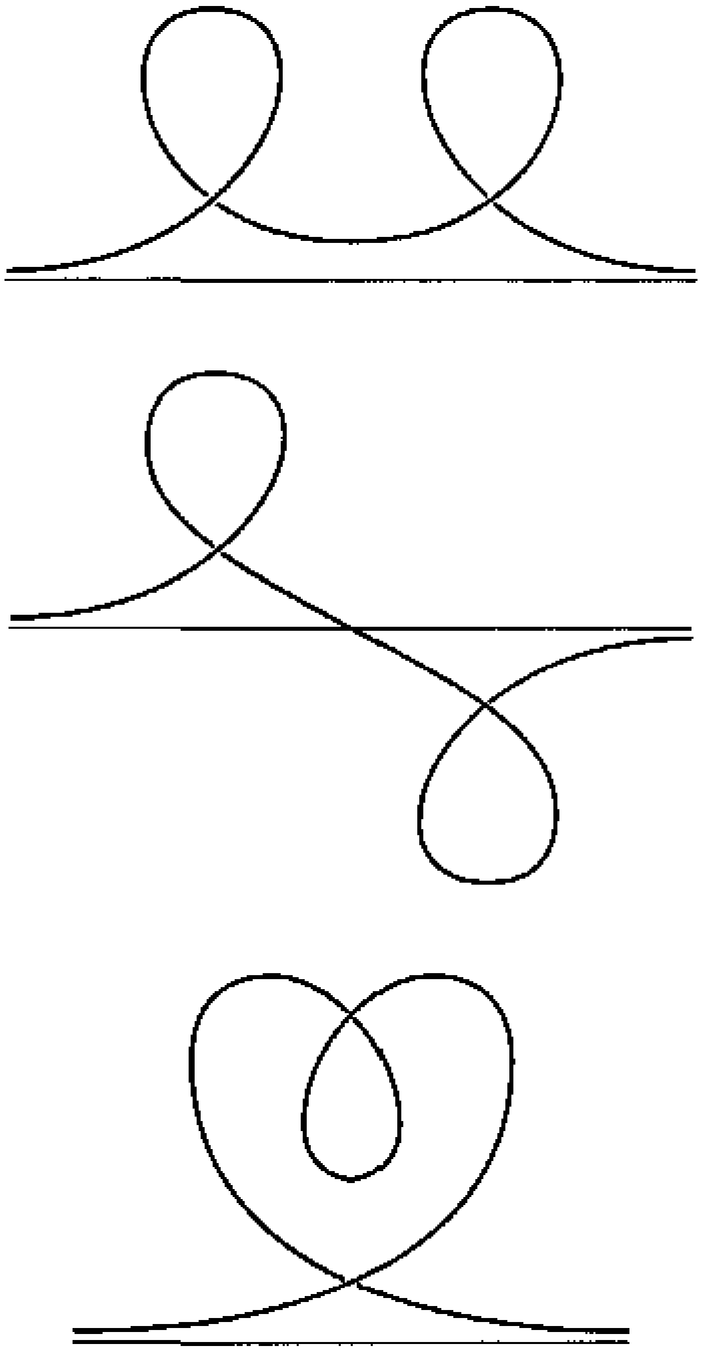


Figure 6.8 Euler's soliton (a) and antisoliton (b)



(a)

(b

(c)

Frenkel - Kontorova soliton
(“sine – Gordon” soliton)

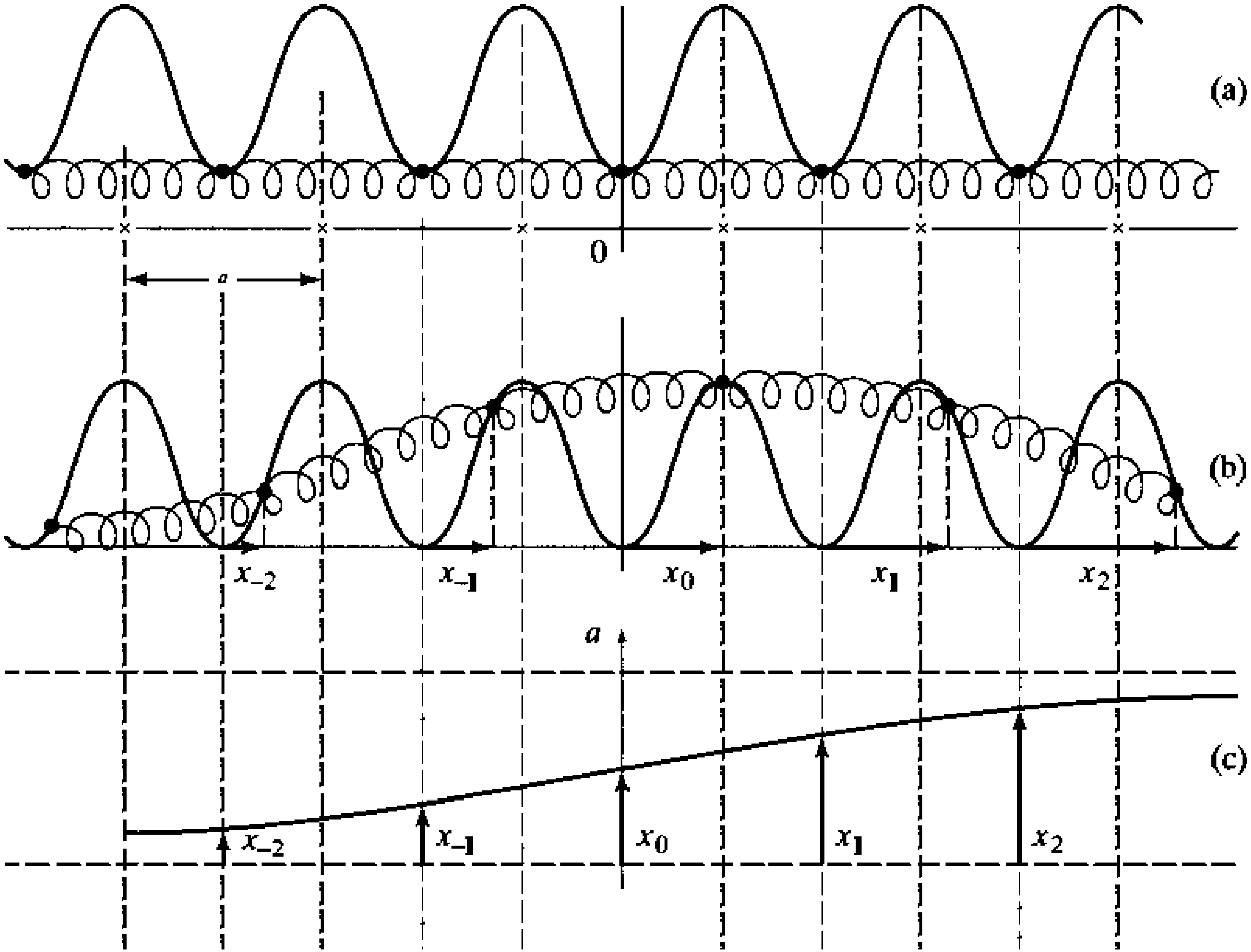
$$m\ddot{y}_n = -f_0 \sin(2\pi y_n/a) + k(y_{n+1} - 2y_n + y_{n-1}). \qquad (6.1)$$

Equation (6.1) is the main equation of the Frenkel—Kontorova model. We will now find the solution of this equation, which describes one moving dislocation.

With the notation $2\pi y_n(t)/a = \psi(t,x), \quad x = na$

in the continuum limit a —> 0, we have the nonlinear equation

$$\ddot{\psi} = v_0^2 \psi'' - \omega_0^2 \sin\psi$$



A model of dislocation

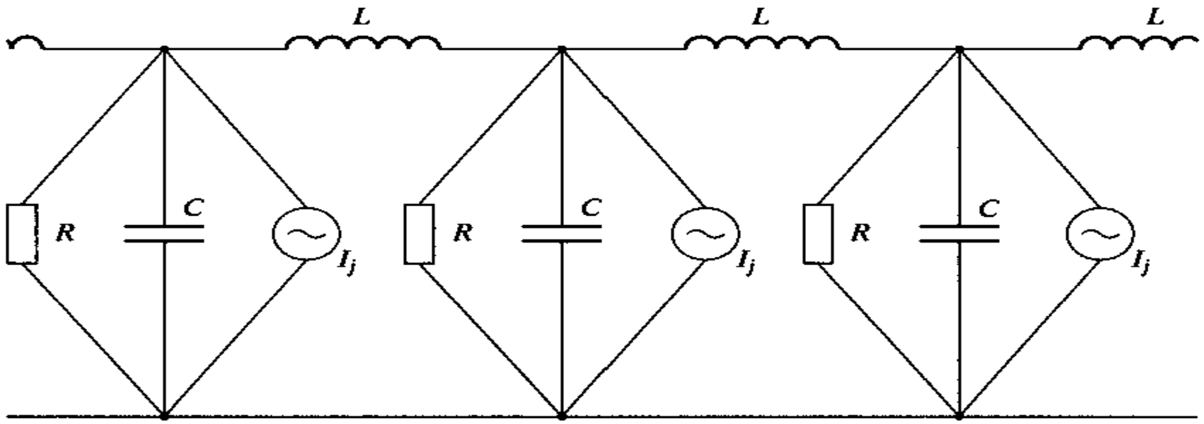


Figure 8.4 Electric scheme of a chain of the Josephson junctions

$$I = I_c \sin\phi, \qquad V = (\Phi_0/2\pi c)\dot{\phi}, \qquad \ddot{\phi} + (RC)^{-1}\dot{\phi} + j_c \sin\phi = j,$$

