INTEGRABLE HIERARCHIES I

Scalar Ansatz (KP hierarchy)

Probably, the most efficient way to deal with integrable nonlinear equations is the Lax representation

$$\partial_t L = [M, L]\,. \tag{1}$$

This readily produces a formally infinite set of conserved quantities $I_n \sim \mathrm{tr}\, L^n$:

$$(1) \quad \Rightarrow \quad \partial_t L^n = [M, L^n] \quad \Rightarrow \quad \partial_t \,\mathrm{tr} L^n = \mathrm{tr}[M, L^n] = 0\,, \tag{2}$$

and enables us to view the t-evolution of L as a similarity transformation,

$$\partial_t T = MT\,, \quad \partial_t T^{-1} = -T^{-1}M \quad \Rightarrow \quad L(t) = TL(0)T^{-1}\,, \tag{3}$$

which implies isospectrality of the corresponding eigenvalue problem:

$$L(t)\psi(t) = z\,\psi(t) \quad \Rightarrow \quad \partial_t z = 0\,, \quad \psi(t) = T\psi(0)\,, \quad \partial_t \psi = M\psi\,. \tag{4}$$

To find a Lax (or $L - M$) pair for a given equation is an extremely difficult (and generally unsolvable) task. If one, on the contrary, starts with some reasonable Ansatz for L, it appears fairly possible to discover an infinite set $\{M_n\}$ of appropriate M-operators. If, moreover, the flows given by the corresponding equations (1) commute (so that the t_n-evolutions may be considered as independent, and the t_n themselves – as a new set of coordinates on the phase space), the whole set of equations (1) with M_n is called an (integrable) hierarchy.

In practice, integrable hierarchies are highly symmetric, infinite sets of nonlinear evolution equations of the Lax type for infinitely many functions u_i of infinitely many variables t_n, $n = 1, 2, \ldots$. We use the notation $x \equiv t_1$, $\partial \equiv \partial_x \equiv \partial_1$, $\partial_n \equiv \frac{\partial}{\partial t_n}$. When needed, we will specially indicate the operator nature of a derivative, e.g., $\partial_n \circ B = \partial_n B + B\,\partial_n$. Formal sums of the type

$$P(x, \partial) = \sum_m a_m(x)\,\partial^m\,, \qquad a_m(x) = 0 \;\text{ for }\; m \gg 0 \tag{5}$$

are called pseudodifferential operators (in a variable x. Other possible arguments of a_m are treated as parameters). As usual,

$$P_+ = \sum_{m\geqslant 0} a_m(x)\,\partial^m\,, \quad P_- = P - P_+ = \sum_{m<0} a_m(x)\,\partial^m\,, \quad \mathrm{Res} P = a_{-1}(x)\,, \tag{6}$$

so that P_+ is a proper differential part of P. Operator ∂^{-1} is defined by $\partial\,\partial^{-1} = \partial^{-1}\,\partial = 1$. For any integer m, the following holds:

$$\partial^m \circ f = \sum_{k\geqslant 0} \binom{m}{k} \partial^k f\, \partial^{m-k}\,. \tag{7}$$

With this rule, pseudodifferential operators form an associative algebra. This is shown straightforwardly, as well as the following useful relation:

$$\mathrm{Res}[P, Q] = \partial(\ldots)\,. \tag{8}$$

The Kadomtsev-Petviashvili (KP) hierarchy is generated by the following (scalar) Ansatz for the Lax operator:

$$L = \partial + \sum_{i=1}^{\infty} u_i \partial^{-i} . \tag{9}$$

Unknown functions u_i may depend on all arguments t_n, or on a part of them. We use

$$B_n \doteq (L^n)_+ \equiv L^n_+ \qquad (B_1 = \partial\,, \;\; B_2 = \partial^2 + 2u_1\,, \;\; B_3 = \partial^3 + 3\partial_x u_1 + 3u_1\partial + 3u_2\,, \;\ldots) \tag{10}$$

to formulate the infinite system of equations of the KP hierarchy:

$$\partial_n L = [B_n, L] \quad \text{or} \quad [\partial_n - B_n\,, L] = 0 \qquad (n = 1, 2, \ldots) \tag{11}$$

These are nonlinear equations in u_i. Their integrability is based on the Lax form of (11) and on the following "zero-curvature" property,

$$[\partial_m - B_m\,, \partial_n - B_n] = \partial_n B_m - \partial_m B_n + [B_m, B_n] = 0\,. \tag{12}$$

From this, one easily deduces that the flows generated by different B_n commute,

$$\partial_n [B_m, L] = \partial_m [B_n, L]\,. \tag{13}$$

To derive (12) itself, we observe that (11) entails $\partial_n L^m = [B_n, L^m]$, that $[B_n, L^m] = -[L^n_-, L^m]$, and then do the following calculations:

$$\begin{aligned} \partial_n L^m - \partial_m L^n &= [B_n, L^m] - [B_m, L^n] = [B_n, L^m] + [L^m_-, L^n] \\ &= [B_n, B_m] + [B_n, L^m_-] + [L^m_-, L^n] = [B_n, B_m] + [L^m_-, L^n_-]\,. \end{aligned} \tag{14}$$

A positive part of this equality is exactly (12).

The KP equation for $u_1 = u(x, y, t)$, $y = t_2$, $t = t_3$,

$$3u_{yy} + \partial_x(-4u_t + u_{xxx} + 12\, uu_x) = 0\,, \tag{15}$$

which gave its name to the whole hierarchy, is extracted either from (12) for $m = 2, n = 3$, or from (11) with $n = 2, 3$.

