INTRODUCTION TO SUPERSYMMETRIC FIELD THEORY

Literature

Books

1. J. Wess, J. Bagger, Supersymmetry and Supergravity, 1983

2. S.J. Gates, M.T. Grisaru, M. Roček, W. Siegel, Superspace or One Thousand and One Lessons in Supersymmetry, 1983

3. P. West, Introduction to Supersymmetry and Supergravity, 1986

4. D. Bailin, A. Love, Supersymmetric Gauge Field Theory and String Theory, 1994

5. I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace, 1995, 1998

6. A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001

Reviews and Lecture Notes

1. M. Sohnius, Introducing Supersymmetry, Phys. Repts, **128**, 39, 1985

2. S.J. Gates, Basic Canon in $D=4, N=1$ Superfield Theory, hep-th/9809064

3. A. Bilal, Introduction to Supersymmetry, hep-th/0101055

4. J.M. Figueroa-O'Farrill, BUSSTEPP Lectures on Supersymmetry, hep-th/0109172

Preface

The course Introduction to supersymmetric field theory is devoted to some basic notions of so called $N=1$, $D=4$ supersymmetry (SUSY). This course can be also entitled as Introductory lectures on supersymmetry or Elementary introduction to supersymmetry or First notions of supersymmetric field theory or Pedagogical introduction to supersymmetry or Supersymmetry for beginners. I want to emphasize, the course assumes the audience never studied supersymmetry before it and now you face this subject for the first time. I am going to consider only basic simple aspects, the more advanced aspects of supersymmetry will be discussed in the course of Professor Ivanov and in the other courses.

General Idea of Supersymmetry

Supersymmetry in physics means a hipotetical symmetry of Nature relating the bosons and fermions. In its essence, the supersymmetry is extension of special relativity symmetry. One can say, the supersymmetry is a special relativity symmetry completed by symmetry between bosons and fermions.

Main idea of supersymmetry in field theory can be explained as follows. Let us consider some model of field theory. Any such a model is given in terms of action functional depending on set of bosonic fields $b(x)$ and set of fermionic fields $f(x)$. The action is $S[b, f]$. Consider the infinitesimal transformations of the fields $b(x)$ and $f(x)$ of the form

$$b \to b + \delta b, \qquad \delta b \sim f$$
$$f \to f + \delta f, \qquad \delta f \sim b$$

Let the action $S[b, f]$ is invariant under these transformations. Then the field model under consideration is called supersymmetric. The transformations $\delta b \sim f$, $\delta f \sim b$ are called the supersymmetry transformations or supertransformations.

Of course, the above considerations look like very schematic and naive while we have no answers the questions:

1. What is an explicit ~~form~~ set of the fields b and f in concret model?
2. What is an explicit form of the transformations $\delta b \sim f$, $\delta f \sim b$?
3. What is an explicit form of invarian action?

Supersymmetry was ~~discovered~~ proposed in 1971 by Yu. Gol'fand and E. Likhtman from Lebedev Physical Institute. Then it was ~~discovered~~ proposed again in some another form in 1972 by D. Volkov and V. Akulov from Khar'kov Institute of Physics and Technology. In 1974 J. Wess and B. Zumino from Karlsruhe University and CERN respectively have constructed first supersymmetric model of four dimensional field theory. After that, a number of papers on supersymmetry became to increase as rolling snow ball. Also it is worth pointing out ~~discovery~~ a proposal of so called two-dimensional supersymmetry in 1971 in context of string theory (P. Ramond, A. Neveu, J. Schwarz, J. Gervais, B. Sakita). Further, I will discuss only four-dimensional supersymmetry.

If the supersymmetry is true symmetry of Nature it immediately leads to fundamental physical consequences. According to Standard Model all elementary particles ~~are~~ form two classes: the particles of matter and the particles mediators of fundamental interactions. All particles of matter are the fermions. All particles mediators of fundamental interactions are the bosons. However, if supersymmetry is true, classification of elementary particlse on bosons and fermions is relative. Supersymmetry ~~transformation~~ transforms bosons into fermions and fermions into bosons. It means, ~~that exists~~ for each boson there should exist its superpartner - fermion and for each fermion there exist ~~and~~ its superpartner - boson. Hence, for each fermionic matter particle there must exist ~~the~~ bosonic matter particle and for each bosonic particle mediator of interaction ~~[illegible]~~ there must exist a fermionic particle mediator of interaction. As a result one gets a beautiful symmetric picture of Nature on fundamental level. ~~[illegible]~~ As I already noted, up to now the supersymmetry is hypothetical symmetry. There is no experimental evidence of supersymmetry at present. However the experiments for searching the superpartners of known particles have already planned for nearest years. Further I am not going to discuss the experimental aspects of supersymmetry.

~~Let us return back to~~ ~~The~~ As I pointed out, the supersymmetry is extension of special relativity symmetry. Special relativity expresses an invariance of physical phenomena under Lorentz transformations and four-dimensional translations $x'^m = x^m + a^m$, where x^m are the Minkowski space coordinates, ~~and~~ a^m is a constant four-vector and $m = 0,1,2,3$. Supersymmetry includes special relativity symmetry and additional ~~transformations~~ symmetry ~~under the~~ associated with transformations $\delta b \sim f$, $\delta f \sim b$. There is a universal procedure to realize the special relativity transformations ~~and~~ together with transformations $\delta b \sim f$, $\delta f \sim b$. Such a procedure is based on notions of superspace and superfield. Aim of my course is to explain a sense of these notions and to show how the supersymmetric field models are constructed in terms of superspace and superfield.

1. Lorentz and Poincare Groups.

1.1 Basic definitions.

Let us consider the four-dimensional Minkowski space with coordinates x^m $(m = 0,1,2,3)$ and metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \equiv \equiv \eta_{mn} dx^m dx^n \qquad (1.1)$$

It is easy to see that form of the metric (1.1) is invariant under the following linear inhomogeneous transformations

$$x'^m = \Lambda^m{}_n x^n + a^m \qquad (1.2)$$

where a^m is constant four-vector and $\Lambda \equiv (\Lambda^m{}_n)$ is a special matrix with constant elements. To find the restrictions on this matrix one considers the form invariance condition

$$\eta_{mn} dx'^m dx'^n = \eta_{pq} dx^p dx^q \qquad (1.3)$$

Substituting Eq (1.2) one gets

$$\eta_{mn} \Lambda^m{}_p \Lambda^n{}_q \, dx^p dx^q = \eta_{pq} dx^p dx^q$$

This means

$$\eta_{mn} \Lambda^m{}_p \Lambda^n{}_q = \eta_{pq} \qquad (1.4)$$

Let Λ^T is a transpose matrix with the elements $(\Lambda^T)_n{}^m = \Lambda^m{}_n$. Then

Eq(1.4) leads to

$$\Lambda^T \eta \Lambda = \eta \qquad (1.5)$$

The transformations (1.2) where the matrix Λ satisfies Eq(1.5) are called the inhomogeneous Lorentz transformations.

Further we denote the transformations (1.2) as (a, Λ). A set of transformations (a, Λ) has two evident subsets first the transformations (a, I) realizing the coordinate transformations

$$x'^m = x^m + a^m \qquad (1.6)$$

I means unit 4x4 matrix. The transformations (1.6) are called the space-time translations. second The transformations $(\Lambda, 0)$ realizing the coordinate transformations

$$x'^m = \Lambda^m{}_n x^n \qquad (1.7)$$

with Λ satisfying Eq(1.5). The transformations (1.7) are called Lorentz rotations or homogeneous Lorentz transformations.

Let us consider two inhomogeneous Lorentz coordinate transformations, one after another. This leads to relation

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1) \qquad (1.8)$$

One can show, if the matrices Λ_1, Λ_2 satisfy Eq(1.5), then their product $\Lambda_2 \Lambda_1$ satisfy Eq(1.5) as well.

It is easy to show that a set of transformations (a, Λ) forms a group where a multiplication low is given by Eq (1.8). Here, the unit group element is $(0, I)$ and the element inverse to the element (a, Λ) is $(-\Lambda^{-1}a, \Lambda^{-1})$ where Λ^{-1} is the inverse matrix, satisfying the relations

$$\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = I \qquad (1.9)$$

$$\Lambda^{m}{}_{p}(\Lambda^{-1})^{p}{}_{n} = (\Lambda^{-1})^{m}{}_{p}\Lambda^{p}{}_{n} = \delta^{m}{}_{n}$$

This group is called the Poincare group. One can show that subset of transformations $(0, \Lambda)$ forms the group which is called the Lorentz group. ~~can subset of transformations (a, ...) forms the group which is called group of translations~~ Subset of transformations (a, I) forms the group which is called translation group. Two these groups are the subgroups of the Poincare group. One can show the following relation

$$(a, \Lambda) = (a, I)(0, \Lambda) \qquad (1.10)$$

Further, we will use only infinitesimal form of inhomogeneous Lorentz transformations. To do that we write $\Lambda = I + \omega$ where $\omega \equiv (\omega^{m}{}_{n})$ is matrix with infinitesimal elements. In this case, the basic reation (1.5) takes the form

$$(I+\omega)^{T}\eta(I+\omega) = \eta$$

or

$$\omega_{mn} + \omega_{nm} = 0 \qquad (1.11)$$

where $\omega_{mn} = \eta_{mp}\omega^p{}_n$. Thus, the matrix ω_{mn} has six independent real elements. This means, the Lorentz group is six parametric Lie group and Poincare group is ten parametric Lie group.

1.2. Proper Lorentz group and $SL(2|\mathbb{C})$ group.

Basic relation ~~of~~ for matrices Λ (1.5) leads to $\det\Lambda^T \det\eta \det\Lambda = \det\eta$. or

$$\det\Lambda = \pm 1 \tag{1.12}$$

In particular, this means that inverse matrix exists. Moreover, one can show that $\Lambda^0{}_0$ can be or positive or negative, ~~[illegible]~~

$$\text{sign}\,\Lambda^0{}_0 = \pm 1 \tag{1.13}$$

Eqs. (1.12) and (1.13) show that set of matrices Λ is divided into four subsets. We consider only the subset $L_+^\uparrow$: $\det\Lambda = 1$, $\text{sign}\Lambda^0{}_0 = 1$. One can show that the set of transformations $(0,\Lambda)$, $\Lambda \in L_+^\uparrow$ forms a group which is called the proper Lorentz group.

Further we prove that proper Lorentz group allows us to introduce the specific objects which are called the two-component spinors.

First of all, we introduce a set of 2×2 complex matrices N with unit determinant. One can show that the set of such matrices forms a group where a multiplication law is defined as an ordinary matrix product.

This group is called the two-dimensional special linear complex group and is denoted as $SL(2|\mathbb{C})$.

~~We will~~ One can show that for each matrix $N \in SL(2|\mathbb{C})$ there exists the matrix $\Lambda \in L_+^\uparrow$ such that

a $\Lambda(N_1 N_2) = \Lambda(N_1)\Lambda(N_2)$

b $\Lambda(N_1) = \Lambda(N_2)$ if and only if $N_1 = \pm N_2$.

The construction is formulated as follows

1. Introduce a linear space of Hermitean 2×2 matrices X, $X^+ = X$. The basis in this space is given by the four matrices $\sigma_m \equiv (\sigma_0, \vec{\sigma})$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.14)$$

The arbitrary matrix X is written as

$$X = X^m \sigma_m \qquad (1.15)$$

where X^m are the real numbers. It is easy to check that

$$\mathrm{tr}\, \sigma_m \sigma_n = +2\, \delta_{mn} \qquad (1.16)$$

Therefore

$$X^m = +\tfrac{1}{2}\, \mathrm{tr}\, X \sigma_m \qquad (1.17)$$

2. Let $N \in SL(2|\mathbb{C})$. Consider the matrix

$$X' = N X N^+ \qquad (1.18)$$

Since $\det N = 1$ one get

$$\det X' = \det X \qquad (1.19)$$

3. Using Eqs (1.19), (1.19) one obtains

$$X'^m = \frac{1}{2} tr X' \sigma_m = \frac{1}{2} tr N X N^+ \sigma_m =$$

$$= \frac{1}{2} tr (\sigma_m N \sigma_n N^+) X^n \equiv$$

$$= \Lambda^m{}_n X^n \qquad (1.20)$$

where

$$\Lambda^m{}_n = \Lambda^m{}_n (N) = \frac{1}{2} tr (\sigma_m N \sigma_n N^+) \qquad (1.21)$$

4. Using the basis matrices (1.14) one can rewrite the expansion $X = X^m \sigma_m$ as follows

$$X = \begin{pmatrix} X^0 + X^3 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - X^3 \end{pmatrix} \qquad (1.22)$$

Therefore

$$\det X = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = - \eta_{mn} X^m X^n \qquad (1.23)$$

The same consideration leads to

$$\det X' = - \eta_{mn} X'^m X'^n \qquad (1.24)$$

However $\det X = \det X'$ (1.19). Hence

$$\eta_{mn} X^m X^n = \eta_{mn} X'^m X'^n \qquad (1.25)$$

Substituting Eq (1.20) one gets

$$\Lambda^T \eta \Lambda = \eta$$

where Λ is given by (1.21). This means, the matrix Λ (1.21) corresponds to Lorentz rotation.

5. One can show that Eq (1.21) leads to

$$\Lambda^0{}_0 > 0, \quad \det \Lambda = 1$$

Thus, the matrices Λ (1.21) correspond to proper Lorentz group.

As a result, the proper Lorentz group is associated with $SL(2|\mathbb{C})$ group.

1.3. Two-component spinors.

Complex 2×2 matrices $N \in SL(2|\mathbb{C})$ act in two dimensional complex space. We denote the vectors of this space as ψ_α, $\alpha = 1, 2$. Action of these matrices look like

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta \tag{1.26}$$

Since each matrix $N \in SL(2|\mathbb{C})$ is associated with some matrix from proper Lorentz group one can say that Eq (1.26) is transformation law of the two-dimensional complex vector under the Lorentz transformations. The vectors ψ_α with the transformation law (1.26) are called left Weyl spinors. The indices α, β are the spinor ones.

Let us introduce the matrix ε with the elements $\varepsilon_{\alpha\beta}$ by the rule

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.27}$$

and inverse matrix ε^{-1} with the elements $\varepsilon^{\alpha\beta}$, that is $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$, $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha{}_\gamma$ and

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.28}$$

One can ~~show that~~ prove the following identities

$$N \varepsilon N^T = \varepsilon, \quad N^T \varepsilon^{-1} N = \varepsilon^{-1} \qquad (1.29)$$

This means that the matrix ε as well the matrix ε^{-1} are the invariant tensors of the $SL(2|C)$ group. The matrices $\varepsilon, \varepsilon^{-1}$ are used to raise and lower the spinor indices:

$$\varphi^\alpha = \varepsilon^{\alpha\beta} \varphi_\beta, \quad \varphi_\alpha = \varepsilon_{\alpha\beta} \varphi^\beta \qquad (1.30)$$

Moreover, one can show that the expression $\varphi_1^\alpha \varphi_{2\alpha}$ is the Lorentz invariant, that is

$$\varphi'^{\alpha}_1 \varphi'_{2\alpha} = \varphi_1^\alpha \varphi_{2\alpha} \qquad (1.31)$$

where φ'_α is given by Eq (1.26)

Let $N \in SL(2|C)$ and let N^* is the corresponding complex conjugate matrix, of course $N^* \in SL(2|C)$ as well. We denote the elements of the matrix N^* as $N^*{}_{\dot\alpha}{}^{\dot\beta}$; $\dot\alpha, \dot\beta = \dot 1, \dot 2$. This matrix acts in complex space of two-dimensional vectors $\chi_{\dot\alpha}$ by the rule

$$\chi'_{\dot\alpha} = N^*{}_{\dot\alpha}{}^{\dot\beta} \chi_{\dot\beta} \qquad (1.32)$$

The two-component complex vectors $\chi_{\dot\alpha}$ with the transformation law (1.32) are called the right Weyl spinors. The indices $\dot\alpha, \dot\beta$ are called the spinor indices. Analogously to previous discussion, one introduces the matrices

$$\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\varepsilon_{\dot\alpha\dot\beta} \varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}{}^{\dot\gamma}$$

allowing us to raise and lower the indices. One can ~~show~~ prove the following identity

$$\chi'_{1\dot\alpha} \chi'^{\dot\alpha}_2 = \chi_{1\dot\alpha} \chi_2^{\dot\alpha} \qquad (1.33)$$

where $\chi^!_{\dot\alpha}$ is given by Eq (1.32). Hence, the expression $\chi_{1\dot\alpha}\chi_2^{\dot\alpha}$ is the Lorentz invariant.

Sometimes, the spinors $\chi_{\dot\alpha}$ are called the dotted spinors while the spinors φ_α are called undotted ones.

