Quasinormal modes and stability of higher-dimensional black holes

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- The space-time of non-rotating black holes in D-space-time dimensions are described by the Reissner-Nordstrom-de Sitter metric,

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2, \qquad (1)$$

$$f(r) = 1 - \frac{2M}{r^{D-3}} + \frac{Q^2}{r^{2D-6}} - \frac{2\Lambda r^2}{(D-2)(D-1)}, \qquad (2)$$

$\Lambda > 0 \quad - de\ Sitter,\ \Lambda < 0 \quad - anti - de\ Sitter$

$$d\Omega_{D-2}^2 = d\chi_2^2 + \sin^2\chi_2^2 d\chi_3^2 + ... + \prod_{i=2}^{D-2} \sin_i^2 d\chi_{D-1}^2$$

F. R. Tangherlini, Nuovo Cim. 27, 636 (1963).

- **Why D-dimensional black holes are interesting:**
  ♢ They appear in brane world theories, where quantum gravity shows itself at Tev energies (RS or ADD models), so that according to those theories with extra dimensions mini black holes might be produced at particle collisions in Large Hadron Collider (LHC).
  ♢ Asymptotically AdS black holes appear in the well-known AdS/CFT correspondence: a large AdS black hole in D-dimensional gravity corresponds to an approximately thermal state in the dual conformal field theory in D-1 dimensions at strong coupling.
  P. Kanti, "Black holes in theories with large extra dimensions: A review," Int. J. Mod. Phys. A19, 4899 (2004)

- The scalar field equation in curved space-time with the metric $g_{\mu\nu}$,

$$\Box\Phi \equiv \frac{1}{\sqrt{-g}} \left(g^{\mu\nu} \sqrt{-g} \Phi_{,\mu} \right)_{,\nu} = 0 \qquad (3)$$

- After separation of angular and time variables

$$\Psi = e^{-i\omega t} R(r) P_{\ell m}(\{\chi_i\}),$$

perturbations of a scalar field of the D-dimensional spherically symmetric solutions of the electro-vacuum Einstein equations with a cosmological constant can be reduced to a single wave-like equations of the form,

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V(r_*))\Psi(r_*) = 0, \qquad (4)$$

$$dr_* = \frac{dr}{f(r)}, \quad r = (r_h, +\infty), \quad r_* = (-\infty, +\infty) \qquad (5)$$

r_* is a "tortoise coordinate".

- The gravitational perturbations of the "background" $g_{\mu\nu}$:

$$g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \tag{6}$$

$\delta g_{\mu\nu}$ are small perturbations. The perturbed Einstein equation has the form,

$$\delta R_{\mu\nu} = k\delta T_{\mu\nu}. \tag{7}$$

- For gravitational perturbations all the perturbed Einstein equations can be split into the three types of perturbations which can be treated *independently* of each other.

- In the end we have a few equations for unknown $\delta g_{\mu\nu}$ where one can distinguish
- 4 scalar components - s (called "polar"when $D = 4$),
- $4(D-2)$ vector components - v (called "axial"when $D = 4$)
- $(D-2)^2$ tensor components - t (coincide with perturbations of test scalar field)

$$\delta g_{\mu\nu} = \begin{pmatrix} s & s & v & v & v & v & v \\ s & s & v & v & v & v & v \\ v & v & t & t & t & t & t \\ v & v & t & . & . & . & t \\ v & v & t & . & . & . & t \\ v & v & t & . & . & . & t \\ v & v & t & t & t & t & t \end{pmatrix} \quad (8)$$

- Using the decomposition into scalar, vector and tensor spherical harmonics, and eliminating non-physical degrees of freedom with the help of the gauge transformations

$$x_\mu \to x_\mu + \xi_\mu, \quad \Longrightarrow \delta g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \qquad (9)$$

A. Ishibashi and H. Kodama (Prog. Theor. Phys. 110, 701 (2003)) reduced the perturbation equations for the $D \geq 4$ Reissner-Nordstrom-de Sitter background to a set of independent wave-like equations

$$\frac{d^2\Psi_i}{dr_*^2} + (\omega^2 - V(r_*)_i)\Psi(r_*)_i = 0, \qquad (10)$$

$i = s$ for scalar $i = +, -$ for vector, and $i = t$ for tensor perturbations. For $D = 4$ case decoupling of the perturbed equations was done by *T. Regge and J. Wheeler (Phys. Rev. 108, 1063 (1957))*.