The main equations (11) can be viewed as the compatibility condition of the linear system

$$L\psi = z\psi\,, \qquad \partial_n \psi = B_n \psi\,. \tag{16}$$

Here $\psi(t, z) \equiv \psi(\{t_n\}, z)$ is known as Baker's function. It admits an expansion

$$\ln\psi = \sum_{n=1}^{\infty} z^n t_n + \sum_{i=1}^{\infty} z^{-i} \psi_i(t)\,. \tag{17}$$

To show this, we note that ∂, and B_m in general, can be re-expanded in powers of L:

$$\partial = L + \sum_{i=1}^{\infty} \sigma_i^{(1)} L^{-i}\,, \tag{18}$$

$$B_m = L^m + \sum_{i=1}^{\infty} \sigma_i^{(m)} L^{-i}\,, \tag{19}$$

all $\sigma_i^{(m)}$ being expressible in $\sigma_i^{(1)}$. From (16) we see that $L^m\psi = z^m\psi$ and

$$\partial_m\psi = (z^m + \sum_{i=1}^{\infty} z^{-i}\sigma_i^{(m)})\psi\,, \tag{20}$$

which agrees with (17) if we fix

$$\partial_m\psi_n = \sigma_n^{(m)}\,. \tag{21}$$

Further, we observe that

$$\partial_k\sigma_n^{(m)} = \partial_m\sigma_n^{(k)} = \partial_k\partial_m\psi_n\,, \tag{22}$$

and, as a corollary, $\partial_k\sigma_n^{(1)} = \partial(\ldots)$, i.e., $\sigma_n^{(1)}$ are conserved currents. Another set of such currents is provided by $\sigma_1^{(m)} = -\mathrm{Res}L^m$ (due to (19)):

$$\partial_k\sigma_1^{(m)} = -\mathrm{Res}\partial_k L^m = -\mathrm{Res}[B_k, L^m] = \partial(\ldots)\,. \tag{23}$$

These two sets are not independent: it can be shown that

$$\sigma_1^{(n)} - n\sigma_n^{(1)} = \sum_{m=1}^{n-1}\partial_m\sigma_{n-m}^{(1)}\,. \tag{24}$$

Let us briefly describe the derivation. Using in $L^{m+1} = LL^m$ eq. (18) for L and (19) for L^{m+1} and L^m, and collecting only positive contributions, we come to

$$B_{m+1} = \partial\circ B_m - \sum_{i=1}^{m}\sigma_i^{(1)}B_{m-i} - \sigma_1^{(m)}\,. \tag{25}$$

Then, acting by both sides upon ψ, we get rid of B and ψ via (16) and (20), and arrive at a general recursion relation for $\sigma_i^{(m)}$:

$$\sigma_i^{(m+1)} = \partial_m\sigma_i^{(1)} + \sigma_{i+1}^{(m)} + \sigma_{m+i}^{(1)} - \sum_{j=1}^{m-1}\sigma_j^{(1)}\sigma_i^{(m-j)} + \sum_{j=1}^{i-1}\sigma_j^{(1)}\sigma_{i-j}^{(m)}\,. \tag{26}$$

Setting $i = n - m$ and summing up in m cancels bilinear parts and yields exactly (24).

An important corollary of (23) is that for any k, m

$$\partial_k\partial_m\psi_1 = \partial_k\sigma_1^{(m)} = \partial(\ldots)\,, \tag{27}$$

and we are prompted to make a definition

$$\psi_1(t) \doteq -\partial\ln\tau(t)\,. \tag{28}$$

This is the first appearance of the famous τ-function. We see that

$$\sigma_1^{(m)} = \partial_m\psi_1 = -\partial_m\partial\ln\tau\,, \qquad u_1 = -\sigma_1^{(1)} = \partial^2\ln\tau\,. \tag{29}$$

Moreover, the recursion relation (24) can be explicitly resolved for $\sigma_n^{(1)}$ in terms of τ-function via the Schur polynomials $p_n(t)$, which are defined (for natural n) as follows:

$$e^{\xi(t,z)} \doteq \sum_{n=0}^{\infty} z^n p_n(t)\,, \qquad \xi(t,z) = \sum_{i=1}^{\infty} z^i t_i\,. \tag{30}$$

In particular, $p_0(t) = 1$, $p_1(t) = t_1$, $p_2(t) = \frac{1}{2}t_1^2 + t_2$, $p_3(t) = \frac{1}{6}t_1^3 + t_1t_2 + t_3$. The following properties are readily deduced:

$$\partial_m p_n(t) = p_{n-m}(t)\,, \qquad n p_n(\tilde{t}) = \sum_{m=1}^{n} t_m p_{n-m}(\tilde{t})\,, \tag{31}$$

where tilde traditionally indicates a special scaling transformation of (infinite number of) arguments of p_n,

$$\tilde{t} = \{t_1, \frac{t_2}{2}, \frac{t_3}{3}, \ldots\}\,, \qquad \tilde{\partial} = \{\partial_1, \frac{\partial_2}{2}, \frac{\partial_3}{3}, \ldots\}\,, \tag{32}$$

so that

$$\exp\left(\sum_{n=1}^{\infty} \frac{t_n}{n} z^n\right) = \sum_{n=0}^{\infty} p_n(\tilde{t}) z^n\,. \tag{33}$$

The second equality in (31) follows from an observation that multiplying (33) by $1+\xi(t,z)$ is equivalent to differentiating it by $z\frac{d}{dz}$.

The explicit solution of (24) mentioned above reads

$$\sigma_n^{(1)} = p_n(-\tilde{\partial})\,\partial \ln \tau\,. \tag{34}$$

To verify it, we replace t_m in (31) by $-\partial_m$,

$$n p_n(-\tilde{\partial}) = -\sum_{m=1}^{n-1} \partial_m p_{n-m}(-\tilde{\partial}) - \partial_n\,, \tag{35}$$

apply this to $\partial \ln \tau$, and then compare with (24).

