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4. Frame-like formulation of free higher-spin dynamics
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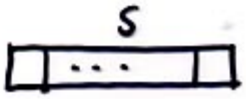
Higher Spin Gauge Fields



C. Fronsdal (1978)

$$\varphi_{n_1 \dots n_s}(x) \qquad\qquad \varphi^{m}{}_{m}{}^{r}{}_{r\, n_5 \dots n_s} = 0$$

HS symmetry

$$\delta \varphi_{n_1 \dots n_s}(x) = \partial_{\{n_1} \varepsilon_{n_2 \dots n_s\}}(x) \qquad \varepsilon^{n}{}_{n\, m_3 \dots m_{s-1}} = 0$$

Invariant action

$$S^s = \frac{(-1)^s}{2} \int d^d x \Big\{ \partial_n \varphi_{m_1 \dots m_s} \partial^n \varphi^{m_1 \dots m_s}$$

$$- \frac{1}{2} s(s-1) \partial_n \varphi^{r}{}_{r\, m_3 \dots m_s} \partial^n \varphi^{k}{}_{k}{}^{m_3 \dots m_s}$$

$$+ s(s-1) \partial_n \varphi^{r}{}_{r\, m_3 \dots m_s} \partial_k \varphi^{n k m_3 \dots m_s}$$

$$- s\, \partial_n \varphi^{n}{}_{m_2 \dots m_s} \partial_r \varphi^{r m_2 \dots m_s}$$

$$- \frac{1}{4} s(s-1)(s-2) \partial_n \varphi^{r}{}_{r}{}^{n}{}_{m_4 \dots m_s} \partial_p \varphi^{q}{}_{q}{}^{p m_4 \dots m_s} \Big\}$$

$s = 1$ Maxwell

$s = 2$ Einstein

$s > 2$ Unifying gauge principle ?!

HS fermions are described analogously

Fang, Fronsdal (78)

Motivation

HS symmetries

SUGRA: $\begin{array}{c} d \le 11 \\ (\mathcal{N} \le 8) \end{array} \Longleftrightarrow s \le 2$

Superstring:

Stueckelberg symmetries as spontaneously broken HS symmetries

$$\delta \varphi_{n_1, n_2, n_3} = \partial_{n_1} \varepsilon_{n_2, n_3 \dots} + \partial_{n_2} \varepsilon_{n_1, n_3 \dots} + \dots$$

$$\delta \varsigma_{n_1, n_2 \dots} = \varepsilon_{n_1, n_2 \dots}$$

Stueckelberg fields

HS gauge theory as a symmetric phase of superstring ?!

Sundborg-Witten limit

$\lambda = g^2 N \to 0 \qquad \ell^2_{str} \Lambda_{AdS} \to \infty$

Sundborg (2001)
Witten (2001)
Klebanov, Polyakov (2002)

Infinite set of currents in the conformal boundary theory ~ HS symmetries in the bulk ?!

Anomalous currents - broken HS symmetries

Higher Spin Problem

To find a nonlinear HS gauge theory

- Correct free field limit
- Unbroken HS gauge symmetries
- Non-Abelian global HS symmetry of a vacuum solution

Difficulties

- Coleman – Mandula S-matrix argument

  Coleman, Mandula (1967)

- HS-gravity interaction problem

  Aragon, Deser (1979)

$$\partial_n \to D_n = \partial_n - \Gamma_n \qquad [D_n, D_m] = R_{nm\ldots}$$

$$\delta \varphi_{nm\ldots} \to D_n \varepsilon_{m\ldots}$$

$$\delta S = \int \underset{\substack{\uparrow \\ \text{Weyl tensor for } s>2}}{R_{\ldots}} D_{\ldots} \varphi_{\ldots} \varepsilon_{\ldots} \neq 0 \quad ?!$$

Resolution

- Some consistent cubic HS interactions (no gravity)

  A. Bengtsson, I. Bentsson, L. Brink (1983)

  Berends, Burgers, van Dam (1984, 1985)

- $\Lambda \neq 0$ allows consistent HS theory with gravity included

  E. S. Fradkin, M. V. (1987)

AdS background:

- resolves Coleman - Mandula argument (no S-matrix in AdS)
- allows new interactions with dimensionful coupling constant built from Λ: higher spins - higher derivatives
- fits the AdS/CFT conjecture

AdS Background and HS Interactions

$$R_{nm,uv} = R_{nm,uv} - \Lambda (g_{nu} g_{mv} - g_{nv} g_{mu})$$

(A) dS: $R_{nm,uv} = 0$

Fluctuations

$R_{nm,uv}$ is small

$$R_{nm,uv} = \Lambda (g_{nu} g_{mv} - g_{nv} g_{mu}) + o(1)$$

$$[D_n, D_m] \sim \Lambda \cancel{+} o(1)$$

Action

$$S = S_2^{s\,cov} + \Delta S^s$$

$$\Delta S^s = \sum_{\substack{p,q \\ p+q \le s}} \Lambda^{-[\frac{p+q}{2}]} \underbrace{(D)^p \varphi \; (D)^q \varphi \; R}_{\text{cubic}}$$

$$\delta \varphi = D \varepsilon + R^{HS} \varepsilon$$

$$\delta S = O(\varphi^3)$$

Coefficients of ΔS^s are fixed uniquely modulo total derivatives

Towards HS Algebras

HS algebras are certain $*$-product algebras (algebras of oscillators)

M.V. (1988)

- AdS subalgebra ($\sim$ vacuum solution)
- Infinite towers of spins
- Higher derivatives in interactions as a result of $*$-product quantum-mechanical nonlocality

How to find?

HS gauge fields are 1-form connections for a HS algebra

Lower spin examples

$s = 1$ A_n $n = 0, 1, \dots d-1$ 1-form

$s = 3/2$ $\Psi_{n\alpha}$ 1-form carrying a spinor index α

$s = 2$

$$g_{nm} = g_{mn} \Rightarrow e_n{}^{a} \, dx^n$$

$$\square\!\square \Rightarrow \square \otimes \square = \square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

general symm. asymm.

Antisymmetric part $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(e_{n\alpha} - e_{\alpha n})$ is unphysical and should be pure gauge: extra gauge symmetry with antisymmetric gauge parameters

$$\varepsilon^{ab}(x) = -\varepsilon^{ba}(x) \qquad \delta e_n{}^{a}(x) = \varepsilon^{a}{}_{b}(x) \, e_n{}^{b}(x)$$

Local Lorentz symmetry.

Gauge field - Lorentz connection 1 form

$$dx^n \omega_n{}^{ab} = - dx^n \omega_n{}^{ba}$$

$$s = 2 : \quad e_n{}^{a}, \; \omega_n{}^{ab}$$

$$g_{nm} = e_n{}^{c} e_m{}^{a} \eta_{ca}$$

Cartan formulation of gravity

$$\left.\begin{array}{ll} dx^n e_n{}^a & P^a \\ dx^n \omega_n{}^{ab} & M^{ab} \end{array}\right\} \text{ Poincare}$$

$$R^a = de^a + \omega^a{}_b e^b \stackrel{d}{=} D^\omega e^a$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$$

R^a, R^{ab} Yang-Mills field strengths for the Poincare algebra

$R^a = 0$ – "zero torsion condition"

$\boxminus \otimes \square$ expresses $\omega^{ab} \Rightarrow \omega^{ab}(e, \partial e)$

$\square \otimes \boxminus$

$$R^{ab}(\omega(e)) = dx^n \wedge dx^m R_{nm}{}^{ab}(\omega(e))$$

Riemann tensor

$$\det |e_n{}^a| \neq 0$$

$$g_{nm} = e_n{}^a e_m{}^b \eta_{ab}$$

$$\left.\begin{array}{l} R^a = 0 \\ R^{ab} = 0 \end{array}\right\} \text{ Flat space-time}$$

Poincare symmetry is by construction!