- The effective potential V has the following form for the test scalar field (and for the tensor type of gravitational perturbations for $D > 4$) is

$$V_{testscalar}(r) =$$

$$f(r)\left(\frac{\ell(\ell + D - 3)}{r^2} + f'(r)\frac{D-2}{2r} + \frac{f(r)(D-4)(D-2)}{4r^2}\right) \quad (11)$$

- For other types of gravitational perturbations the effective potentials have the form,

$$V(r) = f(r)\left(\frac{\ell(\ell + D - 3)}{r^2} + something\right) \quad (12)$$

ℓ is the multi-pole number that runs the valus $\ell = 2, 3, ...$ for gravitational perturbations of the Schwarzschild black hole, while $\ell = 0$(monopole), 1(*dipole*) perturbations are not dynamical and obey the Birkgoeff theorem.

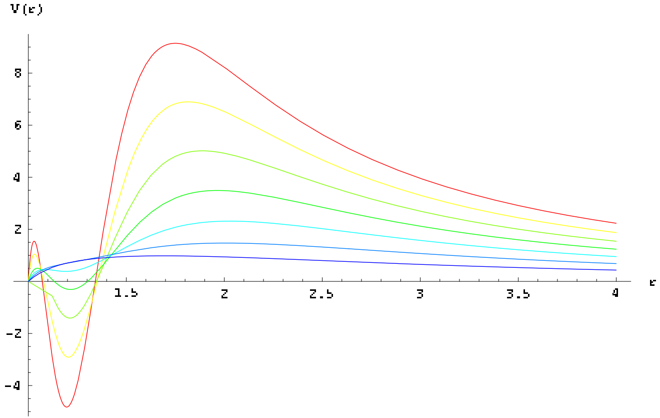


Figura: Effective potentials for gravitational perturbations of scalar type, $D = 5$ (blue)... $D = 11$ (red) ($l = 2$, $Q = 0$, $\Lambda = 0$). For higher D both the peak and the negative gap of the potential increase.

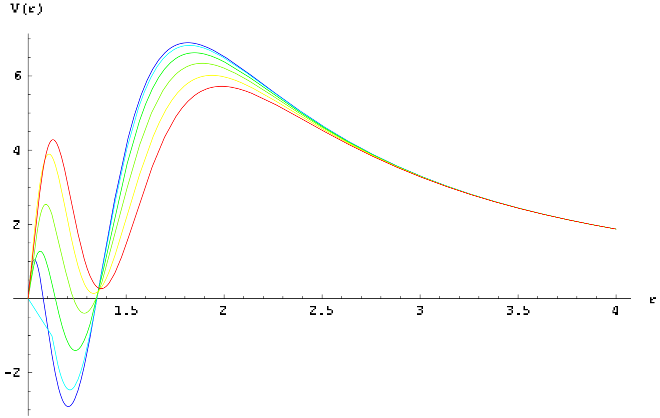


Figura: Effective potentials for gravitational perturbations of scalar type, $Q = 0$ (blue), $Q = 0.2$ (light blue), $Q = 0.4$ (green), $Q = 0.6$ (light green), $Q = 0.8$ (yellow), $Q = 0.98$ (red) ($l = 2$, $D = 10$, $\Lambda = 0$). Increasing of the charge Q cause the negative gap to move upwards. For some Q the minimum of the potential becomes positive.

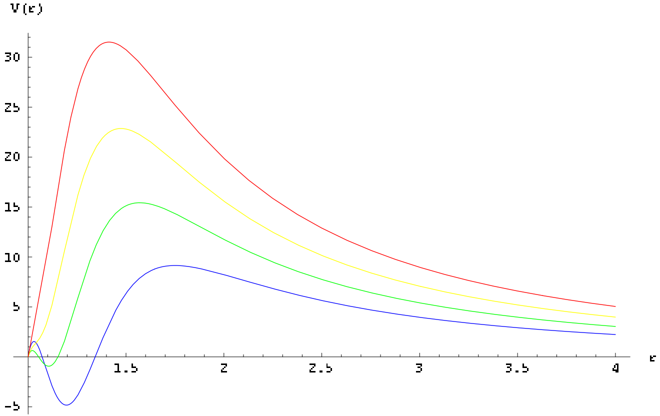


Figura: Effective potentials for gravitational perturbations of scalar type, $l = 2, 3, 4, 5$ (blue, green, yellow, red) ($D = 11$, $Q = 0$, $\Lambda = 0$). For high multipole numbers the potential minimum disappears. Thus for $8 \leq D \leq 10$ the negative gap exists only for the lowest multipole number $l = 2$.