Now we see from $\partial \psi_i = \sigma_i^{(1)}$ that, up to irrelevant x-independent terms,

$$\ln \psi(t,z) = \sum_{n=1}^{\infty} z^n t_n + \sum_{i=1}^{\infty} z^{-i} p_i(-\tilde{\partial}) \ln \tau = \xi(t,z) + \ln \tau(\{t_m - \frac{1}{mz^m}\}) - \ln \tau(t) \tag{36}$$

because

$$f(t_1 + \frac{1}{z}, t_2 + \frac{1}{2z^2}, \ldots) = \exp\left(\sum_{n=1}^{\infty} \frac{\partial_n}{nz^n}\right) f(t) = \sum_{n=0}^{\infty} z^{-n} p_n(\tilde{\partial}) f(t)\,. \tag{37}$$

Thus, we are able to express the Baker function ψ in terms of τ:

$$\psi(t,z) = e^{\xi(t,z)}\,\frac{\tau(\{t_m - \frac{1}{mz^m}\})}{\tau(t)}\,. \tag{38}$$

Using again (36) and (35), we can also 'invert' this relation and express derivatives of τ in terms of ψ:

$$\partial_n \ln \tau(t) = -\mathrm{Res}_z\, z^n (\sum_{m=1}^{\infty} z^{-m-1}\partial_m - \partial_z)\, \ln w(t,z)\,, \tag{39}$$

where Res_z means the coefficient of z^{-1}, and

$$\psi(t,z) = w(t,z)\, e^{\xi(t,z)}\,, \qquad w(t,z) = 1 + \sum_{i=1}^{\infty} w_i(t) z^{-i}\,. \tag{40}$$

It should be noted here that a class of valid objects to act upon with pseudodifferential operators according to the rules $\partial^m e^{\xi(t,z)} = z^m e^{\xi(t,z)}$ and (7) is given by $(\sum_{n\leqslant N} z^n b_n(t))\, e^{\xi(t,z)}$.

Due to (40), it seems now natural to try the following representation for ψ:

$$\psi(t,z) = W e^{\xi(t,z)}\,, \tag{41}$$

where the *dressing operator* W is of the form

$$W = 1 + \sum_{i=1}^{\infty} w_i(t)\, \partial^{-i}\,. \tag{42}$$

Owing to $\partial_n e^{\xi(t,z)} = \partial^n e^{\xi(t,z)} = z^n e^{\xi(t,z)}$, we see from (17), (36), (41) and (42), that w_i, ψ_i and derivatives of τ are related by

$$\sum_{i=1}^{\infty} z^{-i}\psi_i = \ln\,(1 + \sum_{i=1}^{\infty} z^{-i} w_i) = \sum_{i=1}^{\infty} z^{-i} p_i(-\tilde\partial)\ln\tau\,. \tag{43}$$

To guarantee (16), operator W should satisfy the following “dressing” relations:

$$L = W\partial\circ W^{-1}\,, \qquad \partial_n - B_n = W(\partial_n - \partial^n)\circ W^{-1}\,. \tag{44}$$

The last of them reveals the nature of commutativity of the operators $\partial_n - B_n$: they are nothing more than dressed counterparts of the (trivially commuting) operators $\partial_n - \partial^n$. Eqs. (44) can also be rewritten in the form involving W only,

$$\partial_n W = -(W\partial^n \circ W^{-1})_-\, W\,. \tag{45}$$

It looks quite feasible that an analogous equation for the τ-function in closed form also exists. This is really so: one can obtain a fundamental *bilinear identity*

$$\mathrm{Res}_z\, \tau(\{t_m - \frac{1}{mz^m}\})\, \tau(\{t'_m + \frac{1}{mz^m}\})\, e^{\xi(t-t',z)} = 0\,. \tag{46}$$

To derive (46), it is useful to introduce a so-called adjoint Baker function

$$\psi^*(t,z) \doteq (W^*)^{-1} e^{-\xi(t,z)} = w^*(t,z)\, e^{-\xi(t,z)}\,. \tag{47}$$

In W^*, the asterisk (formal conjugation) changes sign of ∂ and the order of all operators,

$$W^* = 1 + \sum_{i=1}^{\infty}(-\partial)^{-i}\circ w_i(t)\,, \qquad (W^*)^{-1} = 1 + \sum_{i=1}^{\infty} w_i^*(t)\,(-\partial)^{-i}\,, \tag{48}$$

whereas w^* is a mere notation. In analogy with the above reasoning, one can derive

$$w^*(t,z) = 1 + \sum_{i=1}^{\infty} w_i^*(t)\, z^{-i} = \frac{\tau(\{t_m + \frac{1}{mz^m}\})}{\tau(t)}\,, \tag{49}$$

and thus rewrite the bilinear identity as follows:

$$\mathrm{Res}_z\, \psi(t,z)\, \psi^*(t',z) = 0\,. \tag{50}$$

Bearing in mind the Taylor expansion in $t - t'$, it is evidently sufficient to show that

$$\mathrm{Res}_z\, \partial^m\psi(t,z)\, \psi^*(t,z) = 0\,, \qquad m \geqslant 0\,, \tag{51}$$

because other derivatives ($\partial_{n>1}$) can be treated via (16). First, for any two pseudodifferential operators $P = \sum a_i \partial^i$ and $Q = \sum b_i \partial^i$ one proves by direct calculation that

$$\mathrm{Res}_z (Pe^{xz})(Qe^{-xz}) = \mathrm{Res} PQ^* = \sum_{i+j=-1} (-)^j a_i b_j \tag{52}$$

(recall that $\mathrm{Res} = \mathrm{Res}_\partial$). Now (51) follows:

$$\begin{aligned}\mathrm{Res}_z\, \partial^m \psi(t,z)\, \psi^*(t,z) &= \mathrm{Res}_z\, \partial^m (We^{\xi(t,z)})\, (W^*)^{-1} e^{-\xi(t,z)} \\ &= \mathrm{Res}_z\, (\partial^m \circ We^{xz})\, (W^*)^{-1} e^{-xz} = \mathrm{Res}\, \partial^m \circ WW^{-1} = \mathrm{Res}\, \partial^m = 0\,. \end{aligned} \tag{53}$$

Conversely, (16) can be derived from (50). In brief: given ψ, we reconstruct W by (42), define L by (44), show that $L\psi = z\psi$ and $(\partial_n - B_n)\psi = (\partial_n - L^n + L^n_-)\psi = e^\xi \mathcal{O}(z^{-1})$. But, due to $\mathrm{Res}_z (\partial_n - B_n)\psi(t,z) \cdot \psi^*(t',z) = 0$, this implies $(\partial_n - B_n)\psi = 0$ (it is shown by setting $t' = t$ after differentiating by t' $0, 1, \ldots$ times). Thus, bilinear identities (46), (50) turn out to be equivalent to entire KP hierarchy (11). We shall see below that bilinear relations of this kind are naturally interpreted within the fermionic representation of the τ and Baker functions.

