H. B. G. Casimir (1948)

$$F_c = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} = -\frac{0.013}{(a_{\mu m})^4}$$

$\boxed{dyn/cm^2}$

conducting plates

attractive force; for $a = 0.5\ \mu m$ $F_c = 0.2\cdot 10^{-5}\frac{N}{cm^2}$

Model of an electron; H. B. G. Casimir (1953)

(numerical coefficient

$$E_c = \frac{\alpha}{r}\hbar c = \frac{e^2}{r} \quad \Longrightarrow \quad \alpha = \frac{e^2}{\hbar c}$$

could be fixed!

Casimir energy — electrostatic energy

charged sphere
the net charge e

H. Boyer (1968)

The Casimir forces exerted on a sphere are repulsive

One-dimensional version of the Casimir effect

$\varphi(t,x)$ is the displacement of the string in transverse direction

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=0 \qquad \varphi(t,0)=\varphi(t,a)=0$$

$$\mathcal{L}=\frac{1}{2}\left(\varphi_t^2-\varphi_x^2\right) \mapsto S=\frac{1}{2}\int_{t_1}^{t_2}dt\int_0^a\left(\varphi_t^2-\varphi_x^2\right)dx$$

action

Equations of motion $\partial_i\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\right)=0$

Hamiltonian description

$$\pi(t,x)=\frac{\partial\mathcal{L}}{\partial\varphi_t}=\varphi_t$$

$$H=\dot{q}_j p_j-L \Longrightarrow H=\int_0^a dx\left(\dot{\varphi}\pi-\mathcal{L}\right)=$$

$$=\frac{1}{2}\int_0^a\left(\pi^2+\varphi_x^2\right)dx$$

Solution of the eqs. of motion

$$\varphi(t,x)=X(x)\cdot T(t) \qquad \ddot{T}X-TX''=0 \quad \Big|\ \frac{1}{TX}$$

$$\frac{\ddot{T}}{T}=\frac{X''}{X}=-\omega^2\ (\text{constant}) \qquad X(x)=A\cos(\omega x)+B\sin(\omega x)$$

boundary conditions imply $A=0$ and

$$\omega\Rightarrow\omega_n=\frac{n\pi}{a}$$

$$T(t)=Ce^{i\omega_n t}+De^{-i\omega_n t}$$

$$\varphi(t,x)=\sum_{n=1}^{\infty}\left(e^{i\omega_n t}a_n+a_n^+e^{-i\omega_n t}\right)\sin\left(\frac{n\pi}{a}x\right)$$

Quantization

$$[q_i, p_j] = i\hbar\,\delta_{ij} \Longmapsto [\varphi(t,x), \pi(t,x')] = i\hbar\,\delta(x-x')$$

$$\Downarrow \qquad [a_n, a_m] = [a_n^+, a_m^+] = 0$$

$$[a_n, a_m^+] = \delta_{nm}$$

$$H = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n (a_n^+ a_n + a_n a_n^+) = \sum_{n=1}^{\infty} \omega_n \left(a_n^+ a_n + \frac{1}{2}\right)$$

$$E_c = \langle 0|H|0\rangle = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n = \frac{\hbar c_s \pi}{2a} \sum_{n=1}^{\infty} n$$

Here c_s is the sound velocity along the string. $e^{-\varepsilon n}$, $\varepsilon > 0$

regularization

$$\sum_{n=1}^{\infty} e^{-\varepsilon n} \cdot n = -\frac{\partial}{\partial \varepsilon} \sum_{n=1}^{\infty} e^{-\varepsilon n} = -\frac{\partial}{\partial \varepsilon}\left(e^{-\varepsilon} + e^{-2\varepsilon} + \dots\right.$$