- **Definition of quasinormal modes for asymptotically flat or asymptotically de Sitter space-times.**
  Quasinormal modes are solutions of the wave equations ($\Psi \sim e^{-i\omega t}$)

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V(r_*))\Psi(r_*) = 0, \qquad (13)$$

  which satisfy the following boundary conditions

$$\Psi \sim e^{\pm i\omega r_*}, \quad r_* \to \pm\infty \qquad (14)$$

$$\omega = \omega_{Re} - i\omega_{Im}.$$

  These are pure out-going (spherical plane) waves at spatial infinity and pure in-going waves at the event horizon. Therefore *no waves come from the infinity or from the event horizon*. For asymptotically de Sitter black holes we just replace infinity by a de Sitter horizon; ω_{Re} defines the real oscillation frequency, ω_{Im} defines the damping rate of a mode.

- **Definition of quasinormal modes for asymptotically anti-de Sitter space-times.**
  Quasinormal modes are solutions of the wave equations ($\Psi \sim e^{-i\omega t}$)

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V(r_*))\Psi(r_*) = 0, \tag{15}$$

which satisfy the following boundary conditions

$$\Psi \sim e^{-i\omega r_*}, \quad r_* \to -\infty \tag{16}$$

$$\Psi(r = +\infty) = 0, \quad r_* \in (-\infty, 0) \tag{17}$$

These are pure in-going waves at the event horizon and Dirichlet boundary condition at spatial infinity. *The Dirichlet boundary condition is natural because the effective potential diverges at the infinity, creating an effective confining box there.*

- **Motivations to study quasinormal modes**

  ♢ Possible observation of gravitational waves from colliding black holes or neutron stars with the help of a new generation of gravitational antennas.

  ♢ AdS/CFT interpretation: quasinormal modes of large asymptotically AdS black holes coincide with poles of the temperature Green function in the dual conformal field theory. Hawking temperature of a black hole is the temperature in the dual thermal field theory.

  ♢ Too optimistic Loop Quantum Gravity interpretation: Some people believe that $Re(\omega)$ in the limit $n \to \infty$ is connected somehow with the so-called Barbero-Immirzi parameter, which is connected somehow, according to other people's belief, with the reproducing in LQG correct formula for the entropy.

  ♢ Test of stability of a metric: in many cases to prove that the space-time is stable (or unstable) is much more difficult than to check that all quasinormal modes are damping (or to find a growing one in case of instability).

- **Main properties of quasinormal modes for asymptotically flat or de-Sitter space-times:**
  ◊ QNMs do not depend on a way of perturbations but only on black hole parameters, thereby they are "characteristic sound"of black holes.
  ◊ QNMs, although are analogues of normal modes in a closed system, are complex, i.e. decaying and radiating away energy.
  ◊ The full spectrum of the QNMs do not form the complete set, so that solution of the wave equation can be expanded in a set of quasinormal modes only during some intermediate late time period, called *quasinormal ringing*, which follows the initial outburst of perturbation. At $t \to$, these quasinormal modes are usually suppressed by *the exponential or power-low tails*.
  ◊ The full non-linear solution of the perturbed Einstein equation (beyond approximation of small perturbations) gives basically the same QN frequencies, what confirms the validity of the linear approximation.

