↓ Recommended reading ↓

A Romance of
Many Dimensions
E. Abbott

$\boxed{1884}$

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A.T. Filippov: 14-15 July, 2005, DIAS-TH

Low dimensional gravity { hep-th/0504101, hep-th/0505060 }

(also - some transparences presented to the Th. Div. Conf. at Lebedev Inst. apr.-May 2005

| some techn details in hep-th/0307269

• Philosophy (general motivation)

• $\mathcal{L}$ and $\mathcal{H}$ dynamics MORE IMPORTANT than (R.) geometry. Geometric variables are shown to be DYNAMICAL VARIABLES also.

• The aim is NOT ONLY to derive a metric but to construct $\mathcal{L}$ and $\mathcal{H}$ and to solve DYNAMICAL EQUATIONS → → and then TO QUANTIZE if possible

• To quantize we have to find → → an EXPLICITLY INTEGRABLE approxim, (like OSCILLATOR in Q.F.T.: in gravity candidates are B.H., Cosmologies, some Waves (?) – almost all related to the Liouville Equation)

• Today we do not know Quant. Grav. effects (accessible to direct experim.) The best quantum connection – through String Theory, having no direct connection to experim., the only hope – Quantum Gravity

$$\begin{tikzcd} Gr. \arrow[r, bend left, "quant."] & S.T. \arrow[l, bend left, "exptl."] \end{tikzcd}$$

Possibly, one of the best things to do is To develop INTUITION on simplest objects. B.H. -- Cosmologies -- Waves

• (Low Dim. Dynamics desribing H.D. world)

• Detailed motivation

- B.H., Cosm., Waves from H.D. SUGRA by dimensional Reduction. Novel feat: many (abelian) gauge fields, scalar fields (as well as H.Rank forms)

KK-MF reduct.

In L.D. (spherical, cylindrical ... symm.) $\longrightarrow$ Potentials $V(\varphi, \psi)$, scalar field ψ

dilaton

- Traditionally, B.H., C., W. are treated separately and differently (and by different people as well!)

In reality, there is a STRONG CONNECTION between these objects, possibly, they are related physically,

Soliton-like waves?

something like(?)

$$\begin{tikzcd} B.H. \arrow[r, squiggly, "W"] \arrow[d, squiggly] & Cosm \arrow[d, squiggly, "W"] \arrow[dl, squiggly] \\ W \arrow[r, squiggly] & B.H. \end{tikzcd}$$

- To study such thing we need Integrable Models (2+1, 1+1 $\nearrow$ 0+1 B.H., $\searrow$ 1+0 C.)

Axial symm.: 2 + 1 or 1+1 (stationary)

Spherical, flat, cylindrical: 1 + 1 (more symmetries!)

- Dim. red. $1+1 \begin{smallmatrix}\nearrow 0+1 \\ \searrow 1+0\end{smallmatrix}$ most interesting!

! Naive approach may be WRONG or INCOMPLETE

More general: separation of r and t as dimensional reduction.

■

3. Integrable low-dimensional theories of gravity originate from H-D grav., supergravity, superstrings

$$11\text{-dim} \longrightarrow 10\text{-dim} \longrightarrow 8\text{-dim} \longrightarrow 1+1 \text{ dim}$$

$8\text{-dim} \dashrightarrow 1+2$

$1+1 \to 0+1$ 'cosmol.' ; $1+1 \to 1+0$ 'black holes'

$(1+1) \sim (t, r)$, $0+1 \sim t$ depend., $1+0 \sim r$ depend.

- $(0+1)/(1+0)$ theories : dynamical (NL) systems with 1 constraint ■
- $(1+1)$ theories: field theories with 2 constraints (Energy $= 0 =$ Moment)
- Integrable LD models are produced by dim. red. of Gravity + abelian g.f. + higher rank forms + scalar matter.

Non abelian $\not\Rightarrow$ integrable th.

In dim $1+1$ ($0+1$, $1+0$) the dyn. var.

g_{ij} $(i = 0, 1)$, φ (dilaton = metr. in H.D.), scalar fields ($\psi^{(n)}$)

$$\mathcal{L} = \sqrt{-g}\Big(\varphi R^{(2)}(g) + V(\varphi, \psi) + \sum_n Z^{(n)}(\nabla\psi^{(n)})^2 + W(\varphi)(\nabla\varphi)^2 + (\text{total derivatives})\Big)$$

boundary terms

$Z^{(n)}(\varphi, \psi)$

appendix 3A

$\mathcal{D}=2$ FLATLAND

- $ds^2 = g_{00}(dx^0)^2 + 2g_{10}\,dx^0 dx^1 + g_{11}(dx^1)^2$

$(x^0, x^1) = (t, r) \qquad (g_{01} = g_{10})$

$g = \det(g_{ij}) = g_{00}\,g_{11} - g_{01}^2 < 0$

Diagonal metric: $\bullet\ ds^2 = e^{2\alpha}dr^2 - e^{2\gamma}dt^2$

$$R^{(2)} = \sum_i g^{ii}R_{ii} = 2e^{-2\gamma}(\ddot\alpha + \dot\alpha^2 - \dot\alpha\dot\gamma) - 2e^{-2\alpha}(\gamma'' + \gamma'^2 - \gamma'\alpha')$$

General expression (Gauß)