Since the dotted spinors transform with help of conjugate matrix N^* we can define a conjugate operation for the spinors by the rule

$(\varphi_\alpha)^* = \bar\varphi_{\dot\alpha}$

I would like to pay attention to the important properties

$$(\varphi_1\varphi_2) \equiv \varphi_1^\alpha\varphi_{2\alpha} = -\varphi_2^\alpha\varphi_{1\alpha} = -(\varphi_2\varphi_1) \qquad (1.34)$$

$$(\chi_1,\chi_2) \equiv \chi_{1\dot\alpha}\chi_2^{\dot\alpha} = -\chi_{2\dot\alpha}\chi_1^{\dot\alpha} = -(\chi_2,\chi_1)$$

Let us consider the matrices σ_m (1.14). Their matrix elements are denoted as $(\sigma_m)_{\alpha\dot\alpha}$. Also we introduce the matrix with upper indices

$$(\sigma_m)^{\alpha\dot\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma_m)_{\beta\dot\beta} \equiv (\tilde\sigma_m)^{\dot\alpha\alpha} \qquad (1.35)$$

One can show, using the explicit forms of the matrices σ_m (1.14) and $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$, that

$$\tilde\sigma_m \equiv (\sigma_0, -\vec\sigma) \qquad (1.36)$$

The matrices σ_m, $\tilde\sigma_m$ possess the many useful properties, for example

$$(\sigma_m\tilde\sigma_n + \sigma_n\tilde\sigma_m)_\alpha{}^\beta = -2\eta_{mn}\delta_\alpha{}^\beta \qquad (1.37a)$$

$$(\tilde\sigma_m\sigma_n + \tilde\sigma_n\sigma_m)^{\dot\alpha}{}_{\dot\beta} = -2\eta_{mn}\delta^{\dot\alpha}{}_{\dot\beta} \qquad (1.37b)$$

$$tr\,\sigma_m\tilde\sigma_n = -2\eta_{mn} \qquad (1.37c)$$

$$(\sigma^m)_{\alpha\dot\alpha}(\tilde\sigma_m)^{\dot\beta\beta} = -2\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta} \qquad (1.37d)$$

$$\sigma_m = N\sigma_n N^+ (\Lambda^{-1}(N))^n{}_m \qquad (1.37e)$$

Here $\Lambda(N)$ is given by Eq(1.21). Eq(1.37e) means that the matrices σ_m are the invariant tensors of $SL(2|C)$ group.

$\tilde\sigma$ – σ_m tilde

Let us consider the expressions

$$v^m = (\bar{\varphi}\tilde{\sigma}^m \varphi) \equiv \bar{\varphi}_{\dot{\alpha}} (\tilde{\sigma}^m)^{\dot{\alpha}\alpha} \varphi_\alpha$$

$$u_m = (\varphi \sigma_m \bar{\varphi}) \equiv \varphi^\alpha (\sigma_m)_{\alpha\dot{\alpha}} \bar{\varphi}^{\dot{\alpha}} \qquad (1.38)$$

One can show, using the transformation laws for the spinors φ_α, $\bar{\varphi}_{\dot{\alpha}}$ and Eq. (1.37e), that v^m is a contravariant vector and u_m is covariant one.

1.4. Lie algebra of the Poincare group.

We consider the lie algebra of the Poincare group on the base of one special example. One can show that its form does not depend on example we begin with.

As we pointed out, the infinitesimal coordinate transformations corresponding to inhomogeneous Lorentz transformations have the form

$$x'^m = x^m + \omega^m{}_n + a^m = x^m + \delta x^m \qquad (1.39)$$

where $\omega^m{}_n$ and a^m are the infinitesimal parameters, $\omega_{mn} = -\omega_{nm}$. Let $t^m(x)$ is a vector field which is determined by transformation law under inhomogeneous Lorentz transformations

$$t'^m(x') = \frac{\partial x'^m}{\partial x^n} t^n(x) \qquad (1.40)$$

where x'^m is given by (1.39). Hence

$$t'^m(x+\delta x) = \Lambda^m{}_n t^n(x)$$

and $\Lambda^m{}_n = \delta^m{}_n + \omega^m{}_n$

Therefore

$$t'^m(x+\omega x+a) = (\delta^m{}_n + \omega^m{}_n)\, t^n(x)$$

Or

$$t'^m(x) + \omega^n{}_k x^k \partial_n t^m(x) + a^n \partial_n t^m(x) =$$
$$= t^m(x) + \omega^m{}_n t^n(x)$$

denoting $\delta t^m(x) = t'^m(x) - t^m(x)$ one gets

$$\delta t^m(x) = -a^n \partial_n t^m(x) + \omega^m{}_n t^n(x) - \omega^n{}_k x^k \partial_n t^m(x) \qquad (1.41)$$

This relation can be rewritten as follows

$$\delta t^m(x) = -i a^r (P_r)^m{}_n t^n + \frac{i}{2}\omega^{rs} (J_{rs})^m{}_n t^n(x) \qquad (1.42)$$

where

$$(P_r)^m{}_n = \delta^m{}_n(-i\partial_r)$$
$$(J_{rs})^m{}_n = \eta_{rk} x^k (P_s)^m{}_n - \eta_{sk} x^k (P_r)^m{}_n + (M_{rs})^m{}_n \qquad (1.43)$$
$$(M_{rs})^m{}_n = i(\delta^m{}_s \eta_{rn} - \delta^m{}_r \eta_{sn})$$

Using the explicit form of the operators P_r, J_{rs} one can calculate the commutation relations among these operators. The result is written as follows

$$[P_r, P_s] = 0$$
$$[J_{rs}, P_m] = i(\eta_{rm} P_s - \eta_{sm} P_r) \qquad (1.44)$$
$$[J_{mn}, J_{rs}] = i(\eta_{mr} J_{ns} - \eta_{ms} J_{nr} + \eta_{ns} J_{mr} - \eta_{nr} J_{ms})$$

The expressions (1.43) are called the generators of the Poincare group in vector representation. The relations (1.44) form the commutation relations of the generators. Usually the relations (1.44) are called the Poincare algebra. One can show, that the infinitesimal transformations under the inhomogenous Lorentz transformations of any tensor or spinor fields are written in form analogous to (1.42), that is

$$\delta \Phi(x) = -i a^m P_m \Phi(x) + \frac{i}{2} \omega^{mn} J_{mn} \Phi(x) \qquad (1.45)$$

with some generators P_m and J_{mn}. The operators P_m are called the translation generators and J_{mn} are called the Lorentz rotation generators. The commutation relations among the operators P_m, J_{mn} have the form (1.44) independently on the types of fields $\Phi(x)$ in (1.45). I would like to emphasize that the algebra (1.44) is mathematical expression of special relativity symmetry.

2. Superspace and Superfields

2.1 Supersymmetry algebra

We will discuss now how to extend the Poincare algebra (1.44) by means of new generators which can provide a symmetry between bosons and fermions. A general idea of such an extension is based on use of undotted and dotted spinor generators $Q_\alpha{}^I$, $\bar{Q}_{\dot\alpha}{}^I$ where $I = 1, 2, .., N$. Integer N numerates a number of new generators.

It is postulated that the generators Q and $\bar{Q}$ posses a fermionic nature so that their commutation relations are given in terms of anticommutators. We preserve the algebra (1.44) for P_m and J_{mn} and should find the following commutators and anticommutators

$$[P_m, Q_\alpha{}^I], \quad [P_m, \bar{Q}_{\dot\alpha}{}^I] \tag{2.1a}$$

$$[J_{mn}, Q_\alpha{}^I], \quad [J_{mn}, \bar{Q}_{\dot\alpha}{}^I] \tag{2.1b}$$

$$\{Q_\alpha{}^I, Q_\beta{}^J\}, \quad \{\bar{Q}_{\dot\alpha}{}^I, \bar{Q}_{\dot\beta}{}^J\} \tag{2.1c}$$

$$\{Q_\alpha{}^I, \bar{Q}_{\dot\alpha}{}^J\} \tag{2.1d}$$

where $\{A, B\}$ means the anticommutator.

Then, one assumes that the right hand sides of above commutators and anticommutators should be the linear combinations of all generators $P_m, J_{mn}, Q_\alpha{}^I, \bar{Q}_{\dot\alpha}{}^J$ with some coefficients. The Lorentz covariance imposes the conditions on these coefficients, they can be constructed only from invariant tensors η_{mn}, $\varepsilon_{\alpha\beta}$ $\varepsilon_{\dot\alpha\dot\beta}$, $(\sigma_m)_{\alpha\dot\alpha}$, $(\tilde\sigma_m)^{\dot\alpha\alpha}$.

Therefore the most general forms for the commutators and anticommutators (2.1) are written as follows

$$
\begin{aligned}
[P_m, Q_\alpha{}^I] &= c_1 (\sigma_m)_{\alpha\dot\alpha} \bar{Q}^{\dot\alpha I} \\
[P_m, \bar{Q}_{\dot\alpha}{}^I] &= c_2 (\tilde\sigma_m)_{\dot\alpha\alpha} Q^{\alpha I} \\
[J_{mn}, Q_\alpha{}^I] &= c_3 (\sigma_{mn})_\alpha{}^\beta Q_\beta{}^I \\
[J_{mn}, \bar{Q}_{\dot\alpha}{}^I] &= c_4 (\tilde\sigma_{mn})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}{}^I \\
\{Q_\alpha{}^I, Q_\beta{}^J\} &= c_5 \varepsilon_{\alpha\beta} Z^{IJ} + \tilde{c}_5 (\sigma^{mn})_{\alpha\beta} J_{mn} X^{IJ} \\
\{\bar{Q}_{\dot\alpha}{}^I, \bar{Q}_{\dot\beta}{}^J\} &= c_6 \varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ} + \tilde{c}_6 (\tilde\sigma^{mn})_{\dot\alpha\dot\beta} J_{mn} \bar{X}^{IJ} \\
\{Q_\alpha{}^I, \bar{Q}_{\dot\alpha}{}^J\} &= c_7 \, 2 (\sigma^m)_{\alpha\dot\alpha} P_m \delta^{IJ}
\end{aligned}
\qquad (2.2)
$$

where $c_1, \ldots, c_7$ are some numerical coefficients, $Z^{IJ} = -Z^{JI}$, $\bar{Z}^{IJ} = -\bar{Z}^{JI}$, $X^{IJ} = X^{JI}$, $\bar{X}^{IJ} = \bar{X}^{JI}$ are some matrices with numerical coefficients. Here

$$
\begin{aligned}
(\sigma_{mn})_{\alpha\beta} &= -\tfrac{1}{4} (\sigma_m \tilde\sigma_n - \sigma_n \tilde\sigma_m)_{\alpha\beta} \\
(\tilde\sigma_{mn})_{\dot\alpha\dot\beta} &= -\tfrac{1}{4} (\tilde\sigma_m \sigma_n - \tilde\sigma_n \sigma_m)_{\dot\alpha\dot\beta}
\end{aligned}
\qquad (2.3)
$$

To fix the coefficients in algebra (2.2) one uses the Jacobi identities which are written in terms of double commutators and anticommutators. We will use the standard terminology when the operators P_m, J_{mn} are called bosonic and the operators $Q_\alpha{}^I, \bar{Q}_{\dot\alpha}{}^I$ are called fermionic.

let B is some bosonic generator and F is some fermionic generator. The Jacobi identities have the form

$$[B_1,[B_2,B_3]]+[B_2,[B_3,B_1]]+[B_3,[B_1,B_2]]=0$$
$$[B,\{F_1,F_2\}]+\{F_1,[F_2,B]\}-\{F_2,[B,F_1]\}=0$$
$$[B_1,[B_2,F]]+[B_2,[F,B_1]]+[F,[B_1,B_2]]=0 \qquad (2.4)$$
$$[F_1,\{F_2,F_3\}]+[F_2,\{F_3,F_1\}]+[F_3,\{F_1,F_2\}]=0$$

Substituting the generators P_m, J_{mn} instead of B and the generators $Q_\alpha{}^I$, $\bar{Q}_{\dot\alpha}{}^I$ instead of operators F to the identities (2.4) and using the relations (2.2) one fixes the all numerical coefficients in Eqs (2.2). The results are $c_1=c_2=0$, $c_3=i$, $c_4=i$, $c_5=c_6=1$, $\hat{c}_5=\hat{c}_6=0$, $c_7=1$. The quantities Z^{IJ}, $\bar{Z}^{IJ}$ must commute with all generators. Thus, one gets the algebra (2.2) in the final form

[illegible]

$$[P_m, Q_\alpha{}^I]=0, \qquad [P_m, \bar{Q}_{\dot\alpha}{}^I]=0$$
$$[J_{mn}, Q_\alpha{}^I]=i(\sigma_{mn})_\alpha{}^\beta Q_\beta{}^I \qquad (2.5)$$
$$[J_{mn}, \bar{Q}_{\dot\alpha}{}^I]=i(\tilde{\sigma}_{mn})_{\dot\alpha}{}^{\dot\beta}\bar{Q}_{\dot\beta}{}^I$$
$$\{Q_\alpha{}^I, Q_\beta{}^J\}=\varepsilon_{\alpha\beta}Z^{IJ}, \qquad \{\bar{Q}_{\dot\alpha}{}^I, \bar{Q}_{\dot\beta}{}^J\}=\varepsilon_{\dot\alpha\dot\beta}\bar{Z}^{IJ}$$
$$\{Q_\alpha{}^I, \bar{Q}_{\dot\beta}{}^J\}=2(\sigma^m)_{\alpha\dot\beta}P_m\delta^{IJ}$$

The Eq (2.5) together with (1.44) form the so called Poincare superalgebra. The fermionic generators $Q_\alpha{}^I$, $\bar{Q}_{\dot\alpha}{}^I$ are called the supercharges, the generators Z^{IJ}, $\bar{Z}^{IJ}$ are called the central charges. If $N=1$, the supersymmetry is called simple or $N=1$ supersymmetry. In this case all central charges are absent. If $N>1$, the supersymmetry is called extended or N-extended supersymmetry. Statement that the relations (2.5) are the most general extension of Poincaré algebra (1.44) by means of fermionic generators is called the Haag, Łopuszański, Sohnius theorem which has been proved in 1975.

Further we will consider only $N=1$ supersymmetry.

2.2. Anticommuting variables

Supersymmetric fields theories (as well as all fermionic field theories) are formulated naturally with help of notions of Grassmann algebra which operates with so called anticommuting variables or anticommuting numbers. We discuss some simplest properties of anticommuting variables.

Why we have to think about anticommuting variables in the case under consideration? We know that generators of Poincaré algebra (1.44) can be associated with coordinate transformations of the Minkowski space. It is naturally to expect that the generators of supersymmetry algebra can also be associated with transformations of coordinates of some space. Since the generators P_m, J_{mn} included into supersymmetry algebra, this new space should include the coordinates x^m of the Minkowski space. But the supersymmetry algebra contains the supercharges Q_α, $\bar{Q}_{\dot\alpha}$ satisfying the relations in terms of anticommutators. It is natural to assume that the supercharges should be associated with some new coordinates. Since the supercharges satisfy the anticommuting relations one can assume that the corresponding coordinates should be anticommuting. Moreover, since the supercharges are the undotted and dotted spinors, it is natural to assume that the anticommuting coordinates should be the undotted and dotted spinors.

Further I will follow some pragmatic point of view. This means I am going to avoid any strict definitions and describe only the rules of operations with anticommuting variables. In principle, all necessary material concerning anticommuting variables is given in standard texts in quantum field theory and I assume it is known.

