Lecture #3
The Universe: from today back to the primordial nucleosynthesis having formulae at hand

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Outline

1. The dynamics of the expanding Universe
   - Examples of cosmological solutions
   - Particles horizon
   - Events horizon
2. The present Universe
   - MD/Λ transition
   - RD/MD transition
   - The age and horizon of the Universe
   - Brightness–redshift dependence
3. Recombination

Friedman equation (00) : $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\varkappa}{a^2}$

$$\nabla_\mu T^{\mu 0} = 0 \longrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

the equation of state

$$p = p(\rho)$$

many-component liquid, in case of termal equilibrium

other equations

$$-3d(\ln a) = \frac{d\rho}{p+\rho} = d(\ln s)$$

entropy is conserved in a comoving frame

$$sa^3 = \text{const}$$

Examples of cosmological solutions

$$\varkappa = 0 \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho$$

dust: $p = 0$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^3}\,, \quad a(t) = const \cdot (t - t_s)^{2/3}\,, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{2}{3t}\,, \quad \rho = \frac{3}{8\pi G} H^2 = \frac{1}{6\pi G}\frac{1}{t^2}$$

the Universe is too young

$$t_0 = \frac{2}{3H_0} = 0.93 \cdot 10^{10}\ \text{yr} \quad (h = 0.7)$$

Cosmological (particle) horizon $l_H(t)$

distnace covered by photons emitted at $t = 0$

the size of causal patch — the size of the visible part of the Universe

in conformal coordinates: $ds^2 = 0 \longrightarrow |d\mathbf{x}| = d\eta$
coordinate size of the horizon equals $\eta(t)$

$$l_H(t) = a(t)\eta(t) = a(t)\int_0^t \frac{dt'}{a(t')}$$

dust

$$l_H(t) = 3t = \frac{2}{H(t)}\,, \quad l_{H,0} = 2.7 \cdot 10^{28}\text{cm} \ \ (h = 0.7)$$

Examples of cosmological solutions

radiation: $p = \frac{1}{3}\rho$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) = const \cdot (t - t_s)^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

$$l_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} = 2t = \frac{1}{H(t)}\,.$$

In case of termal equilibrium

$$\rho_b = \frac{\pi^2}{30}g_b T^4\,, \quad \rho_f = \frac{7}{8}\frac{\pi^2}{30}g_f T^4$$

$$\rho = \frac{\pi^2}{30}g_* T^4\,, \quad g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f$$

$$\rho = \frac{\pi^2}{30} g_* T^4\,, \qquad g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\varkappa}{a^2} \quad \longrightarrow \quad H = \frac{T^2}{M_{Pl}^*}$$

$$M_{Pl}^* = \sqrt{\frac{90}{8\pi^3 g_*}} M_{Pl} = \frac{1}{1.66\sqrt{g_*}} M_{Pl}\,.$$

$g_* = g_*(T)$

$$T(t) \approx \frac{\text{const}}{a(t)}$$

Examples of cosmological solutions

vacuum: $T_{\mu\nu} = \rho_{vac}\eta_{\mu\nu}$ $\qquad p = -\rho$

$$S_G = -\frac{1}{16\pi G}\int R\sqrt{-g}d^4x\,, \quad S_\Lambda = -\Lambda\int\sqrt{-g}d^4x\,.$$

$$a = \text{const}\cdot e^{H_{\rm dS}t}\,, \quad H_{\rm dS} = \sqrt{\frac{8\pi}{3}G\rho_{vac}}$$

de Sitter space: space-time of constant curvature

$$ds^2 = dt^2 - e^{2H_{dS}t}d\mathbf{x}^2$$

$\ddot{a} > 0$, no initial singularity

$ds^2 = dt^2 - e^{2H_{dS}t} d\mathbf{x}^2$

no cosmological horizon: $l_{\mathrm{H}}(t) = e^{H_{dS}t} \int_{-\infty}^{t} dt' e^{-H_{dS}t'} = \infty$

de Sitter (events) horizon ($\mathbf{x} = 0$, t):

from which distnace $l(t)$ one can detect light emitted at t?