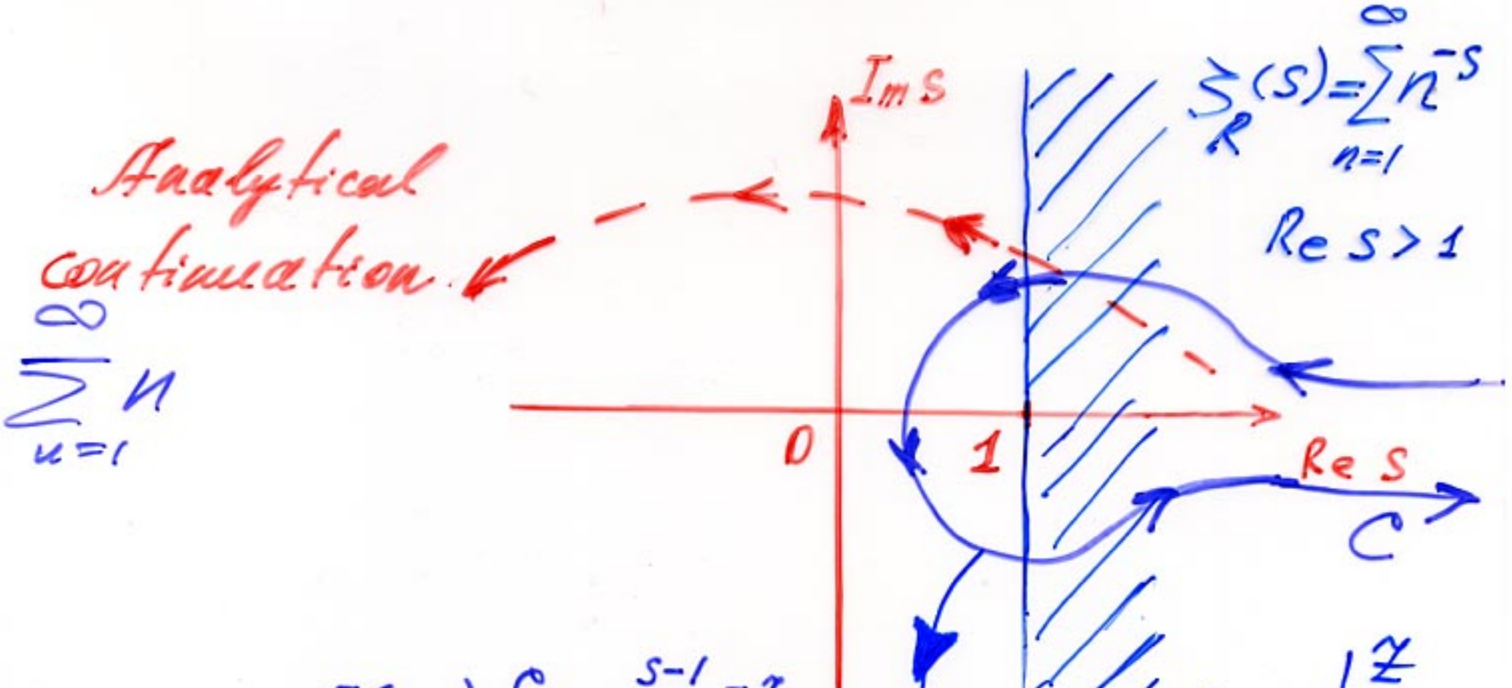
$$= -\frac{\partial}{\partial \varepsilon} e^{-\varepsilon} \underbrace{\left(1 + e^{-\varepsilon} + e^{-2\varepsilon} + \dots\right)}_{\frac{1}{1-e^{-\varepsilon}}} =$$

$$= -\frac{\partial}{\partial \varepsilon} \frac{e^{-\varepsilon}}{1-e^{-\varepsilon}} = -\frac{\partial}{\partial \varepsilon} \frac{1}{e^{\varepsilon}-1} = \frac{e^{\varepsilon}}{(e^{\varepsilon}-1)^2} =$$

$$\overset{\varepsilon \to 0}{=} \frac{1}{\varepsilon^2} - \frac{1}{12} + O(\varepsilon)$$