We denote the anticommuting variables as $\theta_\alpha, \bar\theta_{\dot\alpha}$. This means that θ_α is left spinor and $\bar\theta_{\dot\alpha}$ is right spinor. We can raise the indeces and obtain $\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta$, $\bar\theta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}$. Basic relations expressing the fundamental property of the anticommuting variables have the form

$$\begin{aligned} &\theta_\alpha\theta_\beta + \theta_\beta\theta_\alpha = 0 \\ &\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} + \bar\theta_{\dot\beta}\bar\theta_{\dot\alpha} = 0 \\ &\theta_\alpha\bar\theta_{\dot\alpha} + \bar\theta_{\dot\alpha}\theta_\alpha = 0 \end{aligned} \qquad (2.6)$$

These relations show that

$$\begin{aligned} &(\theta_\alpha)^2 = 0 \qquad \alpha = 1,2 \\ &(\bar\theta_{\dot\alpha})^2 = 0 \qquad \dot\alpha = \dot 1, \dot 2 \\ &\theta_\alpha\theta_\beta\theta_\gamma = 0, \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\bar\theta_{\dot\gamma} = 0 \\ &\alpha,\beta,\gamma = 1,2, \qquad \dot\alpha,\dot\beta,\dot\gamma = \dot 1,\dot 2 \end{aligned} \qquad (2.7)$$

The basic relations (2.6) allows us to simplify the expressions $\theta_\alpha\theta_\beta$ and $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}$. Let us consider $\theta_\alpha\theta_\beta$, this is antisymmetric 2×2 matrix. Any such a matrix should be proportional to $\varepsilon_{\alpha\beta}$.. Hence $\theta_\alpha\theta_\beta = c\,\varepsilon_{\alpha\beta}$. Therefore $\varepsilon^{\beta\alpha}\theta_\alpha\theta_\beta = c\,\varepsilon^{\beta\alpha}\varepsilon_{\alpha\beta} = c\,\delta^\beta{}_\beta = 2c$. This means $c = \frac{1}{2}\varepsilon^{\beta\alpha}\theta_\alpha\theta_\beta$. The $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}$ can be considered analogously. As a result

$$\theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta^2, \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\theta^2$$
$$\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2, \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2 \qquad (2.8)$$
$$\theta^2 \equiv \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha, \qquad \bar\theta^2 \equiv \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}$$

Due to the relations (2.6, 2.7) any function of anticommuting variables can be only a polynomial.

$$F(\theta,\bar\theta) = a + b^\alpha\theta_\alpha + \tilde b_{\dot\alpha}\bar\theta^{\dot\alpha} + c_{\alpha\dot\alpha}\theta^\alpha\bar\theta^{\dot\alpha} + \qquad (2.9)$$
$$+ d\,\theta^2 + \tilde d\,\bar\theta^2 + f_\alpha\theta^\alpha\bar\theta^2 + \tilde f_{\dot\alpha}\bar\theta^{\dot\alpha}\theta^2 + g\,\theta^2\bar\theta^2$$

where $a, b_\alpha, \tilde b_{\dot\alpha}, c_{\alpha\dot\alpha}, d, \tilde d, f_\alpha, \tilde f_{\dot\alpha}, g$ are real or complex numbers. Sometimes, the expressions (2.9) are called the superfunctions and the numbers $a, b_\alpha, \tilde b_{\dot\alpha}, c_{\alpha\dot\alpha}, d, \tilde d, f_\alpha, \tilde f_{\dot\alpha}, g$ are called the components of the superfunctions.

The superfunctions (2.9) can be differentiated and integrated. The derivatives with respect to anticommuting variables are denoted as

$$\frac{\partial}{\partial\theta^\alpha} \equiv \partial_\alpha, \qquad \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \equiv \bar\partial_{\dot\alpha}$$

and satisfy the following rules

1. The derivatives are the linear operations

2. $\partial_\alpha \theta^\beta = \delta_\alpha^\beta \quad , \quad \bar\partial_{\dot\alpha} \bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$ (2.10).

3. $\partial_\alpha (\theta^\beta \theta^\gamma) = \delta_\alpha^\beta \theta^\gamma - \delta_\alpha^\gamma \theta^\beta$ (2.11)

$\bar\partial_{\dot\alpha} (\bar\theta^{\dot\beta} \bar\theta^{\dot\gamma}) = \delta_{\dot\alpha}^{\dot\beta} \bar\theta^{\dot\gamma} - \delta_{\dot\alpha}^{\dot\gamma} \bar\theta^{\dot\beta}$

4. $\partial_\alpha \bar\theta^{\dot\beta}(\ldots) = -\bar\theta^{\dot\beta} \partial_\alpha(\ldots)$ (2.12)

$\bar\partial_{\dot\alpha} \theta^\beta(\ldots) = -\theta^\beta \bar\partial_{\dot\alpha}(\ldots)$

Using these rules one can show

$$\partial_\alpha \partial_\beta + \partial_\beta \partial_\alpha = 0$$
$$\bar\partial_{\dot\alpha} \bar\partial_{\dot\beta} + \bar\partial_{\dot\beta} \bar\partial_{\dot\alpha} = 0 \quad (2.13)$$
$$\partial_\alpha \bar\partial_{\dot\beta} + \bar\partial_{\dot\beta} \partial_\alpha = 0$$

All the derivatives are anticommuting. In ~~[illegible]~~ particular

$$(\partial_\alpha)^2 = 0, \quad (\bar\partial_{\dot\alpha})^2 = 0 \quad (2.14)$$

$\alpha = 1, 2 \qquad \dot\alpha = \dot 1, \dot 2$

In principle, there are two types of derivatives with respect to anticommuting variables, left and right. We consider only left derivative.

Using the relations (2.13) one can show analogously to (2.8) that the expressions $\partial_\alpha \partial_\beta$ and $\bar\partial_{\dot\alpha} \bar\partial_{\dot\beta}$ can be simplified as follows

$$\partial_\alpha \partial_\beta = \frac{1}{2} \varepsilon_{\alpha\beta} \partial^2, \qquad \bar\partial_{\dot\alpha} \bar\partial_{\dot\beta} = -\frac{1}{2} \varepsilon_{\dot\alpha\dot\beta} \bar\partial^2$$
$$\partial^\alpha \partial^\beta = -\frac{1}{2} \varepsilon^{\alpha\beta} \partial^2, \qquad \bar\partial^{\dot\alpha} \bar\partial^{\dot\beta} = \frac{1}{2} \varepsilon^{\dot\alpha\dot\beta} \bar\partial^2 \quad (2.15)$$
$$\partial^2 \equiv \partial^\alpha \partial_\alpha = \varepsilon^{\alpha\beta} \partial_\beta \partial_\alpha, \qquad \bar\partial^2 = \bar\partial_{\dot\alpha} \bar\partial^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \bar\partial_{\dot\alpha} \bar\partial_{\dot\beta}$$

Let us consider now integration over anticommuting variables. Since $(\partial_\alpha)^2=0$, $(\bar\partial_{\dot\alpha})^2=0$, an integral can not be defined as operation inverse to differentiation. Correct definition of the integral over anticommuting variables has been given F. Berezin. ~~Definition~~ Integration is based on the following rules

1. Integral is linear operation
2. Let $d\theta_\alpha, d\bar\theta^{\dot\alpha}$ are the extra ~~anticommuting~~ parameters anticommuting among themselves and with $\theta_\alpha, \bar\theta_{\dot\alpha}$. Then

$$\int d\theta_\alpha F = \partial_\alpha F, \quad \int d\bar\theta^{\dot\alpha} G = \varepsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta} G \qquad (2.16)$$

3. Multiple integral is defined as repeated.

According to these rules

$$\int d\theta_\alpha \theta^\beta = \delta_\alpha{}^\beta, \quad \int d\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta} \qquad (2.17)$$

We define

$$d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta} d\theta_\alpha d\theta_\beta, \quad d^2\bar\theta = \frac{1}{4}\varepsilon_{\dot\alpha\dot\beta} d\bar\theta^{\dot\alpha} d\bar\theta^{\dot\beta} \qquad (2.18)$$

Then

$$\int d^2\theta\, \theta^2 = 1, \qquad \int d^2\bar\theta\, \bar\theta^2 = 1 \qquad (2.19)$$

Using the rules (2.16) and Eqs (2.18) it is easy to see

$$\int d^2\theta\, \partial_\alpha F = 0, \qquad \int d^2\bar\theta\, \bar\partial_{\dot\alpha} G = 0 \qquad (2.20)$$

The relations (2.19) have the very clear interpretation. Let consider

$$\int d^2\theta\, \theta^2 F(\theta) = \int d^2\theta\, \theta^2(a + b_\alpha\theta^\alpha + d\theta^2) =$$
$$= \int d^2\theta\, \theta^2 a = a\int d^2\theta\, \theta^2 = a = F(0) \qquad (2.21)$$

where we have used the property $\theta_\alpha \theta_\beta \theta_\gamma = 0$.
It is worth pointing out that Eq (2.21) is analogous to known property of δ-function,
$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$. Therefore one introduces so called Grassmann δ-function

$$\delta^2(\theta) = \theta^2, \qquad \delta^2(\bar\theta) = \bar\theta^2 \qquad (2.22)$$

Then

$$\int d^2\theta\, \delta^2(\theta) = 1, \quad \int d^2\bar\theta\, \delta^2(\bar\theta) = 1 \qquad (2.23)$$

$$\int d^2\theta\, \delta^2(\theta) F(\theta) = F(0), \quad \int d^2\bar\theta\, \delta^2(\bar\theta) G(\bar\theta) = G(0)$$

Also we note

$$\delta^2(\theta)\big|_{\theta=0} = 0, \qquad \delta^2(\bar\theta)\big|_{\bar\theta=0} = 0 \qquad (2.24)$$

We define also

$$\delta^4(\theta) = \delta^2(\theta)\delta^2(\bar\theta), \quad d^4\theta = d^2\theta\, d^2\bar\theta \qquad (2.25)$$

Then

$$\int d^4\theta\, \delta^4(\theta) F(\theta, \bar\theta) = F(\theta, \bar\theta)\big|_{\theta=0,\, \bar\theta=0} \qquad (2.26)$$

2.3. Superspace

Let us $\theta_\alpha, \bar\theta_{\dot\alpha}$ are the anticommuting variables. We will assume that they are conjugate to each other

$$(\theta_\alpha)^* = \bar\theta_{\dot\alpha}, \quad (\bar\theta_{\dot\alpha})^* = \theta_\alpha \qquad (2.27)$$

We define also the conjugate rule for products anticommuting variables as follows

$$(\theta_\alpha \theta_\beta)^* = \bar\theta_{\dot\beta} \bar\theta_{\dot\alpha}, \quad (\bar\theta_{\dot\beta} \bar\theta_{\dot\alpha})^* = \theta_\alpha \theta_\beta, \quad (\theta_\alpha \bar\theta_{\dot\beta})^* = \theta_\beta \bar\theta_{\dot\alpha} \qquad (2.28)$$

$$(\theta \sigma^m \bar\theta)^* = (\theta \sigma^m \bar\theta)$$

Superspace is defined as a manifold parametrized by the variables $x^m, \theta_\alpha, \bar\theta_{\dot\alpha}$. Here x^m are the coordinates of Minkowski space. As a result, superspace contains the commuting coordinates x^m and anticommuting coordinates $\theta_\alpha, \bar\theta_{\dot\alpha}$. Usually x^m are called the bosonic coordinates and $\theta_\alpha, \bar\theta_{\dot\alpha}$ are called fermionic coordinate. Dimension of superspace is 8, 4 bosonic dimensions and 4 fermionic dimensions.

Any function defined on superspace is called a superfield, $V = V(x,\theta,\bar\theta)$. Taking into account that any superfunction is a finite polynomial in anticommuting variables, one can write

$$V(x,\theta,\bar\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha} + \theta^2F(x) + \bar\theta^2G(x) + (\theta\sigma^m\bar\theta)A_m(x) + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha} + \theta^2\bar\theta^2D(x) \quad (2.29)$$

The coefficients at anticommuting coordinates in superfield (2.29) are called the component fields. All component fields are the usual fields depending on Minkowski space coordinates. We see the unique superfield includes automatically a lot of conventional fields. We will consider further only the superfields which are the scalars under Lorentz rotations. In this case $A(x)$, $F(x)$, $G(x)$, $D(x)$ are the scalar fields, $A_m(x)$ is a vector field, $\psi_\alpha(x)$, $\lambda_\alpha(x)$ are the left spinor fields and $\bar\psi^{\dot\alpha}(x)$, $\bar\eta^{\dot\alpha}(x)$ are the right spinor fields. It is possible to impose a reality condition $V^* = V$ where conjugation of the anticommuting variables is given by Eqs (2.27, 2.28). Then $A(x)$, $D(x)$ are real scalar fields, $A_m(x)$ is a real vector field and $G(x) = F^*(x)$. In addition

$\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}, \quad \bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^* = \bar{\lambda}_{\dot{\alpha}}$

Therefore the real scalar superfield is

$$V(x,\theta,\bar\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha} + \theta^2 F(x) + \bar\theta F^*(x) + (\theta\sigma^m\bar\theta)A_m(x) + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha} + \theta^2\bar\theta D(x) \quad (2.30)$$

Our next purpuse is to realize the supercharges as the operators acting on superfields. We consider the superspace coordinate transformations of the form

$$x'^m = x^m - i(\epsilon\sigma^m\bar\theta - \theta\sigma^m\bar\epsilon)$$
$$\theta'^\alpha = \theta^\alpha + \epsilon^\alpha \quad (2.31)$$
$$\bar\theta_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\epsilon_{\dot\alpha}$$

These transformations are called the supertranslations. Here ϵ^α, $\bar\epsilon_{\dot\alpha}$ are anticommuting transformation parameters. We will show that infinitesimal transformation of scalar superfield under supertranslations is written as (2.32)

$$\delta V = i(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})V$$

where Q_α, $\bar Q_{\dot\alpha}$ are some operators satisfying the algebra

$$\{Q_\alpha, Q_\beta\} = 0, \quad \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \quad (2.34)$$
$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^m{}_{\alpha\dot\alpha} P_m$$

what coincides with anticommutators of the supercharges in the supersymmetry algebra (2.5) for $N=1$.

The superfield V is called scalar if it transforms under supertranslations (2.31) as follows

$$V'(x',\theta',\bar\theta') = V(x,\theta,\bar\theta) \quad (2.35)$$

Substituting Eq (2.31) one gets

$$V'(x+\delta x,\ \theta+\epsilon,\ \bar\theta+\bar\epsilon) = V(x,\theta,\bar\theta)$$

where $\delta x^m = -i(\epsilon\sigma^m\bar\theta - \theta\sigma^m\bar\epsilon)$. Expanding up to first order in parameters $\epsilon, \bar\epsilon$ one obtains

$$V'(x,\theta,\bar\theta) + \delta x^m\partial_m V + \epsilon^\alpha\partial_\alpha V + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} V = V(x,\theta,\bar\theta)$$

Therefore

$$\delta V(x,\theta,\bar\theta) = V'(x,\theta,\bar\theta) - V(x,\theta,\bar\theta) =$$

$$= -\epsilon^\alpha\partial_\alpha V - \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} V - \delta x^m\partial_m V =$$

$$= i(i\epsilon^\alpha\partial_\alpha V) - i(-i\bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} V) +$$

$$+ i\epsilon^\alpha\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m V - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\partial_m V =$$

$$= i\epsilon^\alpha(i\partial_\alpha + \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m)V -$$

$$- i\bar\epsilon^{\dot\alpha}(-i\bar\partial_{\dot\alpha} - \theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m)V$$

Comparison with Eq (2.32) leads to

$$Q_\alpha = i\partial_\alpha + \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \qquad (2.36)$$

$$\bar Q_{\dot\alpha} = -i\bar\partial_{\dot\alpha} - \theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m$$

Now it is easy to check that the operators (2.36) satisfy the relations (2.34) that is these operators carry the representation of the supersymmetry algebra (2.5) on superfields.

The relations (2.32, 2.36) allow us to find the transformation laws of the component fields under the supertranslations. ~~[illegible]~~ To do that we have to substitute $V(x,\theta,\bar\theta)$ (2.29) into Eq.(2.32) and use Eqs (2.36). Further we consider only the real superfield V

$$\delta A + \theta^\alpha \delta\psi_\alpha + \bar\theta_{\dot\alpha}\delta\bar\psi^{\dot\alpha} + \theta^2\delta f + \bar\theta^2 \delta F^* +$$
$$+ \theta^\alpha\bar\theta^{\dot\alpha}\,\delta A_{\alpha\dot\alpha} + \bar\theta^2\theta^\alpha\delta\lambda_\alpha + \theta^2\bar\theta_{\dot\alpha}\delta\bar\eta^{\dot\alpha} + \theta^2\bar\theta^2\delta D =$$

$$= \{ i\epsilon^\alpha(i\partial_\alpha + \bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}) - i\bar\epsilon^{\dot\alpha}(-\bar\partial_{\dot\alpha} - \theta^\alpha\partial_{\alpha\dot\alpha})\} \times$$
$$\times(A + \theta^\beta\psi_\beta + \bar\theta_{\dot\beta}\bar\psi^{\dot\beta} + \theta^2 f + \bar\theta^2 F^* + \theta^\beta\bar\theta^{\dot\beta}A_{\beta\dot\beta} +$$
$$+ \theta^2\theta^\beta\lambda_\beta + \bar\theta^2\bar\theta_{\dot\beta}\bar\eta^{\dot\beta} + \theta^2\bar\theta^2 D) \qquad (2.37)$$

~~From we have~~ Here

$$\partial_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}\partial_m, \qquad (2.38)$$
$$A_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha} A_m$$

Then we have to compute all derivatives with respect to anticommuting variables in right hand side, take into account that $\theta_\alpha\theta_\beta\theta_\gamma = 0$, $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\bar\theta_{\dot\gamma} = 0$ and use Eqs (2.8). The result is written as

$$\delta A = -\epsilon^\alpha\psi_\alpha - \bar\epsilon_{\dot\alpha}\bar\psi^{\dot\alpha}$$
$$\delta\psi_\alpha = -2\epsilon_\alpha F - \bar\epsilon^{\dot\alpha}A_{\alpha\dot\alpha} - i\bar\epsilon^{\dot\alpha}\partial_{\alpha\dot\alpha}A$$
$$\delta F = -\epsilon^\alpha\lambda_\alpha$$
$$\delta A_{\alpha\dot\alpha} = 2(\bar\epsilon_{\dot\alpha}\lambda_\alpha - \epsilon_\alpha\bar\lambda_{\dot\alpha}) - 2i(\epsilon_\alpha\partial_{\beta\dot\alpha}\psi^\beta + \bar\epsilon_{\dot\alpha}\partial_{\alpha\dot\beta}\bar\psi^{\dot\beta}) +$$
$$+ i\partial_{\alpha\dot\alpha}(\bar\epsilon_{\dot\beta}\bar\psi^{\dot\beta} - \epsilon^\beta\psi_\beta) \qquad (2.39)$$

$$\delta\lambda_\alpha = -2\epsilon_\alpha D - i\epsilon_\alpha\partial_m A^m - 2i\bar\epsilon^{\dot\alpha}\partial_{\alpha\dot\alpha}F$$
$$\delta D = -\frac{i}{2}\partial_{\alpha\dot\alpha}(\epsilon^\alpha\bar\lambda^{\dot\alpha} + \bar\epsilon^{\dot\alpha}\lambda^\alpha)$$

As we know from ~~our~~ course of quantum field theory, the scalar and vector fields describe the bosonic particles and are called the bosonic fields. The spinor fields describe the fermionic particles and are called the fermionic fields. Eqs (2.39) show that supersymmetry transforms the bosonic fields into fermionic and fermionic fields into bosonic.