Fermionic Fock space

Remarkably, central objects of the KP hierarchy considered above (Baker's and τ-functions, bilinear equations, and some others) can be represented, and efficiently dealt with, in the framework of free fermionic picture, via so-called Fermi-Bose correspondence. This also reveals the underlying group theory structure of integrable hierarchies.

We begin with description of an appropriate fermionic Fock space. Consider an algebra of the free fermion operators,

$$[\psi_m, \psi_n]_+ = [\bar\psi_m, \bar\psi_n]_+ = 0\,, \qquad [\psi_m, \bar\psi_n]_+ = \delta_{mn}\,, \tag{54}$$

with arbitrary integer indices m, n. We shall call ψ_n ($\equiv \psi_n^+$) the wedging (or creation) and $\bar\psi_n$ ($\equiv \psi_n^-$) the contraction (or annihilation) operators. Of course, these are generators of the Clifford algebra:

$$e_n = \psi_n + \bar\psi_n\,, \quad e_{\bar n} = i(\psi_n - \bar\psi_n)\,, \quad [e_\alpha, e_\beta]_+ = 2\delta_{\alpha\beta}\,. \tag{55}$$

Let $|\,\Omega\rangle$ (fake vacuum) be a state annihilated by each $\bar\psi_n$, and $|\,0\rangle$ (physical vacuum) the following formal object:

$$|\,0\rangle = \psi_{-1}\psi_{-2}\ldots|\,\Omega\rangle\,, \qquad \bar\psi_n|\,\Omega\rangle = 0\,. \tag{56}$$

Then

$$\psi_{n<0}|\,0\rangle = \bar\psi_{n\geqslant 0}|\,0\rangle = 0\,, \tag{57}$$

and we can interpret the free fermionic operators as follows:

$\psi_{n\geqslant 0}$ – creation of a particle
$\bar\psi_{n\geqslant 0}$ – annihilation of a particle
$\bar\psi_{n<0}$ – creation of a hole
$\psi_{n<0}$ – annihilation of a hole

It is also useful to introduce, for any integer m, the vectors

$$|\,m\rangle = \psi_{m-1}\psi_{m-2}\ldots|\,\Omega\rangle\,, \tag{58}$$

so that

$$\psi_{n<m}|\,m\rangle = \bar{\psi}_{n\geqslant m}|\,m\rangle = 0\,. \tag{59}$$

The fermionic Fock space F is spanned by vectors $|\,u\rangle$ which are obtained by application of a finite number of $\psi_m, \bar{\psi}_n$ operators to $|\,0\rangle$. So, $|\,\Omega\rangle$ is not in F . We also introduce the notion of *charge* to characterize the total number of operators ψ used (in comparison with the vacuum $|\,0\rangle$) . Thus, the charges of $\psi_n, \bar{\psi}_n$ and $|\,m\rangle$ are $+1, -1$ and m , respectively.

Let us now define the fermion fields as the following formal expansions:

$$\psi(z) = \sum_n z^n \psi_n = \psi_-(z) + \psi_+(z)\,, \quad \psi_-(z) = \sum_{n<0} z^n \psi_n\,, \quad \psi_+(z) = \sum_{n\geqslant 0} z^n \psi_n\,, \tag{60}$$

$$\bar{\psi}(z) = \sum_n z^{-n-1} \bar{\psi}_n = \bar{\psi}_+(z) + \bar{\psi}_-(z)\,, \quad \bar{\psi}_+(z) = \sum_{n<0} z^{-n-1} \bar{\psi}_n\,, \quad \bar{\psi}_-(z) = \sum_{n\geqslant 0} z^{-n-1} \bar{\psi}_n\,,$$

$$[\psi(z)\,,\,\bar{\psi}(w)]_+ = \delta(z-w) = \sum_n w^n z^{-n-1}\,. \tag{61}$$

We see that the $(-)$ part of a field corresponds to negative, whereas the $(+)$ part to non-negative powers of z , but a more deep mnemonic rule is that $+$ corresponds to creation and $-$ to annihilation:

$$\psi(z)|\,0\rangle = \psi_+(z)|\,0\rangle\,, \qquad \bar{\psi}(z)|\,0\rangle = \bar{\psi}_+(z)|\,0\rangle\,. \tag{62}$$

The duality (bra vs ket) properties of F are self-evident, if we accept ψ_n and $\bar{\psi}_n$ to be mutually Hermitean conjugate. We denote $\langle 0\,|\ldots|\,0\rangle$ by $\langle\ldots\rangle$ and use $\langle m\,|\,m\rangle = 1$. Obviously,

$$\langle\bar{\psi}_m\psi_n\rangle = -\langle\psi_n\bar{\psi}_m\rangle + \delta_{mn} = \delta_{mn}\theta(n)\,, \qquad \theta(n) = \begin{cases} 1 & (n \geqslant 0) \\ 0 & (n < 0) \end{cases} \tag{63}$$

Another useful example of a matrix element is

$$\langle\psi(z)\bar{\psi}(w)\rangle = \langle\bar{\psi}(z)\psi(w)\rangle = (z-w)_w^{-1} = \sum_{n\geqslant 0} w^n z^{-n-1}\,. \tag{64}$$

The normal product is defined as usual: annihilation (with respect to the physical vacuum $|\,0\rangle$) operators should not stand to the left of the creation ones. For φ and χ linear in $\psi, \bar{\psi}$, it is

$$:\varphi\chi: \doteq \varphi_+\chi - \chi\varphi_- = \varphi\chi - \langle\varphi\chi\rangle = \varphi\chi - \langle\varphi_-\chi_+\rangle\,. \tag{65}$$

In particular,

$$:\bar{\psi}_m\psi_n: = -:\psi_n\bar{\psi}_m: = \bar{\psi}_m\psi_n - \delta_{mn}\theta(n)\,. \tag{66}$$

Now we introduce a bilinear current

$$J(z) = :\psi(z)\bar{\psi}(z): = \sum_n z^{-n-1} J_n\,, \qquad J_n = \sum_m :\psi_m\bar{\psi}_{n+m}: \tag{67}$$