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}$$

The simplest example of the spectral zeta functions is the Riemannian zeta function



$$\zeta_R(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}e^{-z}}{1-e^{-z}}\,dz$$

The contour C should avoid the points

$$z = \pm 2n\pi i,\quad n = 1, 2, 3, \dots$$

The Riemann reflection formula

$$\zeta_R(z) = \frac{(2\pi)^z}{\pi}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\,\zeta_R(1-z)$$

The function $\zeta_R(z)$ has a simple pole at the point $s = 1$

$$\zeta_R(s) \underset{s\to 1}{\simeq} \frac{1}{s-1} + \gamma + \cdots \quad \gamma \text{ is the Euler constant}$$

$$E_c = \frac{1}{2}\zeta_R(-1); \qquad \zeta_R(-1) = -\frac{1}{2\pi^2}\zeta_R(2) = -\frac{B_2}{2} = -\frac{1}{12}$$

$$E_c = -\frac{1}{24}\quad\left(-\frac{\hbar c_s \pi}{24a}\right)$$

Fermionic oscillator

$$H_B = \frac{1}{2}(p^2+\omega^2 q^2) = \frac{\hbar\omega}{2}(a^+a + aa^+) = \hbar\omega(a^+a + \frac{1}{2}),$$

$$[a, a^+]_- = aa^+ - a^+a = 1.$$

$$H_F = \frac{\hbar\omega}{2}(b^+b - bb^+) = \hbar\omega(b^+b - \frac{1}{2})$$

$$[b, b^+]_+ = bb^+ + b^+b = 1$$

$$E_F^{Cas} = -\sum_s \frac{\hbar\omega_s}{2}$$

Zero point energy in the Bag Model

A. Chodos et al. Phys. Rev D 9, 3471 (1974); Phys. Rev. D10, 2599 (1974); D12, 2060 (1975).

Confined gluons (free ~~q~~ gluons)

Confined virtual quarks

Hamiltonian of the Bag Model

$$H = BV + \sigma A - \frac{Z}{a}$$

B, σ and Z are the constants

The Casimir energy leads to renormalization of these parameters

Общий метод нахождения спектра э-м колебаний

Уравнения Максвелла (гауссовск. система ед.) в вакууме

$$\begin{cases} \mathrm{rot}\,\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \\ \mathrm{div}\,\vec{E} = 4\pi\rho, \end{cases}$$

$$\begin{cases} \mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \\ \mathrm{div}\,\vec{H} = 0. \end{cases}$$

времен. завис. $e^{-i\omega t}$

$\vec{\Pi}'$ – элек. вектор Герца

$\vec{\Pi}''$ – магнит. вектор Герца

$$\Delta\vec{\Pi} - \frac{\partial^2\vec{\Pi}}{c^2\partial t^2} = 0$$

$$\begin{cases} \vec{E} = \vec{\nabla}\times\vec{\nabla}\times\vec{\Pi}' + i\mu\frac{\omega}{c}\vec{\nabla}\times\vec{\Pi}'' \\ \vec{H} = -i\varepsilon\frac{\omega}{c}\vec{\nabla}\times\vec{\Pi}' + \vec{\nabla}\times\vec{\nabla}\times\vec{\Pi}'' \end{cases}$$

$$\Delta\vec{\Pi} = \mathrm{grad}\,\mathrm{div}\,\vec{\Pi} - \mathrm{rot}\,\mathrm{rot}\,\vec{\Pi}$$

$$\vec{\Pi}' = (\underset{x}{0}, \underset{y}{0}, \underset{z}{\Pi'}), \quad \vec{\Pi}'' = (\underset{x}{0}, \underset{y}{0}, \underset{z}{\Pi''})$$

$$\vec{\nabla}\times\vec{\Pi} \equiv \mathrm{rot}\,\vec{\Pi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \Pi \end{vmatrix} = \vec{i}\,\Pi_y - \vec{j}\,\Pi_x$$