To conclude this subsection, one counters a number of bosonic and fermionic components of the real superfield V. The bosonic fields A, F, D, A_m have 8 real components (A, D, A_m are real and F is complex). The fermionic fields ψ_α, λ_α have also 8 real components (both of them are complex and have two components due to index α). A number of bosonic components is equal to a number of fermionic components.

2.4. Supercovariant derivatives.

Supercovariant derivatives D_α, $\bar{D}_{\dot\alpha}$ are defined by the condition

$$\delta D_\alpha V = D_\alpha (\epsilon^\beta Q_\beta + \bar\epsilon_{\dot\beta} \bar Q^{\dot\beta}) V = D_\alpha \delta V \qquad (2.40)$$
$$\delta \bar D_{\dot\alpha} V = \bar D_{\dot\alpha} (\epsilon^\beta Q_\beta + \bar\epsilon_{\dot\beta} \bar Q^{\dot\beta}) V = \bar D_{\dot\alpha} \delta V$$

This means, $D_\alpha V$, $\bar D_{\dot\alpha} V$ transform like V under the supertranslations (2.31). Eqs (2.40) lead to

$$[D_\alpha, \epsilon^\beta Q_\alpha + \bar\epsilon_{\dot\beta} \bar Q_{\dot\beta}] = 0 \qquad (2.41)$$
$$[\bar D_{\dot\alpha}, \epsilon^\beta Q_\beta + \bar\epsilon_{\dot\beta} \bar Q^{\dot\beta}] = 0$$

Since the parameters ϵ^β, $\bar\epsilon^{\dot\beta}$ are anticommuting one gets from (2.41)

$$\{D_\alpha, Q_\beta\} = 0 \quad , \quad \{D_\alpha, \bar Q_{\dot\beta}\} = 0 \qquad (2.42)$$
$$\{\bar D_{\dot\alpha}, Q_\beta\} = 0, \quad \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$$

The explicit forms of the supercharges are known (2.36), then Eqs (2.42) allow us to find the supercovariant derivatives in explicit form

To do that we can, for example, to write

$$D_\alpha = c_1 \partial_\alpha + c_2 \bar\theta^{\dot\alpha} \partial_{\alpha\dot\alpha}$$
$$\bar D_{\dot\alpha} = c_3 \bar\partial_{\dot\alpha} + c_4 \theta^\alpha \partial_{\alpha\dot\alpha} \qquad (2.43)$$

with some unknown coefficients c_1, c_2, c_3, c_4.
Substituting Eqs (2.43) into (2.42) one can fix all the coefficients. The result looks like

$$D_\alpha = \partial_\alpha + i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m = \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \qquad (2.44)$$
$$\bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m = -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha}$$

Sometimes, the supercovariant derivatives are called the spinor derivatives.

The spinor derivatives possess a number of important properties which are used in supersymmetric field theories

It is easy to show that the spinor derivatives satisfy the algebra

$$\{D_\alpha, D_\beta\} = 0, \qquad \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$$
$$[D_\alpha, \partial_m] = 0, \qquad [\bar D_{\dot\alpha}, \partial_m] = 0 \qquad (2.45)$$
$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha} = 2P_{\alpha\dot\alpha}$$

Since $D_\alpha D_\beta$ is antisymmetric 2×2 matrix, we can fulfil the same analysis as for $\theta_\alpha\theta_\beta$.
It leads to

$$D_\alpha D_\beta = \frac{1}{2}\varepsilon_{\alpha\beta} D^2, \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar D^2$$
$$D^\alpha D^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta} D^2, \qquad \bar D^{\dot\alpha}\bar D^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar D^2 \qquad (2.46)$$
$$D_\alpha D_\beta D_\gamma = 0, \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta}\bar D_{\dot\gamma} = 0$$

Here

$$D^2 = D^\alpha D_\alpha, \qquad \bar D^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha} \qquad (2.47)$$
$$D^\alpha = \varepsilon^{\alpha\beta} D_\beta, \qquad \bar D^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\beta}$$

Using the algebra of spinor derivatives (2.45) one can prove

$$D^2 \bar{D}_{\dot\alpha} D^2 = 0 \,, \qquad \bar{D}^2 D_\alpha \bar{D}^2 = 0 \qquad (2.48a)$$

$$D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} \qquad (2.48b)\,.$$

$$D^2 \bar{D}^2 + \bar{D}^2 D^2 - 2 D^\alpha \bar{D}^2 D_\alpha = 16 \Box \qquad (2.48c)$$

$$D^2 \bar{D}^2 D^2 = 16 \Box D^2 , \qquad \bar{D}^2 D^2 \bar{D}^2 = 16 \Box \bar{D}^2 \qquad (2.48d)$$

$$[D^2, \bar{D}_{\dot\alpha}] = -4i \partial_{\alpha\dot\alpha} D^\alpha , \quad [\bar{D}^2, D_\alpha] = 4i \partial_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \qquad (2.48e)$$

2.5. Chiral superfields.

Using the supercovariant derivatives $D_\alpha, \bar{D}_{\dot\alpha}$ one can impose the constraints on the superfields which are consistent with supersymmetry transformations.

Simplest constraints is written as follows

$$\bar{D}_{\dot\alpha} \Phi(x, \theta, \bar\theta) = 0 \qquad (2.49)$$

This constraint can be exactly solved. Let us act on Eq (1.49) by the operator $e^{-i(\theta\sigma^m\bar\theta)\partial_m}$, one gets

$$e^{-i(\theta\sigma^m\bar\theta)\partial_m} \bar{D}_{\dot\alpha} e^{i(\theta\sigma^m\bar\theta)\partial_m} e^{-i(\theta\sigma^m\bar\theta)\partial_m} \phi = 0 \qquad (2.50)$$

Now compute the expression

$$e^{-i(\theta\sigma^m\bar\theta)\partial_m} \bar{D}_{\dot\alpha} e^{i(\theta\sigma^m\bar\theta)\partial_m} =$$

$$= - e^{-i(\theta\sigma^m\bar\theta)\partial_m} (\bar\partial_{\dot\alpha} + i\theta^\alpha \sigma^n{}_{\alpha\dot\alpha} \partial_n) e^{i(\theta\sigma^m\bar\theta)\partial_m} =$$

$$= -\bar\partial_{\dot\alpha} + i \bar\partial_{\dot\alpha}(\theta\sigma^m\bar\theta)\partial_m + i\theta^\alpha \sigma^m{}_{\alpha\dot\alpha}\partial_m =$$

$$= -\bar\partial_{\dot\alpha} + i \theta^\alpha \sigma^m{}_{\alpha\dot\alpha} \partial_m + i\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m = -\bar\partial_{\dot\alpha}$$

Hence Eq (1.50) leads to

$$\bar{D}_{\dot{\alpha}}\left[e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta,\bar{\theta})\right]=0 \qquad (2.51)$$

Hence, the function $e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta,\bar{\theta})$ is independent of $\bar{\theta}_{\dot{\alpha}}$, that is

$$e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta,\bar{\theta})=\Phi(x,\theta) \qquad (2.52)$$

where $\Phi(x,\theta)$ is arbitrary superfield of x^m and θ_α. Therefore

$$\Phi(x,\theta,\bar{\theta})=e^{i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta)=\Phi(x+i\theta\sigma^m\theta,\theta) \qquad (2.53)$$

Eq (1.53) is solution to the constraint (1.49). Superfield satisfying the constraints (1.49) is called chiral one. Eq (1.53) defines a general form of chiral superfield. It is important to emphasize that the chiral superfield is complex.

Analogously, we consider the constraints

$$D_\alpha\bar{\Phi}(x,\theta,\bar{\theta})=0 \qquad (2.54)$$

superfield satisfying this constraint is called antichiral. One can show that a solution to Eq (1.54) has the form

$$\bar{\Phi}(x,\theta,\bar{\theta})=e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\bar{\Phi}(x,\bar{\theta})=\bar{\Phi}(x-i(\theta\sigma\bar{\theta}),\bar{\theta}) \qquad (2.55)$$

Basic features of the chiral and antichiral superfields are the special dependence on anticommuting variables. chiral superfield $\Phi(x,\theta,\bar{\theta})$ depends essentially only on θ, antichiral superfield $\bar{\Phi}(x,\theta,\bar{\theta})$ depends essentially only on $\bar{\theta}$.

Eqs (2.53, 2.55) allow us to find the component structure of chiral and antichiral superfields.

We begin with Eq (2.53). Since $\Phi(x,\theta)$ is $\bar\theta$-independent it is represented in component form as follows

$$\Phi(x,\theta) = A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) \qquad (2.56).$$

Hence

$$\Phi(x,\theta,\bar\theta) = \Phi(x + i(\theta\sigma\bar\theta), \theta) =$$

$$= e^{i(\theta\sigma^m\bar\theta)\partial_m}\left(A + \theta^\alpha\psi_\alpha + \theta^2 F\right) =$$

$$= \left(1 + i(\theta\sigma^m\bar\theta)\partial_m - \frac{1}{2}(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta)\partial_m\partial_n\right)\left(A + \theta^\alpha\psi_\alpha + \theta^2 F\right)$$

$$= A + \theta^\alpha\psi_\alpha + \theta^2 F + i(\theta\sigma^m\bar\theta)\partial_m A +$$

$$+ i(\theta\sigma^m\bar\theta)\theta^\alpha\partial_m\psi_\alpha - \frac{1}{2}(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta)\partial_m\partial_n A$$

Now we have to use the identities (2.8) $\theta^\beta\theta^\alpha = -\frac{1}{2}\varepsilon^{\beta\alpha}\theta^2$, $\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2$ and transform the term $i(\theta\sigma^m\bar\theta)\theta^\alpha\partial_m\psi_\alpha$ to the form $\frac{i}{2}\theta^2\bar\theta_{\dot\alpha}(\tilde\sigma^m)^{\dot\alpha\alpha}\partial_m\psi_\alpha$ and the term $-\frac{1}{2}(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta)\partial_m\partial_n A$ to the form $-\frac{1}{4}\theta^2\bar\theta^2\Box A$. As a result, the component expansion of the chiral superfield is

$$\Phi(x,\theta,\bar\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \theta^2 F(x) + i(\theta\sigma^m\bar\theta)\partial_m A(x) +$$
$$+ \frac{i}{2}\theta^2\bar\theta_{\dot\alpha}(\tilde\sigma^m)^{\dot\alpha\alpha}\partial_m\psi_\alpha(x) - \frac{1}{4}\theta^2\bar\theta^2\Box A(x) \qquad (2.57)$$

Component expansion for antichiral superfield is obtained from (2.57) by conjugation.

We consider also the transformation of the component of chiral superfield under supertranslations

There is a general relation

$$\delta\Phi = i(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\,\Phi$$

or

$$\delta A + i\theta^\alpha\,\delta\psi_\alpha + \theta^2\,\delta F + \ldots =$$

$$= i(\epsilon^\alpha Q_\alpha + \bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha})(A + \theta^\beta\psi_\beta + \theta^2 F + \ldots)$$

where .... means the other components which are expressed via A, ψ_α, F according to Eq (1.57). Then we have to substitute the explicit forms of the supercharges $Q_\alpha, \bar Q_{\dot\alpha}$ (2.36) and compute the derivatives with respect to $\theta_\alpha, \bar\theta_{\dot\alpha}$. As a result one gets

$$\delta A(x) = -\epsilon^\alpha\psi_\alpha(x)$$

$$\delta\psi_\alpha(x) = -2\epsilon_\alpha F(x) - 2i\bar\epsilon^{\dot\alpha}\partial_{\alpha\dot\alpha}A(x) \qquad (2.58)$$

$$\delta F(x) = -i\,\bar\epsilon_{\dot\alpha}(\tilde\sigma^m)^{\dot\alpha\alpha}\partial_m\psi_\alpha(x)$$

• The same consideration for antichiral superfield gives

$$\delta\bar A(x) = -\bar\epsilon_{\dot\alpha}\bar\psi^{\dot\alpha}(x)$$

$$\delta\bar\psi_{\dot\alpha}(x) = -2\bar\epsilon_{\dot\alpha}\bar F(x) - 2i\,\epsilon^\alpha\partial_{\alpha\dot\alpha}\bar A(x) \qquad (2.59)$$

$$\delta\bar F(x) = +i\,\epsilon^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar\psi^{\dot\alpha}$$

To conclude this subsection one points out that any function only of chiral superfield also is a chiral superfield. The same is valid for function only of antichiral superfield. Indeed,

$$\bar D_{\dot\alpha} f(\Phi) = f'(\Phi)\,\bar D_{\dot\alpha}\Phi = 0$$

3. Superfield Models

3.1. Superfield action.

As known, a field model is given by a set of fields and the corresponding action functional depending on these fields. In addition, an action is written as an integral over Minkowski space of Lorentz invariant Lagrangian. We are going to develop a formulation of supersymmetric field theory where a set of fields consists of superfields and action is written as integral over superspace of superfield Lagrangian.

As a set of fields we consider only the real scalar superfield $V(x,\theta,\bar\theta)$ and chiral and antichiral superfields $\Phi(x,\theta,\bar\theta)$, $\bar\Phi(x,\theta,\bar\theta)$.

· First of all, one introduces the definitions

1. Superspace coordinates are denoted as

$$z^M \equiv (x^m, \theta^\alpha, \bar\theta^{\dot\alpha}) \quad (3.1)$$

Then $V = V(z)$, $\Phi = \Phi(z)$, $\bar\Phi = \bar\Phi(z)$

2. Supercovariant derivatives are denoted as

$$D_M \equiv (\partial_m, D_\alpha, \bar D_{\dot\alpha}) \quad (3.2)$$

3. $$d^8 z = d^4x\, d^2\theta\, d^2\bar\theta = d^4x\, d^4\theta \quad (3.3)$$
$$d^6 z = d^4x\, d^2\theta, \quad d^6\bar z = d^4x\, d^2\bar\theta$$

4. Superspace δ-functions

$$\delta^8(z-z') = \delta^4(x-x')\delta^4(\theta-\theta') \quad (3.4)$$
$$\delta^4(\theta-\theta') = \delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta') = (\theta-\theta')^2(\bar\theta-\bar\theta')^2$$

The most general action functional, which can be written as an integral over superspace, has the form

$$S = \int d^8 z\, \mathcal{L} + \int d^6 z\, \mathcal{L}_c + \int d^6 \bar{z}\, \bar{\mathcal{L}}_c \qquad (3.5)$$

Here $\mathcal{L}$ is a real scalar superfield, ~~and~~ $\mathcal{L}_c$ is a chiral superfield and $\bar{\mathcal{L}}_c$ is antichiral superfield. First terms in (3.5) is called the integral over full superspace, second term is called the integral over chiral superspace and last one is called the integral over antichiral superspace.

