Boundary conditions at spatial infinity for fields in Casimir calculations

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I. Introduction

The starting point of all the Casimir calculations is the spectral sum (or the spectral integral):

$$E_0 = -\frac{1}{2}\sum_n \omega_n$$

Both bounded and unbounded configuration manifolds are considered.

$\{\omega_n\}$

Differential equation

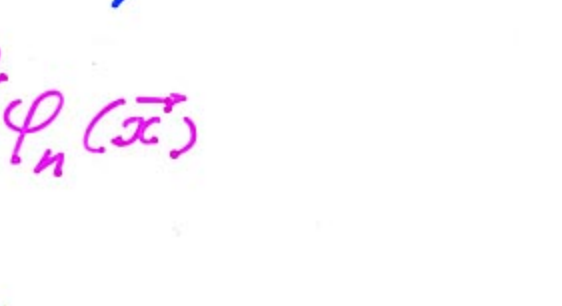
$$\left(-c^2(\vec{x})\,\Delta + \ldots\right)\varphi_n(\vec{x}) = \omega_n^2\,\varphi_n(\vec{x})$$

$\oplus$

Boundary conditions

1) $\hat{B}_1\varphi_n(\vec{x}) = 0, \ \vec{x}\in\Sigma$

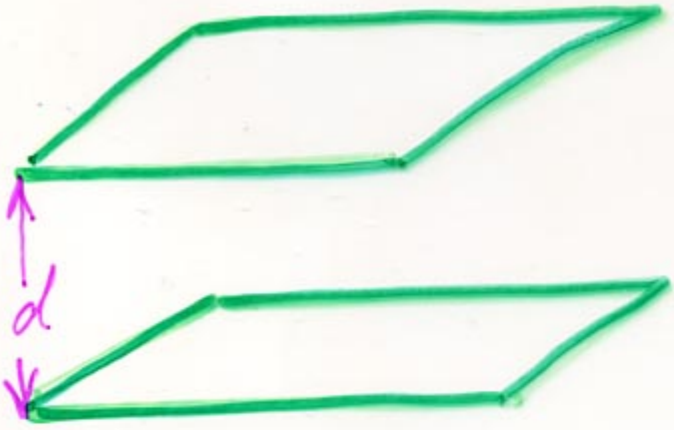
2) $\hat{B}_2\varphi_n(\vec{x}) = 0, \ |\vec{x}|\to\infty$



When the Green's function technique is used the natural modes $\{\varphi_n(\vec{x})\}$ are also considered: $E_0 = \int T_{00}(\vec{x})\,d^3\vec{x}$

$T_0(\vec{x}) \sim \partial\partial G(\vec{x},\vec{x}'), \ \vec{x}=\vec{x}'; \ G(\vec{x},\vec{x}') \sim \sum \varphi_n(\vec{x})\varphi_n(\vec{x}')$

Casimir (1948)



$$E_0 = \frac{1}{2}\sum_n \omega_n$$

$$\sin kd = 0$$

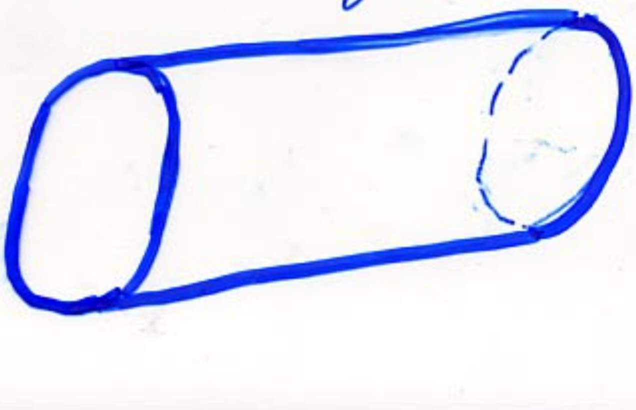
$$\omega_n = \frac{n\pi}{d}$$

Boyer (1968)
perfectly conducting sphere

(inside) + (outside)

dielectric ball

Milton, deRaad, Schwinger
(cylinder, perfectly conducting and dielectric)



(inside) + (outside)

2. Open systems

(quasi-normal modes) → noncompact M

Nontrivial dynamics is located in a compact region however it is not restricted by boundaries.

Such systems are open because the energy can be emited to the infinity.