- **Brief review of methods for calculations of the quasinormal modes:**
  ♢ *the Poschl-Teller method*
  Because of "symmetric"boundary conditions on $+\infty$ and $-\infty$ one can "put upside down"the Poshl-Teller potential

$$V_{PT}(r_*) = V_0 cosh^{-2}(r_*/b), \tag{18}$$

for which the Schrodinger equation () can be solved exactly. Then one fits the height and the curvature of the potential at its maximum to get QNMs:

$$\omega = \frac{1}{b}\left(\sqrt{V_0 b^2 - \frac{1}{4}} - \left(n + \frac{1}{2}\right) i\right). \tag{19}$$

H-J. Blome, R. Mashhoon, Phys. Lett. A, 231 (1984)

♢ *Chandrasekhar-Detweiler method*

By the substitution

$$\Psi = exp\left(i\int^{r_*}\Phi dr_*\right), \tag{20}$$

the wave equation reduces to the Riccati equation

$$id\Phi/dr_* + \omega^2 - \Phi^2 - V(r_*). \tag{21}$$

Then one need perform numerical integration of the Riccati equation in order to obtain the QNMs.

S.Chandrasekhar, S. Detweiler, Proc. R. Soc. Lond. A 344, 441 (1975)

◇ *WKB approach*

Based on the expansion into the Taylor series between the turning points of the effective potential $Q(r_*) = \omega^2 - V = 0$ and matching the solution between the turning points with WKB solutions in the far regions on the left and on the right of the maximum of the effective potential. The final formula reads

$$\imath \frac{\omega^2 - V_0}{\sqrt{-2V_0''}} - \Lambda_2 - \Lambda_3 - \Lambda_4 - \Lambda_5 - \Lambda_6 = n + \frac{1}{2}, \qquad (22)$$

where the correction terms Λ_2, ...Λ_6 depend on the value of the effective potential and its derivatives in the maximum respectively r_* coordinate, n is the overtone number.

B. Schutz, C. Will, Astrophys. J. Lett. 291, L33 (1985) - first order.

S.Iyer and C.M.Will, Phys. Rev. D35 3621 (1987) - Λ_2, Λ_3.

R. Konoplya, Phys. Rev. D68, 024018 (2003) - Λ_4, Λ_5, Λ_6.

♢ *Frobenius method*

One can separate the singular factor of the solution that satisfies the QNMs boundary conditions, and expand the remaining part into the Frobenius series that are convergent in the R-region (between the event horizon and the infinity). The appropriate series, for example, for the $D = 4$ Schwarzschild black hole are:

$$\Psi(r) = e^{i\omega r} r^{(2iM\omega)} \left(1 - \frac{2M}{r}\right)^{-2iM\omega} \sum_n a_n \left(1 - \frac{2M}{r}\right)^n, \quad (23)$$

Substituting the 23 into the wave equation, we obtain a three-term recurrence relation for the coefficients a_n:

$$a_0 a_1 + b_0 a_0 = 0; a_n a_{n+1} + b_n a_n + \gamma_n a_{n-1} = 0, \qquad n > 0, \quad (24)$$

The above three-term recurrence relation can be treated with the so-called *continued fractions* in order to get the coefficients a_n. Thus we have equation with respect to the eigenvalue ω:

$$b_n - \frac{a_{n-1}\gamma_n}{b_{n-1} - \frac{a_{n-2}\gamma_{n-1}}{b_{n-2} - a_{n-3}\gamma_{n-2}/\ldots}} = \frac{a_n\gamma_{n+1}}{b_{n+1} - \frac{a_{n+1}\gamma_{n+2}}{b_{n+2} - a_{n+2}\gamma_{n+3}/\ldots}}, \quad (25)$$

that can be solved numerically.
E. Leaver, Proc. Roy. Soc. Lond. A402, 285 (1985) - Frobenius method for not very high overtones ($n \lesssim 50$)
H. Nollert, Phys. Rev. D47, 5253 (1993) - improvement of the numerical scheme of Leaver for very high overtones
L. Motl, Adv. Theor. Math. Phys. 6, 1135 (2003) - analytical tool to analyze limit $n \to \infty$