Why not to write the following term ~~additions~~

$\int d^8 z\, \mathcal{L}_c$? We will show that is zero using the ~~de~~ properties of the Berezin integral one writes

$$\int d^8 z\, \mathcal{L}_c = \int d^4x\, d^2\theta \left(-\tfrac{1}{4}\right) \varepsilon^{\dot\alpha\dot\beta}\, \bar\partial_{\dot\alpha} \bar\partial_{\dot\beta}\, \mathcal{L}_c =$$

$$= \int d^4 x\, d^2\theta \left(-\tfrac{1}{4}\right) \bar\partial_{\dot\alpha} \bar\partial^{\dot\alpha} \mathcal{L}_c$$

From the other side, one considers

$$\bar D_{\dot\alpha} \bar D^{\dot\alpha} = (-\bar\partial_{\dot\alpha} - i\theta^\alpha \partial_{\alpha\dot\alpha})(-\bar\partial^{\dot\alpha} - i\theta^\alpha \partial_\alpha{}^{\dot\alpha}) =$$

$$= \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha} + \text{total space-time derivatives}$$

Therefore we can write under the integral over d^4x

$$\int d^4x\, d^2\theta \left(-\tfrac{1}{4}\right) \bar\partial_{\dot\alpha} \bar\partial^{\dot\alpha} \mathcal{L}_c = \int d^4x\, d^2\theta \left(-\tfrac{1}{4} \bar D_{\dot\alpha} \bar D^{\dot\alpha}\right) \mathcal{L}_c$$

Now one remembers the definition of chiral superfield $\bar D_{\dot\alpha} \mathcal{L}_c = 0$. Therefore

$$\int d^4x\, d^4\theta\, \mathcal{L}_c = 0 \qquad (3.6)$$

Analogously

$$\int d^4x\, d^4\theta\, \bar{\mathcal{L}}_c = 0 \qquad (3.7)$$

In principle, we can integrate over all anticommuting variables in action (3.5) and get action in ~~standard~~ conventional form as integral over Minkowski space. Indeed

$$S = \int d^4x \left(-\tfrac{1}{4}\right) D^2 \left(-\tfrac{1}{4}\right) \bar{D}^2 \mathcal{L} + \int d^4x \left(-\tfrac{1}{4} D^2\right) \mathcal{L}_c +$$
$$+ \int d^4x \left(-\tfrac{1}{4} \bar{D}^2\right) \bar{\mathcal{L}}_c$$

One can see that

$\partial^2 = D^2 +$ total space-time derivatives

$\bar{\partial}^2 = \bar{D}^2 +$ total space-time derivatives

Therefore, under the integral over d^4x one gets

$$S = \int d^4x \left(\tfrac{1}{16} D^2 \bar{D}^2 \mathcal{L} - \tfrac{1}{4} D^2 \mathcal{L}_c - \tfrac{1}{4} \bar{D}^2 \bar{\mathcal{L}}_c \right) \qquad (3.8)$$

This relation is useful for finding a component form of the action.

The main ~~problem~~ objects ~~of~~ of classical field theory are ~~[illegible]~~ the equations of motion. In the case under consideration the equations of motion can be written completely in superfield form. To do that, it is useful to derive some identities. To be more precise we need the superfield form of the known relation

$$\frac{\delta \varphi(x)}{\delta \varphi(x')} = \delta^{(4)}(x - x')$$

We consider the real scalar superfield V, chiral scalar superfield Φ and antichiral superfield $\bar{\Phi}$. It is clear that non-zero variational derivatives are

$$\frac{\delta V(z)}{\delta V(z')}, \quad \frac{\delta \Phi(z)}{\delta \Phi(z')}, \quad \frac{\delta \bar{\Phi}(z)}{\delta \bar{\Phi}(z')}$$

In first case everything is trivial

$\delta V(z) = \int d^8 z' \, \delta^8(z-z') \, \delta V(z')$

Hence

$$\frac{\delta V(z)}{\delta V(z')} = \delta^8(z-z') \qquad (3.9)$$

Situation for $\frac{\delta \Phi(z)}{\delta \Phi(z')}$ is not so simple since $\Phi(z)$ is constrained superfield, $\bar{D}_{\dot\alpha}\Phi(z)=0$.

First, we point out that

$\int d^6 z \, \Phi(z) = \int d^6 z \, e^{i(\theta\sigma^m\bar\theta)\partial_m} \, \phi(x,\theta) =$

$= \int d^6 z \, [\phi(x,\theta) + \text{total space-time derivative}] = \int d^6 z \, \phi(x,\theta)$

Then, one defines the derivative $\frac{\delta\Phi(z)}{\delta\Phi(z')}$ as follows

$$\delta\Phi(z) = \int d^6 z' \, \frac{\delta\Phi(z)}{\delta\Phi(z')} \, \delta\Phi(z') = \int d^8 z' \, \delta^8(z-z') \, \delta\Phi(z')$$

Last term can be transformed to

$$\int d^8 z' \, \delta^8(z-z') \, \delta\Phi(z') = \int d^6 z' \left(-\tfrac{1}{4}\bar{D}^2_{z'}\right) \delta^8(z-z') \, \delta\Phi(z') =$$

$$= \int d^6 z' \left[\left(-\tfrac{1}{4}\bar{D}^2_z\right) \delta^8(z-z')\right] \delta\Phi(z')$$

Hence

$$\frac{\delta\Phi(z)}{\delta\Phi(z')} = -\frac{1}{4}\bar{D}^2 \, \delta^8(z-z') \qquad (3.10)$$

Analogously

$$\frac{\delta\bar\Phi(z)}{\delta\bar\Phi(z')} = -\frac{1}{4}D^2 \, \delta^8(z-z') \qquad (3.11)$$

It is evident that the derivatives (3.10), (3.11) are the chiral and antichiral superfields respectively in each their argument.

Explicit constructions of the Lagrangians is an art. We consider further the construction of Lagrangians in most popular supersymmetric models.

3.2. Wess - Zumino model.

The Wess - Zumino model describes a dynamics of chiral and antichiral superfields. The most general action of such superfields without higher space-time derivatives looks like

$$S[\Phi,\bar\Phi] = \int d^8z\, K(\Phi,\bar\Phi) + \int d^6z\, W(\Phi) + \int d^6\bar z\, \bar W(\bar\Phi) \quad (3.12)$$

where $K(\Phi,\bar\Phi)$ is a real function of complex superfields $\Phi, \bar\Phi$, $W(\Phi)$ is a function of chiral superfield Φ, $\bar W(\bar\Phi)$ is a function conjugate to $W(\Phi)$. You can ask me, where are the derivatives here? The terms in action containing $W(\Phi)$ and $\bar W(\bar\Phi)$ include only $\Phi(x,\theta)$ and $\bar\Phi(x,\bar\theta)$ without any derivatives. However the derivatives included in definitions of chiral and antichiral superfields

$$\Phi(z) = e^{i(\theta\sigma^m\bar\theta)\partial_m}\,\Phi(x,\theta)$$

$$\bar\Phi(z) = e^{-i(\theta\sigma^m\bar\theta)\partial_m}\,\bar\Phi(x,\bar\theta)$$

Hence the term $K(\Phi,\bar\Phi)$ contains the space-time derivatives. Therefore, the $K(\Phi,\bar\Phi)$ is called the kinetic term, the $W(\Phi)$ is called the chiral superpotential, and $\bar W(\bar\Phi)$ is called the antichiral superpotential. Wess - Zumino model was a first supersymmetric field theory. In this case

$$K(\Phi,\bar\Phi) = \bar\Phi(z)\Phi(z) \quad (3.13)$$

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3, \quad \bar W(\bar\Phi) = \frac{m}{2}\bar\Phi^2 + \frac{\lambda}{3!}\bar\Phi^3$$

where m is a mass and λ is a coupling constant. At present, the chiral superfield theories can be obtained as low-energy limit of superstring theory. In this case the forms of $K(\Phi,\bar{\Phi})$, $W(\Phi)$ are more complicated then in Wess-Zumino model.

We consider now a derivation of equations of motion for the theory with action (3.12). Since $\bar{\Phi}(z)$ essentially depends only on $x, \bar{\theta}$ the variational derivative of action with respect to $\bar{\Phi}(z)$ is defined as

$$\delta_{\bar{\Phi}} S = \int d^6\bar{z} \, \frac{\delta S}{\delta \bar{\Phi}(z)} \, \delta\bar{\Phi}(\bar{z}) \qquad (3.14)$$

In the case of (3.12) one gets

$$\delta_{\bar{\Phi}} S = \int d^8 z \, \frac{\partial K(\Phi,\bar{\Phi})}{\partial \bar{\Phi}(z)} \, \delta\bar{\Phi}(\bar{z}) +$$

$$+ \int d^6\bar{z} \, \frac{\partial \bar{W}}{\partial \bar{\Phi}} \, \delta\bar{\Phi}(z)$$

First integral is transformed as follows

$$\int d^8 z \, \frac{\partial K}{\partial \bar{\Phi}} \delta\bar{\Phi} = \int d^6 z \left(-\frac{1}{4} D^2 \frac{\partial K}{\partial \bar{\Phi}}\right) \delta\bar{\Phi}$$

we pay an attention that $D_\alpha \delta\bar{\Phi} = 0$

Therefore

$$\delta_{\bar{\Phi}} S = \int d^6 z \left\{ -\frac{1}{4} D^2 \frac{\partial K}{\partial \bar{\Phi}} + \frac{\partial \bar{W}(\bar{\Phi})}{\partial \bar{\Phi}} \right\} \delta\bar{\Phi}$$

Hence

$$\frac{\delta S}{\delta \bar{\Phi}(z)} = -\frac{1}{4} D^2 \frac{\partial K(\Phi,\bar{\Phi})}{\partial \bar{\Phi}(z)} + \frac{\partial \bar{W}(\bar{\Phi})}{\partial \bar{\Phi}(z)} \qquad (3.15)$$

Analogously

$$\frac{\delta S}{\delta \Phi(z)} = -\frac{1}{4} \bar{D}^2 \frac{\partial K(\Phi,\bar{\Phi})}{\partial \Phi(z)} + \frac{\partial W(\Phi)}{\partial \Phi(z)} \qquad (3.16)$$

It is evident, that

$$D_\alpha \frac{\delta S}{\delta \bar{\Phi}} = 0, \qquad \bar{D}_{\dot{\alpha}} \frac{\delta S}{\delta \Phi} = 0$$

As a result, the ~~equations~~ superfield equations of motion have the form

$$-\frac{1}{4}\bar{D}^2\frac{\partial K(\Phi,\bar{\Phi})}{\partial\bar{\Phi}(z)}+\frac{\partial\bar{W}(\bar{\Phi})}{\partial\bar{\Phi}(z)}=0$$

$$-\frac{1}{4}D^2\frac{\partial K(\Phi,\bar{\Phi})}{\partial\Phi(z)}+\frac{\partial W(\Phi)}{\partial\Phi(z)}=0 \qquad (3.17)$$

Further we consider only the Wess-Zumino case, then (3.17) look like

$$-\frac{1}{4}D^2\Phi+m\bar{\Phi}+\frac{\lambda}{2}\bar{\Phi}^2=0$$

$$-\frac{1}{4}\bar{D}^2\bar{\Phi}+m\Phi+\frac{\lambda}{2}\Phi^2=0 \qquad (3.18)$$

Next problem we are going to discuss is ~~to obtain~~ a component form of the Wess-Zumino model. We write

$$\Phi(z)=e^{i\theta\sigma^m\bar{\theta}\partial_m}(A+\theta^\alpha\psi_\alpha+F)$$

$$\bar{\Phi}(z)=e^{-i\theta\sigma^m\bar{\theta}\partial_m}(\bar{A}+\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}+\bar{F})$$

Therefore

$$S=\int d^8z\,\bar{\Phi}(x,\bar{\theta})e^{2i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta)+$$

$$+\left\{\int d^6z\left(\frac{m}{2}\Phi(x,\theta)+\frac{\lambda}{3}\Phi^3(x,\theta)\right)+\text{complex conjugate}\right\}$$

Here we transferred the factor $e^{-i(\theta\sigma^m\bar{\theta})\partial_m}$ from $\bar{\Phi}$ to Φ ~~by~~ using integration by parts. As to ~~the~~ integral over chiral and antichiral superspaces, any factors $e^{\pm i(\theta\sigma^m\bar{\theta})\partial_m}$ can be omitted. Then we use the relations (3.8). All other is only direct computations. These calculations can be simplified if we take into account the following properties of Berezin integral. The integral $\int d^8z$ (superfield) is coefficient at $\theta^2\bar\theta^2$ in expansion of (superfield), analogously, the integral $\int d^6z$ (chiral superfield) is coefficient at θ^2 in expansion of (chiral superfield). Therefore to find the component form of $\int d^8z\, \bar\Phi\Phi$ it is sufficient to multiply $\bar\Phi$ by Φ and extract the only term with $\theta^2\bar\theta^2$. To find the component form of $\int d^6z\,(\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3)$ it is sufficient to ~~~~ in expression ~~~~ extract the only term with θ^2 in expression $\frac{m}{2}\Phi^2 + \frac{1}{3!}\Phi^3$. The final result has the form

$$S=\int d^4x\,\{-\partial^m\bar A\partial_m A-\frac{i}{2}\psi^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m\bar\psi^{\dot\alpha}+\bar FF+$$

$$+F(mA+\frac{\lambda}{2}A^2)+\bar F(m\bar A+\frac{\lambda}{2}\bar A^2)-$$

$$-\frac{1}{4}(m+\frac{\lambda}{2}A)\psi^\alpha\psi_\alpha-\frac{1}{4}(m+\lambda\bar A)\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha}\}\qquad(3.19)$$

In principle, the action (3.19) can be expressed in terms of standard four-component spinors instead of two-component ones. To do that one introduces the four-component spinors

$$\Psi=\frac{1}{\sqrt2}\begin{pmatrix}\psi_\alpha\\ \bar\psi^{\dot\alpha}\end{pmatrix}$$

and Dirac γ-matrices

$$\gamma^m=\begin{pmatrix}0&\sigma^m\\ \tilde\sigma^m&0\end{pmatrix}$$

This is purely technical manipulations and I would not like to do that.

It is easy to derive the equations of motion corresponding to component action (3.19)

$$\Box \bar{A} + F(m+\lambda A) - \frac{1}{4}\lambda\, \psi^{\alpha}\psi_{\alpha} = 0$$

$$i\,\sigma^{m}{}_{\alpha\dot{\alpha}}\,\partial_m \bar{\psi}^{\dot{\alpha}} - (m+\lambda A)\,\psi_{\alpha} = 0 \qquad (3.20)$$

$$\bar{F} + (mA + \frac{\lambda}{2}A^2) = 0$$

We see that the component fields F and $\bar{F}$ have no non-trivial dynamics. These fields can be expressed ~~from the~~ algebraically from the equations of motion

$$F = -(m\bar{A} + \frac{\lambda}{2}\bar{A}^2) \qquad (3.21)$$

$$\bar{F} = -(mA + \frac{\lambda}{2}A^2)$$

Such fields are called auxiliary. Let us substitute the auxiliary fields (3.21) into the action (3.19) and introduce the four-component spinors. The result looks like

$$\tilde{S} = \int d^4x \Big\{ \bar{A}(-\Box + m^2)A + \frac{1}{2}\bar{\Psi}(i\gamma^m\partial_m + m)\Psi +$$

$$+ \frac{1}{2}m\lambda(A+\bar{A})\bar{A}A + \frac{1}{4}\lambda^2(\bar{A}A)^2 +$$

$$+ \frac{1}{4}\lambda(A+\bar{A})\bar{\Psi}\Psi + \frac{1}{4}\lambda(A-\bar{A})\bar{\Psi}\gamma_5\Psi \Big\} \qquad (3.22)$$

where as usual $\bar{\Psi} = \Psi^+\gamma^0$ and $\gamma_5 = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix} = -i\gamma^0\gamma^1\gamma^2\gamma^3$!

Thus, the model under consideration is a theory of complex scalar field A and Majorana spinor field Ψ with special cubic and quartic scalar self-interactions and special Yukawa coupling.

To conclude this ~~section~~ subsection we discuss briefly a role of auxiliary fields. We see, the theory can be formulated without these fields. Why do we need them? To ~~c~~ clarify this question let us write the supersymmetry transformations of the component fields (2.58, 2.59)

$$\delta A = -\epsilon^{\alpha}\psi_{\alpha}$$
$$\delta\psi_{\alpha} = -2\epsilon_{\alpha}F - 2i\bar{\epsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha}A$$
$$\delta F = -i\bar{\epsilon}^{\dot\alpha}(\tilde{\sigma}^{m})^{\dot\alpha\alpha}\partial_m\psi_{\alpha}$$
$$\delta\bar{A} = -\bar{\epsilon}_{\dot\alpha}\bar{\psi}^{\dot\alpha}$$
$$\delta\bar{\psi}_{\dot\alpha} = -2\bar{\epsilon}_{\dot\alpha}\bar{F} + 2i\epsilon^{\alpha}\partial_{\alpha\dot\alpha}\bar{A}$$
$$\delta\bar{F} = -i\epsilon^{\alpha}(\sigma^{m})_{\alpha\dot\alpha}\partial_m\bar{\psi}^{\dot\alpha}$$

Now substitute the auxiliary fields from equations of motion (3.21) to above relations, one gets

$$\delta A = -\epsilon^{\alpha}\psi_{\alpha}, \quad \delta\psi_{\alpha} = 2\epsilon_{\alpha}(m\bar{A} + \frac{\lambda}{2}\bar{A}^2) - 2i\bar{\epsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha}A \qquad (3.23)$$
$$\delta\bar{A} = -\bar{\epsilon}_{\dot\alpha}\bar{\psi}^{\dot\alpha}, \quad \delta\bar{\psi}_{\dot\alpha} = 2\bar{\epsilon}_{\dot\alpha}(mA + \frac{\bar\lambda}{2}A^2) + 2i\epsilon^{\alpha}\partial_{\alpha\dot\alpha}\bar{A}$$

The action $\tilde{S}$ (3.22) is automatically invariant under the transformations (3.23). However, let us calculate the commutators of the ~~#~~ transformations (3.23) with two different parameters ϵ_1 and ϵ_2. The result has the form

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = a^m\partial_m A$$
$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_{\alpha} = a^m\partial_m\psi_{\alpha} + i a_{\alpha\dot\alpha}\frac{\delta\tilde{S}}{\delta\bar{\psi}_{\dot\alpha}} \qquad (3.24)$$

where

$$a^m = 2i(\epsilon_1^{\alpha}\sigma^m{}_{\alpha\dot\alpha}\bar{\epsilon}_2^{\dot\alpha} - \epsilon_2^{\alpha}\sigma^m{}_{\alpha\dot\alpha}\bar{\epsilon}_1^{\dot\alpha})$$
$$a_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}a_m$$

Second of the Eqs (3.24) ~~contains~~ shows that the supersymmetry algebra becomes broken when the auxiliary fields are eliminated. This algebra is closed only under equations of motion for spinor fields. Supersymmetry algebra in the theory without auxiliary fields is closed only on shell. Thus, we understand a role of auxiliary fields, they are responsible for off shell clousere of the supersymmetry algebra.