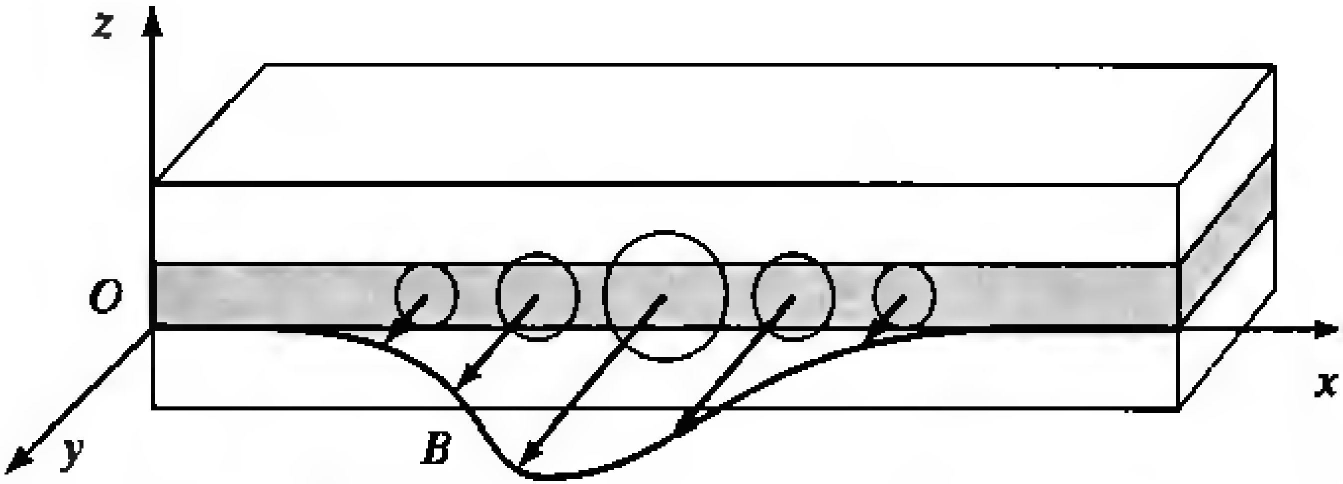
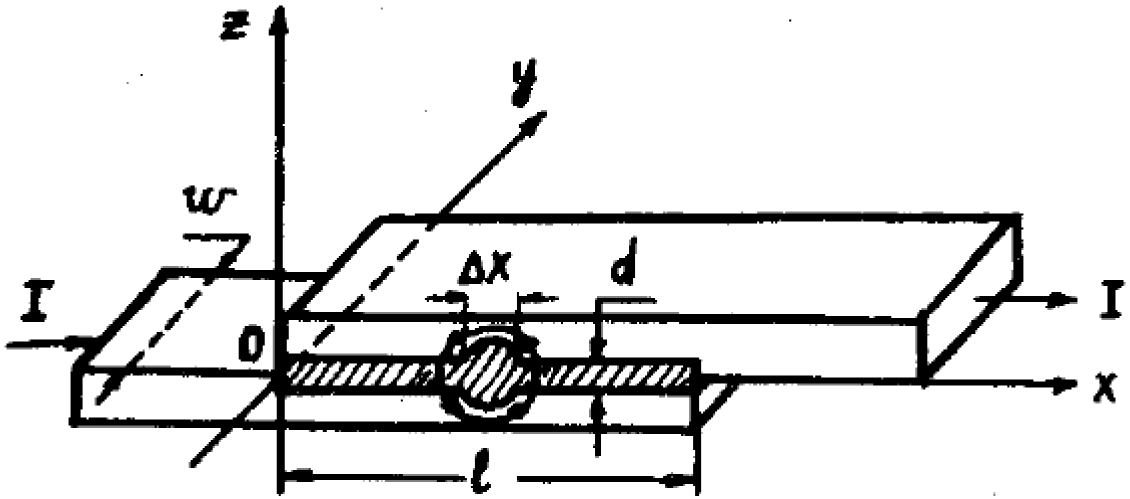


Figure 8.5 Soliton in a long Josephson junction

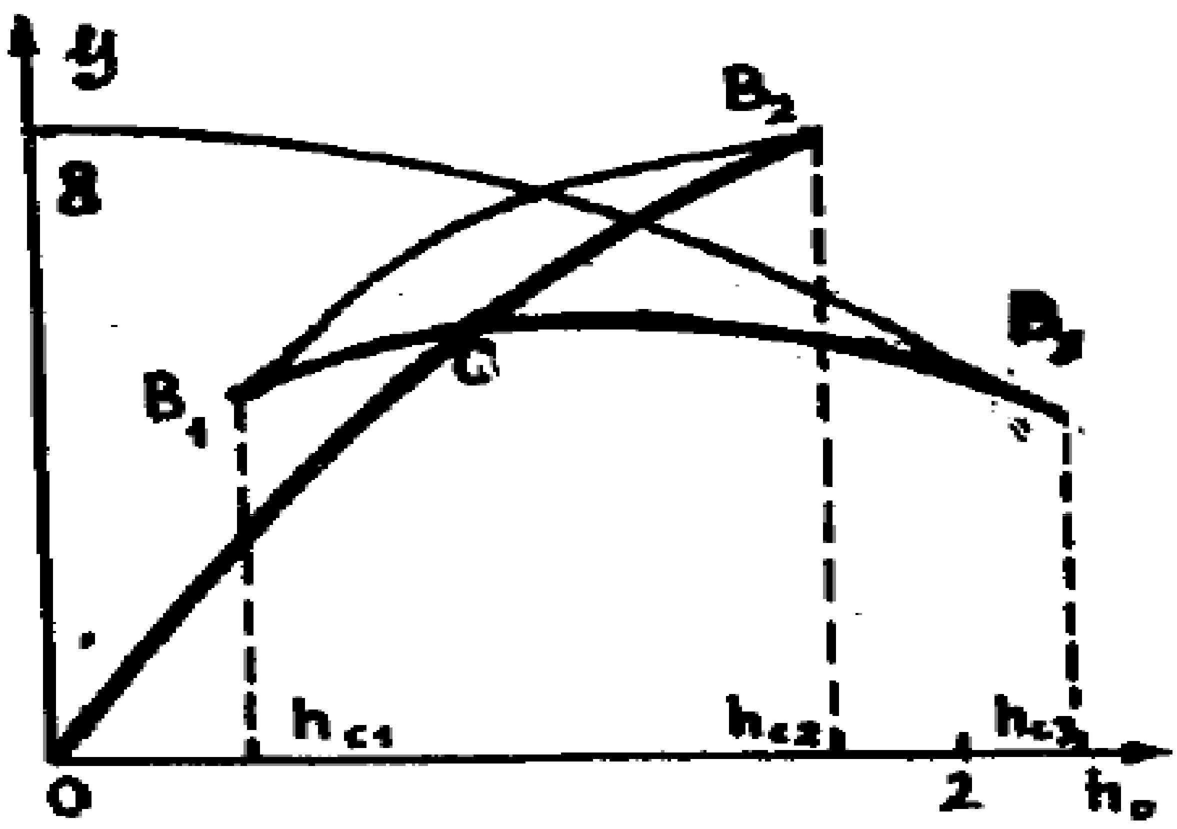


$$L\,(L^{-1}\,\phi_x)_x - LC\phi_{tt} - \lambda_J^{-2}(x)\sin\phi(x,t) = \bar{\alpha}\phi_t$$

$$\phi = 2\pi\Phi(x,t)\,/\,\Phi_0$$

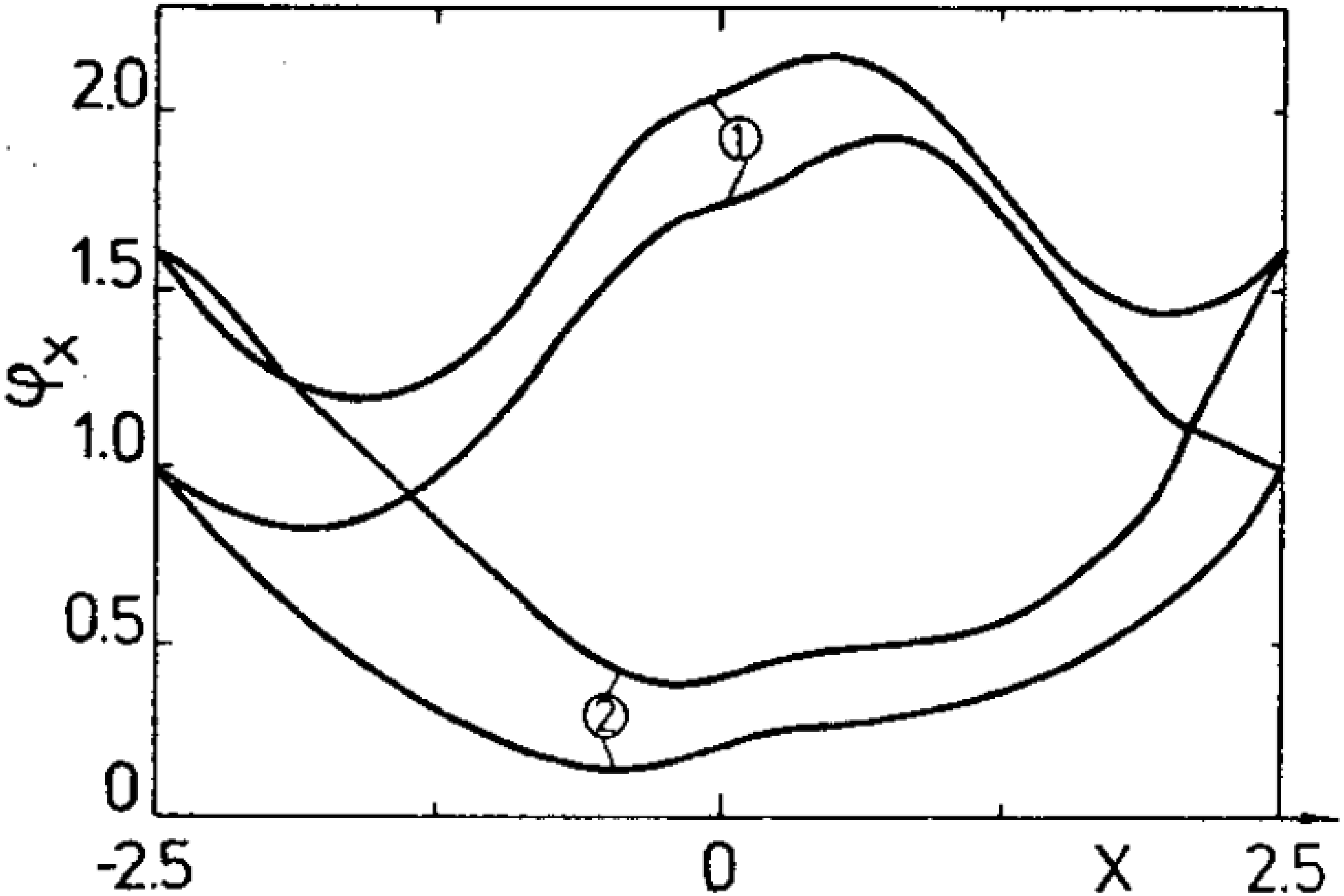
$$\phi_{xx} - \phi_{tt} - [1 - \sum_{i=1}^{n}\mu_i\,\delta(x - x_i)]\sin\phi(x,t) = \alpha\phi_t$$