$$\Pi_x \equiv \frac{\partial}{\partial x}\Pi$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{\Pi} \equiv \mathrm{rot}\,\mathrm{rot}\,\vec{\Pi} = \vec{i}\,\Pi_{xz} + \vec{j}\,\Pi_{yz} - \vec{k}(\Pi_{xx} + \Pi_{yy})$$

$$\Pi' = \vec{e}_z e^{i\vec{k}\vec{x}}\phi(z), \quad \Pi'' = \vec{e}_z e^{i\vec{k}\vec{x}}\psi(z)$$

$$\vec{x} \equiv \vec{s} = (x, y) \quad \vec{k} \equiv (k_x, k_y)$$

TM modes ($\vec{\Pi}'$)

$$\begin{cases} \vec{E} = i(\vec{e}_x k_x + \vec{e}_y k_y) e^{i\vec{k}\vec{s}} \phi'(z) + \vec{e}_z k^2 e^{i\vec{k}\vec{s}} \phi(z) \\ \vec{H} = \varepsilon \frac{\omega}{c} (k_y \vec{e}_x - k_x \vec{e}_y) e^{i\vec{k}\vec{s}} \phi(z) \end{cases}$$

TE-modes ($\vec{\Pi}''$)

$$\begin{cases} \vec{E} = \mu \frac{\omega}{c} (-k_y \vec{e}_x + k_x \vec{e}_y) e^{i\vec{k}\vec{s}} \psi(z) \\ \vec{H} = i(k_x \vec{e}_x + k_y \vec{e}_y) e^{i\vec{k}\vec{s}} \psi'(z) + \vec{e}_z k^2 e^{i\vec{k}\vec{s}} \psi(z) \end{cases}$$

$$-\phi''(z) = \left(\varepsilon(z)\mu(z)\frac{\omega^2}{c^2} - k_{\parallel}^2\right)\phi(z);$$

$$-\psi''(z) = \left(\varepsilon(z)\mu(z)\frac{\omega^2}{c^2} - k_{\parallel}^2\right)\psi(z)$$

В случае пластин

$0 \le z \le a$, $0 \le k^2 < \infty$

Идеально проводящие пластины,

Π' – условия Дирихле
Π'' – условия Неймана] Доказать!

$$\Pi'_z(\vec{x}, z) = e^{i\vec{k}\vec{x}} \sin\left(\frac{n\pi z}{a}\right), \quad n = 1, 2, 3, \ldots$$

$$\Pi''_z(\vec{x}, z) = e^{i\vec{k}\vec{x}} \cos\left(\frac{n\pi z}{a}\right), \quad n = 0, 1, 2, \ldots$$

$$\omega_n^2(\vec{k}) = c^2\left[\vec{k}^2 + \left(\frac{n\pi}{a}\right)^2\right] \quad \vec{x} = (x, y), \quad \vec{k} = (k_x, k_y)$$

$$E_0 = \frac{\hbar}{2}\zeta\left(s = -\frac{1}{2}\right)$$

$$\zeta(s) = \frac{L_x L_y}{c^{2s}} \int \frac{d^2k}{(2\pi)^2} \left\{ 2\sum_{n=1}^{\infty} \left[\vec{k}^2 + \left(\frac{n\pi}{a}\right)^2\right]^{-s} + (k^2 + \mu^2)^{-s} \right\}$$

μ – инфракрасная регуляризация

$$\zeta(s) = \frac{L_x L_y}{2\pi c^{2s}} \left[\left(\frac{\pi}{a}\right)^{2-2s} \frac{\zeta_R(2s-2)}{s-1} + \frac{1}{2}\frac{\mu^{2-2s}}{s-1} \right]$$

$$s = 1/2$$

$$E_0 = -\frac{c\hbar\pi^2}{720}\frac{L_x L_y}{a^3}$$

$$\frac{E_0}{V} = -\frac{c\hbar\pi^2}{720a^4}, \quad V = a L_x L_y$$

плотность энергии