Let us return back to superfield formulation of the theory. It has two attractive points in compare with component formulation:

1. We work with unique object $\Phi(z)$ instead to work with a set of component fields A, ψ_α, F.
2. Supersymmetry algebra is automatically off-shell closed.

3.3. Supersymmetric sigma-model

Conventional sigma-model is defined as a scalar field theory with the following action

$$S[\varphi] = -\frac{1}{2}\int d^4x \; g_{ij}(\varphi)\,\partial_m\varphi^i\,\partial^m\varphi^j \qquad (3.25)$$

Here φ^i is a set of scalar fields, $i=1,2,\dots,n$. These fields are considered as the coordinates on Riemann manifolds with metric $g_{ij}(\varphi)$. Action is invariant under the transformations

$$\varphi'^i = f^i(\varphi)$$
$$g'_{ij}(\varphi') = \frac{\partial \varphi^k}{\partial \varphi'^i}\frac{\partial \varphi^l}{\partial \varphi'^j}\, g_{kl}(\varphi) \qquad (3.26)$$

We consider a supersymmetric generalization of this model.

Supersymmetric sigma model is a dynamical theory of a set of chiral $\Phi^i(z)$ and antichiral $\bar{\Phi}^i(z)$ superfields. Its action has the form

$$S[\Phi,\bar{\Phi}] = \int d^8 z \; K(\Phi,\bar{\Phi}) \qquad (3.27)$$

where $K(\Phi,\bar{\Phi})$ is a real functions of n complex variables and their conjugates. The function $K(\Phi,\bar{\Phi})$ is defined up to transformation

$$K^*(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \qquad (3.28)$$

where $\Lambda(\Phi)$ is an arbitrary holomorphic function of n complex variables. The model (3.27) has been sugested by B. Zumino. It is a generalization of kinetic term of Wess-Zumino model: one considers n chiral superfields instead one and the action includes the arbitrary function K instead of special $K = \bar{\Phi}\Phi$.

To understand which relation the model (3.27) has to sigma-model (3.25) one finds a component form of action (3.27). Let

$$\Phi^i = A^i + \theta^\alpha \psi_\alpha{}^i + F^i + \ldots \qquad (3.29)$$

$$\bar{\Phi}^i = \bar{A}^i + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha} i} + \bar{F}^i + \ldots$$

General procedure of geting the component form of superfield action is discussed in subsection 3.1. This leads to

$$S = \int d^4x \, \frac{1}{16} D^2 \bar{D}^2 K(\Phi,\bar{\Phi}) = \int d^4x \, L \qquad (3.30)$$

Then we should make the direct calculations taking into account the $\bar{D}_{\dot{\alpha}} \Phi^i = 0$, $D_\alpha \bar{\Phi}^i = 0$. The final result is written as follows

$$L = -K_{i\underline{j}}\left(\partial^m \bar{A}^{\underline{j}} \partial_m A^i - \bar{F}^{\underline{j}} F^i + \frac{i}{4} \psi^{i\alpha} \sigma^m_{\alpha\dot{\alpha}} \overleftrightarrow{\partial}_m \bar{\psi}^{\underline{j}\dot{\alpha}}\right) -$$
$$- \frac{1}{4} K_{ij\underline{i}}\left(\bar{F}^{\underline{i}} \psi^{i\alpha}\psi^j_\alpha - i\partial_m A^i \psi^{j\alpha}\sigma^m_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\underline{i}}\right) -$$
$$- \frac{1}{4} K_{i\underline{i}\underline{j}}\left(F^i \bar{\psi}^{\underline{i}}_{\dot{\alpha}} \bar{\psi}^{\underline{j}\dot{\alpha}} + i\partial_m \bar{A}^{\underline{i}} \psi^{i\alpha}\sigma^m_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\underline{j}}\right) +$$
$$+ \frac{1}{16} K_{ij\underline{i}\underline{j}} \psi^{i\alpha}\psi^j_\alpha \bar{\psi}^{\underline{i}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\underline{j}} \qquad (3.31)$$

where

$$K_{i_1\ldots i_p \underline{i}_1\ldots \underline{i}_q} = \frac{\partial^{p+q} K(A,\bar{A})}{\partial A^{i_1}\ldots\partial A^{i_p}\partial\bar{A}^{\underline{i}_1}\ldots\partial\bar{A}^{\underline{i}_q}} \qquad (3.32)$$

Comparison with (3.25) shows that in purely boson sector we get a sigma-model of complex scalar fields A^i where the metric is

$$g_{i\underline{j}}(A,\bar{A}) = \frac{\partial^2 K(A,\bar{A})}{\partial A^i \partial \bar{A}^{\underline{j}}} \qquad (3.33)$$

The metric of the form (3.33) defines the so called Kählerian geometry. The function $K(A,\bar{A})$ is called the Kählerian potential.

The model (3.31) is described by complex scalar fields A^i, ~~the~~ ~~...~~ the spinor fields ψ^i_α and auxiliary fields F^i. It is convenient to rewrite the Lagrangian (3.31) in some another form. Let us introduce the fields

$$\mathcal{F}^i = F^i - \frac{1}{4}\Gamma^i_{jk}\psi^j\psi^k \qquad (3.34)$$

where Γ^i_{jk} are the Christoffel ~~...~~ symbols calculated for the metric $g_{i\underline{j}}$ (3.33) where all derivatives are taken with respect to A^i but not to $\bar{A}^{\underline{i}}$.

Then one gets

$$L = -g_{i\underline{j}}\left(\partial^m \bar{A}^{\underline{j}} \partial_m A^i - \bar{\mathcal{F}}^{\underline{j}} \mathcal{F}^i + \frac{i}{4}\psi^{i\alpha}\sigma^m_{\alpha\dot{\alpha}} \overleftrightarrow{\nabla}_m \bar{\psi}^{\underline{j}\dot{\alpha}}\right) + \frac{1}{16} R_{i\underline{j}k\underline{l}}\, \psi^{i\alpha}\psi^k_\alpha \bar{\psi}^{\underline{j}}_{\dot{\alpha}}\bar{\psi}^{\underline{l}\dot{\alpha}} \qquad (3.35)$$

Here $R_{i\bar{j}k\bar{l}}$ is a curvature tensor which can be calculated in the form

$$R_{i\bar{j}k\bar{l}}(A,\bar{A}) = \frac{\partial^2 g_{i\bar{j}}}{\partial A^k \partial \bar{A}^{\bar{l}}} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial A^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{A}^{\bar{l}}} =$$

$$= K_{i\bar{j}k\bar{l}} - g^{p\bar{q}} K_{ik\bar{q}} K_{p\bar{j}\bar{l}} \qquad (3.36)$$

The ∇_m denote so called targen space covariant derivatives

$$\nabla_m \psi^i_\alpha = \partial_m \psi^i_\alpha + \Gamma^i_{jl} (\partial_m A^j) \psi^l_\alpha$$
$$\nabla_m \bar{\psi}^{\bar{i}}_{\dot{\alpha}} = \partial_m \bar{\psi}^{\bar{i}}_{\dot{\alpha}} + \Gamma^{\bar{i}}_{\bar{j}\bar{l}} (\partial_m \bar{A}^{\bar{j}}) \bar{\psi}^{\bar{l}}_{\dot{\alpha}} \qquad (3.37)$$

We see that supersymmetric sigma-model is also expressed completely in geometrical terms as well as the conventical sigma-model.

3.4. Supersymmetric Yang-Mills theories

Let we have a set of chiral Φ^i and antichiral $\bar{\Phi}_i$ superfields where $\bar{\Phi}_i = (\Phi^i)^*$ and action

$$S = \int d^8 z\, \bar{\Phi}_i \Phi^i \qquad (3.38)$$

We assume that these fields carry a representation of compact Lie group. This means that these fields transform as follows

$$\Phi'^i = (e^{i\Lambda})^i{}_j \Phi^j$$
$$\bar{\Phi}'_i = \bar{\Phi}_j (e^{-i\Lambda})^j{}_i \qquad (3.39)$$

where $\Lambda = \Lambda^I T^I$, $(T^I)^i{}_j$ are the Hermitean generators satisfying the standard relation

$[T^I, T^J] = i f^{IJK} T^K$, and $\Lambda^I =$ const. It is evident that the action

(3.38) is invariant under the transformations (3.39)

Now let us try to clarify if this action is invariant under transformation of the type (3.39) but with local parameters $\Lambda = \Lambda(x) = \Lambda^I(x) T^I$. We pay attention that ϕ and $\bar\Phi$ are the chiral and antichiral superfields. To preserve this property we have to consider the transformations in the form

$$\phi'^i = (e^{i\Lambda})^i{}_j \, \phi^j \quad \text{or} \quad \phi' = e^{i\Lambda}\phi \qquad (3.40)$$

$$\bar\Phi'_i = \bar\Phi_j (e^{-i\bar\Lambda})^j{}_i \quad \text{or} \quad \bar\Phi' = \bar\Phi e^{-i\bar\Lambda}$$

where Λ must be chiral superfield and $\bar\Lambda$ must be antichiral superfield,

$$\Lambda = \Lambda^I(z) T^I \qquad (3.41)$$

$$\bar\Lambda = \bar\Lambda^I(z) T^I$$

$$\bar D_{\dot\alpha} \Lambda^I(z) = 0, \qquad D_\alpha \bar\Lambda^I(z) = 0$$

However, the action (3.38) is not invariant under the transformations (3.40, 3.41). Indeed

$$S' = \int d^8z \, \bar\Phi' \phi' = \int d^8z \, \bar\Phi e^{-i\bar\Lambda} e^{i\Lambda} \phi \neq \int d^8z \, \bar\Phi \phi$$

To provide an invariance under the transformations (3.40, 3.41) we modify the action by introduction of gauge superfield. This means we consider the new action

$$S = \int d^8z \, \bar\Phi_i (e^{2V})^i{}_j \phi^j \equiv \int d^8z \, \bar\Phi e^{2V} \phi \qquad (3.42)$$

where $V = V(z)^I T^I$ and $V^I(z)$ is real scalar superfield. Transformation law of the superfield V is fixed by requirement that action (3.42) is invariant.

$$\int d^8z\, \bar{\Phi}' e^{2V'} \Phi' = \int d^8z\, \bar{\Phi} e^{2V} \Phi$$

Or

$$\int d^8z\, \bar{\Phi} e^{-i\bar{\Lambda}} e^{2V'} e^{i\Lambda} \Phi = \int d^8z\, \bar{\Phi} e^{2V} \Phi$$

Hence

$$e^{2V'} = e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda} \qquad (3.43)$$

Thus, if the superfield V transforms according (3.43) the action (3.42) will be invariant. The superfield V is called gauge superfield and the transformations (3.40, 3.41, 3.43) are called the supergauge transformations. Also V is called Yang-Mill superfield.

Since we have the new field in the theory, we should find action for this field. Let us introduce the superfields

$$W_\alpha = -\frac{1}{8} \bar{D}^2 (e^{-2V} D_\alpha e^{2V}) \qquad (3.44)$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{8} D^2 (e^{2V} \bar{D}_{\dot\alpha} e^{-2V})$$

It is clear that W_α is chiral and $\bar{W}_{\dot\alpha}$ is antichiral superfields

$$\bar{D}_{\dot\alpha} W_\alpha = 0 \quad , \quad D_\alpha \bar{W}_{\dot\alpha} = 0 \qquad (3.45)$$

One can show that W_α and $\bar{W}_{\dot\alpha}$ are Lie-algebra-valued superfields,

$$W_\alpha = W_\alpha^I T^I \qquad (3.46)$$

$$\bar{W}_{\dot\alpha} = \bar{W}_{\dot\alpha}^I T^I$$

$W_\alpha, \bar{W}_{\dot\alpha}$ change covariantly under the supergauge transformations (3.43). Indeed

$$W'_\alpha = -\frac{1}{8} \bar{D}^2 (e^{-2V'} D_\alpha e^{2V'}) =$$

$$= -\frac{1}{8} \bar{D}^2 (e^{i\Lambda} e^{-2V} e^{i\bar{\Lambda}} D_\alpha e^{-i\bar{\Lambda}} e^{2V} e^{-i\Lambda}) =$$

$= e^{i\Lambda}\left(-\frac{1}{8}\bar{D}^2 e^{-2V} D_\alpha e^{2V}\right) e^{-i\Lambda} +$

$-\frac{1}{8} e^{i\Lambda} \bar{D}^2 D_\alpha e^{-i\Lambda} = e^{i\Lambda} W_\alpha e^{-i\Lambda} - \frac{1}{8} e^{i\Lambda} \bar{D}^2 D_\alpha e^{-i\Lambda}$.

$e^{-i\Lambda}$ is a chiral superfield, $\bar{D}_{\dot\alpha} e^{-i\Lambda} = 0$. Solution of this equation has the form $e^{-i\Lambda} = \bar{D}^2 U$, where U some superfield. Therefore $\bar{D}^2 D_\alpha e^{-i\Lambda} = \bar{D}^2 D_\alpha \bar{D}^2 U = 0$ where we have used the identity (2.48a). As a result

$$W'_\alpha = e^{i\Lambda} W_\alpha e^{-i\Lambda} \quad (3.47)$$

Analogously

$$\bar{W}'_{\dot\alpha} = e^{i\bar\Lambda} \bar{W}_{\dot\alpha} e^{-i\bar\Lambda} \quad (3.48)$$

The relations (3.47, 3.48) show that the quantities

$tr(W^\alpha W_\alpha)$, $tr(\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha})$,

where tr denotes matrix trace, are invariant under the supergauge transformations. Therefore we can define action for gauge superfield as follows

$$S_{SYM} = \frac{1}{4g^2}\int d^6z\, tr\, W^\alpha W_\alpha + \frac{1}{4g^2}\int d^6\bar{z}\, tr\, \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \quad (3.49)$$

Here g is a coupling constant. The superfields (3.47, 3.48) are called the superfield strengths, action (3.49) is considered as action of supersymmetric Yang-Mills theory. Moreover, one can show that

$$\int d^6z\, tr\, W^\alpha W_\alpha = \int d^6\bar{z}\, tr\, \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \quad (3.50)$$

Therefore, the final form of action of super Yang-Mills theory is

$$S_{SYM}[V] = \frac{1}{2g^2}\int d^6 z \, tr W^\alpha W_\alpha \qquad (3.51)$$

As a result, action of super Yang-Mills theory coupled to supersymmetric matter is written as

$$S[V, \Phi, \bar{\Phi}] = \frac{1}{2g^2}\int d^6 z \, tr W^\alpha W_\alpha + \int d^8 z \, \bar{\Phi} e^{2V} \Phi \qquad (3.52)$$

Further we discuss two point. First, we find the superfield equations of motion for the theory (3.52) and component form of the action (3.52).