Yu.Galpern, ATF, Soliton bound states in long Josphson junctions, 1983, JETP



$$\mathcal{G} = \mathcal{E} + h_0 \phi_0 - h_\ell \phi_\ell .$$

$$\mathcal{E} = \int_0^\ell dx \left\{ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + \left[1 - \Sigma\, \mu_i\, \delta(x - x_i)\right] (1 - \cos\phi) \right\},$$

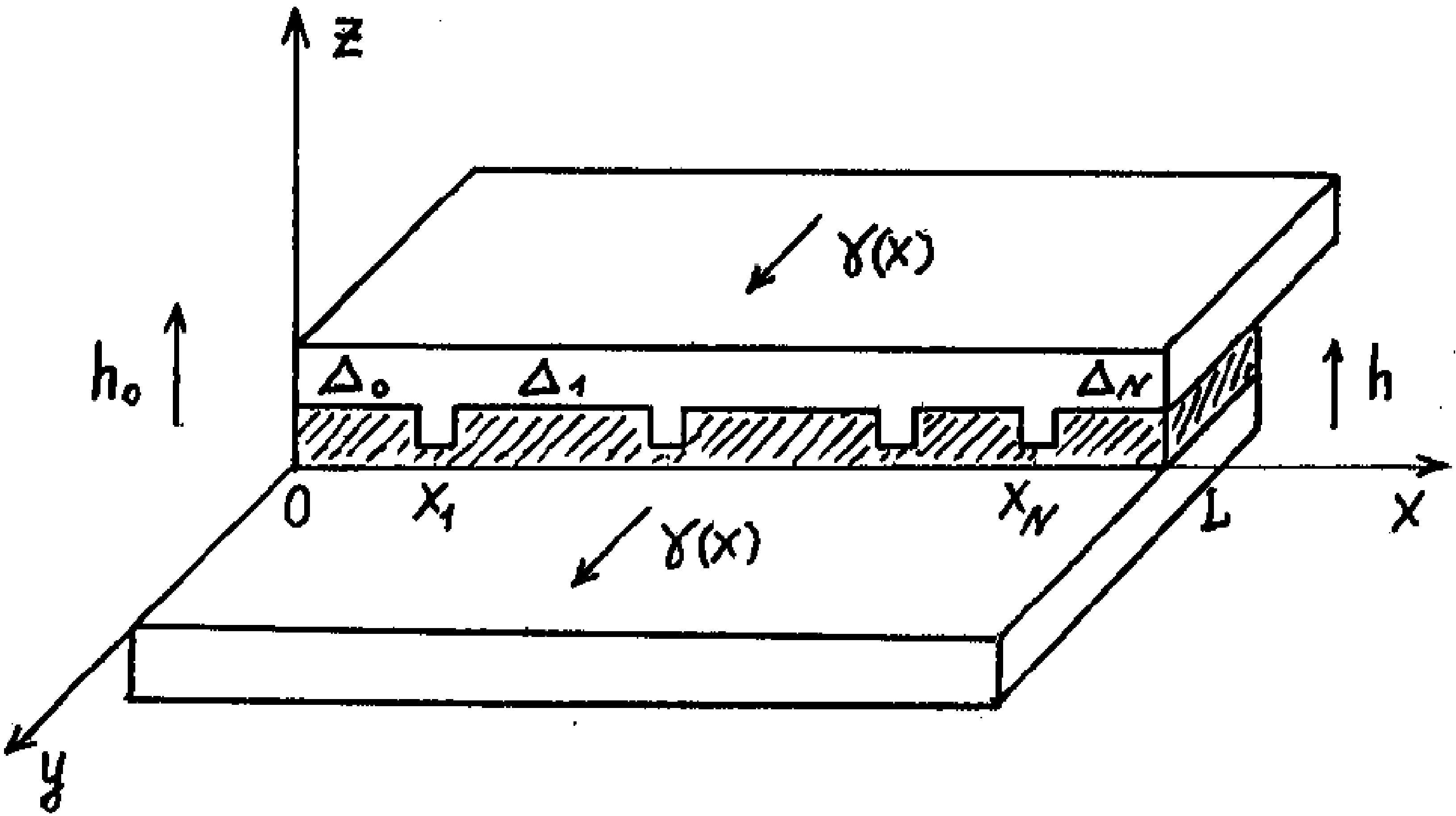


. The distribution of φ_x in the junction of length $2\ell = 5$ with $\mu = 1$, i.e. $j_{sm} = \tanh^2(2x)$. 1. Soliton localized on inhomogeneity. 2. The state with a maximum value of $\gamma_c(0)$.

ATF, Galpern, Boyadjiev, Puzynin. 1986

$$\phi'' = \sum \mu_n \delta(x - x_n) \sin \phi(x) + \gamma(x),$$

$$\phi'(0) = h_0, \ \phi'(l) = h.$$



The Josephson map and Chaos

A.T.Filippov and Yu.S.Gal'pern, Phys. Lett. A **172** (1993),471

magnetic flux distribution $\phi(x)$ in a homogeneous lattice may be described by the equation

$$\frac{d^2}{dx^2}\phi(x) = \sum_n \mu\delta(x - x_n)\sin\phi(x) + \gamma. \quad (1)$$

Then, the flux distribution at the $n-th$ interval $x_n < x < x_{n+1}$ is expressed as

$$\phi_n(x) = a_n + b_n(x - x_n) + \frac{1}{2}\gamma(x - x_n)^2, \quad (2)$$

Assuming the lattice to be periodic $(x_{n+1} - x_n = \Delta)$, we can determine the coefficients a_n and b_n by solving the equation

$$\begin{aligned} a_n &= a_{n-1} + \bar{b}_{n-1} + \frac{1}{2}\bar{\gamma}, \\ \bar{b}_n &= \bar{b}_{n-1} + \bar{\mu}\sin a_n + \bar{\gamma} \end{aligned} \quad (3)$$