in conformal coordinates: $ds^2 = 0 \longrightarrow |d\mathbf{x}| = d\eta$
coordinate size: $\eta(t \to \infty) - \eta(t) = \int_t^{\infty} \frac{dt'}{a(t')}$
physical size: $l_{dS} = a(t) \int_t^{\infty} \frac{dt'}{a(t')} = \frac{1}{H_{dS}}$

observer will never be informed what happens at distances larger $l_{dS} = H_{dS}^{-1}$

marginal matter: gas of straight strings

$p = w\rho$, $\quad w = -\frac{1}{3}$, $\quad p = -\frac{1}{3}\rho$

Friedmann equaiton for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \left(\rho_{\mathrm{M}} + \rho_{rad} + \rho_{\Lambda} + \rho_{\mathrm{curv}}\right)$$

$$\frac{8\pi}{3} G \rho_{\mathrm{curv}} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{\mathrm{M},0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\mathrm{GeV}}{\mathrm{cm}^3}\,, \qquad \text{for } h = 0.7$$

$$\Omega_X \equiv \frac{\rho_X}{\rho_c}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \rho_c \left[\Omega_{\mathrm{M}} \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda} + \Omega_{curv} \left(\frac{a_0}{a}\right)^2\right]$$

Deceleration–accelaration transition

$$\dot{a}^2 = \frac{8\pi}{3} G\rho_c \left(\frac{\Omega_{\rm M} a_0^3}{a} + \Omega_\Lambda a^2 \right) \qquad \ddot{a} = a \frac{4\pi}{3} G\rho_c \left(2\Omega_\Lambda - \Omega_{\rm M} \left(\frac{a_0}{a} \right)^3 \right)$$

$$z_{acc} = \left(\frac{2\Omega_\Lambda}{\Omega_{\rm M}} \right)^{1/3} - 1 \approx 0.85$$

$z \equiv a_0/a - 1$ at $z \gg 1$ dust-dominated

matter-radiation (RD/MD) transition

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4\right]$$

RD: $a(t) \propto \sqrt{t}$ MD: $a(t) \propto t^{2/3}$

$$z_{eq} + 1 = \frac{a_0}{a_{eq}} \sim \frac{\Omega_{\rm M}}{\Omega_{rad}} \sim 10^4\,, \qquad T_{eq} = T_0(1+z_{eq}) \sim 10^4\ \mathrm{K} \sim 1\ \mathrm{eV}\,.$$

More accurately: $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$$\rho_\nu = 3\cdot 2\cdot \frac{7}{8}\frac{\pi^2}{30} T_\nu^4\,, \quad \rho_{rad} = \rho_\gamma + \rho_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30} T^4$$

matter-radiation (RD/MD) transition

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4\right]$$

RD: $a(t) \propto \sqrt{t}$ smooth transition MD: $a(t) \propto t^{2/3}$

$$z_{eq} + 1 = \frac{a_0}{a_{eq}} = 0.6\frac{\Omega_{\rm M}}{\Omega_{rad}} = 3.0\cdot 10^3\,, \qquad T_{eq} = T_0(1+z_{eq}) = 0.7\ {\rm eV}$$

More accurately: $t_{eq} = \frac{1}{2H_{eq}} = \frac{M_{Pl}^*}{2T_{eq}^2} = 80$ kyr $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$$\rho_\nu = 3\cdot 2\cdot \frac{7}{8}\frac{\pi^2}{30}T_\nu^4\,, \quad \rho_{rad} = \rho_\gamma + \rho_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30}T^4$$

The age of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right],$$

$$\Omega_{\rm M} + \Omega_\Lambda = 1\,, \quad a(t) = a_0\left(\frac{\Omega_{\rm M}}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{2/3}$$

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda}}\frac{1}{H_0}\text{Arsinh}\sqrt{\frac{\Omega_\Lambda}{\Omega_{\rm M}}} = 1.38\cdot 10^{10}\text{ yr}$$

$$\Omega_{\rm M} = 0.24\,,\ \ \Omega_\Lambda = 0.76\,,\ \ h = 0.73$$

The size of the Horizon: visible part of the Universe

$$l_{\mathrm{H},0} = a_0 \int_0^{t_0} \frac{dt}{a(t)} = \frac{2}{H_0} \cdot 1.8 = 14.8 \text{ Gpc}$$

Brightness–redshift dependence in the Universe

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

coordinate distance $\chi = \int_{t_i}^{t_0} \frac{dt}{a(t)}$ $z(t) = \frac{a_0}{a(t)} - 1$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_{\mathrm{M}}(z'+1)^3 + \Omega_\Lambda + \Omega_{curv}(z'+1)^2}}$$