In fact, the normal ordering is effective only for J_0 :

$$J_0 = \sum_{m\geqslant 0} \psi_m\bar{\psi}_m - \sum_{m<0} \bar{\psi}_m\psi_m\,. \tag{68}$$

The following holds:

$$(J_n)^+ = J_{-n}\,, \qquad J_0|\,m\rangle = m|\,m\rangle\,, \qquad J_{n>0}|\,m\rangle = 0\,, \tag{69}$$

so that J_0 measures the charge. The last equality in (69) is not surprising because $J_{n>0}$ effectively shifts particles by n positions down, which is clearly impossible in a 'filled' state $|\,m\rangle$. Moreover, for any Fock vector $|\,u\rangle$ and sufficiently large n ,

$$J_n|\,u\rangle = 0 \qquad (n \gg 0)\,. \tag{70}$$

Further,

$$[J_n, \psi_m] = \psi_{m-n}\,, \qquad [J_n, \bar{\psi}_m] = -\bar{\psi}_{m+n}\,, \tag{71}$$

$$[J_n, \psi(z)] = z^n\psi(z)\,, \qquad [J_n, \bar{\psi}(z)] = -z^n\bar{\psi}(z)\,. \tag{72}$$

At last, the commutator of the current components has a typical bosonic form:

$$\begin{aligned}
[J_m, J_n] &= [J_m, \sum_{k\geqslant 0}\psi_k\bar{\psi}_{n+k} - \sum_{k<0}\bar{\psi}_{n+k}\psi_k] \\
&= \sum_{k\geqslant 0}(\psi_{k-m}\bar{\psi}_{n+k} - \psi_k\bar{\psi}_{n+m+k}) - \sum_{k<0}(-\bar{\psi}_{n+m+k}\psi_k + \bar{\psi}_{n+k}\psi_{k-m}) \\
&= \sum_{l=-m}^{-1}(\psi_l\bar{\psi}_{l+n+m} + \bar{\psi}_{l+n+m}\psi_l) = m\delta_{m,-n}\,.
\end{aligned} \tag{73}$$

The current $J(z)$ contains infinite summation. To work with such operators in the Fock space, we need some special techniques. By definition, any Fock vector is *finite* in the following sense: it differs from $|\,0\rangle$ in a finite number of positions (filled or empty), or, in other words, the maximum $|n|$ of a position whose status differs from that in $|\,0\rangle$, is finite. When treating Fock vectors as infinite columns (the occupation number representation), it is clear that a finite vector has only finite number of nonzero components. We are interested in the operators which send finite vectors to finite. These are described by so-called finite matrices, which obey $A_{ij} = 0$ for $|i-j| \gg 0$ (so a finite matrix may actually have infinite number of nonzero elements). Multiplication (and commutation) operations respect finiteness.

Now consider a class of operators in F of the form

$$X_A = \sum_{mn} A_{mn} :\psi_m\bar{\psi}_n: \tag{74}$$

with A finite. Such operators carry zero charge, and send Fock vectors to Fock vectors: normal ordering cuts off the regions $n \gg 0$ and $m \ll 0$, and finiteness of A completes the job. For instance,

$$J_n = X_{A_n}\,, \qquad (A_n)_{mk} = \delta_{k,n+m}\,. \tag{75}$$

One easily deduces

$$[X_A, \psi_n] = A_{mn}\psi_m\,, \qquad [X_A, \bar{\psi}_n] = -A_{nm}\bar{\psi}_m\,, \tag{76}$$

and

$$[X_A, X_B] = X_{[A,B]} + \omega(A, B) \tag{77}$$

where the 2-cocycle

$$\omega(A,B) = -\omega(B,A) = \sum_{ij} A_{ij}B_{ji}(\theta(j)-\theta(i)) \tag{78}$$

is well defined for finite A and B. For example,

$$\omega(A_m, A_n) = m\delta_{m,-n}\,. \tag{79}$$

Thus we come to a centrally extended Lie algebra

$$\mathrm{gl}_\infty = \{X_A\} \oplus \mathbb{C}. \tag{80}$$

Really, the basic commutator

$$[\psi_m\bar\psi_n, \psi_p\bar\psi_q] = \delta_{pn}\psi_m\bar\psi_q - \delta_{qm}\psi_p\bar\psi_n \tag{81}$$

copies the commutator of the (gl)-generators $(E_{mn})_{ij} = \delta_{im}\delta_{jn}$.

The corresponding Lie group $\mathbf{G}$ (a subgroup of the Clifford group) consists of invertible elements g of the Clifford algebra obeying

$$gVg^{-1} \subset V\,, \qquad g\bar V g^{-1} \subset \bar V\,, \tag{82}$$

with V being a linear space spanned by ψ_n and $\bar V$ by $\bar\psi_n$. Its connected subgroup containing unity is formed by the elements of the type $\exp(X_A)$.

Consider as an example the Clifford algebra generated by $[a,a^\dagger]_+ = 1$, $aa = a^\dagger a^\dagger = 0$. Its basis is $\{1, a, a^\dagger, a^\dagger a\}$, zero-charge elements are 1 and $a^\dagger a$, whereas V and $\bar V$ are one-dimensional spaces along a and $a^\dagger$, respectively. The elements of the group of our interest are

$$\alpha(1+\beta a^\dagger a) \qquad \text{with} \quad \alpha \neq 0\,,\ \beta \neq -1\,. \tag{83}$$

Namely,

$$(1+\beta a^\dagger a)^{-1} = (1 - \frac{\beta}{1+\beta}a^\dagger a)\,. \tag{84}$$

For $\beta > -1$ elements (83) admit exponential form due to

$$e^{\gamma a^\dagger a} = 1 + (e^\gamma - 1)a^\dagger a\,, \tag{85}$$

otherwise they don't, being in the second connected component (without unity).

We claim that any element of the orbit $\mathbf{G}|\,0\rangle$ obeys bilinear relations specific to the τ-function of the KP hierarchy, and thus is nothing more than a fermionic appearance of this function. To prove this (and other) statements about the properties of τ and Baker functions, and to relate their fermionic representations with more conventional ones, we need to discuss the Fermi-Bose correspondence.