A detailed study of JM

$$X_{n+1} = X_n + P_{n+1}$$

$$P_{n+1} = P_n - \frac{K}{2\pi}\sin(2\pi X_n) + \Gamma$$

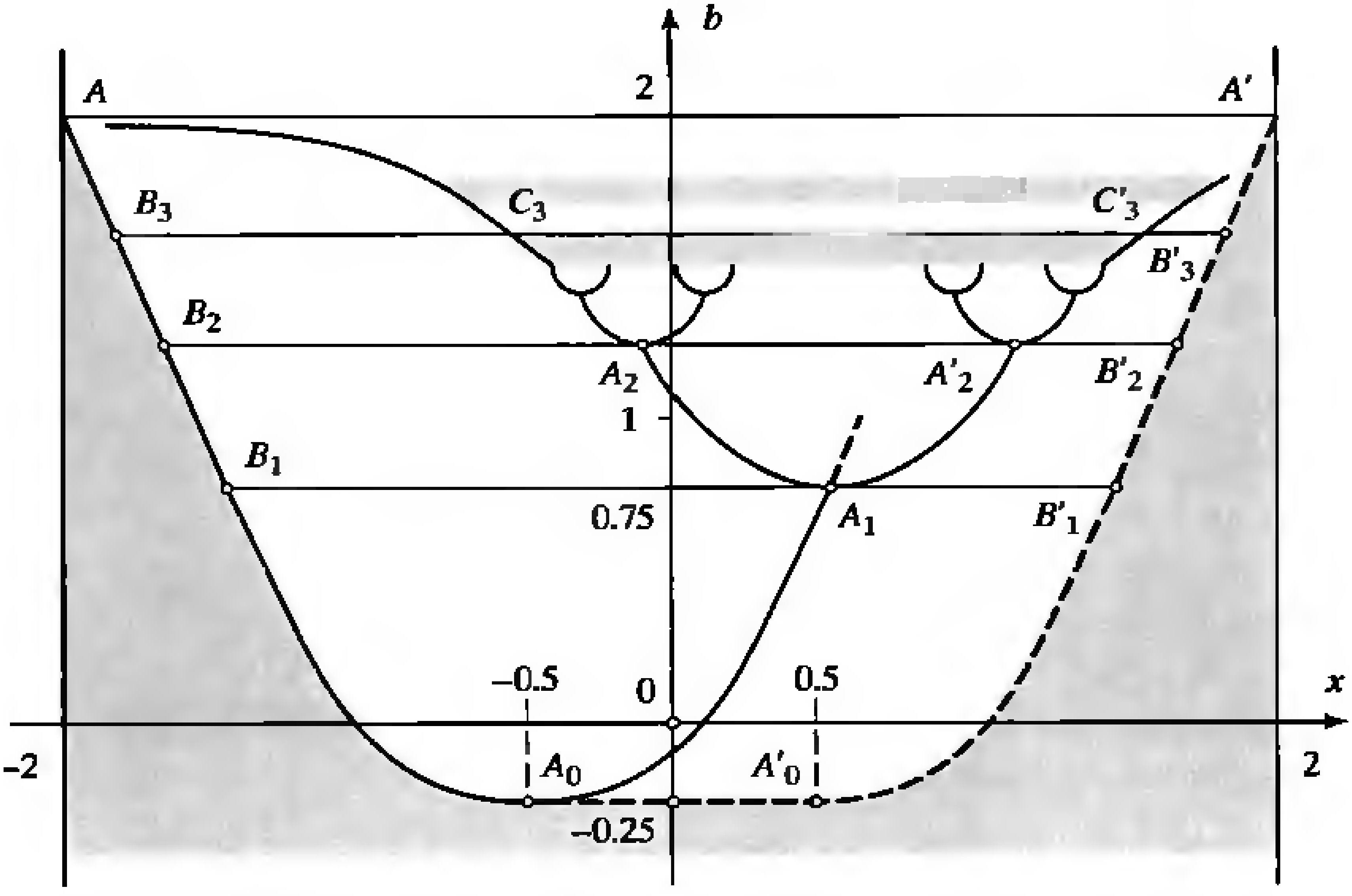
Y. Nomura, Y.H. Ichikawa and A.T. Filippov

Stochasticity in the Josephson Map

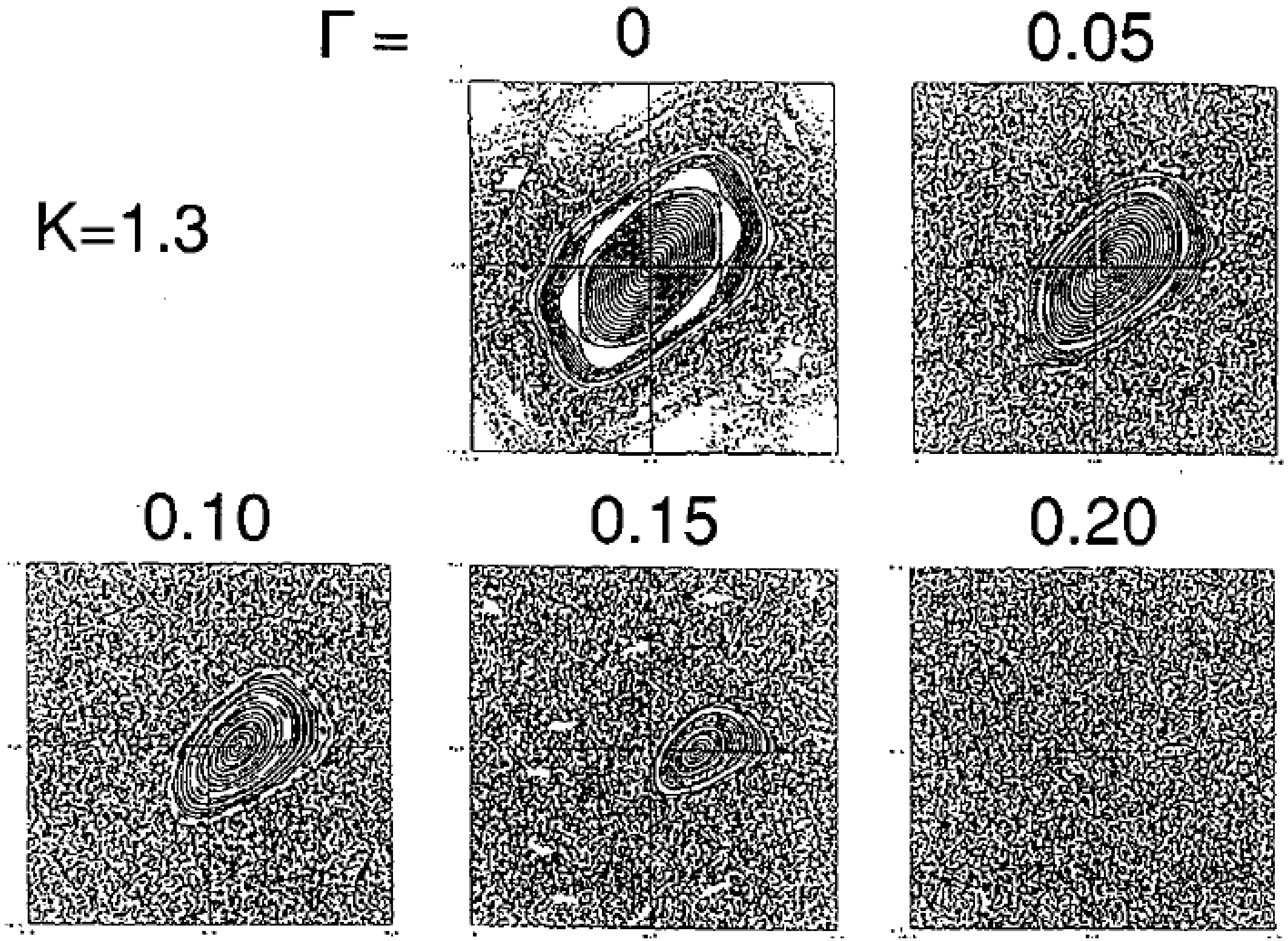
(Received - Apr. 10, 1996)

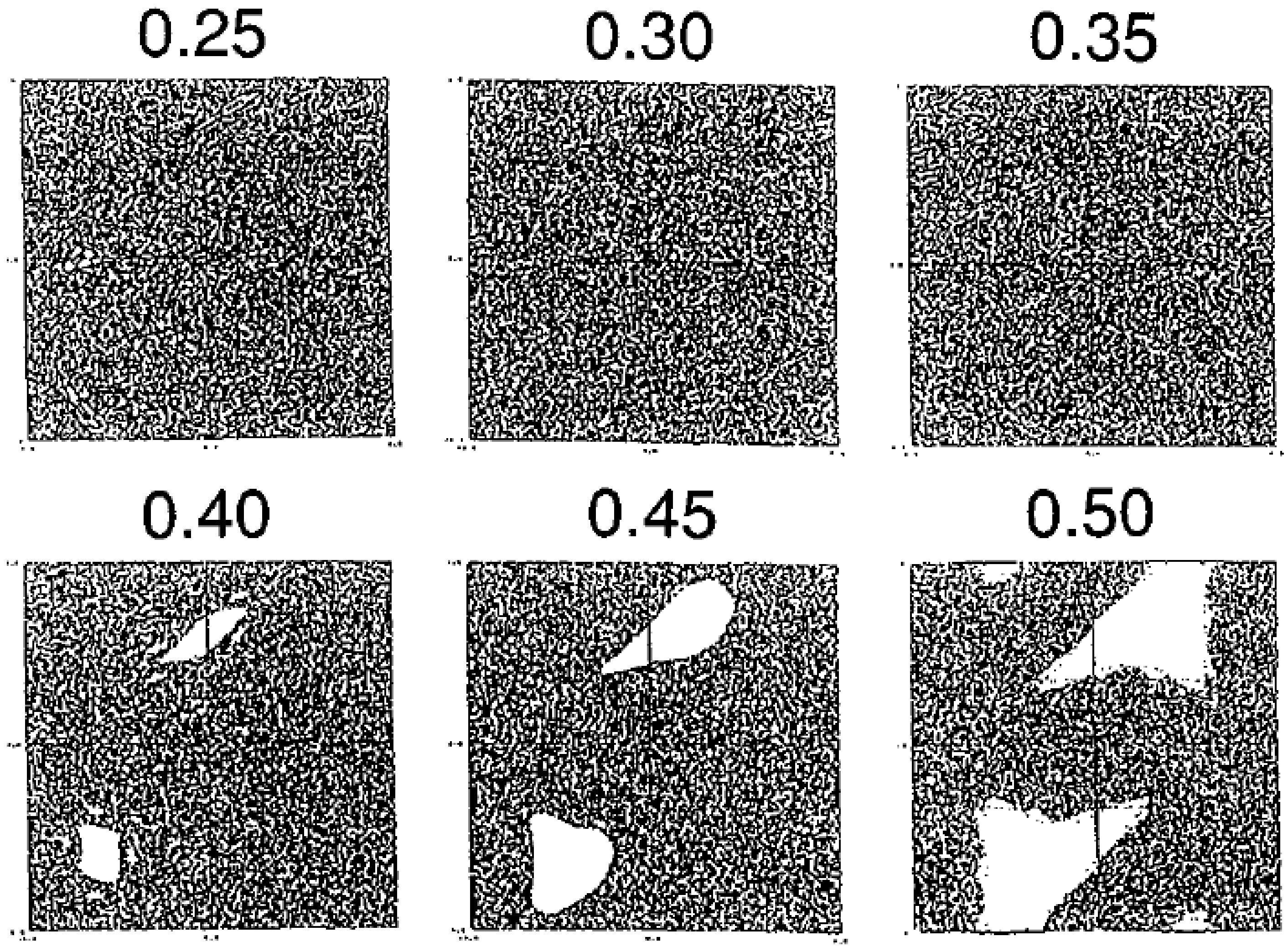
NIFS-413 Apr. 1996

A simple model of Chaos $$x_{n+1} = b - x_n^2$$



Phase space portrait of the Josephson map for the stochastic parameters ; a) $K = 1.3$, b) $K = 2.1$ and c) $K = 3.3$. The number attached to each frame stands for the value the bias Γ.