Boundary conditions at infinity:

i) Sommerfeld radiation cond. (classical math. phys.) or Gamov cond. (QM);

Spectrum (k) is discret but complex

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} u(r) = \mathrm{const}, \quad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u}{\partial r} - iku\right) = 0$$

uniqueness of solutions, Hamiltonian is nonhermitian

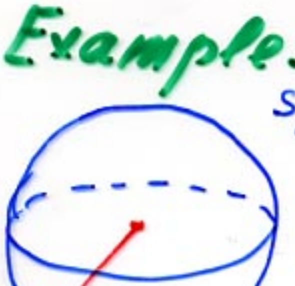
ii) scattering conditions

$$u(r) \underset{r\to\infty}{\sim} e^{ikr} + e^{-ikr}$$

spectrum (k) is positive but continuous

$$u(r) \notin L_2(R)$$

Example: Maxwell eqs. for perfectly conducting sphere with radiation conditions at ∞



$$TE: \begin{cases} j_\ell(a\omega) = 0 & r<a, \\ h_\ell^{(1)}(a\omega) = 0 & r>a; \end{cases} \quad TM \begin{cases} \frac{d}{dr}\left[r j_\ell(\omega r)\right]\Big|_{r=a} = 0, & r<a, \\ \frac{d}{dr}\left[r h_\ell^{(1)}(\omega r)\right]\Big|_{r=a} = 0, & r>a \end{cases}$$

There are two separate spectral problems for $r<a$ and for $r>a$. At $r=a$ we have boundary conditions instead of matching conditions.

$$h_\ell^{(1)}(z) = e^{+iz}\left[P_\ell\left(\frac{1}{z}\right) + Q_\ell\left(\frac{1}{z}\right)\right],$$

with P_ℓ and Q_ℓ being polynomials.

Hence these frequency eqs. for a given ℓ have a finite number of complex roots ($z = a\omega/c$):

$\ell=1$ $$h_1^{(1)}(z) = -\frac{1}{z}e^{iz}\left(1+\frac{i}{z}\right) = 0 \quad \text{(TE-modes)}$$

$$\left(z h_1^{(1)}(z)\right)' = -\frac{i}{z^2}e^{iz}\left(z^2+iz-1\right) = 0 \quad \text{(TM-modes)}$$

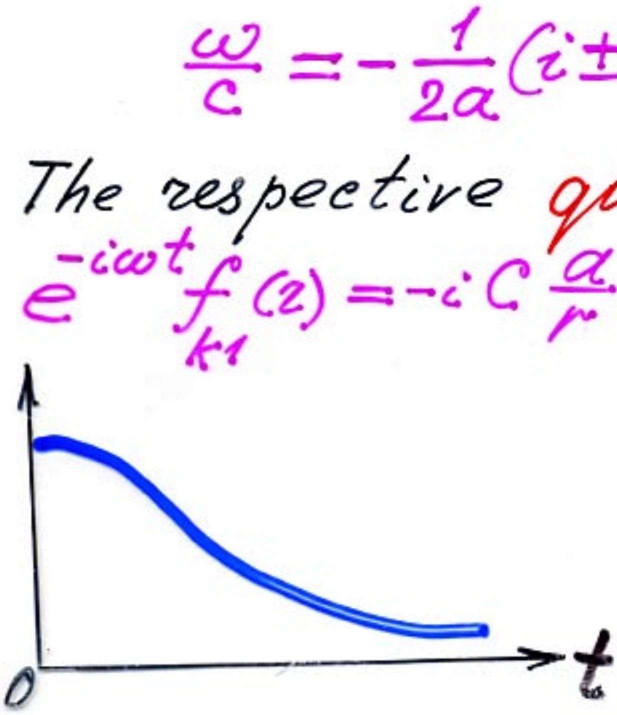
Eigenfrequencies are

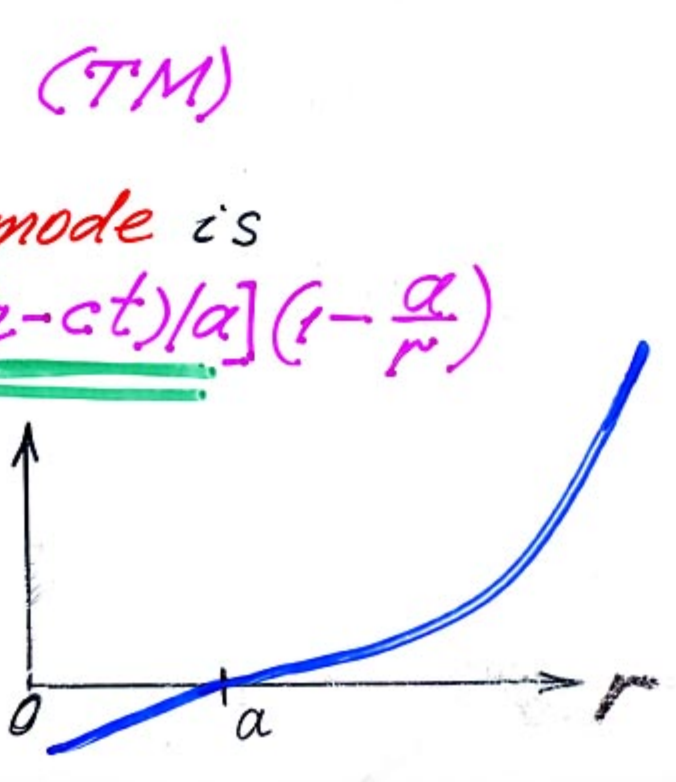
$$\frac{\omega}{c} = -\frac{i}{a} \qquad \text{(TE)}$$