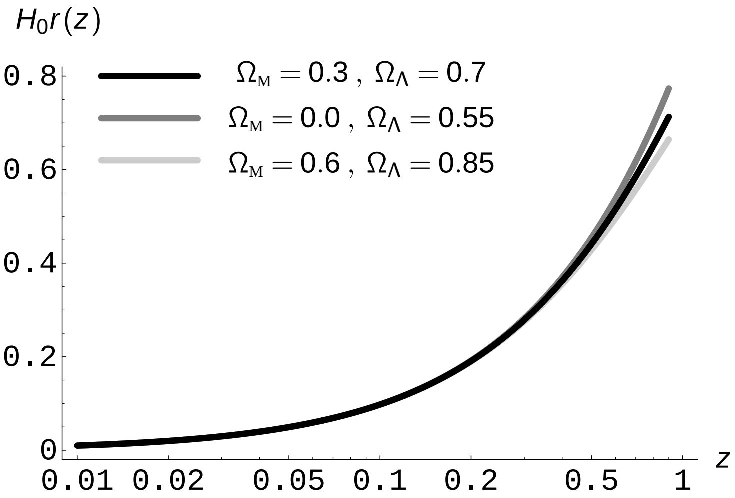
$$a_0^2 H_0^2 \Omega_{curv} = 1\,, \quad \Omega_{\mathrm{M}} + \Omega_\Lambda + \Omega_{curv} = 1$$

$$S(z) = 4\pi r^2(z)\,, \qquad r(z) = a_0 \sinh\chi(z)$$

detector: $N_\gamma \propto S^{-1}$, $\omega = \omega_i/(1+z)$, $dt_0 = (1+z)dt_i$
hence, for the brightness (the flux as mesaured by a detector) one has

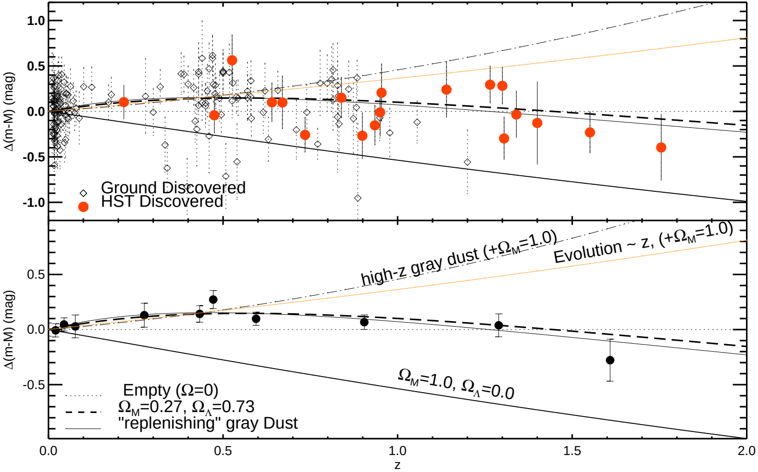
$$J = \frac{L}{(1+z)^2 S(z)} \equiv \frac{L}{4\pi r_{ph}^2}\,, \quad r_{ph} = (1+z)\cdot r(z)$$

Degeneracies in Brightness–redshift dependence



Brightness–redshift dependence: SN Ia

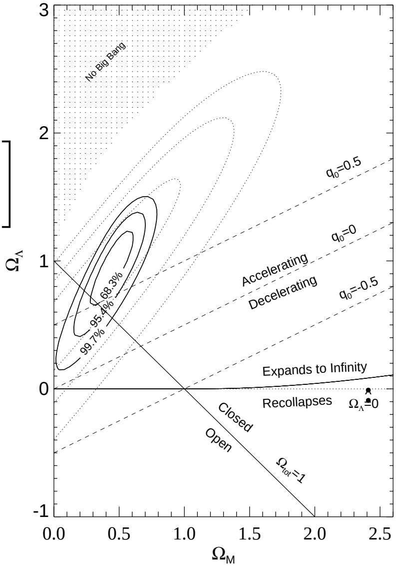
$$\Delta(m - M) = 5 \log \frac{r_{ph}}{r_{ph}(\Omega_{curve} = 1)}$$



Cosmological parameters from SN Ia

$$\chi(z) = \frac{1}{a_0 H_0}\left[z - \frac{z^2}{4}(\Omega_M - 2\Omega_\Lambda) + C_1(\Omega_M - 2\Omega_\Lambda)\cdot z^3 + C_2(2\Omega_M - \Omega_\Lambda)\cdot z^3\right]$$

$$q_0 = -\frac{1}{H_0^2}\left(\frac{\ddot{a}}{a}\right)_0$$



Temperature of the recombination

$$n_e = g_e \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{(\mu_e - m_e)/T} , \quad g_e = 2 ,$$

$$n_p = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} e^{(\mu_p - m_p)/T} , \quad g_p = 2 ,$$

$$n_{\rm H} = g_{\rm H} \left(\frac{m_{\rm H} T}{2\pi}\right)^{3/2} e^{(\mu_{\rm H} - m_{\rm H})/T} , \quad g_{\rm H} = 4$$