In the present studies, we have shown that the external uniform bias imposed on the system described by the standard map gives rise to rich manifestation of the nonlinear behavior, with the sensitive dependence on the control parameters.

Vortices in water, superflud liquids, in Universe…

From MACRO to MICRO and back

A relation between solutions of Einstein's equations and equations of Navier – Stokes, etc…

Unity of UNIVERSE and theories of EVERYTHING

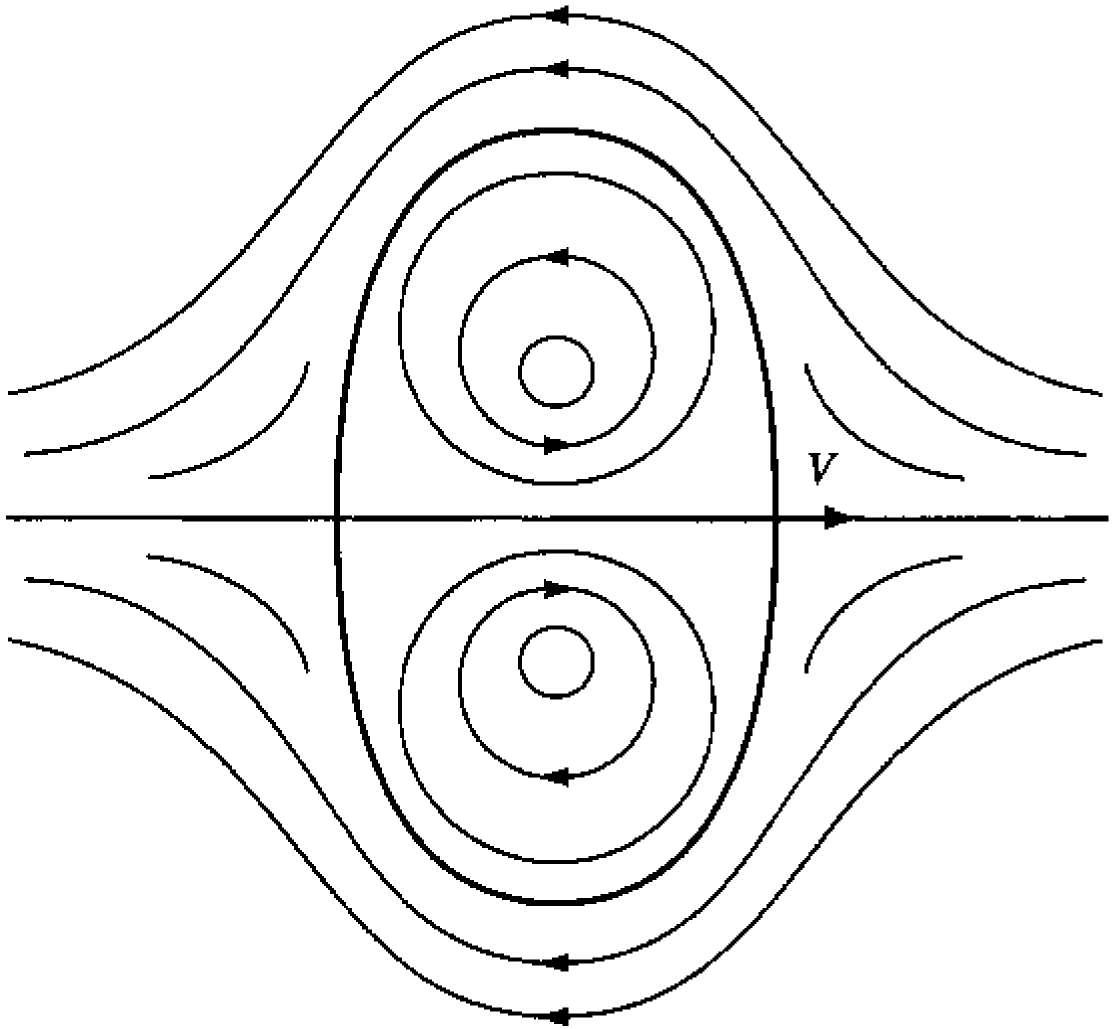


Figure 3.4 Kelvin's oval

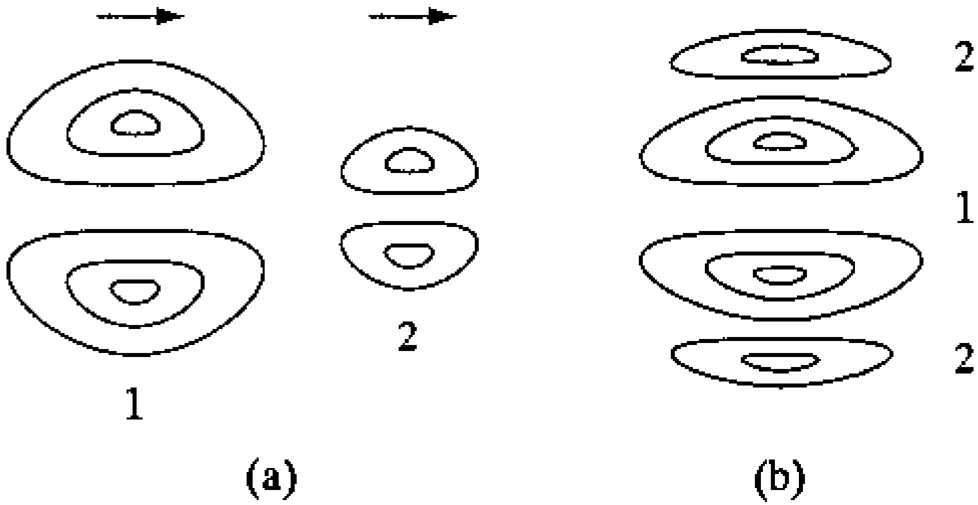


Figure 7.14 Collision of two vortex pairs according to McWilliams and Zabusky

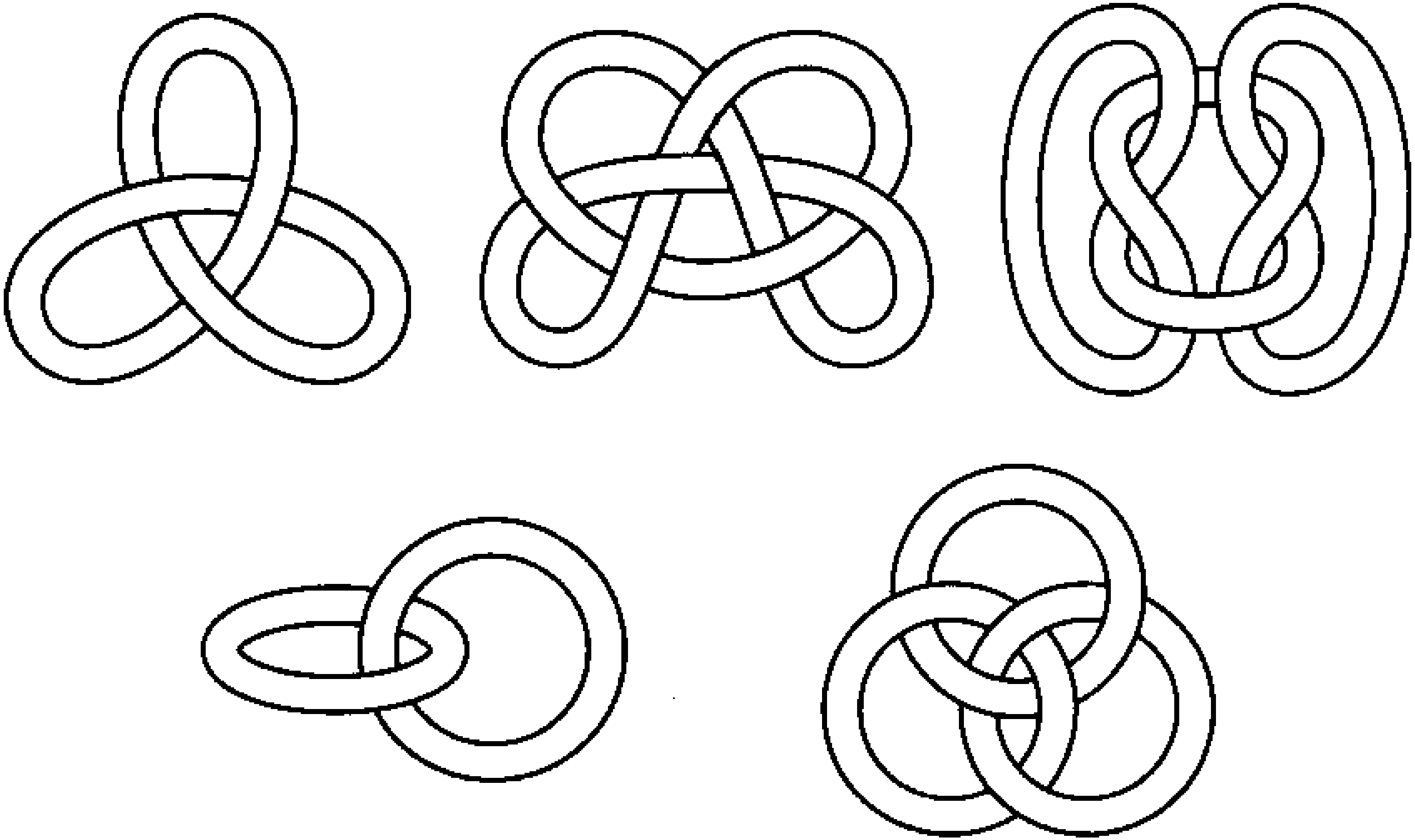


Figure 3.7 Kelvin's idea of vortex atoms. Differently knotted vortex rings represent different atoms