$$\frac{\omega}{c} = -\frac{1}{2a}\left(i \pm \sqrt{3}\right) \qquad \text{(TM)}$$

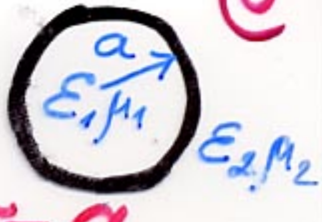
The respective quasi-mode is

$$e^{-i\omega t} f_{k1}(r) = -iC\frac{a}{r}\exp\left[(r-ct)/a\right]\left(1-\frac{a}{r}\right)$$





a compact ball



Matching conditions at $r=a$

$$C_1^{TE}\frac{1}{\sqrt{\varepsilon_1\mu_1}}\hat{j}_\ell(k_1 a) = C_2^{TE}\frac{1}{\sqrt{\varepsilon_2\mu_2}}\hat{h}_\ell^{(1)}(k_2 a),$$

$$C_1^{TE}\frac{1}{\mu_1}\hat{j}_\ell'(k_1 a) = C_2^{TE}\frac{1}{\mu_2}\hat{h}_\ell^{(1)\prime}(k_2 a),$$

$$k_i^2 = \varepsilon_i\mu_i\frac{\omega^2}{c^2},\; i=1,2.$$

$\hat{\ }$ Ricatti-Bessel functions

Friquency equations

$$\Delta_\ell^{TE}(a\omega) \equiv \sqrt{\varepsilon_1\mu_2}\,\hat{j}_\ell'(k_1 a)\hat{h}_\ell^{(1)}(k_2 a) - \sqrt{\varepsilon_2\mu_1}\,\hat{j}_\ell(k_1 a)\hat{h}_\ell^{(1)\prime}(k_2 a) \quad (=0$$

$$\Delta_\ell^{TE}(a\omega) = \sqrt{\varepsilon_2\mu_1}\,\hat{j}_\ell'(k_1 a)\hat{h}_\ell^{(1)}(k_2 a) - \sqrt{\varepsilon_1\mu_2}\,\hat{j}_\ell(k_1 a)\hat{h}_\ell^{(1)\prime}(k_2 a) = 0.$$

Discrete complex valued roots

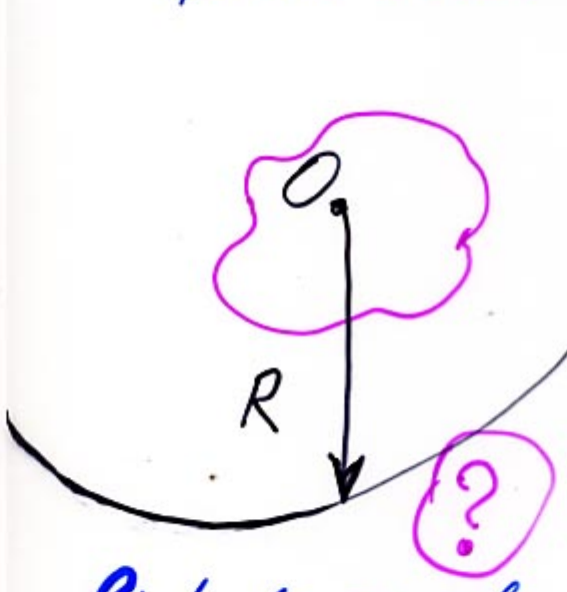
(quasi-normal modes
Debye, 1908 PhD Thes.)

These modes cannot be used for expanding the field and introducing the Fock representation. (they are not complete in a standard formulation)

In order to get the spectral zeta function or the heat kernel one needs to define, in a correct way, summation or integration over the spectrum.