Gravity via gauging of (A)dS group

$$e^a,\ \omega^{ab} \Rightarrow W^{AB} = -W^{BA} \qquad A,B,\ldots = 0,\ldots,d$$

frame 1-form, Lorentz connection $\qquad a,b,\ldots = 0,\ldots d-1$

$$W^{AB} = dx^n W_n{}^{AB} \in \begin{matrix} O(d-1,2) & AdS \\ O(d,1) & dS \end{matrix}$$

Field strength

$$R^{AB} = dW^{AB} + W^{AC} \wedge W^{DB}\, \eta_{CD}$$

$$\eta_{AB} = \begin{matrix} (+--\ldots-+) & AdS_d \\ (+--\cdots-) & dS_d \end{matrix}$$

Compensator

$V^A(x)$ $[L]$

$$V^A V^B \eta_{AB} = \begin{Bmatrix} r^2 & AdS \\ -r^2 & dS \end{Bmatrix} = -\Lambda^{-1}$$

$AdS \quad \Lambda < 0$

$dS \quad \Lambda > 0$

Length of V^A = radius of (A)dS space-time

Standard gauge

$$V^A = r\delta^A_d$$

$$\underset{L^0}{e^a} = \underset{L^{-1}}{W^{aB}} \underset{L}{V_B}$$

$$\underset{L^{-1}}{\omega^{ab}} = {}^{\perp}W^{AB} \qquad {}^{\perp}W^{AB} V_B = 0$$

Covariant definitions

$$E^A = \mathcal{D} V^A = dV^A + W^A{}_B V^B$$

$$E^A V_A = 0 \qquad (V^A V_A = \text{const})$$

$$g_{nm} = E_n{}^A E_m{}^B \eta_{AB}$$

Lorentz connection

$$\omega^{L\,AB} = W^{AB} + \Lambda (E^A V^B - E^B V^A)$$

$$\mathcal{D}^L V^A = dV^A + \omega^{L\,A}{}_B V^B \equiv 0$$

In the standard gauge $V^A = r \delta^A_d$ one recovers usual definitions

Curvature decomposition:

$$R^A = R^{AB} V_B \quad (R^A V_A \equiv 0) \quad \text{torsion}$$

$$^{\perp}R^{AB} : \; ^{\perp}R^{AB} V_B = 0 \quad \text{shifted Riemann}$$

In the standard gauge

$$R_{nm}^{\alpha} = \partial_n e_m{}^\alpha + \omega_n{}^\alpha{}_b e_m{}^b - n \leftrightarrow m$$

$$R_{nm}{}^{ab} = \partial_n \omega_m{}^{ab} + \omega_n{}^a{}_c \omega_m{}^{cb} + \Lambda e_n{}^a e_m{}^b - (n \leftrightarrow m)$$

$$R^A = 0 \Rightarrow \omega = \omega(E, \partial E)$$

$$^{\perp}R^{AB} = 0 , \quad \operatorname{rank} |E_n{}^A| = d \Rightarrow (A)dS_d$$

Vacuum Symmetry

Any vacuum solution

$$R^{AB}(W_0) = 0$$

describes (A)dS$_d$ provided that

$$\mathrm{rank}\,|E_{0\,n}{}^{A}| = d$$

YM curvature R^{AB} transforms homogeneously under the gauge transformation

$$* \quad \delta W^{AB} = D\varepsilon^{AB} \qquad \delta R^{AB} = \varepsilon^{A}{}_{C} R^{CB} + \varepsilon^{B}{}_{C} R^{AC}$$

and diffeomorphisms.

Generically * maps one solution to another

Vacuum symmetry

$$\delta W_0 = D_0 \varepsilon_{gl}^{AB}(x) = 0$$

The equation

$$\frac{\partial}{\partial x^n} \varepsilon_{gl}^{AB}(x) + W_{0\,n}{}^{A}{}_{C}\, \varepsilon_{gl}^{CB}(x) + W_{0\,n}{}^{B}{}_{C}\, \varepsilon_{gl}^{AC}(x) = 0$$

is consistent because $D_0{}^2 = R(W_0) = 0$

It fixes $\varepsilon_{gl}^{AB}(x)$ in terms of $\varepsilon_{gl}^{AB}(x_0)$ at any x_0

012 $\varepsilon_{gl}^{AB}(x)$ are $o(d-1,2)\,(o(d,1))$ global symmetry parameters

Einstein-Hilbert action in the MacDowell-Mansouri-Stelle-West form

$d=4$

MacDowell, Mansouri (78)
Stelle, West (1980)

$$S = -\frac{1}{4\kappa^2|\Lambda|^{1/2}} \int_{M^4} \epsilon_{ABCDE} V^A R^{BC} \wedge R^{DE}$$

Symmetries

(i) diffeomorphisms

(ii) local $O(d-1,2)$ (or $O(d,1)$)

Compensator transforms as

$$\delta V^A = \xi^n \partial_n V^A + \varepsilon^A{}_B V^B = \xi^n D_n V^A + \varepsilon'^A{}_B V^B$$

$$\varepsilon^{AB} = \varepsilon'^{AB} + \xi^n W_n{}^{AB}$$

$$\delta V^A = \xi^n E_n{}^A + \varepsilon'^A{}_B V^B$$

V^A is pure gauge

Using (A)dS symmetry V^A can be fixed arbitrarily. Leftover gauge symmetry diffeomorphisms + local Lorentz

Standard gauge $V^A = r\delta^A_d$

$$S = +\frac{1}{4\kappa^2\Lambda} \int_{M^4} \epsilon_{abcd} R^{ab} \wedge R^{cd}$$

Macdowell, Mansouri (1977)

$$S = \frac{1}{4\varkappa^2\Lambda} \epsilon^{abcd} \times$$

$$\times \int_{M^4} (d\omega_{ab} + \omega_{ae} \wedge \omega^e{}_b + \Lambda e_a \wedge e_b) \wedge (d\omega_{cd} + \omega_{cf} \wedge \omega^f{}_d + \Lambda e_c \wedge e_d)$$

$$S = \Lambda^{-1} S_{-1} + \Lambda^0 S_0 + \Lambda S_1$$

- $S_{-1}(\omega)$: $\frac{\delta S_{-1}}{\delta \omega} \equiv 0$ topological inv.
- $S_0 = \frac{1}{2\varkappa^2} \int \epsilon^{abcd} e_c \wedge e_d \wedge \underbrace{(d\omega_{ab} + \omega_{ae} \wedge \omega^e{}_b)}_{\text{Riemann tensoe}}$

$$\sim \frac{1}{\varkappa^2} \int |e| R \quad \text{Einstein action}$$

- $S_1 = \frac{1}{4\varkappa^2} \Lambda \int_{M^4} \epsilon^{abcd} e_a \wedge e_b \wedge e_c \wedge e_d$

Cosmological term

Λ - cosmological constant

Equations of motion

$$S = -\frac{1}{4\varkappa^2|\Lambda|^{1/2}} \int \epsilon_{ABCDE} V^A R^{BC} \wedge R^{DE}$$

$\delta R^{AB} = \mathcal{D}(\delta W^{AB})$ general variation

$DR^{AB} = 0$ Bianchi identities

$$\delta S = \frac{1}{4\varkappa^2|\Lambda|^{1/2}} \int \epsilon_{ABCDE} \mathcal{D}V^A \wedge R^{BC} \wedge \delta W^{DE}$$

Taking into account

$$\mathcal{D}V^A = E^A$$

Equations of motion

$$\epsilon_{ABCDE} E^A \wedge R^{BC} = 0$$

$R^{AB} = 0$
(A)dS vacuum

Variation with respect to Lorentz connection : V - transversality with respect to $\mathcal{D}, E$ is equivalent to the zero torsion condition $R^{BC} V_C = 0$

Lorentz connection is auxiliary field expressed in terms of $e, \partial e$ by virtue of its field equations