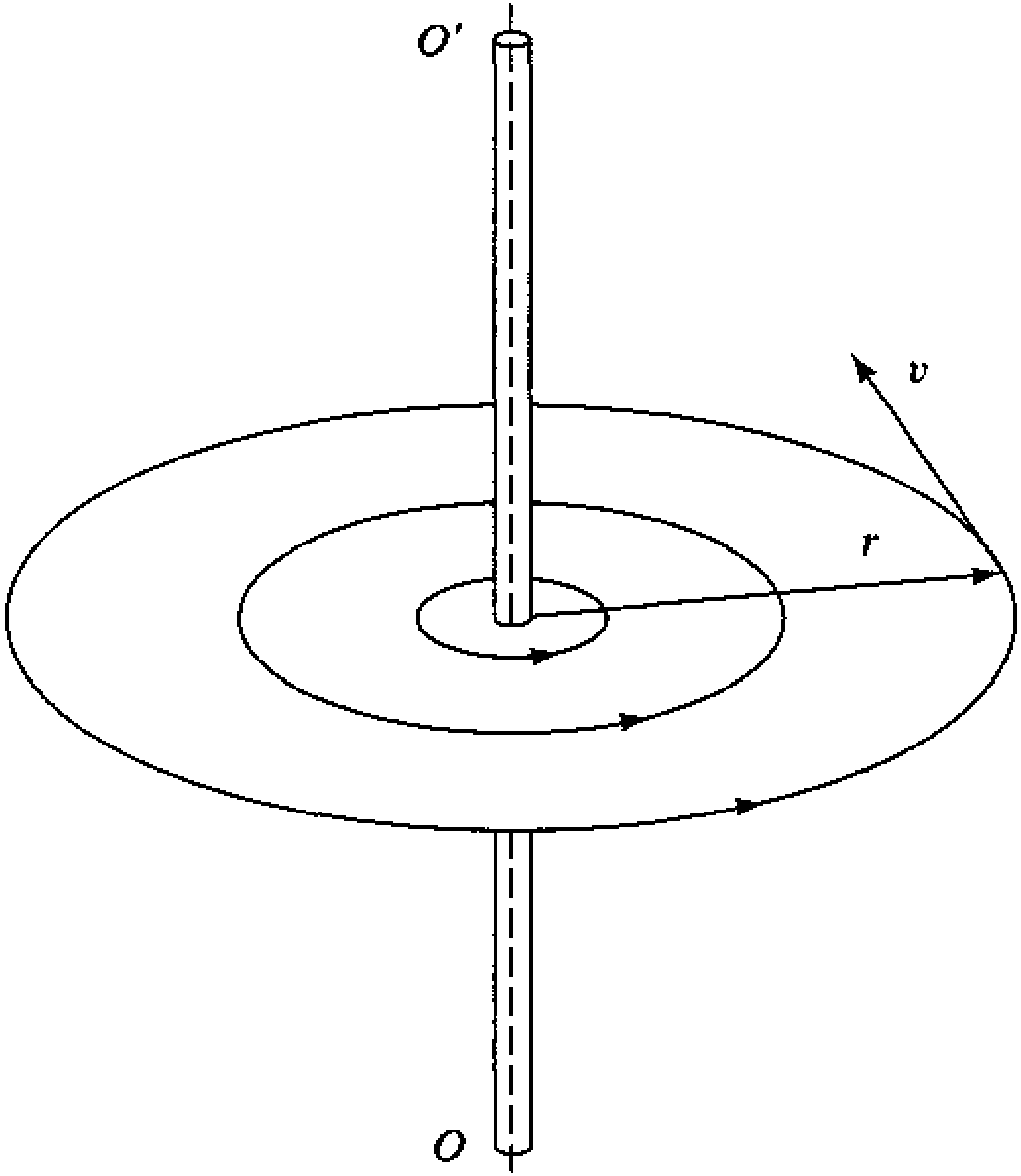


Figure 8.1 Superfluid vortex and its flow environment

The Universe in a Helium Droplet

GRIGORY E. VOLOVIK

Low Temperature Laboratory,
Helsinki University of Technology
and
Landau Institute for Theoretical Physics, Moscow

CONTENTS

Reissner - Nordstroem black hole

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega \ ,$$

$$r_+ r_- = \frac{Q^2}{G^2} \ , \ r_+ + r_- = \frac{2\mathcal{M}}{G} \ .$$

In the laboratory frame the dynamics of quasiparticles, propagating in this velocity field, is given by the line element provided by the effective metric in eqn (5.2):

$$ds^2 = -\left(1 - \frac{v_s^2(r)}{c^2}\right)dt^2 - 2\frac{v_s(r)}{c^2}drdt + \frac{1}{c^2}(dr^2 + r^2 d\Omega^2) \, .$$

If the 'superflow' is inward, and the velocity profile is $v_s(r) = -c(r_h/r)^{1/2}$, this equation corresponds to the line element for the black hole obtained by Painlevé (1921) and Gullstrand (1922). Among the other metrics used for the black hole