How to do this for open systems?

a) Let us put our system in a large box or sphere with $R \to \infty$



$$\hat{B} u \Big|_{\Sigma} = 0$$

countable real spectrum of k depending on R and, in principle, on oper. $\hat{B}$.

b) Let us adress the regorous scattering theory (complex plane of k) and use the spectral density defined here

Some preliminary consideration

Free scalar field $\varphi(\vec{x},t)$

$$\hat{H} = \frac{1}{2}\int d^3x\,(\dot{\varphi}^2 + (\nabla\varphi)^2 + m^2\varphi^2) = \frac{1}{2}\int d^3k\,\omega(\vec{k})\cdot[a^+a + aa^+]$$

$$\boxed{E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int d^3k\,\omega(\vec{k})}$$

$$\varphi(\vec{x},t) = \int d^3k\left[\psi_k(x)\,a(\vec{k}) + \psi_k^*(x)\,a^+(\vec{k})\right]$$

scattering states

First integration over d^3x and after that $\langle 0|\ldots|0\rangle$

In the case of continuous spectrum

$$\omega^2(\vec{k}) = \vec{k}^2 + m^2 \quad , 0 \le k < \infty$$

we have

$$E_0 = 2\pi \int_0^\infty k^2 dk \, (k^2 + m^2).$$

Thus, the information on the potential in the problem under study is los

What is the reason?

Let us do first

$$\langle 0 | \dots | 0 \rangle$$

and after that we evaluate the integral over $d^3\vec{x}$:

$$E_0 = \frac{1}{2}\int d^3k\,\omega(k)\underbrace{\int d^3x\,\psi_k^*(x)\,\psi_k(x)}_{\rho(k)\,\sim\,\delta^{(3)}(\vec{k}-\vec{k})=\delta^{(3)}(0)}$$

$$= \frac{1}{2}\int d^3k\,\omega(k)\rho(k), \quad \text{if } \psi_k(x) \sim e^{ikx}$$

then $\rho(k) = \int d^3x \cdot 1 = \mathrm{Vol} \Rightarrow \infty$

How to calculate $\rho(k)$?

a) "Physical" consideration (large sphere)

depends on $\hat{B}$

$$\psi_\ell(k,r) \underset{r\to\infty}{\sim} \frac{1}{r}\sin(kr + \delta_\ell(k) + \varphi_0)$$

phase shift

(depends on the pot.)

$$\hat{B}\,\psi_\ell(k,r)\Big|_{r=R} = 0$$

$$kR + \delta_\ell(k) + \varphi_0 = n\pi$$

$$\frac{dn}{dk} = \frac{1}{\pi}\frac{d}{dk}\delta_\ell(k) + R, \quad \frac{dn_0}{dk} = R \to \infty$$

$$n(k) \equiv \rho(k)$$

$$\boxed{\rho(k) - \rho_0(k) = \frac{1}{\pi}\frac{d}{dk}\delta_\ell(k)}$$

b) "Mathematical" consideration

Different solutions of the radial Schrödinger equation:

$$\left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r)\right)\psi_k(r) = E\,\psi_k(r)$$

i) regular solution $\varphi(k,0)=0,\ \varphi'(0,k)=1$

ii) physical solution

$$\psi_k(r) = f(k)\varphi(k,r)$$

iii) Jost solutions $f_\ell(k,r)$

$$\lim_{r\to\infty} e^{ikr} f_\ell(k,r) = i^\ell$$

$$\lim_{r\to\infty} f_\ell(k,r) = w_\ell(kr) \text{ free solution}$$

$$w_\ell(kr) \sim i^\ell e^{-ikr}, \quad r\to\infty$$

$$w_\ell(z) = -iz\, h_\ell^{(2)}(z) = -i\left(\frac{1}{2}\pi z\right)^{1/2} H^{(2)}_{\ell+1/2}(z)$$
$$= (-1)^\ell w_\ell(-z)^*$$

iv) 'Principal value' solutions (standing waves)

All these solutions form a complete set of functions in $L_2(R)$.

The function of spectral shift

$$\bar\rho(k) = \rho(k) - \rho_0(k) = Tr[G - G_0]$$

$$G = (H - E + i\varepsilon)^{-1}, \quad G_0 = (H_0 - E + i\varepsilon)^{-1} \qquad E = k^2$$

$$Tr\, G_0 = \int d^3x \int d^3k \frac{\psi^*_{k0}(r)\psi_{k0}(r)}{k^2 - E + i\varepsilon} \underset{\varepsilon\to 0}{\Longrightarrow} \int \psi^*_{k0}(r)\psi_{k0}(r)\, dr = \rho_0(k)$$