Fermi-Bose correspondence

There is an isomorphism between the fermionic Fock space F and the (bosonic Fock) space B of polynomials in t_n, y, y^{-1}. Let us define

$$H(t) = \sum_{n>0} t_n J_n\,. \tag{86}$$

It is easily seen that

$$H(t)|\,m\rangle = 0\,, \qquad [H(t), J_n] = (\theta(n)-1)nt_{-n}\,. \tag{87}$$

Further useful properties of $H(t)$ are formulated in terms of $\xi(t,z)$ (30):

$$[H(t), \psi(z)] = \xi(t,z)\psi(z)\,, \qquad [H(t), \bar{\psi}(z)] = -\xi(t,z)\bar{\psi}(z)\,, \tag{88}$$

and, consequently,

$$e^{H(t)}\psi(z)e^{-H(t)} = e^{\xi(t,z)}\psi(z)\,, \qquad e^{H(t)}\bar{\psi}(z)e^{-H(t)} = e^{-\xi(t,z)}\bar{\psi}(z)\,. \tag{89}$$

With the use of Schur polynomials (30) one can also write

$$e^{H(t)}\psi_n e^{-H(t)} = \sum_{m=0}^{\infty} p_m(t)\psi_{n-m}\,, \tag{90}$$

$$e^{H(t)}\bar{\psi}_n e^{-H(t)} = \sum_{m=0}^{\infty} p_m(-t)\bar{\psi}_{n+m}\,. \tag{91}$$

Now, the *bosonization map* is

$$B(|\,u\rangle) \doteq \sum_m y^m \langle m|e^{H(t)}|\,u\rangle\,. \tag{92}$$

For any Fock vector $|\,u\rangle$, $B(|\,u\rangle)$ is a polynomial in t_n, y, y^{-1}, which is identical zero only for $|\,u\rangle = 0$. Clearly, $B(|\,m\rangle) = y^m$. Also, if $|\,u\rangle$ carries a definite charge m, only the y^m-term of the sum in (92) survives. Furthermore, if $|\,u\rangle$ is a monomial in $\psi, \bar{\psi}$ acting upon $|\,0\rangle$, then its Bose-counterpart can be explicitly written in terms of the Schur polynomials. Really, such a monomial $|\,\varphi\rangle$ is fully characterized by its charge (say, m) and a *partition* λ_φ which is a finite non-increasing sequence of natural numbers $\{\lambda_1, \lambda_2, \ldots\}$, indicating the differences in positions of particles in $|\,\varphi\rangle$ and $|\,m\rangle$, beginning from the top. A general formula

$$B(|\,\varphi\rangle) = y^m\langle m|e^{H(t)}|\,\varphi\rangle = y^m p_{\lambda_\varphi}(t) \tag{93}$$

results from repeated use of (90). The Schur polynomials $p_\lambda(t)$ indexed by partitions are defined through elementary Schur polynomials (30) as follows (we assume $p_{n<0}(t)=0$):

$$p_\lambda(t) = \det(p_{\lambda_i+j-i}(t))\,, \qquad 1 \leqslant i,j \leqslant \sum_k \lambda_k\,. \tag{94}$$

For example, the state $|\,\varphi\rangle = \psi_m\psi_{m-1}|\,m-2\rangle$ has $\lambda_\varphi = \{1,1\}$ and

$$p_{\lambda_\varphi}(t) = \det\begin{pmatrix} p_1(t) & p_2(t) \\ p_0(t) & p_1(t) \end{pmatrix} = p_1^2(t) - p_0(t)p_2(t)\,. \tag{95}$$

Let us now derive, for later use, a formula

$$\langle m+1|\psi(z)|\,\varphi\rangle = z^m p_{\lambda_\varphi}(\{-\frac{1}{nz^n}\}) = z^m\langle m|e^{-H(\{\frac{1}{nz^n}\})}|\,\varphi\rangle\,. \tag{96}$$

To produce a nonzero contribution, a monomial $|\,\varphi\rangle$ (of charge m) should be equal to $|\,m\rangle$, or display exactly one vacancy (hole) in its 'body' to be filled in by some ψ_n from $\psi(z)$. Namely ($N = 0,1,2,\ldots$)

$$|\,\varphi\rangle = \psi_m\psi_{m-1}\ldots\psi_{m-N+1}|\,m-N\rangle \quad \Rightarrow \quad \langle m+1|\psi(z)|\,\varphi\rangle = (-)^N z^{m-N}\,. \tag{97}$$

In other words, a partition λ_φ may contain only 1's. To compute the corresponding determinant in (94), observe that

$$\sum_{n=0}^{\infty} p_n(\{-\frac{1}{mz^m}\})\, w^n = \exp\,(-\sum_{n=1}^{\infty}\frac{w^n}{nz^n}) = \exp\ln(1-\frac{w}{z}) = 1-\frac{w}{z}\,, \tag{98}$$

so

$$p_1(\{-\frac{1}{mz^m}\}) = -\frac{1}{z}\,, \qquad p_n(\{-\frac{1}{mz^m}\}) = 0 \quad \text{for} \;\; n \geqslant 2\,. \tag{99}$$

Now the determinants in p_{λ_φ} in eq. (96) are easily evaluated. They either agree with (97), for appropriate $|\,\varphi\rangle$, or vanish due to (99), in both cases verifying (96). Due to $|\,\varphi\rangle$ being arbitrary monomial we, in fact, have proved the following useful formula:

$$\langle m|\psi(z) = z^{m-1}\,\langle m{-}1|e^{-H(\{\frac{1}{nz^n}\})}\,. \tag{100}$$

Analogously, one derives

$$\langle m|\bar\psi(z) = z^{-m-1}\,\langle m{+}1|e^{H(\{\frac{1}{nz^n}\})}\,. \tag{101}$$

Let us now proceed with bosonization of various fermionic operators. From (87), due to $\langle m|J_0 = m\langle m|\,,\ \langle m|J_{n<0} = 0\,,$ we obtain

$$B(J_0|\,u\rangle) = y\partial_y B(|\,u\rangle) \tag{102}$$

(hence, q^{J_0} bosonizes to the operation $y \to qy$, i.e., $B(q^{J_0}|\,u\rangle)(t,y) = B(|\,u\rangle)(t,qy)$), and

$$B(J_n|\,u\rangle) = \partial_n B(|\,u\rangle)\,, \quad B(J_{-n}|\,u\rangle) = nt_n B(|\,u\rangle) \quad (n>0)\,. \tag{103}$$

One concludes that components of the current $J(z)$ are represented in B as follows $(n>0)$:

$$J_0 \sim y\partial_y = \partial_{\ln y}\,, \qquad J_n \sim \partial_n\,, \qquad J_{-n} \sim nt_n\,, \tag{104}$$

which completely agrees with their commutation rules (73).