PAINLEVÉ–GULLSTRAND METRIC IN SUPERFLUIDS

$$ds^2 = -\,dt^2\,(c^2 - v^2)\; - \;2\,v\,dr\,dt\; + dr^2 + \; r^2 d\Omega^2$$

g_{00} g_{0r}

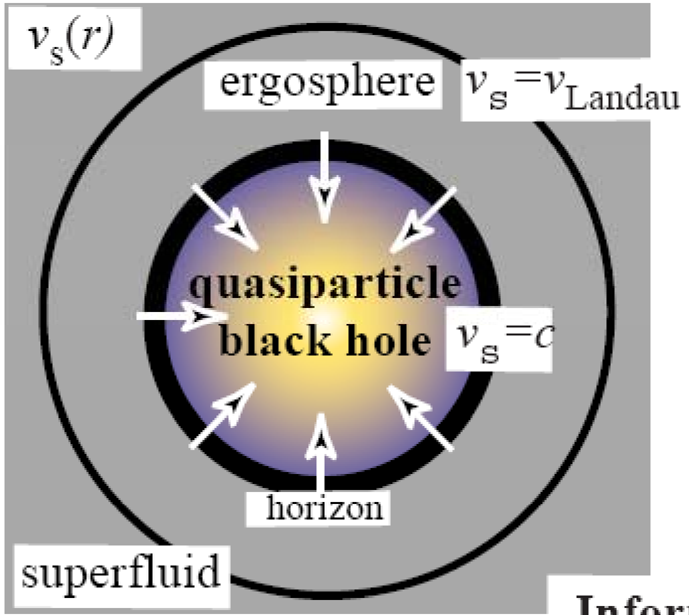
effective metric in superfluids

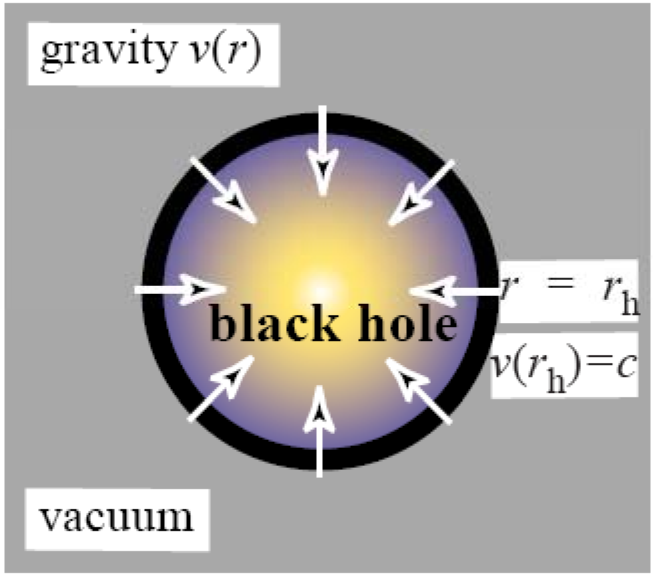
$v(r) = v_s(r)$

Painlevé-Gullstrand metric of black hole

$$v^2(r) = \frac{2GM}{r} = c^2\,\frac{r_h}{r}$$

Kinetic energy of superflow = potential of gravitational field





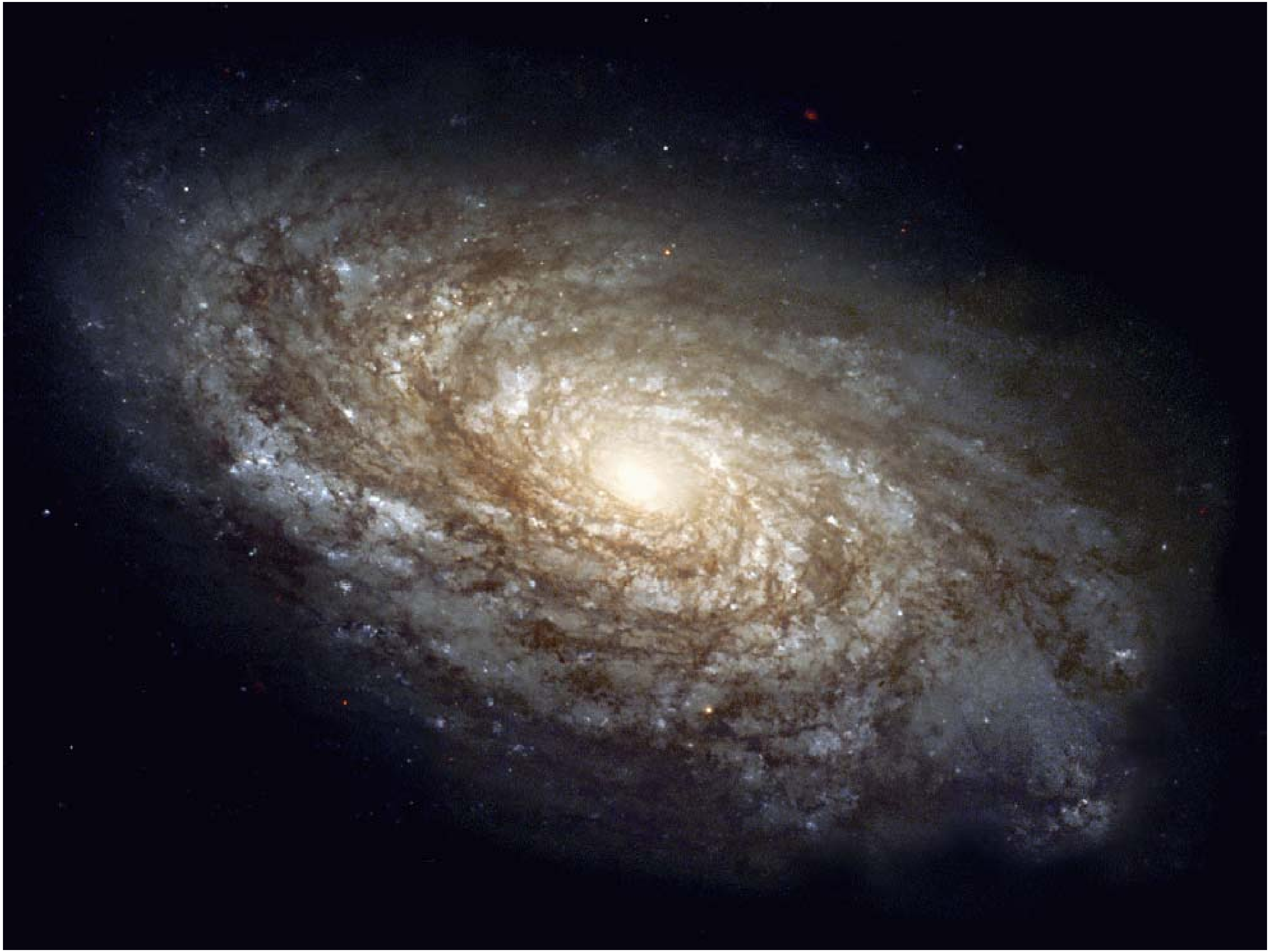
horizon at $v_s(r_h) = c$
c – maximum attainable speed of quasiparticles

Information from interior region cannot be transferred by quasiparticles / matter

horizon at $g_{00} = 0$
or $v(r_h) = c$

Superfluids can also simulate the rotating black hole. An example is shown in Fig. 32.8. The types of the condensed matter black holes, ergoregions and surfaces of the infinite red shift can now be classified in terms of the symmetry of the superfluid velocity field $\mathbf{v}_s$. There are three important elements of discrete symmetries which form the group $Z_2 \times Z_2$. One of them is time reversal symme-

FIG. 32.8. Whirlpool simulating the rotating black hole. The radial velocity of the flow is directed toward the center of the black hole.



A relation between solutions of Einstein's and Navier – Stokes equations.

Black Holes and viscous incompressible hydrodynamics.

Conformal hydrodynamics and all that…

Fluid dynamics of R-charged black holes

Johanna Erdmenger[1,*], Michael Haack[2,†], Matthias Kaminski[1,‡], Amos Yarom[2,§]

[1]Max Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany

[2]Arnold Sommerfeld Center for Theoretical Physics, Ludwig-Maximilians-Universität, Department für Physik, Theresienstrasse 37, 80333 München, Germany

We construct electrically charged AdS_5 black hole solutions whose charge, mass and boost-parameters vary slowly with the space-time coordinates. From the perspective of the dual theory, these are equivalent to hydrodynamic configurations with varying chemical potential, temperature and velocity fields. We compute the boundary theory transport coefficients associated with a derivative expansion of the energy momentum tensor and R-charge current up to second order. In particular, for the current we find a first order transport coefficient associated with the vorticity of the fluid.

August 2008

The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity

Sayantani Bhattacharyya[a,*], **Shiraz Minwalla** [a,†] **and Spenta R. Wadia**[a,‡]

[a]Department of Theoretical Physics,Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400005, India

Abstract: We note that the equations of relativistic hydrodynamics reduce to the incompressible Navier-Stokes equations in a particular scaling limit. In this limit boundary metric fluctuations of the underlying relativistic system turn into a forcing function identical to the action of a background electromagnetic field on the effectively charged fluid. We demonstrate that special conformal symmetries of the parent relativistic theory descend to 'accelerated boost' symmetries of the Navier-Stokes equations, uncovering a conformal symmetry structure of these equations. Applying our scaling limit to holographically induced fluid dynamics, we find gravity dual descriptions of an arbitrary solution of the forced non-relativistic incompressible Navier-Stokes equations. In the holographic context we also find a simple forced steady state shear solution to the Navier-Stokes equations, and demonstrate that this solution turns unstable at high enough Reynolds numbers, indicating a possible eventual transition to turbulence.

TIFR/TH/08-40

FROM NAVIER-STOKES TO EINSTEIN

Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov and Andrew Strominger

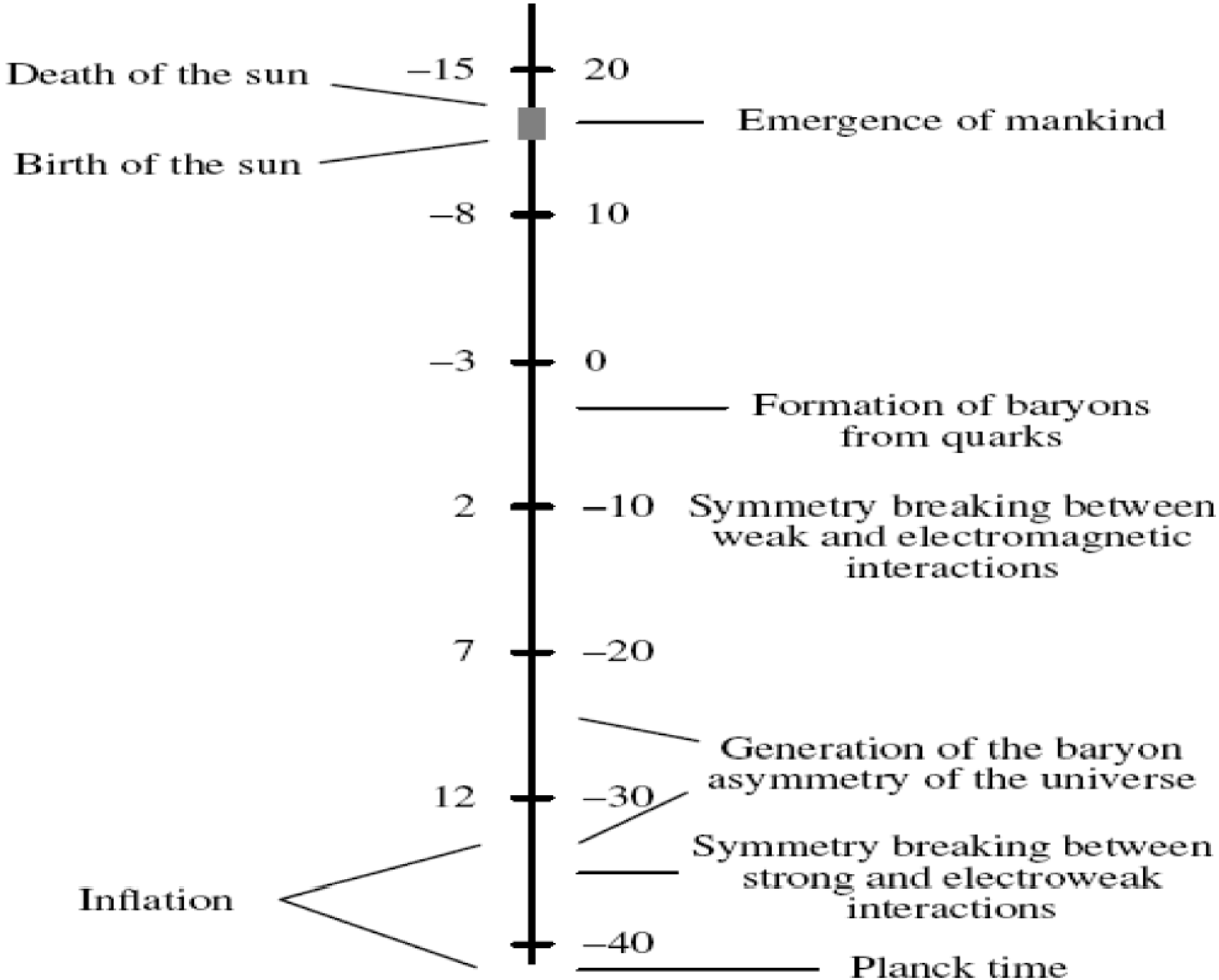
Center for the Fundamental Laws of Nature, Harvard University
Cambridge, MA, 02138