◇ *Numerical integration in time domain*

One can reduce the wave equation to the following form

$$4\frac{\partial^2\Psi(u,v)}{\partial u\partial v}+V(r(u,v))\Psi(u,v)=0 \tag{26}$$

where $u=t-r_\star$, $v=t+r_\star$. We can discretize the equation (26), and then implement a finite differencing scheme to solve it numerically. There are two popular schemes

$$\begin{aligned}\Psi_\ell(N)&=\Psi_\ell(W)+\Psi_\ell(E)-\Psi_\ell(S)\\&-\Delta^2V(S)\frac{\Psi_\ell(W)+\Psi_\ell(E)}{8}\ ,\end{aligned} \tag{27}$$

where $N=(u+\Delta,v+\Delta)$, $W=(u+\Delta,v)$, $E=(u,v+\Delta)$ and $S=(u,v)$. Another possible scheme is

$$\begin{aligned}&\left[1-\frac{\Delta^2}{16}V(S)\right]\Psi_\ell(N)=\Psi_\ell(E)+\Psi_\ell(W)-\Psi_\ell(S)\\&-\frac{\Delta^2}{16}\left[V(S)\Psi_\ell(S)+V(E)\Psi_\ell(E)+V(W)\Psi_\ell(W)\right]\ .\end{aligned} \tag{28}$$

Although the discretization (28) is more time consuming than (27), (28) is more stable for asymptotically AdS geometries. Using either (27) or (28), the algorithm will cover the region of interest in the $u - v$ plane, using the value of the field at three points in order to calculate it at a forth one.

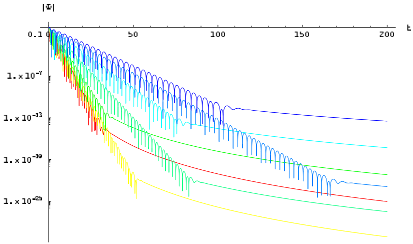
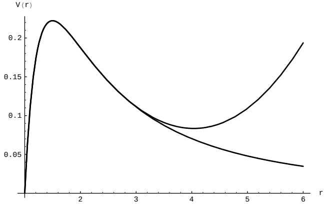


Figura: Time-domain profiles for gravitational perturbations of scalar type ($Q = 0$, $\Lambda = 0$) for $D = 5$ (blue)... $D = 11$ (red) at the same point $r = 2$. Profile for higher D decays quicker.

C. Gundlach, R. H. Price and J. Pullin, Phys. Rev. D49, 883 (1994).

♢ *Fit/interpolation method for potentials unknown in analytic form*
The basic idea is that the low laying QNMs are determined mainly by the behavior of the effective potential near its peak, while the behavior of the potential far from black hole is insignificant. Then fit or interpolation of the numerically given effective potential in some region near a black hole allows to use further WKB or time-domain methods.

Figura: On the figure below we see the potential for $s = 1$ perturbations near the SBH ($\ell = 2$), and the same potential interpolated numerically near its maximum. Despite the behavior of the two potentials are very different in the full region of r, except for a small region near a black hole, the low-laying QNMs for both potentials are almost the same.



R. Konoplya, A. Zhidenko, Phys. Lett. B644, 186 (2007); Phys. Lett. B648, 236 (2007).

♢ *Horowitz-Hubeny method for AdS black holes*
The Schwarzschild-anti-de Sitter metric has the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{29}$$

$$f(r) = 1 - \frac{r_+}{r} - \frac{r_+^3}{rR^2} + \frac{r^2}{R^2}, \tag{30}$$

r_+ is the black hole event horizon. The wave equation can be transformed to the form

$$f(r)\frac{d^2\psi(r)}{dr^2} + (f'(r) - 2i\omega)\frac{d\psi(r)}{dr} - V(r)\psi(r) = 0. \tag{31}$$

By re-scaling of r we can put $R = 1$. R is the anti-de Sitter radius. The potential V diverges at the infinity $r = \infty$.

The main point of the approach is to expand the solution to the wave equation (31) around $x_+ = \frac{1}{r_+}$ $(x = 1/r)$:

$$\psi(x) = \sum_{n=0}^{\infty} a_n(\omega)(x - x_+)^n \tag{32}$$

and to find the roots of the equation $\psi(x = 0) = 0$ following from the Dirichlet boundary condition at infinity ($\psi(r = \infty) = 0$). One has to truncate the sum (32) at some large $n = N$ and check that for greater n the roots converge.