- $ds^2 = A\,dr^2 - C\,dt^2 + 2\mathcal{D}\,dr\,dt, \quad \Delta \equiv \mathcal{D}^2 + AC$

$$\sqrt{\Delta}\,R^{(2)} = \underbrace{\left(\frac{\dot A}{\sqrt\Delta}\right)^{\cdot} - \left(\frac{C'}{\sqrt\Delta}\right)' + \left(\frac{\dot{\mathcal{D}}}{\sqrt\Delta}\right)' + \left(\frac{\mathcal{D}'}{\sqrt\Delta}\right)^{\cdot}}_{\text{total derivative}} + \underbrace{\frac{1}{2\Delta^{3/2}}\begin{vmatrix} A & C & \mathcal{D} \\ A' & C' & \mathcal{D}' \\ \dot A & \dot C & \dot{\mathcal{D}} \end{vmatrix}}_{\Phi};$$

$$\Phi\,dr\wedge dt = -\frac{1}{2}d\left(\frac{\mathcal{D}}{\sqrt\Delta}\,d\ln\frac{A}{C}\right)$$

$\Rightarrow \int \sqrt{\Delta}\,R^{(2)}\,dr\wedge dt$ = Boundary terms

$\mathcal{D}=d$: $\bullet\ ds^2 = \sum_{i=1}^{d}\varepsilon_i e^{2F_i}dx_i^2, \quad \varepsilon_i = \pm 1$

$$R = \sum_{i=1}^{d}\varepsilon_i e^{-2F_i}\left[2F_{i,i}\Sigma_{,i} - 2\Sigma_{,ii} - (\Sigma_{,i})^2 - \bar\Sigma_i\right]$$

$$\Sigma_{,i} = \sum_{m\neq i} F_{m,i}; \quad \bar\Sigma_i = \sum_{\ell\neq i} F_{\ell,i}^2$$

(see Landau, Lifschitz)

Appendix:

$\longrightarrow$ Return to $\underline{D=2}$ formulas

$$\left[\begin{array}{l} \nabla_i \psi \equiv \partial_i \psi \equiv \psi_{,i} \\ \nabla^2 \psi \equiv \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ik} \partial_k \varphi \right) \end{array}\right.$$

Light-cone metric: $\alpha = \gamma$

- $ds^2 = e^{2\alpha}(dr^2 - dt^2)$;

$t = u + v$; $r = u - v$; $e^{2\alpha} = f$

- $ds^2 = -4 f(u,v)\, du\, dv$ (Also possible: $A = C = 0$, $u = r$, $v = t$, $D = -25$
- $R = \frac{1}{f}(\partial_t^2 - \partial_r^2) \ln|f| = \frac{1}{f} \partial_u \partial_v \ln|f|$

$$\nabla^2 \psi = -\frac{1}{f} \varphi_{,uv} \equiv -\frac{1}{f} \partial_u \partial_v \varphi$$

$$R = -\nabla^2 \ln|f|$$

$$\sqrt{|g|}\, R = 2|f| \frac{1}{f} \partial_u \partial_v \ln|f| = \overbrace{2 \varepsilon_f\, \partial_u \partial_v \ln|f|}^{\text{total deriv.}}$$

Exercise: derive the E.O.M. for φ, ψ, f using the above formulas

NB: The only nontriv. thing — to compute the constraints.

- Spherical reduction $d=4 \mapsto d=2$

Diagonal: $ds^2 = e^{2\alpha} dr^2 - e^{2\gamma} dt^2 + \underbrace{e^{2\beta}}_{(\text{radius})^2} d\Omega_2^2$

$\alpha(r,t)$
$\beta(r,t)$
$\gamma(r,t)$

$d\Omega_2^2 = (d\theta^2 + \sin^2\theta \, d\varphi^2)$; metric on $S^{(2)}$

With the above general formula:

$$R^{(4)} = \underbrace{R^{(2)}}_{\text{using coord.}} + \underbrace{2e^{-2\beta}}_{R(S^{(2)})} + 2(\nabla\beta)^2 + \frac{(\text{total deriv})}{\sqrt{|g^{(4)}|}}$$

using coord. invar. we may write this in the general metric!

$$\sqrt{|g^{(4)}|} = \underbrace{e^{\alpha+\gamma}}_{\sqrt{|g^{(2)}|}} \cdot e^{+2\beta} = \sqrt{|g^{(2)}|}\, e^{2\beta}$$

$$\int d^4x \sqrt{|g^{(4)}|}\, R^{(4)} = \int dt\, dr \sqrt{|g^{(2)}|} \underbrace{\int \sin\theta\, d\theta\, d\varphi\, R^{(4)}}_{\text{integrate out } \theta,\varphi}$$

$$\mathcal{L}^{(4)} = \sqrt{|g^{(4)}|}\left(R^{(4)} + V_{(\psi)} + Z(\nabla\psi)^2\right)$$

$$\mathcal{L}^{(2)}_{\text{eff}} = e^{\alpha+\gamma+2\beta}\left[R^{(2)} + 2e^{-2\beta} + 2(\nabla\beta)^2 + V(\psi) + Z(\nabla\psi)^2\right]$$

(here ∇ – w.r.t 2-dim. m.)

$e^{2\beta} \equiv \varphi$ dilaton

- If $V \equiv 0$, $\nabla\psi = 0$ (or $Z \equiv 0$) $\longrightarrow$ pure dilaton gravity giving the Schwarzschild B.H.
- Not difficult to include Maxwell field → Reissner-Nordstrøm B.H.