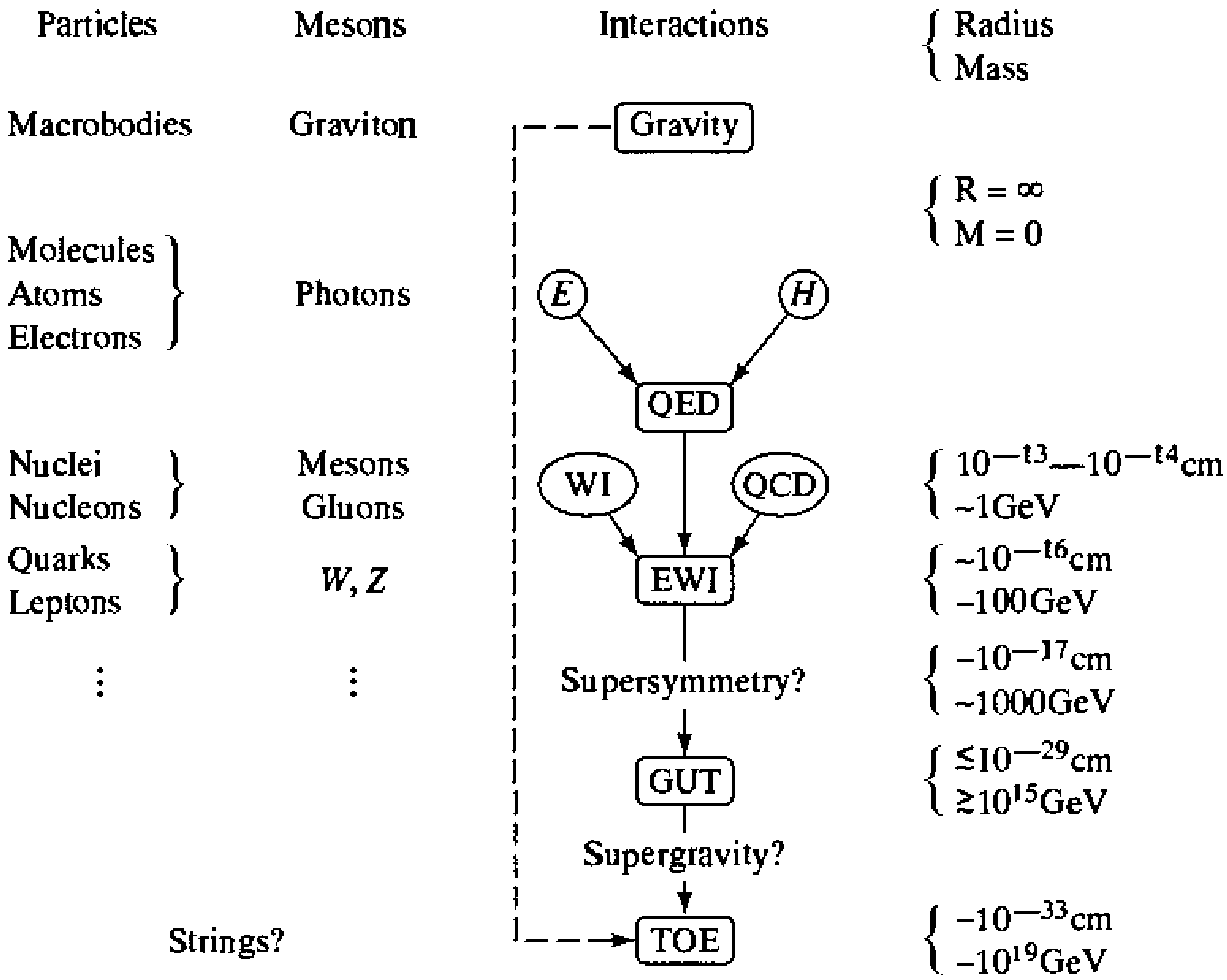
We show by explicit construction that for every solution of the incompressible Navier-Stokes equation in $p+1$ dimensions, there is a uniquely associated "dual" solution of the vacuum Einstein equations in $p+2$ dimensions. The dual geometry has an intrinsically flat timelike boundary segment Σ_c whose extrinsic curvature is given by the stress tensor of the Navier-Stokes fluid. We consider a "near-horizon" limit in which Σ_c becomes highly accelerated. The near-horizon expansion in gravity is shown to be mathematically equivalent to the hydrodynamic expansion in fluid dynamics, and the Einstein equation reduces to the incompressible Navier-Stokes equation. For $p = 2$, we show that the full dual geometry is algebraically special Petrov type II. The construction is a mathematically precise realization of suggestions of a holographic duality relating fluids and horizons which began with the membrane paradigm in the 70's and resurfaced recently in studies of the AdS/CFT correspondence.

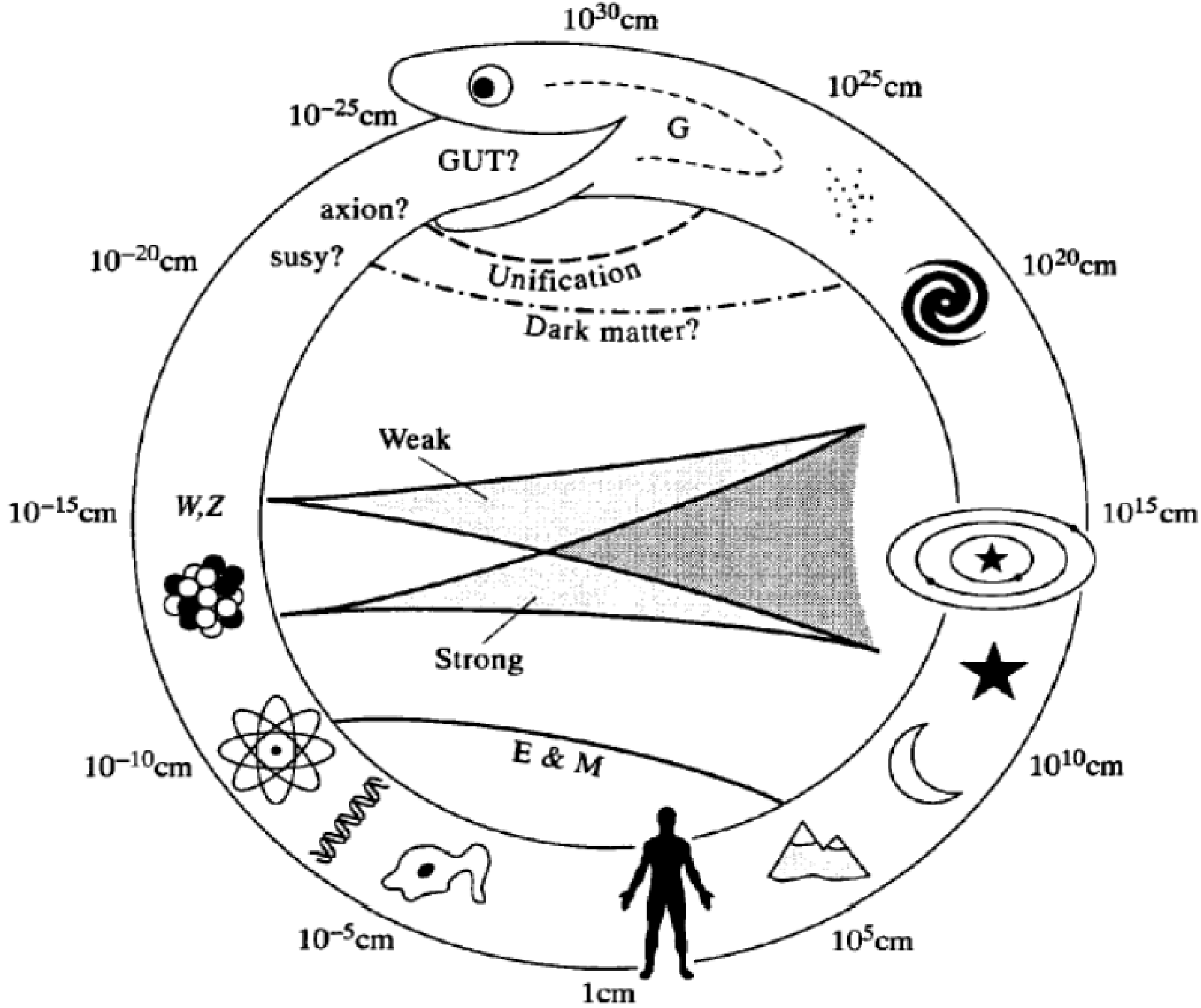
arXiv:1101.2451

Unity of UNIVERSE and theories of EVERYTHING

What is Life from the cosmological point of view?


Death of the sun
Birth of the sun
Emergence of mankind
–15
20
–8
10
–3
0
Formation of baryons from quarks
2
–10
Symmetry breaking between weak and electromagnetic interactions
7
–20
Generation of the baryon asymmetry of the universe
12
–30
Symmetry breaking between strong and electroweak interactions
Inflation
–40
Planck time


Particles
Mesons
Interactions
Radius
Mass
Macrobodies
Graviton
Gravity
R = ∞
M = 0
Molecules
Atoms
Electrons
Photons
E
H
QED
Nuclei
Nucleons
Mesons
Gluons
WI
QCD
10−13—10−14cm
~1GeV
Quarks
Leptons
W, Z
EWI
~10−16cm
~100GeV
Supersymmetry?
~10−17cm
~1000GeV
GUT
≲10−29cm
≳10¹⁵GeV
Supergravity?
Strings?
TOE
~10−33cm
~10¹⁹GeV



10^{30}cm
10^{25}cm
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G
GUT?
axion?
susy?
10^{-20}cm
Unification
Dark matter?
10^{20}cm
Weak
10^{-15}cm
W,Z
10^{15}cm
Strong
E & M
10^{-10}cm
10^{10}cm
10^{-5}cm
10^{5}cm
1cm

a temporal

END