Consider

$$\delta S_{SYM}[V] = \frac{1}{2g^2}\int d^6 z \, tr(\delta W^\alpha W_\alpha + W^\alpha \delta W_\alpha) =$$

$$= \frac{1}{g^2}\int d^6 z \, tr \, \delta W^\alpha W_\alpha =$$

$$= -\frac{1}{8g^2}\int d^6 z \, \bar{D}^2 tr \delta(e^{-2V} D^\alpha e^{2V}) W_\alpha =$$

$$= \frac{1}{2g^2}\int d^8 z \, tr \, \delta(e^{-2V} D^\alpha e^{2V}) W_\alpha .$$

Using the identity $e^{-2V} e^{2V} = 1$ one gets

$$\delta e^{-2V} = - e^{-2V}(\delta e^{2V}) e^{-2V}$$

Hence

$$\delta S_{SYM}[V] = \frac{1}{2g^2}\int d^8 z \, tr\{- e^{-2V}(\delta e^{2V}) e^{-2V} D^\alpha e^{2V} +$$

$$+ e^{-2V}(D^\alpha \delta e^{2V})\} W_\alpha$$

Using the identity $tr \, e^{-2V}(D^\alpha \delta e^{2V}) W_\alpha =$
$= tr(D^\alpha \delta e^{2V}) W_\alpha e^{-2V}$ and integrating by parts one gets

$$\delta S_{SYM}[V] = -\frac{1}{2g^2}\int d^8z\, tr\{(e^{-2V}\delta e^{2V})(e^{-2V}D^\alpha e^{2V})W_\alpha +$$
$$+ \delta e^{2V} D^\alpha(W^\alpha e^{-2V})\} =$$
$$= -\frac{1}{2g^2}\int d^8z\, tr\{(e^{-2V}\delta e^{2V})(e^{-2V}D^\alpha e^{2V})W_\alpha +$$
$$+ e^{2V}e^{-2V}\delta e^{2V} D^\alpha(W_\alpha e^{-2V})\} =$$
$$= -\frac{1}{2g^2}\int d^8z\, tr(e^{-2V}\delta e^{2V})[(e^{-2V}D^\alpha e^{2V})W_\alpha + D^\alpha(W_\alpha e^{-2V})e^{2V}] =$$
$$= -\frac{1}{2g^2}\int d^8z\, tr(e^{-2V}\delta e^{2V})[(e^{-2V}D^\alpha e^{2V})W_\alpha +$$
$$+ D^\alpha W_\alpha + W_\alpha(D^\alpha e^{-2V})e^{2V}] =$$
$$= -\frac{1}{2g^2}\int d^8z\, tr(e^{-2V}\delta e^{2V})[D^\alpha W_\alpha + (e^{-2V}D^\alpha e^{2V})W_\alpha +$$
$$+ W_\alpha(e^{-2V}D^\alpha e^{2V})] \qquad (3.53)$$

We define the operator $\mathcal{D}_\alpha$ by the rule

$$\mathcal{D}_\alpha = D_\alpha + i\Gamma_\alpha \qquad (3.54)$$

where

$$i\Gamma_\alpha = e^{-2V}(D_\alpha e^{2V}) \qquad (3.55)$$

This operator acts on W_α as follows

$$\mathcal{D}^\alpha_\alpha W_\alpha = D^\alpha W_\alpha + i\{\Gamma^\alpha, W_\alpha\} \qquad (3.56)$$

Eq (3.53) shows that the equations of motion of super Yang-Mills theory have the form

$$\mathcal{D}^\alpha W_\alpha = 0 \qquad (3.57)$$

Now we show that the equations are invariant under supergauge transformations

Indeed,

$$\mathcal{D}'^{\alpha} W'_{\alpha} = D^{\alpha} W'_{\alpha} + e^{-2V'}(D^{\alpha} e^{2V'}) W'_{\alpha} + W'_{\alpha} e^{-2V'}(D^{\alpha} e^{2V'}) =$$

$$= D^{\alpha}(e^{i\Lambda} W_{\alpha} e^{-i\Lambda}) + e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}} D^{\alpha}(e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda}$$

$$+ e^{i\Lambda} W_{\alpha} e^{-i\Lambda} e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}} D^{\alpha}(e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}) =$$

$$= (D^{\alpha} e^{i\Lambda}) W_{\alpha} e^{-i\Lambda} + e^{i\Lambda}(D^{\alpha} W_{\alpha}) e^{-i\Lambda} + \underline{e^{i\Lambda} W_{\alpha} (D^{\alpha} e^{-i\Lambda})} +$$

$$+ e^{i\Lambda} e^{-2V} D^{\alpha}(e^{2V} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda} +$$

$$+ e^{i\Lambda} W_{\alpha} e^{-2V} D^{\alpha}(e^{2V} e^{-i\Lambda}) =$$

$$= (D^{\alpha} e^{i\Lambda}) W_{\alpha} e^{-i\Lambda} + e^{i\Lambda}(D^{\alpha} W_{\alpha}) e^{-i\Lambda} - e^{i\Lambda} W_{\alpha} (D^{\alpha} e^{-i\Lambda}) +$$

$$+ e^{i\Lambda} e^{-2V}(D^{\alpha} e^{2V}) W_{\alpha} e^{-i\Lambda} + e^{i\Lambda}(D^{\alpha} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda} +$$

$$+ e^{i\Lambda} W_{\alpha} e^{-2V}(D^{\alpha} e^{2V}) e^{-i\Lambda} + \underline{e^{i\Lambda} W_{\alpha} (D^{\alpha} e^{-i\Lambda})} =$$

$$= e^{i\Lambda}[D^{\alpha} W_{\alpha} + e^{-2V}(D^{\alpha} e^{2V}) W_{\alpha} + W_{\alpha} e^{-2V}(D^{\alpha} e^{2V})] +$$

$$+ (D^{\alpha} e^{i\Lambda}) W_{\alpha} e^{-i\Lambda} + e^{i\Lambda}(D^{\alpha} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda}$$

Last term can be trasformed as follows

$$e^{i\Lambda}(D^{\alpha} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda} = -(D^{\alpha} e^{i\Lambda}) W_{\alpha} e^{-i\Lambda}$$

Therefor the sum of two last term is

$$(D^{\alpha} e^{i\Lambda}) W_{\alpha} e^{-i\Lambda} + e^{i\Lambda}(D^{\alpha} e^{-i\Lambda}) e^{i\Lambda} W_{\alpha} e^{-i\Lambda} = 0$$

Thus

$$\mathcal{D}'^{\alpha} W'_{\alpha} = e^{i\Lambda}(\mathcal{D}^{\alpha} W_{\alpha}) e^{-i\Lambda} \qquad (3.58)$$

As a result, if $\mathcal{D}^{\alpha} W_{\alpha} = 0$ then $\mathcal{D}'^{\alpha} W'_{\alpha} = 0$, equations of motion are gauge invariant.

To conclude this subsection we find a component form of the action S_{SYM} (3.51). The component expansion of the superfield V is

$$V(z) = A + \theta^\alpha \psi_\alpha + \bar\theta_{\dot\alpha} \bar\psi^{\dot\alpha} + \theta^2 f + \bar\theta^2 \bar f + (\theta \sigma^m \bar\theta) A_m + \bar\theta^2 \theta^\alpha \lambda_\alpha + \theta^2 \bar\theta_{\dot\alpha} \bar\lambda^{\dot\alpha} + \theta^2 \bar\theta^2 D \qquad (3.59)$$

First of all one shows that the component form (3.59) can be simplified with help of gauge transformations. These transformations are

$$e^{2V'} = e^{i\bar\Lambda} e^{2V} e^{-i\Lambda}$$

or in the infinitesimal form

$$e^{2(V+\delta V)} = e^{2V} + i\bar\Lambda e^{2V} + e^{2V} i\Lambda$$

Hence

$$\delta V = \frac{i}{2}(\bar\Lambda - \Lambda) + O(V) \qquad (3.60)$$

Let

$$\Lambda = e^{i\theta\sigma^m\bar\theta \partial_m} (u(x) + \theta^\alpha \rho_\alpha(x) + \theta^2 f(x))$$

Then

$$\delta V = \frac{i}{2}(\bar u - u) - \frac{i}{2}\theta^\alpha \rho_\alpha + \frac{i}{2}\bar\theta_{\dot\alpha}\bar\rho^{\dot\alpha} - \frac{i}{2}\theta^2 f + \frac{i}{2}\bar\theta^2 \bar f + \frac{1}{2}(\theta\sigma^m\bar\theta)\partial_m(u + \bar u) + \cdots \qquad (3.61)$$

Comparision of Eq (3.59) and Eq (3.61) shows that the components A, ψ_α, $\bar\psi^{\dot\alpha}$, F and $\bar F$ in expansion (3.59) can be done arbitrary. This means, there exists a gauge where these components are equal to zero. The other words, the above components can be gauged away. As a result, the V ~~reduces~~ is reduced to

$$V = (\theta\sigma^m\bar\theta) A_m + \bar\theta^2 \theta^\alpha \lambda_\alpha + \theta^2 \bar\theta_{\dot\alpha} \bar\lambda^{\dot\alpha} + \theta^2\bar\theta^2 D \qquad (3.62)$$

This gauge fixing is called the Wess-Zumino gauge. It can be written as

$$V| = 0, \qquad D_\alpha V| = 0, \qquad D^2 V| = 0$$

It is important to emphasize that in Wess-Zumino gauge, the e^{2V} is reduced to finite order polynomial. That allows us to find the W_α in the form

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{4}\bar{D}^2 [V, D_\alpha V] \qquad (3.63)$$

After that we use the general rule

$$\int d^6 z \, \mathcal{L}_c = \int d^4x \left(-\frac{1}{4} D^2\right) \mathcal{L}_c$$

where $\mathcal{L}_c = \frac{1}{2g^2} W^\alpha W_\alpha$ and W_α is given by Eq (3.63).
Now we have to fulfill the straightforward calculations. The final result has the form

$$S_{SYM} = \frac{1}{g^2}\int d^4x \, tr\Big\{-\frac{1}{4} G^{mn} G_{mn} - i\lambda^\alpha \sigma^m{}_{\alpha\dot{\alpha}} \nabla_m \bar{\lambda}^{\dot{\alpha}} + 2D^2 \Big\} \qquad (3.64)$$

where

$$G_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n] \qquad (3.65)$$

$$\nabla_m \bar{\lambda}^{\dot{\alpha}} = \partial_m \bar{\lambda}^{\dot{\alpha}} + i[\bar{\lambda}^{\dot{\alpha}}, A_m]$$

We see that the super Yang-Mills theory includes vector field A_m, Majorana spinor $\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}$ and auxiliary field D. It is worth pointing out that the Wess-Zumino gauge does not fix completely a gauge freedom of the theory. A residual gauge symmetry just corresponds to gauge freedom in conventional Yang-Mills theory coupled to Majorana spinor.

4. Superfield Perturbation Theory

4.1 Scheme of perturbative expansion in quantum field theory.

The purpose of this subsection is to remember a construction of perturbative expansion in quantum field theory. Basic notion of such a construction is the generating functional of Green functions in terms of path integral.

Let φ is a set of fields in the model with action $S[\varphi]$. The generating functional of Green functions is written as follows

$$Z[J] = \int \mathcal{D}\varphi \, e^{i(S[\varphi] + \int dx\, \varphi(x) J(x))} \qquad (4.1)$$

The external classical field $J(x)$ is called a source. The Green functions are calculated on the base of $Z[J]$ in the form

$$G_n(x_1, \ldots, x_n) = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta i J(x_1) \ldots \delta i J(x_n)} \Big|_{J=0} \qquad (4.2)$$

To develop the perturbative expansion of Green functions one writes the action as a sum of quadratic part $S_0[\varphi]$, ~~and~~ which is called the free action, and interaction $S_{int}[\varphi]$ including higher powers of fields. Then

$$Z[J] = e^{i S_{int}[\frac{\delta}{i\delta J}]} Z_0[J] \qquad (4.3)$$

where

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{i(S_0[\varphi] + \int dx\, \varphi(x) J(x))} = e^{\frac{i}{2}\int dx dx' J(x) G(x,x') J(x')} \qquad (4.4)$$

where $D(x,y)$ is the Feynmann propagator.

Perturbative series for Green functions is obtained after substitute Eqs (4.3), (4.4) into Eq (4.2) and expand $e^{iS_{int}[\frac{\delta}{\delta iJ}]}$ in power series in $S_{int}[\frac{\delta}{\delta iJ}]$. and fulfill the differentiations with respect to source $J(x)$ The result is described by Feynmann diagrams, the propagators are defined by quadratic part of action and the vertices are created by $S_{int}[\varphi]$

For further it will be useful the following observation illustraiting how to derive equation for Feynmann propagator.

Let

$$S_0[\varphi] = \frac{1}{2}\int d^4x\ \varphi(x)\Lambda_x\varphi(x)$$

where Λ_x is some proper differential operator acting on coordinate x. Write the equations of motion in the theory with action

$$S_0[\varphi] + \int dx\ \varphi(x)J(x)$$

They have the form

$$\Lambda_x\varphi_J(x) + J(x) = 0 \qquad (4.5)$$

$\varphi_J(x)$ means that it is the field in the theory with source J. Feynmann propagator $D(x,x')$ is defined as follows

$$D(x,x') = \frac{\delta\varphi_J(x)}{i\delta J(x)} \qquad (4.6)$$

Then, the Eq (4.5) leads to

$$\Lambda_x D(x,x') = i\delta^4(x-x') \qquad (4.7)$$

Namely such a definition of propagator will be used in superfield theories.

4.2 Superpropagators in the Wess–Zumino model

We consider the Wess–Zumino model with the action

$$S[\Phi,\bar{\Phi}] = \int d^8z\, \bar{\Phi}\Phi + \int d^6z \left(\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3\right) + \int d^6\bar{z}\left(\frac{m}{2}\bar{\Phi}^2 + \frac{\lambda}{3!}\bar{\Phi}^3\right) =$$

$$= S_0[\Phi,\bar{\Phi}] + S_{int}[\Phi,\bar{\Phi}] \qquad (4.5)$$

where

$$S_0[\Phi,\bar{\Phi}] = \int d^8z\, \bar{\Phi}\Phi + \int d^6z\, \frac{m}{2}\Phi^2 + \int d^6\bar{z}\, \frac{m}{2}\bar{\Phi}^2 \qquad (4.6)$$

$$S_{int}[\Phi,\bar{\Phi}] = \frac{\lambda}{3!}\int d^6z\, \Phi^3 + \frac{\lambda}{3!}\int d^6\bar{z}\, \bar{\Phi}^3$$

To find the propagators in this theory we will follow the procedure pointed out in previous subsection. First of all we have to write the free action in the theory coupled to sources. Since there are chiral and antichiral superfields in the model, the sources also should be chiral and antichiral superfields. Hence, the free action in the theory with sources has the form

$$\int d^8z\, \bar{\Phi}\Phi + \int d^6z\, \frac{m}{2}\Phi^2 + \int d^6\bar{z}\, \frac{m}{2}\bar{\Phi}^2 + \int d^6z\, \Phi J + \int d^6\bar{z}\, \bar{\Phi}\bar{J} \qquad (4.7)$$

Here $J(z)$ is chiral external superfield and $\bar{J}(z)$ is antichiral one.

Equations of motion corresponding to the action (4.6) are

$$-\frac{1}{4}\bar{D}^2\Phi_J + m\bar{\Phi}_J + \bar{J} = 0 \qquad (4.8)$$

$$-\frac{1}{4}D^2\bar{\Phi}_J + m\Phi_J + J = 0$$

The theory under consideration is characterized by matrix propagator with elements

$$G_{++}(z,z') = \frac{\delta \Phi_J(z)}{i\delta J(z')}, \quad G_{+-}(z,z') = \frac{\delta \Phi_J(z)}{i\delta \bar{J}(z')}$$

$$G_{-+}(z,z') = \frac{\delta \bar{\Phi}_J(z)}{i\delta J(z')}, \quad G_{--}(z,z') = \frac{\delta \bar{\Phi}_J(z)}{i\delta \bar{J}(z')} \qquad (4.9)$$

Index + means that the propagator is chiral superfield on given argument, index − means that the propagator is antichiral superfield on given argument. For example

$$\bar{D}_{\dot\alpha z}\, G_{++}(z,z') = 0, \qquad D_{\alpha z'}\, G_{--}(z,z') = 0.$$

Differentiating the equations (4.8) with respect to $J(z)$ and $\bar{J}(z)$ one gets

$$-\tfrac{1}{4}\bar{D}^2 G_{++}(z,z') + m\, G_{-+}(z,z') = 0$$
$$-\tfrac{1}{4}\bar{D}^2 G_{+-}(z,z') + m\, G_{--}(z,z') = i\,\delta_-(z,z')$$
$$-\tfrac{1}{4}\bar{D}^2 G_{-+}(z,z') + m\, G_{++}(z,z') = i\,\delta_+(z,z') \qquad (4.10)$$
$$-\tfrac{1}{4}\bar{D}^2 G_{--}(z,z') + m\, G_{+-}(z,z') = 0$$

Here $\delta_+(z,z')$ is a chiral δ-function

$$\delta_+(z,z') = -\tfrac{1}{4}\bar{D}^2 \delta^4(x-x')\,\delta^4(\theta-\theta') = -\tfrac{1}{4}\bar{D}^2\delta^8(z-z') \qquad (4.11)$$

and $\delta_-(z,z')$ is antichiral δ-function

$$\delta_-(z,z') = -\tfrac{1}{4}D^2\delta^4(x-x')\,\delta^4(\theta-\theta') = -\tfrac{1}{4}D^2\delta^8(z-z') \qquad (4.12)$$

$$\bar{D}_{\dot\alpha}\,\delta_+ = 0, \qquad D_\alpha\,\delta_- = 0.$$

Eqs (4.10) allows us to find the components G_{+-}, G_{-+} algebraically

$$G_{-+} = \frac{1}{4m} D^2 G_{++}$$
$$G_{+-} = \frac{1}{4m} \bar{D}^2 G_{--} \qquad (4.13)$$

Substituting Eqs (4.13) into the other Eqs (4.10) one gets

$$-\frac{1}{16m} D^2 \bar{D}^2 G_{--} + m G_{--} = i\,\delta_-$$
$$-\frac{1}{16m} \bar{D}^2 D^2 G_{++} + m G_{++} = i\,\delta_+ \qquad (4.14)$$

Now one uses the identities for antichiral and chiral superfields (see Eqs (2.48c))

$$\frac{1}{16} D^2 \bar{D}^2 G_{--} = \Box G_{--}, \quad \frac{1}{16} \bar{D}^2 D^2 G_{++} = \Box G_{++}$$

As a result, one finds

$$(\Box - m^2) G_{--} = -im\,\delta_-$$
$$(\Box - m^2) G_{++} = -im\,\delta_+ \qquad (4.15)$$

Hence

$$G_{++} = \frac{-im}{\Box - m^2 + i\varepsilon}\,\delta_+$$
$$G_{--} = \frac{-im}{\Box - m^2 + i\varepsilon}\,\delta_- \qquad (4.16)$$

Here we have used the Feynmann prescription defining the causal Green function. Then, we have to substitute Eqs (4.16) into Eqs (4.13) and find G_{-+}, G_{++}. Thus, the matrix propagator in the model under consideration is

$$G = \frac{-i}{\Box - m^2 + i\varepsilon} \begin{pmatrix} m\,\delta_+ & \frac{1}{4}\bar{D}^2 \delta_- \\ \frac{1}{4} D^2 \delta_+ & m\,\delta_- \end{pmatrix} \qquad (4.17)$$

Expression (4.17) is called the superpropagator in the Wess-Zumino model.