It appears useful to introduce in F an operator Y which augments charge by one and bosonizes to multiplication by y: $B(Y|\,u\rangle) = yB(|\,u\rangle)$. This is achieved via the definition

$$Y\,\psi_n = \psi_{n+1}\,Y\,, \qquad Y^+ = Y^{-1} \tag{105}$$

that leads to the following relations:

$$Y\bar\psi_n = \bar\psi_{n+1}Y\,, \qquad Y|\,m\rangle = |\,m+1\rangle\,, \qquad \langle m|\,Y = \langle m-1|\,, \tag{106}$$

$$[J_n, Y] = 0 \quad (n\neq 0)\,, \qquad [J_0, Y] = Y\,, \qquad q^{J_0}Y = q\,Y\,q^{J_0}\,. \tag{107}$$

Taken in the form $[J_0\,, \ln Y] \sim [\partial_{\ln y}\,, \ln y] = 1$, this suggests to interpret $\ln Y$ as a creation and J_0 as an annihilation operator.

Now we are in a position to bosonize fermion fields.

$$\begin{aligned} B(\psi(z)|\,u\rangle) &= \sum_m y^m\langle m|e^{H(t)}\psi(z)|\,u\rangle = e^{\xi(t,\,z)}\sum_m y^m\langle m|\psi(z)e^{H(t)}|\,u\rangle \\ &= e^{\xi(t,\,z)}\sum_m y^m z^{m-1}\langle m{-}1|e^{H(\{t_n-\frac{1}{nz^n}\})}|\,u\rangle = e^{\xi(t,\,z)}e^{-\xi(\tilde\partial,\,z^{-1})}\sum_m y^m z^{m-1}\langle m{-}1|e^{H(t)}|\,u\rangle \\ &= e^{\xi(t,z)}e^{-\xi(\tilde\partial,\,z^{-1})}y\sum_m (zy)^m\langle m|e^{H(t)}|\,u\rangle = e^{\xi(t,\,z)}e^{-\xi(\tilde\partial,\,z^{-1})}\,y\,z^{y\partial_y}B(|\,u\rangle)\,. \end{aligned} \tag{108}$$

Performing a similar calculation for $\bar{\psi}$ we conclude that

$$\psi^{\pm}(z) \sim e^{\pm\xi(t,\,z)} e^{\mp\xi(\tilde{\partial},\,z^{-1})}\, y^{\pm 1}\, z^{\pm y\partial_y} = e^{\pm\sum_{n>0} z^n t_n}\, e^{\mp\sum_{n>0}\frac{\partial_n}{nz^n}}\, y^{\pm 1}\, z^{\pm y\partial_y}\,. \tag{109}$$

In view of the bosonized form of J_n , this implies

$$\psi^{\pm}(z) = \exp(\pm\sum_{m=1}^{\infty}\frac{z^m}{m}J_{-m})\,\exp(\mp\sum_{n=1}^{\infty}\frac{J_n}{nz^n})\; Y^{\pm 1}\, z^{\pm J_0} \tag{110}$$

and may be considered as properly ordered bosonic counterpart of $:e^{\pm\varphi(z)}:$ with

$$\varphi(z) = \sum_{n\neq 0}\frac{J_n}{-n}z^{-n} + J_0\ln z + \ln Y\,, \qquad \frac{d}{dz}\varphi(z) = J(z)\,. \tag{111}$$

Clearly, in our derivation (108) of the bosonization formulas (109),(110) for the fermion fields, the relations (100),(101) play the crucial role. It should be noted that the inverted calculation, i.e., deducing (100),(101) from (110), is even simpler. So, in principle, all the construction could be started with deriving (110) from the 'first principles': namely, from the commutators (72). For example, the fermionic counterpart of $e^{-\xi(t,z)}\psi(z)\, e^{\xi(\tilde{\partial},\,z^{-1})}$ proves to commute with J_n for $n \neq 0$ and is thus equal to $Yf(z, J_0)$ (because in B this means commutativity with t_n and ∂_n , and, as a result, independence of both) . A function f is found (say, using $\langle m+1|\psi(z)|\, m\rangle = z^m$) to be z^{J_0}, which justifies (110).

Exponential operators like those encountered in (109) and (110) ,

$$V^{\alpha}(z) = \exp(\alpha\sum_{n>0} z^n t_n)\,\exp(-\alpha\sum_{n>0}\frac{\partial_n}{nz^n})\, y^{\alpha}\, z^{\alpha y\partial_y} \tag{112}$$

as well as their F-counterparts, are known as *vertex operators*. Of course, $\psi^{\pm}(z) \sim V^{\pm 1}(z)$. From (107) and the well-known identity

$$[a,b] = \lambda \in \mathbb{C} \quad\Longrightarrow\quad e^a e^b = e^{\lambda} e^b e^a \tag{113}$$

we find $z^{\alpha y\partial_y}\, y^{\beta} = z^{\alpha\beta}\, y^{\beta}\, z^{\alpha y\partial_y}$,

$$\left[-\alpha\sum_{m=1}^{\infty}\frac{\partial_m}{mz^m}\,,\,\beta\sum_{n=1}^{\infty}w^n t_n\right] = -\alpha\beta\sum_{n=1}^{\infty}\frac{w^n}{nz^n} = \alpha\beta\ln(1-\frac{w}{z})\,,$$

$$\exp(-\alpha\sum_{m=1}^{\infty}\frac{\partial_m}{mz^m})\,\exp(\beta\sum_{n=1}^{\infty}w^n t_n) = \frac{(z-w)_w^{\alpha\beta}}{z^{\alpha\beta}}\exp(\beta\sum_{n=1}^{\infty}w^n t_n)\,\exp(-\alpha\sum_{m=1}^{\infty}\frac{\partial_m}{mz^m})\,,$$

and finally obtain

$$\begin{aligned} V^{\alpha}(z)\,V^{\beta}(w) &= (z-w)_w^{\alpha\beta}\;:V^{\alpha}(z)\,V^{\beta}(w): \\ &= (z-w)_w^{\alpha\beta}\, e^{\alpha\xi(t,\,z)+\beta\xi(t,\,w)}\, e^{-\alpha\xi(\tilde{\partial},\,z^{-1})-\beta\xi(\tilde{\partial},\,w^{-1})}\, y^{\alpha+\beta}(z^{\alpha}w^{\beta})^{y\partial_y}\,. \end{aligned} \tag{114}$$