$V^E \epsilon_{ABCDE} E^A \wedge R^{BC} = 0$ Einstein equations

Towards HS symmetries

Idea: to find an algebra

$$g \supset h = (A)dS \text{ algebra}$$

such that the gauge fields

$$W \in g \qquad\qquad R = dW + W \wedge W$$

allow for a MacDowell-Mansouri type formulation

$$S = \int_{M^d} V \; R_1 \wedge R_1 \wedge E \wedge E \ldots$$

equivalent to Fronsdal formulation in the free field limit

$$W = W_0 + W_1 \qquad\qquad R(W_0) = 0$$

$W_0 \in h$ background (A)dS field

$W_1 \in g$ HS fields

$$R_1 = \mathcal{D}_0 W_1$$

Manifest HS gauge invariance

$$\delta W_1 = \mathcal{D}_0 \mathcal{E} \qquad \mathcal{D}_0^2 = 0 \qquad \mathcal{E} \in g$$

g is the HS vacuum global symmetry

Young Tableaux

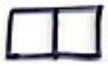
gl_M tensors

$$A^{a_1;a_2;a_3\dots} \qquad a_i = 1,\dots,M$$

Different symmetry properties

Rank-2 symmetric:

$$A^{a_1;a_2} - A^{a_2;a_1} = 0$$



Rank-2 antisymmetric:

$$A^{a_1;a_2} + A^{a_2;a_1} = 0$$

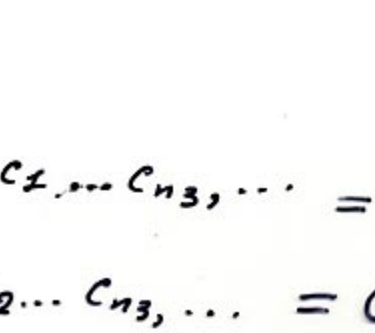
General case: symmetric basis

$$A^{a_1\dots a_{n_1},\, b_1\dots b_{n_2},\, c_1\dots c_{n_3},\dots}$$

$$n_1 \geq n_2 \geq n_3 \geq \dots$$

$$\begin{cases} A^{a_1\dots a_{n_1},\, a_{n_1+1} b_2\dots b_{n_2},\, c_1\dots c_{n_3},\dots} = 0 \\ A^{a_1\dots a_{n_1},\, b_1\dots b_{n_2},\, b_{n_2+1} c_2\dots c_{n_3},\dots} = 0 \\ \vdots \end{cases}$$

$Y(n_1, n_2, n_3, \dots)$:



A number of rows $\leq M$

Bosonic generating function

$$A(Y) = \sum_{n_1, n_2, \dots} A^{a_1 \dots a_{n_1},\, b_1 \dots b_{n_2},\, c_1 \dots c_{n_3}} \times$$
$$\times Y^1_{a_1} \dots Y^1_{a_{n_1}} Y^2_{b_1} \dots Y^2_{b_{n_2}} Y^3_{c_1} \dots Y^3_{c_{n_3}} \dots$$

Symmetrization is manifest

$$[Y^\alpha_a , Y^\beta_b] = 0 \qquad a = 1, \dots, M$$
$$\alpha = 1, \dots P$$

Young conditions

$$* \begin{cases} Y^\alpha_a \dfrac{\partial}{\partial Y^\beta_a} A = 0 & \alpha < \beta \\ Y^\alpha_a \dfrac{\partial}{\partial Y_a{}^\beta} A \Big|_{\alpha=\beta} = n_\alpha A \end{cases}$$

Howe dual algebras

$$gl_M : \qquad t_a{}^b = Y^\alpha_a \frac{\partial}{\partial Y^\alpha_b}$$

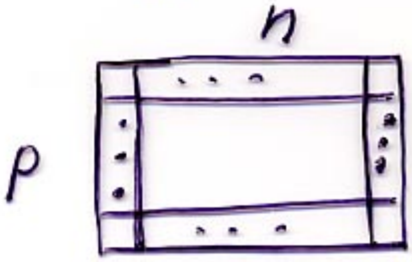
$$gl_P : \qquad l^\alpha{}_\beta = Y^\alpha_a \frac{\partial}{\partial Y^\beta_a}$$

$$[l, t] = 0$$

Young conditions * are highest weight conditions with respect to Howe dual gl_P

Blocks

Rectangular tableax



$n_\alpha = n$

are invariants of Howe dual sl_p

$$\left(Y^\alpha{}_a \frac{\partial}{\partial Y_a{}^\beta} - \frac{1}{p}\delta^\alpha{}_\beta\, Y^\gamma{}_a \frac{\partial}{\partial Y^\gamma{}_a}\right) A(Y) = 0$$

$$Y^\gamma{}_a \frac{\partial}{\partial Y^\gamma{}_a} A(Y) = nA(Y)$$

$n_\alpha = n$ implies that weights of sl_p are zero

Rows in the block tableax can be interchanged with the sign factor $(-1)^n$

For the block tableax, the sl_p lowest weight conditions are consequences of the sl_p highest weight conditions because polynomials $A(Y)$ form finite-dimensional modules over sl_p

$O(M)$ invariant traceless tensors

$$A(Y) = \sum_{n_1, n_2, \dots} A^{a_1 \dots a_{n_1},\, b_1 \dots b_{n_2},\, c_1 \dots c_{n_3} \dots} \times$$

$$Y^1_{a_1} \dots Y^1_{a_{n_1}}\, Y^2_{b_1} \dots Y^2_{b_{n_2}}\, Y^3_{c_1} \dots Y^3_{c_{n_3}}$$

Contraction of any two indices with the $O(M)$ invariant metric η_{ab} gives zero:

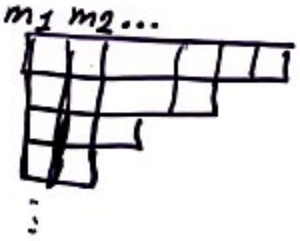
In addition to Young condition one imposes

$$\eta_{ab} \frac{\partial^2}{\partial Y^\alpha_a \partial Y^\beta_b} A(Y) = 0$$

Traceless Young tableaux

$$Y^{tr}(n_1, n_2, n_3, \dots)$$

$m_1\ m_2 \dots$



n_1
n_2
n_3

$$n_1 \geq n_2 \geq n_3 \dots \qquad\qquad m_1 + m_2 \leq M$$

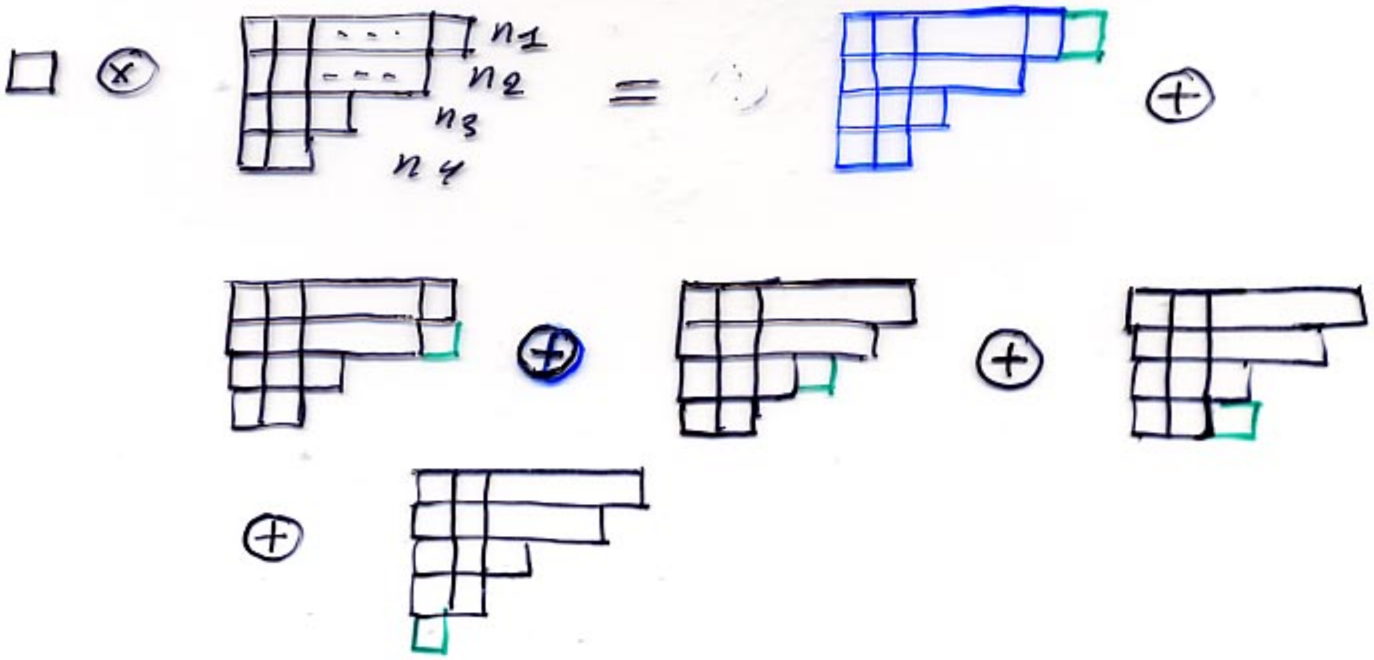
$$\epsilon^{a_1 \dots a_M} \epsilon^{b_1 \dots b_M} = \eta^{a_1 b_1} \dots \eta^{a_M b_M} + \dots$$