$$k^2 - E + i\varepsilon = (k + \sqrt{E})(k - \sqrt{E})$$

$$\frac{1}{x + i\varepsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

In the same way we have

$$Tr\, G_E = \rho(k), \quad k^2 = E$$

Regorous scattering theory

H is a complete Hamiltonian

$$H = H_0 + V, \quad H|E\rangle = E|E\rangle,$$

where H_0 is free Hamiltonian

$$H_0 = \frac{p^2}{2m}, \quad H_0|E_0\rangle = E_0|E_0\rangle$$

$R(z)$ and $R_0(z)$ are the relevant resolvent operators

$$R(z) = (H - z)^{-1}, \qquad R_0(z) = (H_0 - z)^{-1}$$

Two-body time delay operator

$$q(E) = -i S^+(E) \frac{d}{dE} S(E)$$

Important spectral property

$$2\,\mathrm{Im}\,\mathrm{Tr}\left[R(E+i0) - R_0(E+i0)\right] = \mathrm{tr}\left[q(E)\right]$$

$$\hat{I} = \int dE' \, |E'\rangle\langle E'| = \int dE_0' |E_0'\rangle\langle E_0'|$$

$$R(z) = \int dE' \frac{|E'\rangle\langle E'|}{E' - z}, \quad R_0(z) = \int dE_0' \frac{|E_0'\rangle\langle E_0'|}{E_0' - z}$$

$$\frac{1}{x - i0} = \mathcal{P}\frac{1}{x} + i\pi\delta(x)$$

$$2\,\mathrm{Im}\,\mathrm{Tr}\,R(E+i0) = 2\,\mathrm{Im}\int d^3x \int dE' \frac{\langle x|E'\rangle\langle E'|x\rangle}{E' - E - i0}$$
$$= 2\pi \int \langle x|E\rangle\langle E|x\rangle = 2\pi \int \psi_E^*(x)\, \psi_E(x)\, d^3x.$$

$$2\pi \int d^3x \left[\overset{\rho_{11}(E)}{\psi^*_E(x)\psi_E(x) - \psi^*_{0E}(x)\psi_{0E}(x)} \right] = \mathrm{tr}\, q(E)$$

$$\mathrm{tr}\, q(E) = 2 \sum_l (2l+1) \frac{d}{dE} \delta_l(E)$$

$$S(E) = e^{2i\delta_l(E)}$$

The function of the spectral shift (Russian terminology)

$$\Delta\rho(E) \equiv \rho(E) - \rho_0(E) = \frac{1}{\pi} \sum_l (2l+1) \frac{d}{dE} \delta_l(E)$$

in terms of Green functions

$$\Delta\rho(E) = \frac{1}{\pi} \mathrm{Im} \int d^3x \left[G(x, x; E+i0) - G_0(x, x; E+i0) \right]$$

$$E \to k \quad (E \sim k^2)$$

Contour rotation ($\omega \to i\omega$)