4. • $\mathcal{L}_{eff} = \sqrt{-g}\left(\varphi \cdot R + V(\varphi,\psi) + Z(\varphi,\psi)\, g^{ij}\partial_i\psi\,\partial_j\psi\right)$

$ds^2 = -4f(u,v)\,du\,dv\,; \qquad g_{ij} = \begin{pmatrix} 0 & -2f \\ -2f & 0 \end{pmatrix}$

(1) $\partial_u\partial_v\varphi + fV = 0\,; \qquad V'_\psi = \partial_\psi(V(\varphi,\psi)), \dots$

(2) $\partial_u(Z\partial_v\psi) + \partial_v(Z\partial_u\psi) + fV'_\psi = Z'_\psi\,\partial_u\psi\,\partial_v\psi$

(3) $\boxed{f\partial_i\left(\frac{\partial_i\varphi}{f}\right) = Z(\partial_i\psi)^2 \quad , \; i = u, v}$ Constraint

If $V'_\psi(\varphi,\psi_0) = 0 \;\Rightarrow\; \exists$ solution with $\psi \equiv \psi_0$

('scalar vacuum')

• Exercise: 1. Using (3) prove that

$\exists\, a(u), b(v):\; f = \varphi_u\, b'(v) = \varphi_v\, a'(u)$

and that $\Rightarrow \quad \varphi(u,v) = \varphi(\underbrace{a(u) + b(v)}_{\equiv\tau}) \equiv \varphi(\tau)$

$\Rightarrow f(u,v) = \varphi'(\tau)\, a'(u)\, b'(v)$

2. Using (1), show that

$\varphi''(\tau) + \varphi'(\tau)\, V(\varphi) = 0 \;\Rightarrow\; \left[\varphi'(\tau) + N(\varphi(\tau))\right]'_\tau = 0$

where $N(\varphi) \overset{\text{def}}{=} \int d\varphi\, V(\varphi,\psi_0)$

$\Rightarrow \quad f(u,v) \equiv h(\tau)\, a'(u)\, b'(v) = [M - N(\varphi)]\, a'(u)\, b'(v)$

$\boxed{h(\tau) = M - N(\varphi)}$

3. $\varphi(\tau)$ can be found from $\boxed{\int \frac{d\varphi}{M - N(\varphi)} = \tau - \tau_0}$,

! Horizon: $\overset{\text{(always exists)}}{\exists}\; M - N(\varphi_h) = 0, \quad \varphi_h \to$ horizon appendix. Weyl in (Flatland) Removing $(\nabla\varphi)^2$ (4a)

$$\mathcal{L} = \sqrt{|g|}\left[\varphi R^{(2)}(g) + V(\varphi,\psi) + W(\varphi)(\nabla\varphi)^2 + Z(\varphi)(\nabla\psi)^2\right]$$

• Show that $\mathcal{L} \to \bar{\mathcal{L}}$ when $g_{ij} = w(\varphi)\bar{g}_{ij}$

(*) $$\begin{cases} \bar{\mathcal{L}} = \sqrt{|\bar{g}|}\left[\varphi R^{(2)}(\bar{g}) + w(\varphi)V(\varphi,\psi) + Z(\varphi)(\nabla\psi)^2\right] \\ \text{if} \quad w(\varphi) = \exp\int d\varphi\,(-W(\varphi)) \end{cases}$$

Thus, the W-term is removed.

Exercise

1. Show that in (u,v) coord.

$$\mathcal{L}_{ef} = \varphi(\log f)_{,uv} + fV - W(\varphi)\varphi_u\varphi_v - Z(\varphi)\psi_u\psi_v$$

2. Show that

$$\mathcal{L}_{ef} = \overbrace{\varphi(\log\bar{f})_{,uv} + \bar{f}w(\varphi)V(\varphi) - Z(\varphi)\psi_u\psi_v}^{\bar{\mathcal{L}}_{ef}} + \left(\varphi\frac{w'}{w}(\varphi)\varphi_u\right)_{,v} - \left[\frac{w'}{w}(\varphi) + W(\varphi)\right]\varphi_u\varphi_v$$

3. This gives $\int\mathcal{L}_{ef} = \int\bar{\mathcal{L}}_{ef}$ + (bound. term

and $\bar{\mathcal{L}}_{ef} = \varphi(\log\bar{f})_{uv} + \bar{f}wV - Z\psi_u\psi_v$

if $w(\varphi) = \exp\int d\varphi\,(-W(\varphi))$

and the ~~[illegible]~~ statement (*) is true in general metric.