are highest weights of $sp(2p)$ which is Howe dual to $O(M)$

$$l^\alpha{}_\beta = Y^\alpha_a \frac{\partial}{\partial Y^\beta_a}$$

$$l^{\alpha\beta} = \eta^{ab} Y^\alpha_a Y^\beta_b \qquad l_{\alpha\beta} = \eta_{ab} \frac{\partial^2}{\partial Y^\alpha_a \partial Y^\beta_b}$$

Multiplication rule



All admissible tableax with one cell added

$$Y(1,0,0,\dots) \otimes Y(n_1, n_2, \dots) =$$

$$Y(n_1+1, n_2, \dots) \oplus Y(n_1, n_2+1, \dots) \oplus \dots$$

For traceless tableax

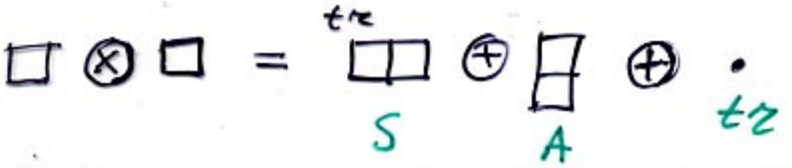
$$Y(1,0,0,\dots) \otimes Y^{tr}(n_1, n_2, n_3, \dots) =$$

$$= Y^{tr}(n_1+1, n_2, n_3, \dots) + Y^{tr}(n_1-1, n_2, n_3 \dots)$$

$$+ Y^{tr}(n_1, n_2+1, n_3, \dots) + Y^{tr}(n_1, n_2-1, n_3 \dots)$$

$$+ Y^{tr}(n_1, n_2, n_3+1, \dots) + Y^{tr}(n_1, n_2, n_3-1, \dots)$$

For example

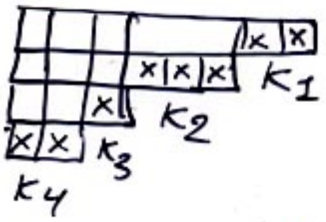


Reduction rule

$Y_M^{(tr)}(n_1, n_2, n_3, \dots)$ is sl_M ($O(M)$) representation

Its content as the $sl_{M-1}(O(M-1))$ representation

$$Y_M^{(tr)}(n_1, n_2, n_3, \dots) = \sum_{k_i=0}^{n_i - n_{i-1}} \oplus Y_{M-1}^{(tr)}(n_2+k_1, n_3+k_2, \dots)$$



One can cut any number of crossed cells

Let $O(M-1)$ be stability algebra of some $O(M)$ vector V^a

Reduction implies decomposition of a tensor into V-transversal and V parallel components.

V-transversal components form representations of $O(M-1)$

Crossed cells denote V-parallel components of tensor indices

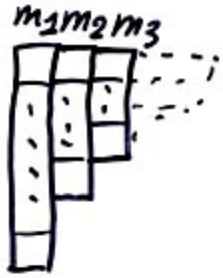
General case: antisymmetric basis

$$A^{[a_1\ldots a_{m_1}],[b_1\ldots b_{m_2}],[c_1\ldots c_{m_3}]}$$

$$M \geq m_1 \geq m_2 \geq m_3 \geq \ldots$$

$$\begin{cases} A^{[a_1\ldots a_{m_1}, a_{m_1+1}][b_2\ldots b_{m_2}],[c_1\ldots c_{m_3}]} = 0 \\ A^{[a_1\ldots a_{m_1}],[b_1\ldots b_{m_2}, b_{m_2+1}],[c_2\ldots c_{m_3}]} = 0 \\ \vdots \end{cases}$$

$$Y[m_1, m_2, m_3, \ldots]$$



Fermionic generating function

$$A(\varphi) = A^{[a_1\ldots a_{m_1}],[b_1\ldots b_{m_2}],[c_1\ldots c_{m_3}]\ldots}$$
$$\varphi^1_{a_1}\ldots\varphi^1_{a_{m_1}}\varphi^2_{b_1}\ldots\varphi^2_{b_{m_2}}\varphi^3_{c_1}\ldots\varphi^3_{c_{m_3}}\ldots$$

$$\varphi^\alpha_a \varphi^\beta_b = -\varphi^\beta_b \varphi^\alpha_a$$

$$\left.\begin{array}{ll} t_a{}^b = \varphi^\alpha_a \dfrac{\partial}{\partial \varphi^\alpha_b} & gl_M \\ l^\alpha{}_\beta = \varphi^\alpha_a \dfrac{\partial}{\partial \varphi^\beta_a} & gl_p \end{array}\right\} \text{Howe dual}$$

Young tableau is a highest weight of gl_p

Frame-like formulation of HS Dynamics

Flat space-time

M.V. (1980)
Lopatin, M.V (198

$$\varphi_{n_1 \dots n_s} \longrightarrow e_n{}^{a_1 \dots a_{s-1}}$$

$$e^{a_1 \dots a_{s-1}} = dx^n \, e_n{}^{a_1 \dots a_{s-1}}$$

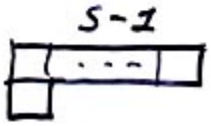
$$e_{n,b}{}^{b a_1 \dots a_{s-3}} = 0$$

$$\varphi_{n_1 \dots n_s} = e_{\{n_1, n_2 \dots n_s\}} \qquad \varphi^n{}_n{}^m{}_{m k_5 \dots k_s} = 0$$

Antisymmetric part in $e_n{}^{a_1 \dots a_{s-1}}$ should be pure gauge analogously to the antisymmetric part of the usual frame $e_{n,a}$

Fronsdal

Traceless tableax



Lorentz-type

Gauge transformation law

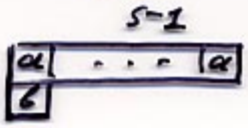
$$\delta e_{n, a_1 \dots a_{s-1}} = \partial_n \underbrace{\varepsilon_{a_1 \dots a_{s-1}}}_{\text{Fronsdal}} + h_n{}^b \overbrace{\underbrace{\varepsilon_{a_1 \dots a_{s-1}, b}}_{\text{Lorentz}}}^{\text{traceless}}$$

$h_n{}^b = \delta_n{}^b$ frame

$$\varepsilon_{\{a_1 \dots a_{s-1}, a_s\}} = 0 \qquad \varepsilon_{a_1 \dots a_{s-2} b}{}^b{}_{,c} = 0$$