$$\Delta\rho_l(k) = \frac{1}{2\pi i} \left[\ln F_l(-k) - \ln F_l(k) \right]$$

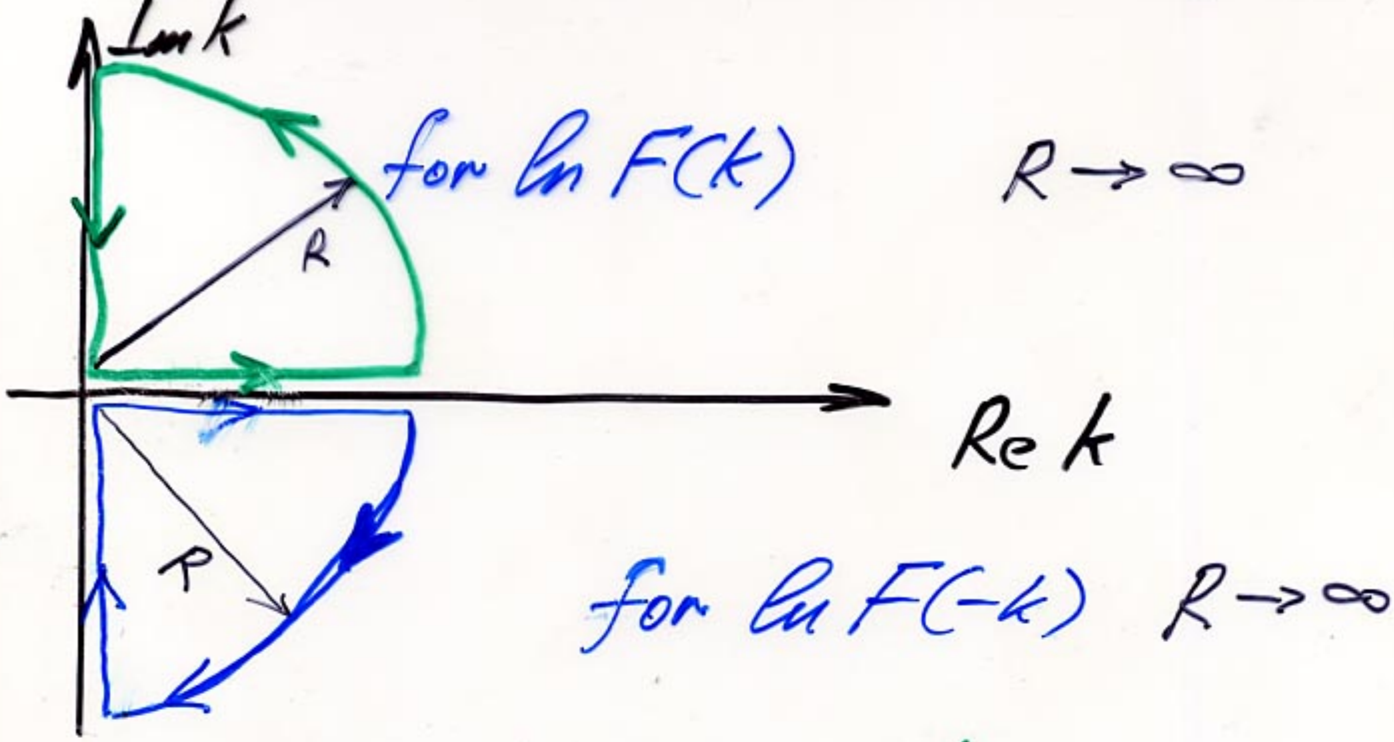
$S(E) = \dfrac{F(-k)}{F(k)}$

$F(k)$ is the Jost function

It is holomorfic for $\mathrm{Im}\, k > 0$

Scalar nonrelativistic scattering

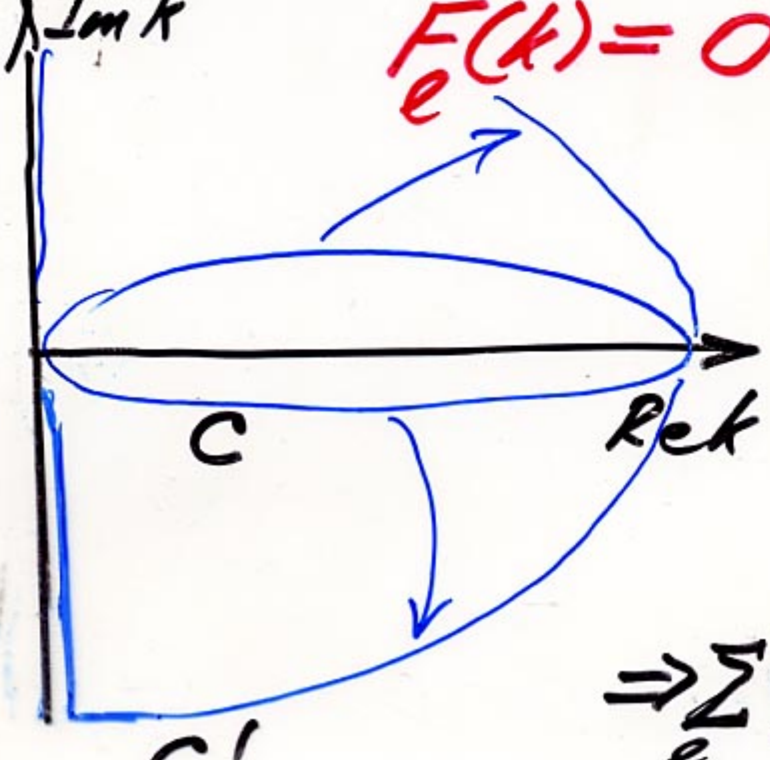
$$\psi_k(r) = \frac{1}{2} i k^{-l-1} \left\{ F_l(k) e^{ikr} - (-1)^l F_l(-k) e^{-ikr} \right\}$$