Thus, spherical grav. Lagr.:

$$\mathcal{L}^{(2)}_{\text{eff}} = \sqrt{|g|}\Big\{\varphi R^{(2)}(g) + \overbrace{2}^{V} + \overbrace{\frac{1}{2\varphi}}^{W}(\nabla\varphi)^2 + Z(\nabla\psi)^2 \Big\}$$

$$\begin{cases} |g| = e^{\alpha+\gamma} \\ \varphi = e^{2\beta} \end{cases}$$

$$\boxed{\frac{w'(\varphi)}{w(\varphi)} = -W(\varphi) = \frac{-1}{2\varphi}}$$ To remove $(\nabla\varphi)^2$-te

$$w(\varphi) = 1/\sqrt{\varphi}$$

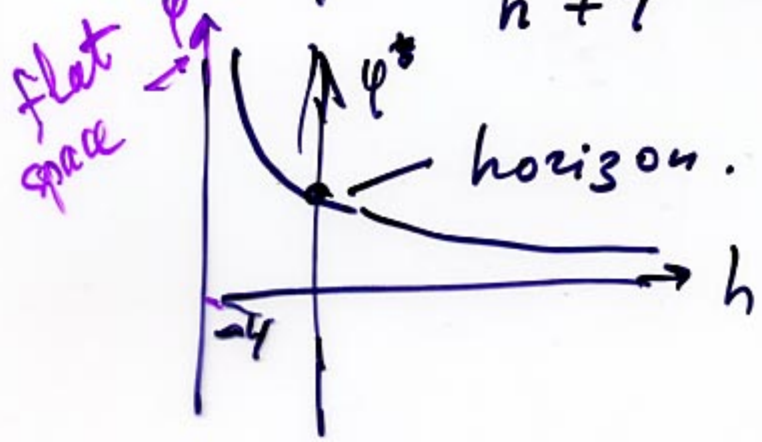
$$\hookrightarrow \bar{\mathcal{L}}^{(2)}_{\text{eff}} = \sqrt{|\bar g|}\Big\{\varphi R^{(2)}(\bar g) + \overbrace{2/\sqrt{\varphi}}^{\bar V(\varphi)} + Z(\nabla\psi)^2\Big\}$$

$$\bar N(\varphi) = \int d\varphi\, \bar V(\varphi) = 4\sqrt{\varphi}$$

$h = w.\bar h$ rem.!

$$\bar h = M - \bar N(\varphi);\quad \boxed{h = \frac{1}{\sqrt{\varphi}}\left(M - 4\sqrt{\varphi}\right)}$$ final form!

$$\Rightarrow \sqrt{\varphi} = \frac{M}{h+4} \text{ (dimensionless param)} \quad \text{or} \quad \varphi = \frac{M^2}{(h+4)^2}$$



Note: $\varphi = r^2$, then

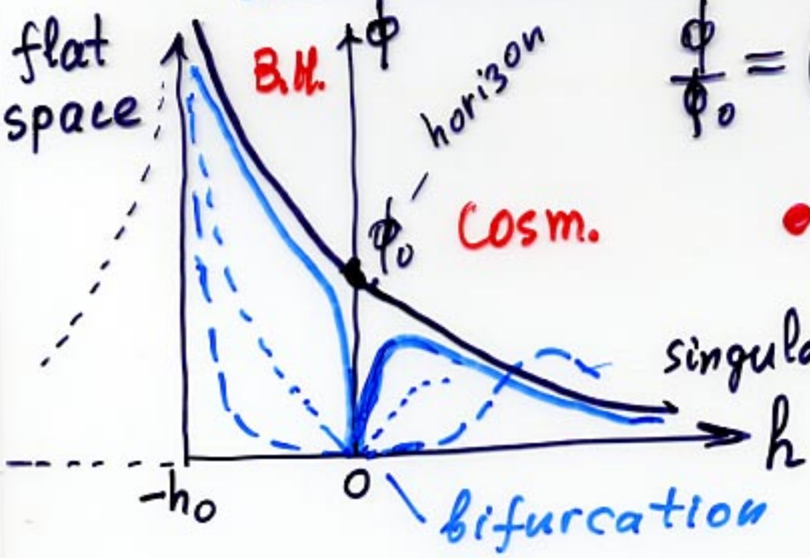
$$\frac{r}{r_0} = \frac{M/M_0}{\frac{h}{h_0} + 1}$$

(returning to dimensional notation)

5.

SUMMARY of 'black holes':

$\phi \equiv \sqrt{\varphi}$ (above)



$$\frac{\phi}{\phi_0} = \left(1+\frac{h}{h_0}\right)^{-1} \quad \text{— Schwarzschild.}$$

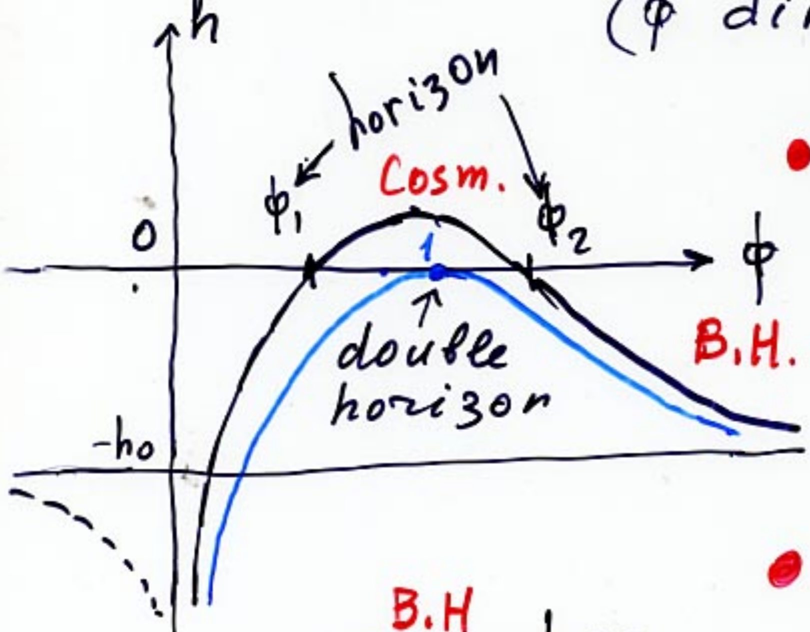
- $$\frac{\phi}{\phi_0} = \left|\frac{h}{h_0}\right|^{2\delta}\left(1+\left|\frac{h}{h_0}\right|^{1+2\delta}\right)^{-1}$$

$$2\delta = \sqrt{1+2C_0^2} - 1$$

$C_0 = Z\dot{\psi}$ (integral)

Reissner – Nordstrøm B.H.