Lorentz connection-type gauge field

$dx^n \omega_{n,\, a_1 \dots a_{s-1},\, b}$



Torsion-type curvature form

$$R_{nm,\, a_1 \dots a_{s-1}} = \partial_n e_{m,\, a_1 \dots a_{s-1}} + h_n{}^c \omega_{m,\, a_1 \dots a_{s-1},\, c} - (n \leftrightarrow m)$$

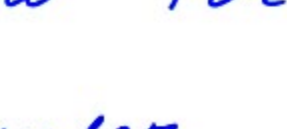
is reconstructed from the transformation law for $e_{n,\, a_1 \dots a_{s-1}}$

It is invariant under

$$\delta e_{n,\, a_1 \dots a_{s-1}} = \partial_n \varepsilon_{a_1 \dots a_{s-1}} + h_n{}^b \varepsilon_{a_1 \dots a_{s-1},\, b}$$
$$\delta \omega_{n,\, a_1 \dots a_{s-1},\, b} = \partial_n \varepsilon_{a_1 \dots a_{s-1},\, b} + h_n{}^c \varepsilon_{a_1 \dots a_{s-1},\, bc}$$

where $\varepsilon_{a_1 \dots a_{s-1},\, b_1 b_2}$ is second-level Lorentz-type symmetry parameter

$\varepsilon_{a_1 \dots a_{s-1},\, b_1 b_2}$



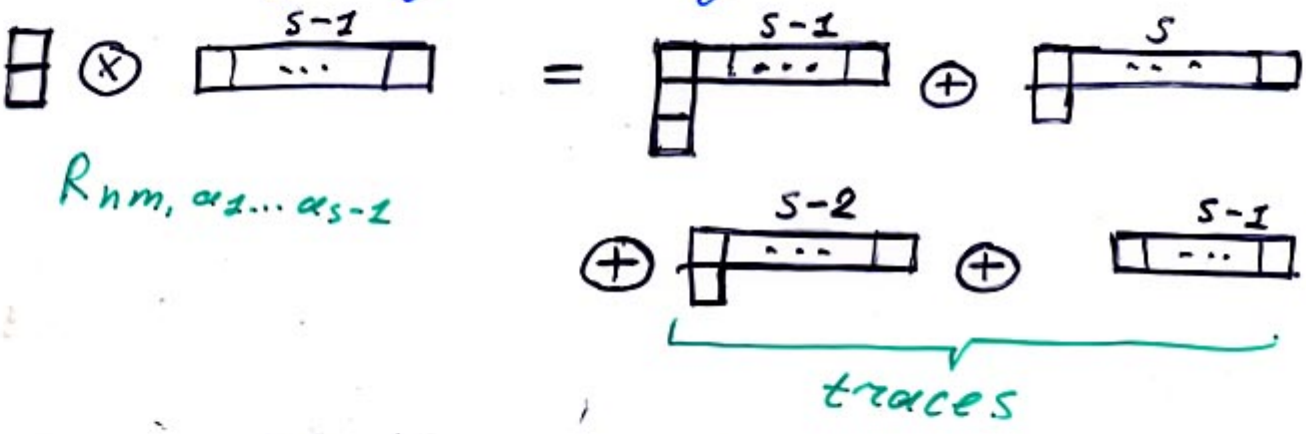
traceless + antisymmetric

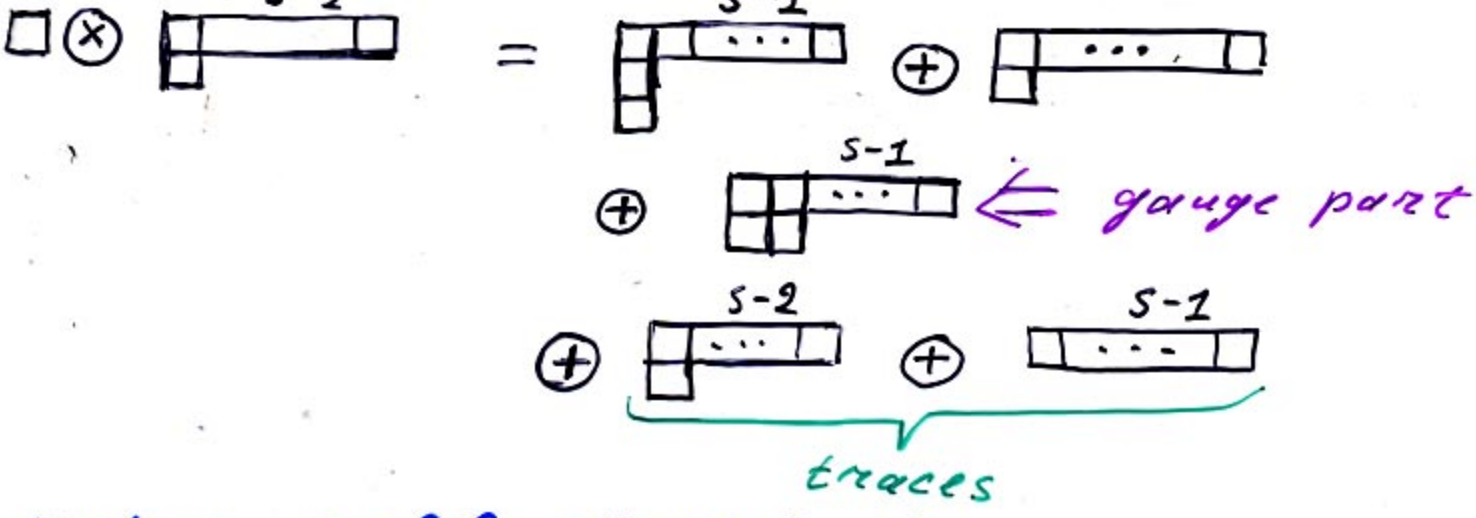
$$\varepsilon_{\{a_1 \dots a_{s-1},\, a_s\} b} = 0$$

Zero-torsion constraint

$$R_{nm,\alpha_1\ldots\alpha_{s-1}} = 0$$

expresses the Lorentz-type field $\omega_{n,\alpha_1\ldots\alpha_{s-1},b}$ via first derivatives of $e_{n,\alpha_1\ldots\alpha_{s-1}}$ modulo pure gauge part with the level-2 local Lorentz symmetry





Extra cell (index) in

$$\omega_{n,\alpha_1\ldots\alpha_{s-1},b} \sim (\partial e)\ldots$$

is for one derivative applied to the dynamical field $e_{n,\alpha_1\ldots\alpha_{s-1}}$

Introducing level-2 Lorentz fields

$$\omega_{n,\, a_1 \dots a_{s-1},\, b_1 b_2}$$

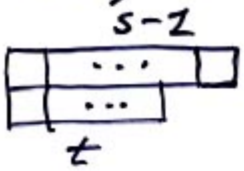
the process continues leading to the linearized curvatures

$$R^1_{nm,\, a_1 \dots a_{s-1},\, b_1 \dots b_t} = \partial_n \omega_{m,\, a_1 \dots a_{s-1},\, b_1 \dots b_t}$$

$$+ h_n{}^c \omega_{m,\, a_1 \dots a_{s-1},\, b_1 \dots b_t c} - (n \leftrightarrow m)$$

for the set of gauge fields

$$\omega_{n,\, a_1 \dots a_{s-1},\, b_1 \dots b_t} \qquad 0 \le t \le s-1$$



traceless in fiber indices a, b

$t = 0$: frame field $e_{n,\, a_1 \dots a_{s-1}}$

- Linearized curvatures are invariant under the gauge transformations

$$\delta \omega_{n,\, a_1 \dots a_{s-1},\, b_1 \dots b_t} = \partial_n \varepsilon_{a_1 \dots a_{s-1},\, b_1 \dots b_t}$$

$$+ h_n{}^c \varepsilon_{a_1 \dots a_{s-1},\, b_1 \dots b_t c}$$

$$\delta R^1 = 0$$

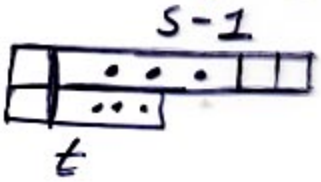
- $\omega_{n,\, a_1 \dots a_{s-1},\, b_1 \dots b_t}$ are expressed in terms of order t derivatives of $e_{n,\, a_1 \dots a_{s-1}}$ by appropriate invariant constraints