Formal prescription

radiation condition implyes

$F_l(k) = 0$



$$E_0 = \frac{1}{2} \sum \left(\text{zeros of } F_l(k) \right)$$

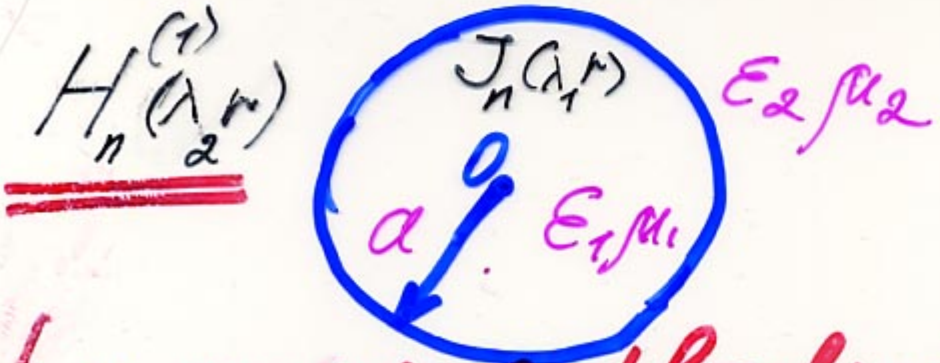
$$\Rightarrow \sum_l \int_C k \frac{d}{dk} \ln F_l(k)$$

$$\Rightarrow \sum_l \int_{C'} k \frac{d}{dk} \ln F_l(k)$$

K. Milton, V.V.N., A.V. Nesterenko, Phys. Rev. D (2001)

5. The importance of specifying the BC at spatial infinity on physical basis

A compact circular infinite cylinder



Let us consider first Radiation condition when $r \to \infty$

$$f(r) = \begin{cases} J_n(\lambda_1 r), & r < a, \\ H_n^{(1)}(\lambda_2 r), & r > a. \end{cases}$$

Frequency eq. (Bordag, Pirozhenko, Milton, Inés Cavero-Peláez)

$$\lambda_1^2 \lambda_2^2 \Delta_n^{TE}(\lambda_1 a, \lambda_2 a) \cdot \Delta_n^{TM}(\lambda_1 a, \lambda_2 a) - n^2 \omega^2 k_z^2 (\varepsilon_1 \mu_1 - \varepsilon_2 \mu_2)^2 \left[J_n(\lambda_1 a) H_n^{(1)}(\lambda_2 a) \right]^2 = 0.$$

where

$$\Delta_n^{TE}(\lambda_1 a, \lambda_2 a) = a \mu_1 \lambda_2 J_n'(\lambda_1 a) H_n^{(1)}(\lambda_2 a) - a \mu_2 \lambda_1 J_n(\lambda_1 a) H_n^{(1)\prime}(\lambda_2 a) = 0,$$

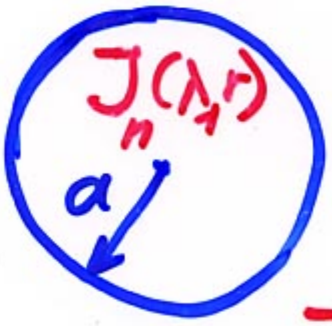
$$\Delta_n^{TM}(\lambda_1 a, \lambda_2 a) =$$

$$= a\varepsilon_1 \lambda_2 J_n'(\lambda_1 a) H_n^{(1)}(\lambda_2 a) - a\varepsilon_2 \lambda_1 J_n(\lambda_1 a) H_n^{(1)\prime}(\lambda_2 a)$$

$$\lambda_i^2 = k_i^2 - k_z^2, \quad k_i^2 = \varepsilon_i \mu_i \frac{\omega^2}{c^2}, \quad i = 1, 2,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Surface waive conditions (dialectric waiveguade)



$$f(r) \sim e^{-\bar{\lambda}_2 r}$$

$$f(r) = C K_n(\bar{\lambda}_2 r)$$

$$\bar{\lambda}_2^2 > 0$$

$$\bar{\lambda}_2^2 \equiv -\lambda_2^2 = k_z^2 - \mu_2 \varepsilon_2 \frac{\omega^2}{c^2}$$

TE-modes

$r \geq a$

$$\left.\begin{aligned} B_z &= J_0(\lambda_1 r), \\ B_r &= -\frac{ik_z}{\lambda_1} J_1(\lambda_1 r), \\ E_\varphi &= \frac{i\omega}{c\lambda_1} J_1(\lambda_1 r) \end{aligned}\right\} r \leq a$$

$$B_z = C K_0(\bar{\lambda}_2 r)$$

$$B_r = \frac{ik_z C}{\bar{\lambda}_2} K_1(\bar{\lambda}_2 r)$$

$$E_\varphi = -\frac{i\omega C}{c\bar{\lambda}_2} K_1(\bar{\lambda}_2 r)$$

Matching conditions ($r=a$)