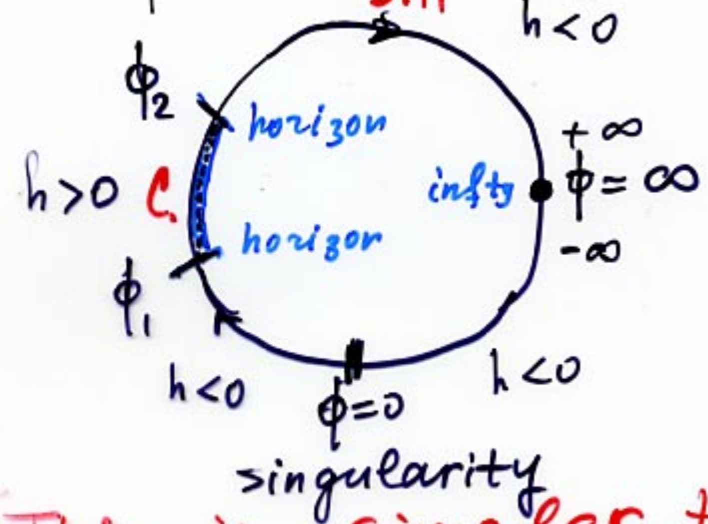
(ϕ dimensionless)



- $$\frac{h}{h_0} = \frac{1}{\phi}\left[\left(\phi_1 + \frac{1}{\phi_1}\right) - \left(\phi + \frac{1}{\phi}\right)\right]$$

Extreme R-N B.H. ($\phi_1 = 1$):

- $$\frac{h}{h_0} = -\left(1-\frac{1}{\phi}\right)^2$$



Simple horizon:

$$\phi - \phi_0 \sim e^{v_0 \tau}, \; v_0\tau \to -\infty$$

Double horizon:

- $$\boxed{\phi - \phi_0 \sim \frac{2}{v_0' \tau}, \quad \tau \to \infty}$$

Degenerate B.H.

[This is simpler than the Szekeres – Kruskal diagram (actually 1+1 dim.)]

6.

Reduction to Lineland from Flatland (or, better, Surfaceland)

* $\mathcal{L}_{g.f} = \varphi F_{,uv} + e^F V - Z \psi_u \psi_v$
* $F \equiv \log|f(u,v)|$ Let φ, F, ψ – fuctions of τ

$\tau = a(u) + b(v)$, or simply $\tau = u + v$

Then $\mathcal{L}_{g.f} \to \underbrace{\varphi \ddot{F}}_{(\varphi\dot{F})^{\bullet} - \dot{\varphi}\dot{F}} - Z\dot{\psi}^2 + e^F V$

$[\dot{\psi} \equiv \partial_\tau \psi = \frac{d}{d\tau}\psi(\tau)]$

$$\mathcal{L}^{eff}_{g.f} = \underbrace{-\dot{\varphi}\dot{F} - Z(\varphi)\dot{\psi}^2}_{} + \underbrace{e^F V(\varphi,\psi)}_{\text{Potential}}$$

$\psi_1 = \frac{1}{2}(F+\varphi)$, $\psi_2 = \frac{1}{2}(F-\varphi) \mapsto$ $Z<0(!)$

$-\dot{\varphi}\dot{F} = -\dot{\psi}_1^2 + \dot{\psi}_2^2$

$\mathcal{L} = T - V$; $\mathcal{H} = T + V = -\dot{\varphi}\dot{F} - Z\dot{\psi}^2 - e^F V$

Restore constraint: $l(\tau)$ – Lagrange multip

$$\boxed{\mathcal{L} = \frac{1}{l}(-\dot{\varphi}\dot{F} - Z\dot{\psi}^2) + l e^F V(\varphi,\psi)}$$

$$\frac{\delta \mathcal{L}}{\delta l} = \frac{1}{l^2}(\dot{\varphi}\dot{F} + Z\dot{\psi}^2) + e^F V \underset{l=1}{=} -\mathcal{H}$$

Instead of $l=1$ one may define

$\underbrace{t = \int_0^\tau l(\tau)\,d\tau}_{\text{invariant evolution param}}$ – the same result with evolution param t instead of τ.