(A)dS Higher Spin Curvatures

The set of HS 1 forms

$\omega^{a_1 \dots a_{s-1}, b_1 \dots b_t} \qquad 0 \le t \le s-1$

of $o(d-1,1)$ Lorentz tensors

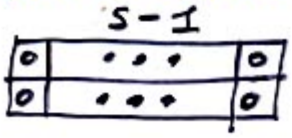


$0 \le t \le s-1$ traceless

is equivalent to a single 1 form

$W^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} \qquad A, B, \dots = 0, \dots d$

taking values in the irreducible representation

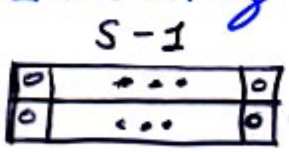
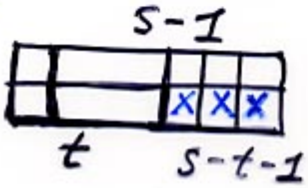


traceless $\quad s=2$: 

$W^{\{A_1 \dots A_{s-1}, A_s\} B_2 \dots B_{s-1}} = 0 \qquad W_C{}^{C A_3 \dots A_{s-1}, B_1 \dots B_{s-1}} = 0$

of AdS algebra $o(d-1,2)$ or dS algebra $o(d,1)$

Lorentz decomposition

 $= \sum_{t=0}^{s-1} \oplus$ 

 - contraction with V^A

 - V^A transversal

V^A - compensator

$e^{A_1 \dots A_{s-1}} = W^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} V_{B_1} \dots V_{B_{s-1}}$

is automatically V^A - transversal

Linearised (A)dS HS curvatures

$$R^{1}{}_{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} = dW_{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$$

$$+(s-1)\Big(W_{0A_1}{}^{C}\wedge W_{CA_2\ldots A_{s-1},\,B_1\ldots B_{s-1}}$$

$$+W_{0B_1}{}^{C}\wedge W_{A_1\ldots A_{s-1},\,CB_2\ldots B_{s-1}}\Big)$$

$W_{0AB}=-W_{0BA}$ is background (A)dS gravitational field which satisfies (A)dS equation

$e_0 + \omega_0$

$$* R_0 = dW_{0AB} + W_{0A}{}^{C}\wedge W_{CB} = 0$$

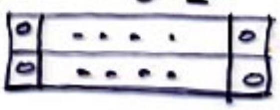
$$R^1 = \mathcal{D}_0 W$$

$$\delta_0 W_{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} = \mathcal{D}_0\left(\varepsilon_{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}\right)$$

$$\delta R^1 = \mathcal{D}_0{}^2\,\varepsilon = 0 \qquad \mathcal{D}_0{}^2 = 0 \quad *$$

$\mathcal{D}_0$ is the covariant derivative of $o(d-1,2)$ or $o(d,1)$ in the representation

$s-1$



Lorentz invariant decomposition in terms of compensator

$$W_0{}^{AB} = \omega_0^{L\,AB} - \Lambda\,(E_0{}^A V^B - E_0{}^B V^A)$$

$$\mathcal{D}_0 W_{A_1\dots A_{s-1},B_1\dots B_{s-1}} = D_0^L W_{A_1\dots A_{s-1},B_1\dots B_{s-1}}$$

$$\Lambda^0 \quad +\Lambda(s-1)\big(V_{A_1} E_0^C \wedge W_{C A_2\dots A_{s-1},B_1\dots B_{s-1}} + V_{B_1} E_0^C \wedge W_{A_1\dots A_{s-1},C B_2\dots B_{s-1}}\big)$$

$$\Lambda^1 \quad -\Lambda(s-1)V^C\big(E_{0A_1}\wedge W_{C A_2\dots A_{s-1},B_1\dots B_{s-1}} + E_{0B_1}\wedge W_{A_1\dots A_{s-1},C B_2\dots B_{s-1}}\big)$$

By appropriate field rescaling the dependence on Λ in the first term can be rescaled away. Then the limit $\Lambda\to 0$, $\mathcal{D}_0^L \to d$ reproduces the Minkowski space-time HS curvatures we started with

- Without rescaling

$$\omega_{a_1\dots a_{s-1},b_1\dots b_t} \sim \left[(\Lambda^{-1/2}\partial)^t e_{a_1\dots a_{s-1}}\right]$$

$t\geq 2$ – "extra fields" – carry higher derivatives and negative powers of Λ. "Extra fields" appear for $s>2$

HS Invariant Action

$$S_2^s = \frac{1}{2}\sum_{p=0}^{s-2} \alpha(s,p)\, \epsilon_{A_1 \dots A_{d+1}} \int_{M^d} E_0^{A_5} \wedge \dots \wedge E_0^{A_d}\, V^{d+1} \times$$

$$V_{C_1} \dots V_{C_{2(s-2-p)}} \wedge R_1^{A_1 B_1 \dots B_{s-2},\, A_2 C_1 \dots C_{s-2-p} D_1 \dots D_p}$$

$$\wedge R_1^{A_3}{}_{B_1 \dots B_{s-2},}{}^{A_4 C_{s-1-p} \dots C_{2(s-2-p)}}{}_{D_1 \dots D_p}$$

where the coefficients $\alpha(s,p)$ are fixed by the condition

$$\frac{\delta S}{\delta W_{A_1 \dots A_{s-1},\, B_1 \dots B_{s-1}}} V^{B_1} V^{B_2} \equiv 0$$

as

$$\alpha(s,p) = \tilde{\alpha}(s)\, \Lambda^{-(s-p-3/2)} \frac{(d-5+2(s-p-2))!!\,(s-p-1)}{(d-3+2(s-2))!!\,(s-p-2)!}$$

to guarantee that S_2^s depends only on the dynamical field $e^{a_1 \dots a_{s-1}}$ and auxiliary field $\omega^{a_1 \dots a_{s-1},\, b}$

By construction, the action is gauge invariant under

$$\delta e^{a_1 \dots a_{s-1}} = d\,\varepsilon^{a_1 \dots a_{s-1}} + h_b\, \varepsilon^{a_1 \dots a_{s-1},\, b}$$

background frame — Lorentz-type

and is, therefore, equivalent to the Fronsdal's action

HS Algebras

Generic element

$$f = \sum_{n=0}^{\infty} f^{A_1 \dots A_n,\, B_1 \dots B_n} \, T_{A_1 \dots A_n,\, B_1 \dots B_n}$$

Generators $T_{A_1 \dots A_n,\, B_1 \dots B_n}$

$$T_{\{A_1 \dots A_n,\, A_{n+1}\} B_2 \dots B_n} = 0 \qquad T^{C}{}_{C A_3 \dots A_n,\, B_1 \dots B_n} = 0$$

$$T_{A,B} = - T_{B,A} : \; o(d-1,2)$$

$$[T_{A,B},\; T_{C_1 \dots C_n,\, D_1 \dots D_n}] = \eta_{B C_1} T_{A C_2 \dots C_n,\, D_1 \dots D_n} + \dots$$

What is a non Abelian algebra with this properties ?!

Are there global symmetries of this type in field theories? Conserved currents?

Yes!