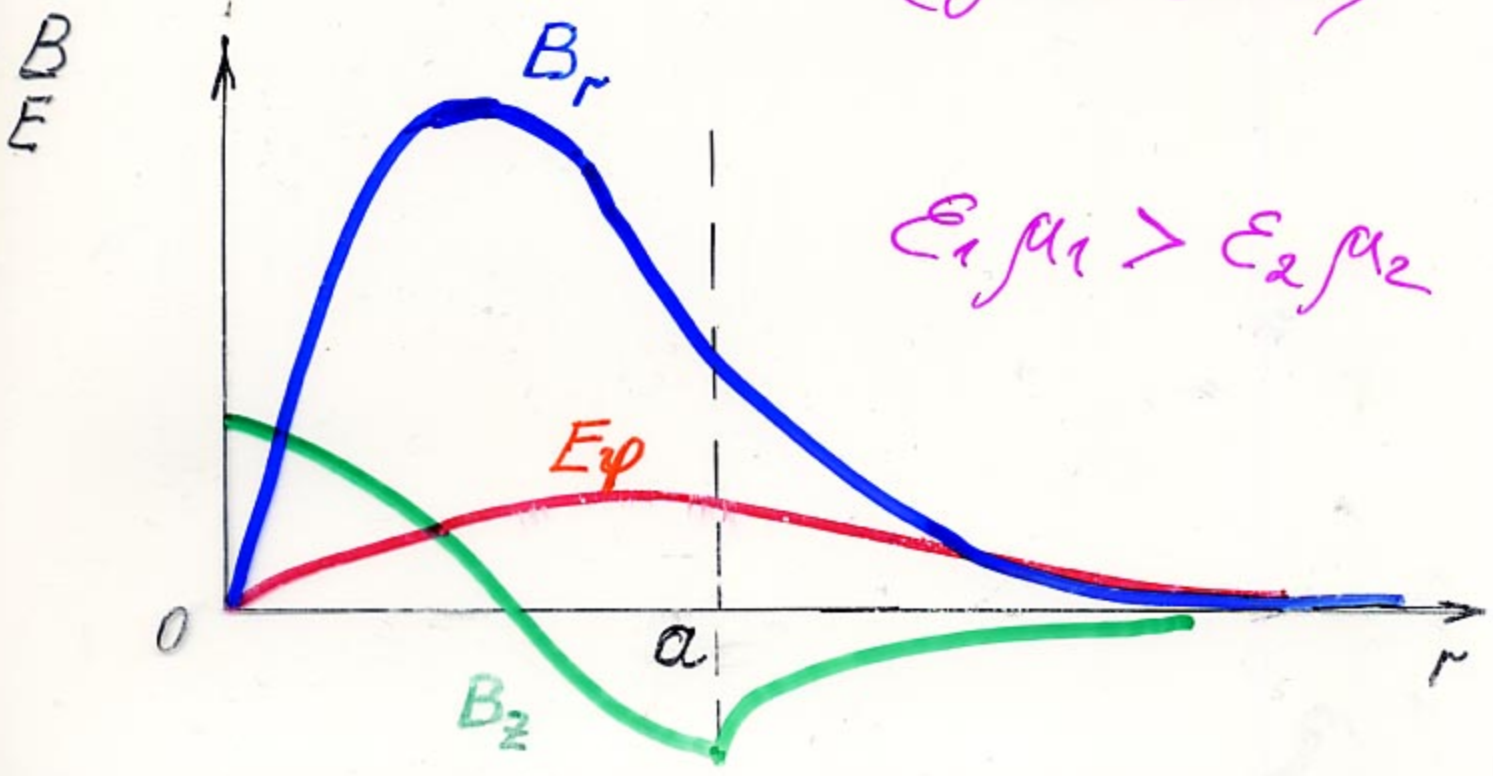
$$\begin{cases} C K_0(\bar\lambda_2 a) = J_0(\lambda_1 a), \\ -\dfrac{C}{\bar\lambda_2} K_1(\bar\lambda_2 a) = \dfrac{1}{\lambda_1} J_1(\lambda_1 a). \end{cases} \qquad n=0$$

Frequency equations

$$\frac{J_1(\lambda_1 a)}{\lambda_1 J_0(\lambda_1 a)} + \frac{K_1(\bar\lambda_2 a)}{\bar\lambda_2 K_0(\bar\lambda_2 a)} = 0 \quad \text{(TE-modes)}$$

$$\frac{J_1(\lambda_1 a)}{\lambda_1 J_0(\lambda_1 a)} + \frac{\varepsilon_2}{\varepsilon_1}\frac{K_1(\bar\lambda_2 a)}{\bar\lambda_2 K_0(\bar\lambda_2 a)} = 0 \quad \text{(TM-modes)}$$

(Jackson)



6. Conclusions

1. All the results obtained in Casimir calculations for open systems by making use of the radiation conditions are valid and there is no need to rederive them a new.

2. The boundary conditions imposed on field at $r \to \infty$ should be chosen proceeding from the physical content of the problem under study.