Dimensional reductions often give

- Integrable theories in $0+1$ dim or $(1+0)$

I. N-Liouville models

consider a class of models

$$\mathcal{L} = \frac{1}{l}\left(-\dot{F}\dot{\varphi} - \sum_{n=3}^{N} z_n \dot{\psi}_n^2\right) + l\varepsilon \sum_{n=1}^{N} \frac{1}{2} g_n e^{q_n}$$

$$\boxed{q_n = \sum_{m=1}^{n} \psi_m a_{mn}}, \quad \psi_1 = \frac{1}{2}(F+\varphi), \quad \psi_2 \equiv \frac{1}{2}(F-\varphi)$$

Suppose that $z_n = -1$

$$\begin{cases} a_{1n} = 1 + a_n \\ a_{2n} = 1 - a_n \end{cases}$$

! (if $z_n = c_n \xi(\varphi)$ we can include $\xi(\varphi)$ in $1/l$)

Suppose that a_{mn} satisfy the relations ($\varepsilon_1 = -1$, $\varepsilon_m = 1$, $m \geq 2$)

$$\boxed{\sum_{m=1}^{N} \varepsilon_m a_{ml} a_{mn} = \frac{1}{\gamma_n} \delta_{ln}}$$ p-orth.

Then the e.o.m. are reduced to N Liouville equations

Horison: $F \to -\infty$, $\tau \to \pm\infty$

$$\boxed{\ddot{q}_n + \bar{g}_n e^{q_n} = 0}, \quad \bar{g}_n = -\frac{g_n}{\gamma_n}\varepsilon, \quad \varepsilon \equiv h/|h|$$ NB

($\bar{g}_n \gtrless 0$ or $\bar{g}_n = 0$)

!

$$\boxed{e^{-q_n} = \frac{|\bar{g}_n|}{2\mu_n^2}\left(e^{\mu_n(\tau-\tau_n)} + e^{-\mu_n(\tau-\tau_n)} + 2\varepsilon_n\right)}$$

$\mu_n^2 \in \mathrm{Re}$ $\quad \varepsilon_n \equiv \bar{g}_n/|\bar{g}_n|$

Exercise: study this solut.!

The constraint: $\boxed{\sum_{n=1}^{N} \gamma_n \mu_n^2 = 0}$

Moduli: $\mu_1, \ldots, \mu_{N-1}; \tau_1, \ldots, \tau_{N-1}$

- One and only one γ_n is < 0 (γ_1)

$$\sum \gamma_n = 0$$

Generalization to $D=2$

1+1 dimensional integrable N-Liouville theories

!? NOT true in spherical theories. Possible for other symm.?

Let $Z_n = \text{const}$ $(\underline{Z_n = -1})$

$$\boxed{\mathcal{L} = \varphi \partial_u \partial_v \ln|f| - \sum_n Z \partial_u \psi^{(n)} \partial_v \psi^{(n)} + f V(\varphi, \psi^{(n)})}$$

$$V = \sum \varepsilon g_n e^{q_n - \ln|f|}$$

$\ln|f| \equiv F$, $\varepsilon = \operatorname{sgn} f$, $q_n = F + a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}$

$$(*) \quad \boxed{C_i \equiv f \partial_i (\partial_i \varphi / f) + \sum_{n=3}^{N} (\partial_i \psi_n)^2 = 0}, \quad i = u, v$$

non-linear!

$$\boxed{\partial_u \partial_v q_n - \tilde{g}_n e^{q_n} = 0} \quad \text{Liouville.}$$

$$\Big[\tilde{g}_n = \varepsilon \gamma_n g_n; \quad \gamma_n^{-1} = \sum \varepsilon_m a_{mn}^2; \quad \begin{cases} \varepsilon_1 = -1 \\ \varepsilon_m = +1, \ m \geq 2 \end{cases}$$

The conditions on a_n, a_{mn} (orthogonality) are the same as in 0+1 dimensional case

Solving the constraints (*) is a non-trivial problem

They can be solved because the norms $\gamma_n^{-1} \equiv \sum \varepsilon_m a_{mn}^2$ satisfy the constraint

$$\boxed{\sum_{n=1}^{N} \gamma_n = 0. \quad (!)}$$

Conformal spin repr.

- Let $e^{-q_n/2} \equiv X_n$ Liouville is equiv. to

$$-X_n \partial_u \partial_v X_n + \partial_u X_n \partial_v X_n = \gamma_n \frac{\varepsilon g_n}{2} \equiv \frac{\tilde{g}_n}{2}$$

$$\boxed{X_n = a_n(u) b_n(v) + \bar{a}_n(u) \bar{b}_n(v)}, \quad a_n, b_n \text{ - arbit}$$

$s = -1/2$ $\quad \bar{a}_n = -a_n(u) \int \frac{w_a^{(u)}}{a_n^2(u)} du \qquad \bar{b}_n = -b_n(v) \int \frac{w_b^{(u)}}{b_n^2(v)} dv$

Constraints: $C_i = \sum_{m=1}^{N} \gamma_m \left(q_{m,i}^2 - 2 q_{m,ii} \right), \quad i = u, v$

$$\boxed{C_u = 4 \sum_{m=1}^{N} \gamma_m \frac{a_m''(u)}{a_m(u)} = -4 C^2(u)} \quad (\text{or, } C_u \equiv 0) \qquad \gamma_m = \frac{1}{\lambda_m}$$

contrib. of free scalar field (not Liouville)

- $C_u = 4 \sum \gamma_m \left[\left(\frac{a_m'}{a_m} \right)' + \left(\frac{a_m'}{a_m} \right)^2 \right]$

Let $\frac{a_n'}{a_n} = \rho_n(u) - X(u)$ Then $C_i = -C^2(u)$

$\uparrow$ arbitrary

if and o. if

$$\boxed{X = \frac{1}{2} \left[\ln \left| \sum \gamma_n \rho_n \right| \right]' + \frac{1}{2} \frac{\sum \gamma_n \rho_n^2}{\sum \gamma_n \rho_n} + C^2}$$

$$\rightarrow \quad a_n = \exp \int [\rho_n(u) - X(u)] \, du$$

(actually, we have $N-1$ arbitrary functions, not N !)