S. Konstein, M.V. V. Zaikin (2000)

Bosonic HS Algebra in AdS_d

$\hat{Y}_i^A$: AdS_d symplectic pair

$$[\hat{Y}_i{}^A, \hat{Y}_j^B] = \epsilon_{ij}\, \eta^{AB} \qquad \begin{matrix} A,B=0\dots d \\ i,j=1,2 \end{matrix}$$

η^{AB} is $O(n,m)$ invariant metric
$n=2,\ m=d-1$ for AdS_d ...
$\epsilon^{ij}=-\epsilon^{ji}$ is $sp(2)$ symplectic form

Symplectic pair $\hat{Y}_1{}^A=\hat{Y}^A,\ \hat{Y}_2{}^A=\hat{P}^A$
$O(n,m)$ ($AdS_d \sim CFT_{d-1}$) generators

$$T^{AB}=-T^{BA}=\frac{1}{2}\{\hat{Y}_i{}^A, \hat{Y}_j^B\}\,\epsilon^{ji}$$

$sp(2)$ generators

$$t_{ij}=t_{ji}=\frac{1}{2}\{\hat{Y}_i{}^A, \hat{Y}_{jA}\}$$

Key fact

$$[T^{AB}, t_{ij}]=0 \qquad \text{Howe duality}$$

HS algebra - t_{ij} quotient of the centralizer of $sp(2)\sim sl_2$

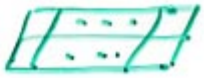
Eastwood (2002)

Generic element

$$f(\hat{Y}) = \sum_{m,n} f_{A_1\ldots A_m,\, B_1\ldots B_n}\, \hat{Y}_1^{A_1}\ldots\hat{Y}_1^{A_m}\hat{Y}_2^{B_1}\ldots\hat{Y}_2^{B_n}$$

$$S:\ [t_{ij}, f] = 0$$

$$t_{ij} = t_{ji} = \frac{1}{2}\{\hat{Y}_i^A, \hat{Y}_{jA}\}$$

$sp(2) \sim sl_2$ – Howe dual 

Ideal

$$I:\ g(\hat{Y}) = t_{ij}\, g^{ij}(\hat{Y})$$

$$[t_{ij}, g^{ke}] = \delta_i^k g_j{}^e + \delta_j^k g^e{}_i + \delta_i{}^e g_j{}^k + \delta_j{}^e g_i{}^k$$

$$g \in I, \quad [t_{ij}, f] = 0$$

$$\Downarrow$$

$fg \in I$

$gf \in I$

I is two-sided ideal

The factor algebra S/I consists of generators described by traces two-row Young tableaux

Lie HS algebra $hu(1/sp(2)[n,m])$

$n=2,\ m=d-1 \to AdS$ $\qquad$ $n=1,\ m=d \to dS$

Star - product

Let $\hat{Y}_\alpha$ $\alpha = 1 \ldots 2p$ be oscillators

$*$ $[\hat{Y}_\alpha, \hat{Y}_\beta] = 2\, C_{\alpha\beta}$ $C_{\alpha\beta} = -C_{\beta\alpha}$

non-degenerate

$$\hat{f}(\hat{Y}) = \sum_n f^{\alpha_1 \ldots \alpha_n} \hat{Y}_{\alpha_1} \ldots \hat{Y}_{\alpha_n}$$

Using $*$ one can achieve that

$f^{\alpha_1 \ldots \alpha_n}$ is totally symmetric:

Weyl ordering

Let Y_α be usual commuting variables

$$[Y_\alpha, Y_\beta] = 0$$

$f(Y) = \sum_n f^{\alpha_1 \ldots \alpha_n} Y_{\alpha_1} \ldots Y_{\alpha_n}$ is Weyl symbol of $\hat{f}(\hat{Y})$

$(f * g)(Y)$ is the symbol of $\hat{f}(\hat{Y})\, \hat{g}(\hat{Y})$

The star product is associative by construction $(f * g) * h = f * (g * h)$

A form of $(f * g)(Y)$ is non-trivial because the Weyl re-ordering is to be performed in $\hat{f}(\hat{Y})\, \hat{g}(\hat{Y})$

Triangle formula

$$(f*g)(Y) = \frac{1}{(2\pi)^{2p}} \int ds\, dt\, \exp(-s_\alpha t_\beta C^{\alpha\beta})$$
$$f(y+s)\, g(y+t)$$

$$C^{\alpha\beta} C_{\gamma\beta} = \delta^\alpha_\gamma \qquad C^{\alpha\beta} = -C^{\beta\alpha}$$

To derive: $\hat{f}_W(Y) = \int ds\, e^{i\hat{Y}_\alpha s^\alpha} \tilde{f}(s)$

$f_W(Y) = \int ds\, e^{iY_\alpha s^\alpha} \tilde{f}(s)$

+ Campbell-Hausdorf formula

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \quad \text{if } [[A,B],A]=0, \; [[A,B],B]=0$$

Moyal product formula

$$(f*g)(Y) = \exp\left[\frac{\partial^2}{\partial Y_{1\alpha} \partial Y_{2\beta}} C_{\alpha\beta}\right] f(Y_1) g(Y_2) \Big|_{Y = Y_1 = Y_2}$$

Properties

(i) associativity

(ii) $1 * g = g * 1 = g$

(iii) $Y_\alpha * f = \left(Y_\alpha + \frac{\partial}{\partial Y^\alpha}\right) f$ $\qquad A^\alpha = C^{\alpha\beta} Y_\beta$

$f * Y_\alpha = \left(Y_\alpha - \frac{\partial}{\partial Y^\alpha}\right) f$ $\qquad A_\alpha = A^\beta C_{\beta\alpha}$

$[Y_\alpha, f]_* = 2 \frac{\partial}{\partial Y^\alpha} f$

(iv) $str(f*g) = (-1)^{\pi(f)\pi(g)} str(g*f)$

$str(f) = f(0) \qquad f(-Y) = (-1)^{\pi(f)} f(Y)$

 odd functions are odd elements

Summary of Lecture I

(i) Gravity as $o(d-1,2)$ $(o(d,1))$ gauge field

$\boxed{}$ $W^{A}{}_{,}{}^{B} = - W^{B}{}_{,}{}^{A}$ $\qquad W^{A,B} = dx^n\, W_n{}^{A,B}$

V^A - compensator

$$V^A V_A = - \Lambda^{-1}$$

Standard gauge

$$V^A = r\, \delta^A_d$$

Usual gravity

$\boxed{}$ $e^a = W^{a}{}_{,}{}^{B} V_B$ $\qquad a, b, c \ldots = 0 - d-1$

$\boxed{}$ $\omega^{ab} = W^{a}{}_{,}{}^{b}$ $\qquad$ are V-transversal

$E^A = \mathcal{D} V^A$ $\qquad \mathcal{D} V^A = dV^A + W^A{}_B V^B$

(A)dS geometry

$R^{AB} = 0$ $\qquad \operatorname{rank} |E_n{}^A| = d$

$$R = dW^{AB} + W^A{}_C \wedge W^{CB}$$

A particular vacuum W_0

$$R(W_0) = 0$$

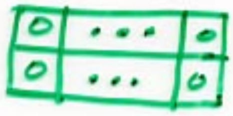
 has by construction (A)dS symmetry

Summary of Lecture I

(ii) Spin-s (totally symmetric)

HS fields belong to irreps of

$O(d-1,2)$ $(O(d,1))$



$$W^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}}$$

$$W^{\{A_1\dots A_{s-1},\, A_s\}B_2\dots B_{s-1}} = 0$$

$$W^{A_1\dots A_{s-1},\, B_1\dots B_{s-3}C}{}_C = 0$$

$$R_1{}^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}} = \mathcal{D}_0 W^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}}$$

$$\delta_0 W^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}} = \mathcal{D}_0 \epsilon^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}}$$

Abelian

$$\delta_0 R_1 = 0$$

$$W^{a_1\dots a_{s-1},\, b_1\dots b_t B_{t+1}\dots B_{s-1}} V_{B_{t+1}}\dots V_{B_{s-1}}$$

a, b - V-transversal

$s-1$

t

-Lorentz HS fields

$$e^{a_1\dots a_{s-1}} = W^{a_1\dots a_{s-1},\, B_1\dots B_{s-1}} V_{B_1}\dots V_{B_{s-1}}$$

dynamical

$\omega^{a_1\dots a_{s-1},\, b_1\dots b_t}$ $\quad t>0$ auxiliary

$$\Lambda^{-\frac{t}{2}}\partial^t e$$

$$S = \int V^{\,\,\dots} (E)^{d-4} R_1^{\,\dots} \wedge R_1^{\,\dots} \qquad \frac{\delta S}{\delta \omega} = 0 \quad t>1$$