The superpropagator (4.17) can be transformed to more useful form. The matter is, the superpropagator contains the various δ-functions. This means, we have to consider separately the chiral vertex $\frac{\lambda}{3!}\int d^6z\, \Phi^3$ and antichiral vertex $\frac{\bar\lambda}{3!}\int d^6\bar z\, \bar\Phi^3$. However, the superpropagator can be rewritten in the form where all its matrix elements will include the same δ-functions $\delta^8(z,z')$ and, moreover, it will include extra D^2, $\bar D^2$ factors allowing us to transform each integral over chiral or antichiral subspaces to integral over full superspace.

First of all, let us substitute into matrix (4.17) the explicit expressions for $\delta_\pm$:

$$\delta_+ = -\frac{1}{4}\bar D^2\delta^8, \qquad \delta_- = -\frac{1}{4}D^2\delta^8.$$

Then

$$G = \frac{-i}{\Box - m^2 + i\varepsilon}\begin{pmatrix} -\frac{m}{4}\bar D^2\delta^8 & -\frac{1}{16}\bar D^2 D^2\delta^8 \\ -\frac{1}{16}D^2\bar D^2\delta^8 & -\frac{m}{4}D^2\delta^8 \end{pmatrix} \qquad (4.18)$$

Now one uses the identities (2.48d)

$$\bar D^2 = \frac{\bar D^2 D^2 \bar D^2}{16\Box}, \qquad D^2 = \frac{D^2\bar D^2 D^2}{16\Box}$$

Therefore

$$G = \frac{-i}{\Box - m^2 + i\varepsilon}\begin{pmatrix} -\frac{m}{4}\frac{\bar D^2 D^2}{16\Box}\bar D^2\delta^8 & -\frac{\bar D^2 D^2}{16}\delta^8 \\ -\frac{D^2\bar D^2}{16}\delta^8 & -\frac{m}{4}\frac{D^2\bar D^2}{16\Box}\delta^8 \end{pmatrix}$$

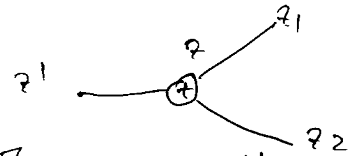
$$= +\frac{1}{16}\begin{pmatrix} \frac{m}{4}\frac{\bar D^2 D^2}{\Box}\bar D^2 & \bar D^2 D^2 \\ D^2\bar D^2 & \frac{m}{4}\frac{D^2\bar D^2}{\Box} \end{pmatrix}\frac{i}{\Box - m^2 + i\varepsilon}\,\delta^8(z,z') \qquad (4.19)$$

In this form the superpropagator contains the same delta-function $\delta^8(z,z')$ in all its elements.

4.3. Superghaphs

The Feynmann diagrams in the theory under considera-tion are constructed on the base of standard rules. The elements of superpropagator matrix (4.19) correspond to lines and the integrals $\frac{\lambda}{3!}\int d^6 z$ and $\frac{\bar\lambda}{3!}\int d^6 \bar z$ correspond to vertices. The only point we have to control is consistency of chirality or antichirality of the propagators with chirality and antichirality of the vertices. It is worth emphasizing that the diagrams are completely formulated in superspace terms. Therefore they called the supergraphs.

Let us consider some chiral vertex within a superghaph



(4.20)

There are three internal lines attached to chiral vertex. Corresponding contribution looks like

$$\int d^6 z \; G_{A+}(z', z)\, G_{+B}(z, z_1)\, G_{+C}(z, z_2) \qquad (4.21)$$

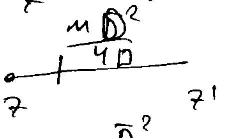
where $A, B, C = +, -$. The propagators G_{AB} are given by Eq (4.19). We see from Eq (4.19) that each propagator $G_{A+}(z', z)$ has a structure $(\dots)_{z'}(-\frac{1}{4}\bar D^2_z)\delta^8(z'-z)$. Factor $-\frac{1}{4}\bar D^2_z$ can be used under the integral (4.20) in order to get full measure $d^8 z$. ~~The same situation will be~~ Analogous situation will be if one of the lines in (4.20) is external. In this case we have the contribution

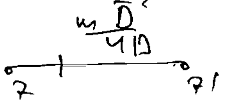
$$\int d^6 z \; \Phi(z)\, G_{+A}(z, z_1)\, G_{+B}(z, z_2) \qquad (4.27)$$

We see from Eq (4.19) that any propagator $G_{+A}(z, z_1)$ has a structure $-\frac{1}{4}\bar{D}^2_z(\ldots)$. Due to chirality $\Phi(z)$ we can put the factor $-\frac{1}{4}\bar{D}^2_z$ so that one gets $\int d^6 z (-\frac{1}{4}\bar{D}^2)\{\ldots\} = \int d^8 z \{\ldots\}$. We again obtain the integral over full superspace. Analogous consideration is obviously valid for antichiral vertex as well. As a result, all vertices in the supergraphs contains the integrals over full superspace only.

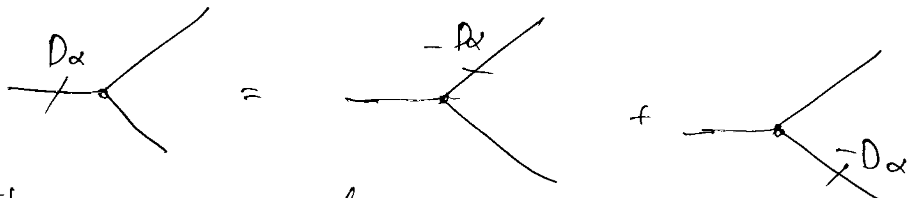
Now, let us pay attention that each matrix element of (4.19) contains the factor $(-\frac{1}{4}D^2)(-\frac{1}{4}\bar{D}^2)$ or the factor $(-\frac{1}{4}\bar{D}^2)(-\frac{1}{4}D^2)$. We will treat them as the operators in vertices acting on internal lines. This means that it is possible to introduce the following rules for the lines

$\Phi\bar{\Phi}$ - line  $= \dfrac{i}{\Box - m^2}\,\delta^8(z-z') = K_{+-}$

$\Phi\Phi$ - line  $= K_{++} = \frac{1}{4}\bar{D}^2 \dfrac{im}{\Box(\Box - m^2)}\,\delta^8(z-z') = \dfrac{m\bar{D}^2}{4\Box} K_{+-}$ (4.28)

$\bar{\Phi}\bar{\Phi}$ - line  $= K_{--} = \frac{1}{4}\bar{D}^2 \dfrac{im}{\Box(\Box - m^2)}\,\delta^8(z-z') = \dfrac{m\bar{D}^2}{4\Box} K_{+-}$

As a result, the lines can contain the factors D_α, $\bar{D}_{\dot\alpha}$ acting on corresponding functions K_{+-}. These K_{++}, K_{+-}, K_{-+}, K_{-+} are called the improved superpropagators. The factors D_α, $\bar{D}_{\dot\alpha}$ can be transfered from one line to another with help of integrating by parts



This manipulation allows us to remove all D-factors from at least one Grassmann δ-function and integrate over one of $d^4\theta$. ~~result we can decrease a number~~ Since, each of the propagators contains the $\delta^4(\theta-\theta')$ we can decrease a number of integration over $d^4\theta$.

As usual we can make transformation to momentum space. It corresponds

$$\partial_m \to i p_m$$
$$D_\alpha \to D_\alpha(p) = \partial_\alpha + (\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} p_m$$
$$\bar D_{\dot\alpha} \to \bar D_{\dot\alpha}(p) = -\bar\partial_{\dot\alpha} - \theta^\alpha \sigma^m_{\alpha\dot\alpha} p_m$$

It is easy to check the relations

$$\{D_\alpha(p), D_\beta(p)\} = 0 \quad , \{\bar D_{\dot\alpha}(p), \bar D_{\dot\beta}(p)\} = 0$$
$$\{D_\alpha(p), \bar D_{\dot\alpha}(p)\} = -2(\sigma^m)_{\alpha\dot\alpha} p_m \qquad (4.24)$$

All further consideration is analogous to conventional field theories and we do not discuss that. It is interesting to study only aspects specific for supersymmetric theories.

4.4. Non-renormalization theorem

The non-renormalization theorem ~~formulated as follows~~ describes a structure of supergraphs for effective action. Let remind that the effective action is constructed from generating functional of Green functions by means of Legendre transformation. In its essence, the effective action is generating function of connected, one-particle irreducible Green functions. In conventional field theory, the effective action has the following general form

$$\Gamma[\varphi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \, \Gamma^{(n)}(x_1, \dots, x_n) \, \varphi(x_1) \dots \varphi(x_n) \qquad (4.25)$$

where $\Gamma^{(n)}(x_1, \dots, x_n)$ are the connected, one-particle irreducible Green functions.

Non-renormalization theorem sounds as follows: each term in the effective action can be expressed as an integral over a single $d^4\theta$. The other words, any supergraph contributing to effective action can be presented in the form with single integral over full superspace. Idea of prove is based on observation that the superpropagator (4.19) contains the Grassmann δ-function $\delta^4(\theta-\theta')$ allowing to remote the integrals over anticommuting variables.

<u>Proof</u> Consider a one-particle irreducible 1-loop supergraph. As we know, each vertex is integrated over $d^4\theta$ and each line includes $\delta^4(\theta-\theta')$ with some number of factors D_α, $\bar{D}_{\dot\alpha}$ acting on this delta-function. All p-dependence is suppressed. Choose in the supergraph a fixed loop involving, say n vertices. It is clear there will be a cycle of propagators, delta-functions associated with propagators,
$\delta^4(\theta_1-\theta_2)\,\delta^4(\theta_2-\theta_3)\dots\delta^4(\theta_{n-1}-\theta_n)\,\delta^4(\theta_n-\theta_1)$
with derivatives D_α, $\bar{D}_\alpha$ acting on delta-functions. Integrating by parts we can transfer all the spinor derivative acting on $\delta^4(\theta_1-\theta_2)$ onto $\delta^4(\theta_2-\theta_3)$ or $\delta^4(\theta_n-\theta_1)$ or external lines to the loop. Then one integrates over θ_2 and removes $\delta^4(\theta_1-\theta_2)$ and replaces θ_2 by θ_1 everywhere.

After than we continue this process n-1 times. As a result, the cycle is reduced to a single delta-function. The expression for the supergraph takes the form

$$\prod_A \int d^4\theta_A \int d^4\theta_1 \; P(\theta_1, \theta_A)\{D \ldots D\bar{D} \ldots \bar{D}\, \delta^4(\theta_n - \theta_1)\}\big|_{\theta_n = \theta_1} \qquad (4.25)$$

Here the ~~table~~ label A enumerates the vertices external to the given loop. Last term in (4.25) can be easily evaluated. As a result of commutation relations among $D_\alpha, \bar{D}_{\dot\alpha}$, any product of the $\bar{D}$ and D factors is reduced to an expression involving no more than four such factors. Since $\delta^4(\theta_n - \theta_1) = (\theta_n - \theta_1)^2(\bar{\theta}_n - \bar{\theta}_1)^2$ we need exactly two D-factors and two $\bar{D}$-factors for the expression (4.25) to be non-zero. In this case we use the identity

$$\frac{1}{16} D_n^2 \bar{D}_n^2\, \delta^4(\theta_n - \theta_1)\big|_{\theta_n = \theta_1} = 1$$

As a result, the loop is contracted to a point in θ-space. Continuing the above procedure loop-by-loop one reduces the supergraph to a point in θ-space and the total contribution takes the form

$$\left(\prod_{i=1}^{L} \int d^4 p_i\right) \int d^4\theta\; \mathcal{F}(p_1, p_2, \ldots, p_L, \theta, \bar{\theta})$$

where p_i are the momenta.

Let us discuss the consequences of this theorem.

1. The general structure of the effective action of the Wess-Zumino model is

$$\Gamma[\Phi,\bar\Phi] = \sum_{n=2}^{\infty} \int d^4x_1 \dots d^4x_n \int d^4\theta \, T_n(x_1,\dots,x_n) F_1(x_1,\theta) \dots F_n(x_n,\theta) \quad (4.26)$$

where T_n are the translationally invariant function of Minkowski space coordinates and $F_1, F_2, \dots, F_n$ are the local functions of superfields $\Phi, \bar\Phi$ and their covariant derivatives

$$F_i = F_i(\Phi_i, \bar\Phi_i, D_M\Phi_i, \bar D_M\bar\Phi_i \dots) \quad (4.27)$$

$$\Phi_i = \Phi(x_i,\theta,\bar\theta), \quad \bar\Phi_i = \bar\Phi_i(x_i,\theta,\bar\theta),$$

2. All vacuum supergraphs vanish. Indeed, according to non-renormalization theorem, every vacuum supergraph is written as follows

$$A \int d^4\theta \cdot 1$$

where A is a loop momentum integral. But $\int d^4\theta \, 1 = 0$.

3. ~~Chiral potential has no~~ There are no chiral divergences in Wess-Zumino model. Indeed, in this case ~~the~~ such a supergraph contribution must be local in ~~coordinat~~ Minkowski ~~coordinat~~ space and hence, must be written as

$$\int d^4x \, d^2\theta \, W(\Phi) + \int d^4x \, d^2\bar\theta \, \bar W(\bar\Phi)$$

but it contradicts to non-renormalization theorem.

~~Sometimes~~ At first sight, the renormalization theorem prohibits any ~~chiral~~ contributions to chiral potential. It is not true, the finite corrections to chiral potential are possible. For example, let us consider the identity

$$\int d^8z \left(-\frac{1}{4}\frac{D^2}{\Box}\right) G = \int d^6z \, G$$

where G is chiral superfield. Therefore, if there is a term of the type $\int d^8 z \left(-\frac{1}{4}\frac{D^2}{\Box}\right) G(\Phi)$ among the others terms in (4.26), it can give contribution to chiral quantum corrections. We emphasize, this term is non-local in x-space. As to the divergences, they always local.

4. Non-renormalization theorem immediately shows that there is the only renormalization constant in Wess-Zumino model.

The standard arguments show that the renormalization of the Wess-Zumino model should have the form

$$\Phi = Z_1^{1/2}\Phi_R, \qquad \bar\Phi = Z_1^{1/2}\bar\Phi_R$$

$$m = Z_m m_R, \qquad \lambda = Z_\lambda \lambda_R$$

wher the label R means the renormalized quantity. The corresponding renormalized action is

$$S_R = \int d^8 z \; Z_1 \bar\Phi_R \Phi_R + \left[\int d^6 z \left(\frac{1}{2} Z_1 Z_m m_R \Phi_R^2 + \right.\right.$$
$$\left.\left. + \frac{1}{3!} Z_\lambda Z_1^{3/2} \lambda_R \Phi_R^3\right) + c.c.\right]$$

where Z_1, Z_m, Z_λ are the renormalization constants. But, according to non-renormalization theorem $\frac{1}{2} m\Phi^2 + \frac{\lambda}{3!}\Phi^3 = \frac{1}{2} m_R \Phi_R^2 + \frac{\lambda_R}{3!}\Phi_R^3$

Hence, $Z_1 Z_m = 1$, $Z_1^{3/2} Z_\lambda = 1$. The model is characterized by the only renormalization constant.