- e.g. for $N=2$, a_1 and a_2 are defined in terms of $\rho_1(u) - \rho_2(u)$

10. One may present the solution in diff. forms!

! The simplest possible repr. of the solutions

(the simpler one may be written with g.f.)

- Take arbitrary $\mu_n(u)$, $\nu_n(v)$ satisfying the constraints

$$\sum \gamma_n \mu_n^2(u) = 0 = \sum \gamma_n \nu_n^2(v)$$

- Then the general solution is:

$$(*)\quad \begin{cases} a_n(u) = (\sum \gamma_m \mu_m)^{-1/2} \exp(\int \mu_n(u)\, du) \\ b_n(v) = (\sum \gamma_m \nu_m)^{-1/2} \exp(\int \nu_n(v)\, dv) \end{cases} !$$

$$e^{-q_n/2} \equiv X_n(u,v) = a_n(u) b_n(v) + \bar{a}_n(u) \bar{b}_n(v)$$

$$\text{where} \begin{cases} \bar{a}_n = C_n a_n(u) \int \frac{du}{a_n^2(u)} \\ \bar{b}_n = D_n b_n(v) \int \frac{dv}{b_n^2(v)} \end{cases}$$

C_n or D_n = fixed, otherwise arbitrary

In terms of a_n, b_n one may give an interesting classification of the Liouville solutions.

They are also convenient for quantizing having a group th. meaning. ($-\frac{1}{2}$ conf. spin)

(*) One can fix coordinates (A,B) instead of (u,v) by choosing $\sum \gamma_m \mu_n = A'(u)$ $\quad \sum \gamma_m \nu_m = B'(v)$

Instead of $\mu_n(u), \nu_n(v)$ we may consider $\xi_n(u), \eta_n(v)$

- Localized (soliton-like) waves.

$$\hat{\xi}_n = \text{const}, \quad \hat{\eta}_n = \text{const}, \quad \hat{\xi}_n \neq \hat{\eta}_n$$

$(n = 2, \dots, N; \ \hat{\xi}_n \sim \mu_n(u)/\mu_1(u))$

$$\boxed{X_n = d_n \, \mathrm{ch}[c_n(r - v_n t)]} \qquad \begin{cases} r = u + v \\ t = u - v \end{cases}$$

$$\begin{cases} c_n = \dfrac{1}{2\hat{g}_n}(\hat{\xi}_n + \hat{\eta}_n); \\[2ex] v_n = \dfrac{\hat{\eta}_n - \hat{\xi}_n}{\hat{\eta}_n + \hat{\xi}_n}; \end{cases} \qquad \underline{\sum \hat{\eta}_n^2 = \sum \hat{\xi}_n^2 = 1}^{\,*)}$$

$n = 2, \dots, N$

Ingoing and outgoing 'spherical' waves are localized.

Waves of matter (scalar) coupled to dilaton gravity exist in N-Liouville model in $d. = 1+1$.

(approximately related to spherical waves)?

Problem: in spherical grav. $Z_n \neq -1$ (in fact, $Z_n \sim \varphi$).

*) These new moduli are introduced by further gauge fixing – introducing $A(u), B(v)$ coord.

12. Cylindrical waves and axisymmetric BH

$$ds^2 = g_{ij}dx^i dx^j + W\left[e^{\psi}(dx^3 + A\,dx^4)^2 + e^{-\psi}(dx^4)^2\right]$$

1. Cylind. $x^1, x^2 = t, r;\ x^3 = z,\ x^4 = \varphi$ — 1+1 metric

$g_{ij}(t,r)$, $W(t,r)$, etc.

2. Axisym. $x^1, x^2 = z, r;\ x^3 = \varphi,\ x^4 = t$ — 0+2 metric

$g_{ij}(z,r)$, $W(z,r)$, etc. (F. Ernst equation)

What is dilaton, what is a 'material' field (write Energy-momentum and try to see)

- A. Einstein, N. Rosen (1937) – special case of cylindr. waves (diagonal ds^2)

The key equation is linear

$$(\partial_r^2 - \partial_t^2)\,\sigma(r,t) + \frac{2\lambda+1}{r}\,\sigma(r,t) = 0, \quad \lambda > -\tfrac{1}{2}$$

Solutions: $\sigma = \int_0^{\pi} d\alpha\, q(t - r\cos\alpha)\sin^{2\lambda}\alpha$

($\lambda = \tfrac{1}{2} \to$ d'Alembert) $\sigma = \int_0^{\infty} d\alpha\, q(t \pm \cosh\alpha)\sinh^{2\lambda}\alpha$ &

using R.S. Ward

- More general case (Woodhouse, (1989)

The key equation is nonlinear

$$\partial_r(r J^{-1}\partial_r J) - \partial_t(r J^{-1}\partial_t J) = 0 \qquad J = 2\times 2 \text{ symm. matrix}$$

('σ-model' equations)

A class of solutions by matrix 'generating' 'functions' (similar to F. Ernst eq.)