Let us consider some particular cases. Using (114) for $\alpha = 1$, $\beta = -1$ together with an identity $(z-w)_w^{-1} - (z-w)_z^{-1} = \delta(z-w)$, one easily checks (61) :

$$[\psi(z)\,,\,\bar{\psi}(w)]_+ \sim e^{\xi(t,\,z)-\xi(t,\,w)} e^{-\xi(\tilde{\partial},\,z^{-1})+\xi(\tilde{\partial},\,w^{-1})}\,(\frac{z}{w})^{y\partial_y}\,((z-w)_w^{-1} + (w-z)_z^{-1}) = \delta(z-w)\,.$$

Analogously, one can verify that, for example,

$$\psi(z)\,\psi(w) \sim (z-w)\, e^{\xi(t,\,z)+\xi(t,\,w)} e^{-\xi(\tilde{\partial},\,z^{-1})-\xi(\tilde{\partial},\,w^{-1})} y^2\,(zw)^{y\partial_y} \tag{115}$$

is manifestly antisymmetric, as it should be due to fermionic nature of ψ .

KP hierarchy via free fermions

Now we are to justify our claim that any $g \in \mathbf{G}$ is the fermionic counterpart of a τ-function, i.e., of some solution of the (bilinear identities of) KP hierarchy. Namely, we offer the following bosonized expressions as candidates for the role of τ and Baker functions:

$$\tau(t) \equiv \tau(t;g) = \langle 0|e^{H(t)}g|\,0\rangle\,, \tag{116}$$

$$\psi(t,z) = \frac{\langle 1|e^{H(t)}\psi(z)g|\,0\rangle}{\tau(t;g)}\,, \tag{117}$$

$$\psi^*(t,z) = \frac{\langle -1|e^{H(t)}\bar{\psi}(z)g|\,0\rangle}{\tau(t;g)}\,. \tag{118}$$

To begin with, we check the relation (38) using (89) and (100):

$$\begin{aligned}\langle 1|e^{H(t)}\psi(z)g|\,0\rangle &= e^{\xi(t,z)}\langle 1|\psi(z)e^{H(t)}g|\,0\rangle = e^{\xi(t,z)}\langle 0|e^{H(t)-H(\{\frac{1}{nz^n}\})}g|\,0\rangle \\ &= e^{\xi(t,z)}\langle 0|e^{H(\{t_n-\frac{1}{nz^n}\})}g|\,0\rangle = e^{\xi(t,z)}\,\tau(\{t_n-\frac{1}{nz^n}\})\,.\end{aligned} \tag{119}$$

Similarly, using (101) serves to confirm (49).

Since the proper relations between the τ and Baker functions are thus verified, it suffices to check the bilinear identity in the form (50):

$$\begin{aligned}\mathrm{Res}_z\,\langle 1|e^{H(t)}\psi(z)g|\,0\rangle\,\langle -1|e^{H(t')}\bar{\psi}(z)g|\,0\rangle &= \sum_n \langle 1|e^{H(t)}\psi_n g|\,0\rangle\,\langle -1|e^{H(t')}\bar{\psi}_n g|\,0\rangle \\ &= \Big(\langle 1|e^{H(t)}\otimes\langle -1|e^{H(t')}\Big)\sum_n \psi_n g|\,0\rangle\otimes\bar{\psi}_n g|\,0\rangle\,.\end{aligned} \tag{120}$$

But any element $|\,u\rangle = g|\,0\rangle$ of the orbit $\mathbf{G}|\,0\rangle$ obeys a bilinear relation

$$\mathrm{Res}_z\,\psi(z)|\,u\rangle\otimes\bar{\psi}(z)|\,u\rangle = \sum_n \psi_n|\,u\rangle\otimes\bar{\psi}_n|\,u\rangle = 0 \tag{121}$$

which is readily shown in the infinitesimal form $g = e^{X_A} \approx 1 + X_A$ with the help of (76):

$$\begin{aligned}\sum_n \psi_n g|\,0\rangle\otimes\bar{\psi}_n g|\,0\rangle &\approx \sum_n \left(\psi_n|\,0\rangle\otimes\bar{\psi}_n|\,0\rangle + \psi_n X_A|\,0\rangle\otimes\bar{\psi}_n|\,0\rangle + \psi_n|\,0\rangle\otimes\bar{\psi}_n X_A|\,0\rangle\right) \\ &= (\mathrm{id}\otimes\mathrm{id} + X_A\otimes\mathrm{id} + \mathrm{id}\otimes X_A)\sum_n \psi_n|\,0\rangle\otimes\bar{\psi}_n|\,0\rangle = 0\end{aligned} \tag{122}$$

because for any n one of $\psi_n, \bar{\psi}_n$ must be an annihilation operator. The bilinear identity (50) is thus proved. Its another version (46), in terms of the τ-function, follows from (121) through the bosonization of fermion fields.

Of course, the orbit $\mathbf{G}|\,0\rangle$ provides a great supply of solutions of bilinear identities (hence, of the KP hierarchy). The simplest one is $\tau = 1$ which corresponds to the vacuum state itself. Further, any zero-charged monomial $|\,\varphi\rangle$ (see (93)) lies in (a completion of) the orbit, therefore all Schur polynomials $p_{\lambda_\varphi}(t)$ are solutions. Another important class of (N-soliton) solutions is produced by the (repeated) action of exponents of vertex operators, related to quadratic combinations of the Fermi fields, upon 1.