T_r: $\quad n_p(T_r) \simeq n_{\rm H}(T_r)$

baryon number conservation

$$n_p + n_{\rm H} = n_{\rm B} , \quad n_{\rm B}(T) = \eta_{\rm B} n_\gamma(T) ,$$

chemical equilibrium

$$p + e \leftrightarrow H + \gamma : \quad \mu_p + \mu_e = \mu_{\rm H}$$

electroneutrality

$$n_p = n_e$$

Temperature of the recombination

$$X_p \equiv \frac{n_p}{n_B}\,, \quad X_H \equiv \frac{n_H}{n_B}\,, \quad \Delta_H \equiv m_p + m_e - m_H = 13.6\ \text{eV}$$

Saha equation

$$X_p + \frac{2\zeta(3)}{\pi^2}\eta_B\left(\frac{2\pi T}{m_e}\right)^{3/2} X_p^2 e^{\frac{\Delta_H}{T}} = 1$$

$$X_H = \frac{2\zeta(3)}{\pi^2}\eta_B\left(\frac{2\pi T}{m_e}\right)^{3/2} X_p^2 e^{\frac{\Delta_H}{T}}$$

recombination: $X_p \sim 1$, $X_H \sim 1$

$$T_r \approx \frac{\Delta_H}{\ln\left(\frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)}\left(\frac{m_e}{\Delta_H}\right)^{3/2}\eta_B^{-1}\right)} \approx 0.38\ \text{eV}$$

Last scattering: $\gamma e \to \gamma e$

$$\sigma_{\rm T} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.67\cdot 10^{-24}\,\text{cm}^2\,, \qquad \tau_\gamma = \frac{1}{\sigma_{\rm T}\cdot n_e(T)}$$

$$n_e^2 = n_p^2 = \left(\frac{m_e T}{2\pi}\right)^{3/2}\frac{2\zeta(3)}{\pi^2}T^3\eta_{\rm B}\mathrm{e}^{-\Delta_{\rm H}/T}$$

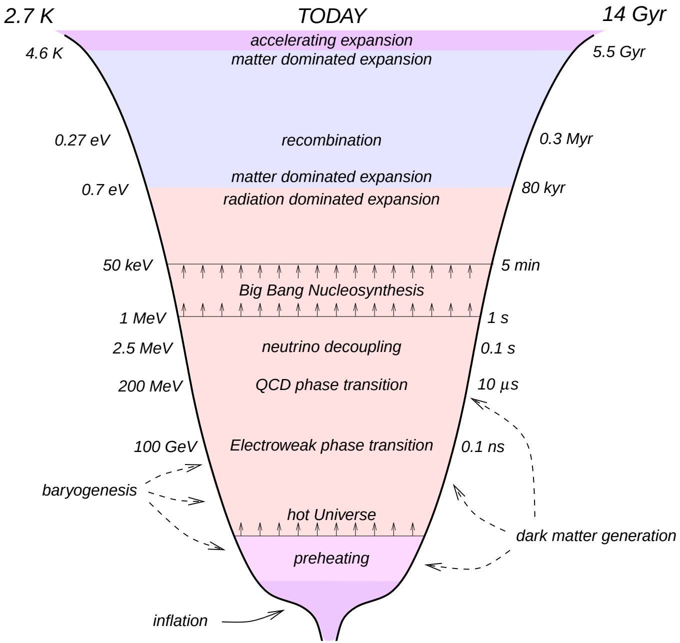
last scattering: $\tau_\gamma(T_f) \simeq H^{-1}(T_f) \simeq t_f$

$$\frac{\Delta_{\rm H}}{T_f} = \ln\left[\sigma_{\rm T}^2 t_r^2\cdot\left(\frac{m_e T_r}{2\pi}\right)^{3/2}\eta_{\rm B}\frac{2\zeta(3)}{\pi^2}T_r^3\right]$$

$$T_f = 0.27\ \text{eV}\,,\quad z = 1100\,,\quad t_f = 2.7\cdot 10^5\ \text{yr}$$

for general processes one should solve kinetic equations

$$\frac{dn_{X_i}}{dt} + 3Hn_{X_i} = \int(production - destruction)$$


2.7 K
TODAY
14 Gyr
4.6 K
accelerating expansion
matter dominated expansion
5.5 Gyr
0.27 eV
recombination
0.3 Myr
0.7 eV
matter dominated expansion
radiation dominated expansion
80 kyr
50 keV
5 min
Big Bang Nucleosynthesis
1 MeV
1 s
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter generation
preheating
inflation