Sol. 1 Very general formula (in our Weyl frame) (19)

$$\boxed{d\tilde{s}^2 = -4\,[M - N(\varphi)]\, da\, db}\,, \quad \tau \equiv a+b$$

- $d\tau = \frac{d\varphi}{M - N(\varphi)}$ $\quad a - b = t,\ a + b = \tau$ $\quad \varphi = \varphi(a+b)$

$$\boxed{d\tilde{s}^2 = [M - N(\varphi)]\,dt^2 - \frac{d\varphi^2}{[M - N(\varphi)]}}$$

φ is like r in general

→ For $\mathcal{E}$-M in dim. $d = n+2$, $\nu \equiv \frac{1}{n}$

$$ds^2 = \frac{d\tilde{s}^2}{\varphi^{1-\nu}} = 8n^2\left(1 - \frac{M/2n^2}{r^{n-1}} - \frac{\bar{\beta}Q^2}{2n(n-1)}\frac{1}{r^{2(n-1)}}\right) da\, db$$

Standard Weyl frame

$(\beta<0)$ $\quad \frac{M}{2n^2} \equiv 2m\,; \quad -\frac{\bar{\beta}Q^2}{2n(n-1)} \equiv q^2$ $\quad$ by def. $\quad \varphi^{2\nu} = r^2$

- $a \stackrel{\text{def}}{=} \lambda \ln \alpha(u)\,, \quad b \stackrel{\text{def}}{=} \lambda \ln \beta(v)\,,$

$$\lambda \ln(\alpha\beta) = \int \frac{d\varphi}{M - N(\varphi)} \equiv \tau\,; \quad \ln\frac{\alpha}{\beta} = \gamma t \quad (\gamma^2 \equiv 1/4n^2)$$

$$\rightarrow \boxed{2\,ds^2 = \left(1 - \frac{2m}{r^{n-1}} + \frac{q^2}{r^{2(n-1)}}\right)^{-1} dr^2 + (1 - \dots)\,dt^2} \quad (RN)$$

With a different choice of $\lambda = -1/V(\varphi_h)$, one may get a Szekeres-Kruskal type coordinates:

$$ds^2 = -4\,\frac{M - N(\varphi)}{\exp\left[-V(\varphi_h)\int\frac{d\varphi}{M - N(\varphi)}\right]}\, d\alpha\, d\beta$$

In $d=4$, $Q=0$ this will give the standard S-K:

$$ds^2 = 2\,\frac{r_h^2}{r}\, e^{-r/r_h}\, d\alpha\, d\beta\,, \quad \text{etc. etc.}$$

VdA, Cav. ATF IJMPD (95-96) Ph.L.B (98)

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REM : ~~HOW~~ IT WORKS. (Appendix)

- K-M-K-F - reduction $(D+\overset{\text{torus}}{p} \mapsto D)$

$$ds^2 = \underbrace{g_{\alpha\beta}\,dx^\alpha dx^\beta}_{D} + h_{ij}(dy^i + A^i_\alpha dx^\alpha)(dy^j + A^j_\beta dx^\beta)$$

$$G_{\alpha\beta} = g_{\alpha\beta} + h_{ij}A^i_\alpha A^j_\beta; \quad G_{\alpha i} = h_{ij}A^j_\alpha, \quad G_{ij} = h_{ij}$$

$$G^{\alpha\beta} = g^{\alpha\beta}; \quad G^{\alpha i} = -g^{\alpha\beta}A^i_\beta, \quad G^{ij} = h^{ij} + A^{i\alpha}A^j_\alpha$$

scalar fields

$$\overset{(D+p)}{R}[G] = \overset{(D)}{R}[g] - \underbrace{\frac{2}{\sqrt{h}}\boxed{g}\sqrt{h}} + \frac{1}{4}\partial_\alpha h^{ij}\partial^\alpha h_{ij} + $$
$$+ \frac{1}{4}(h^{ij}\partial_\alpha h_{ij})(h^{kl}\partial^\alpha h_{kl}) - \frac{h_{ij}}{4}F^i_{\alpha\beta}F^{j\alpha\beta}$$

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha \leftarrow \text{Abelian g. fields}$$

$$\sqrt{G}\,\overset{(D+p)}{R}[G] = \sqrt{-g}\,\sqrt{h}\left\{\overset{(D)}{R}[g] - \frac{2}{\sqrt{h}}\boxed{g}\sqrt{h} + \dots\right\}$$

dilaton factor $\qquad h_{ij}$ - scalar matter f.

- Reduction of other fields:

if $p=1$
$h_{D+1,D+1} \equiv e^{2\sigma}$
$X^{D+1} \equiv \theta$

as $\partial_\theta\psi \equiv 0$, we have for scalar fields $G^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi =$

$$= g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi.$$

For a 3-form $H^{(D+1)}_{\mu\nu\lambda}$

$$H^{(D+1)}_{\mu\nu\lambda}H^{(D+1)\mu\nu\lambda} = \tilde{H}^{(D)\alpha\beta\gamma}\tilde{H}^{(D)}_{\alpha\beta\gamma} + 3e^{-2\sigma}H^{(D)}_{\alpha\beta\theta}H^{(D)\alpha\beta\theta}$$

$$\tilde{H}^{(D)}_{\alpha\beta\gamma} = H^{(D)}_{\alpha\beta\gamma} - (A_\alpha H^{(D)}_{\beta\gamma\theta} + [\alpha\beta\gamma])$$