G. T. Horowitz and V. E. Hubeny, Phys. Rev. D62, 024027 (2000).

- **Methods in the order of growing accuracy for the low-laying modes:** Poschl-Teller $\rightarrow$ Chandrasekhar-Detweiler $\rightarrow$ WKB $\rightarrow$ time domain $\rightarrow$ Fit/interpolation $\rightarrow$ Frobenius (Horowitz-Hubeny for AdS)
  Below are the values of $2M\omega$ found by different methods:
  $0.756 - 0.182i$ - Poschl-Teller
  $0.74734 - 0.17792i$ - Chandrasekhar-Detweiler
  $0.7464 - 0.1784i$ - 3th order WKB
  $0.7472 - 0.1780i$ - 6th order WKB
  $0.74735 - 0.17792i$ - time domain
  $0.747343 - 0.177925i$ - *accurate* Frobenius method
  The frequencies are given in geometrical units. For conversion into kHz multiply by $2\pi(5142 Hz M_{\odot}/M)$. The fundamental mode is approximately $1.2 kHz$ and $0.55 ms$ for a black hole of 10 solar masses.

- **Stability of the higher dimensional black holes**

  If the effective potential V is positive definite, the differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V \qquad (33)$$

  is a positive self-adjoint operator in $L^2(r_*, dr_*)$. Then there are no negative (growing) mode solutions that are normalisable, i.e. for a well-behaved initial data (smooth data of compact support), all solutions are bounded at all time.

  With the help of the so-called Friedrich extension, the operator A can be extended to a self-adjoint operator with the same lower bound of the spectrum of ω (*A. Ishibashi, H. Kodama gr-qc/0312012*). After excluding of all cases which can be proved to be stable with the Friedrich extension, A. Ishibashi and H. Kodama obtained the following table:

Tabela: Stability of static black holes.

		Tensor		Vector		Scalar	
		$Q = 0$	$Q \neq 0$	$Q = 0$	$Q \neq 0$	$Q = 0$	$Q \neq 0$
$K = 1$	$\lambda = 0$	OK	OK	OK	OK	OK	$D = 4, 5$ OK $D \geq 6$?
	$\lambda > 0$	OK	OK	OK	OK	$D \leq 6$ OK $D \geq 7$?	$D = 4, 5$ OK $D \geq 6$?
	$\lambda < 0$	OK	OK	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?

By an extensive search of quasinormal modes, both in time and frequency domains, we have shown that spherically symmetric static black holes with arbitrary charge and positive (de Sitter) lambda-term are stable for $D = 5, 6, > ...11$.
R.A. Konoplya, A. Zhidenko, Nucl. Phys. B777, 182 (2007).
Now the table looks in a different way

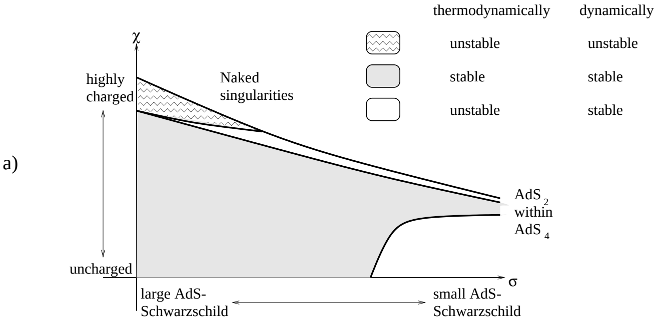
Tabela: Stability of static black holes.

		Tensor		Vector		Scalar	
		$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$
$K=1$	$\lambda=0$	OK	OK	OK	OK	OK	$D\leq 11$ OK
	$\lambda>0$	OK	OK	OK	OK	$D\leq 11$ OK	$D\leq 11$ OK
	$\lambda<0$	OK	OK	OK	OK	$D=4$ OK $D\geq 5$?	$D=4$ OK $D\geq 5$?

And what about AdS cases?

According to *S. Gubser, I. Mitra, JHEP 08, 018 (2001)* we should suspect instability there. Indeed there was considered the $D = 5$ RNAdS black hole and it was found that AdS black holes which lack local thermodynamic stability often also lack stability against small perturbations.

Figura:



Reviews on quasinormal modes of black holes:
"Quasinormal modes of stars and black holes", Kostas D. Kokkotas, Bernd G. Schmidt, Living Rev. Rel. 2,2 (1999). e-Print: gr-qc/9909058
"Quasinormal modes: the characteristic 'sound' of black holes and neutron stars"Hans-Peter Nollert, Class. Quant. Grav. 16 R159 (1999).

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