Summary of Lecture I

Bosonic HS algebra as star-product algebra

$$f(Y) = \sum_{m,n} f_{A_1 \dots A_m, B_1 \dots B_n} Y_1^{A_1} \dots Y_1^{A_m} Y_2^{B_1} \dots Y_2^{B_n}$$

$$f * g(Y) = \frac{1}{(2\pi)^{2(d+1)}} \int dS\, dT\, f(Y+S)\, g(Y+T) \exp(-2 S_i^A T_j^B \epsilon^{ij} \eta_{AB})$$

$$[Y_i^A, Y_j^B]_* = \epsilon_{ij} \eta^{AB} \qquad i, j = 1, 2 \qquad \epsilon_{ij} = -\epsilon_{ji}$$

$o(n, m)$ generators

$$T^{AB} = -T^{BA} = Y_i^A Y_j^B \epsilon^{ji}$$

$sp(2)$ generators

$$t_{ij} = t_{ji} = Y^A_i Y_{jA}$$

$$[t_{ij}, t_{kl}]_* = \epsilon_{ik} t_{jl} + \epsilon_{il} t_{jk} + \epsilon_{jk} t_{il} + \epsilon_{jl} t_{ik}$$

Howe dual pair

$$[t_{ij}, T^{AB}]_* = 0$$

Summary of lecture I

HS algebra is built in two steps

(i) subalgebra of $sp(2)$ singlets

$$S: \qquad [f, t_{ij}]_* = 0$$

equivalent to

$$\left(Y^{Ai}\frac{\partial}{\partial Y^{A}{}_{j}} - Y^{Aj}\frac{\partial}{\partial Y^{A}{}_{i}}\right) f(Y) = 0$$

$$Y^{1A}\frac{\partial}{\partial Y^{A2}} = 0 \qquad\qquad Y^{2A}\frac{\partial}{\partial Y^{1}{}_{A}} = 0$$

$i=1;\ j=1$ $\qquad\qquad$ $i=2\ \ j=2$

$$Y^{A}_{1}\frac{\partial}{\partial Y^{A}_{1}} - Y^{A}_{2}\frac{\partial}{\partial Y_{2}^{A}} = 0$$

$i=1 \quad j=2$

$$f(Y) = \sum_{m=0}^{\infty} f_{A_1\ldots A_m,\, B_1\ldots B_m}\, Y_1^{A_1}\ldots Y_1^{A_m}\, Y_2^{B_1}\ldots Y_2^{B_m}$$

$$f_{\{A_1\ldots A_m,\, A_{m+1}\}B_2\ldots B_m} = 0$$

m

Every tableau appears just once!

(ii) To factor out the two-sided ideal

$$I: \; g(Y) = t_{ij} * g^{ij}(Y)$$

$$[t_{ij}, g^{ke}]_* = \delta_i^k g_j{}^e + \delta_j^k \delta_i^e + \delta_i^e g_j{}^k + \delta_j^e g_i{}^k$$

$$g \in I, \quad [t_{ij}, f]_* = 0$$

$$\Downarrow$$

$$f * g \in I \qquad g * f \in I$$

The factor-algebra

$$S/I = hu(1/sp(2)[n,m])$$

Signature of η

has a representative

$$f(Y) = \sum_{n=0}^{\infty} f_{A_1 \dots A_n, B_1 \dots B_n} Y_1^{A_1} \dots Y_1^{A_n} Y_2^{B_1} \dots Y_2^{B_n}$$

traceless $\quad f_{A_1 \dots A_n, B_1 \dots B_{n-2} C}{}^{C} = 0$

S

traceless  $\quad f_{\{A_1 \dots A_n, A_{n+1}\} B_2 \dots B_n} = 0$

because

$$t_{ij} = Y^A_i \, Y_{jA}$$

contain traces (contractions of (A)dS indices

Higher Spin Gauge Fields

$$\omega(Y|x) = \sum_n \omega_{A_1\dots A_n, B_1\dots B_n}(x) Y_1^{A_1}\dots Y_1^{A_n} Y_2^{B_1}\dots Y_2^{B_n}$$

$$\omega_{\{A_1\dots A_n, A_{n+1}\}B_2\dots B_n} = 0 \qquad \omega_{A_1\dots A_n, B_1\dots B_{n-2}C}{}^{C} = 0$$

$$R(Y|x) = d\,\omega(Y|x) + (\omega \wedge * \omega)(Y|x)$$

$*$ acts on Y_i^A

$$(f * g)(Y) = \frac{1}{(2\pi)^{2(d+1)}} \int dS\, dT\, f(Y+S)\, g(Y+T)$$
$$\exp(-2 S_i^A T_j^B \epsilon^{ij} \eta_{AB}) .$$

$sp(2)$ invariance condition

$$\mathcal{D} t_{ij} \equiv \overset{=0}{d\, t_{ij}} + [\omega, t_{ij}]_* = 0 \qquad t_{ij} = Y_i^A Y_j^B \eta_{AB}$$

Every integer spin $s \geq 1$ appears just once!

$hu(1/sp(2)[2, d-1])$ admits indeed a unitary representation in which all massless fields of integer spins $s = 0, 1, 2 \dots$ appear just once!

(a nontrivial consistency check!)

Unfolded Dynamics : example of gravity

Riemann + torsion tensors

$$R^{A,B} = dx^n \wedge dx^m R_{nm;}{}^{A,B} \qquad \begin{array}{l} n,m = 0\dots d-1 \\ A,B = 0,\dots d \end{array}$$

$$R^{A,B} + R^{B,A} = 0 \qquad \boxed{} \qquad o(d-1,2) \text{ or } o(d,1)$$

Equivalently

$$R^{A,B} = E_C \wedge E_D\, \tau^{C,D;A,B}$$

$$\tau^{C,D;A,B} = -\tau^{D,C;A,B} = -\tau^{C,D;B,A}$$

V^A - transversality conditions

(i) $V_C\, \tau^{C,D;A,B} = 0$ Because $E_C V^C = 0$

V - parallel components donot contribute

(ii) $\tau^{C,D;A,B} V_B = 0 \iff R^{A,B} V_B = 0$

zero-torsion condition

(i)+(ii): $\tau^{C,D;AB}$ is V-transversal,

i.e. Lorentz $(o(d-1,1))$ tensor

(iii) Bianchi identities: $R^{A,B} \wedge E_B = 0$

$$\tau^{[C,D;A]B} = 0 \qquad \boxed{}$$

(iv) Einstein equations

$$\tau^{C,A;}{}_{C,}{}^{B} = 0 \qquad \text{traceless}$$

Einstein equations

- $R^{A,B} = E_C \wedge E_D \, C^{AC,BD}$

$C^{AC,BD}$ is Weyl tensor in the symmetric basis

$$C^{AB,CD} = C^{BA,CD} = C^{AB,DC}$$

(i) V - transversal

(ii) $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$: $C^{\{AB,C\}D} = 0$

(iii) $C^{A}{}_{A,}{}^{BC} = 0$

It is useful to treat $C^{AB,CD}(x)$ as an independent field variable expressed via the gravitational connections (i.e. frame and Lorentz connection) by •

Bianchi Identities

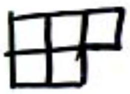
• $R^{A,B} = E_C \wedge E_D \, C^{AC,BD}$

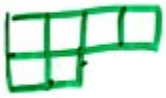
$$\Downarrow$$

$E_C \wedge E_D \wedge [\mathcal{D}^L C^{AC,BD}] = 0$ $\quad D^L$ - Lorentz

$$\Downarrow$$

•• $\mathcal{D}^L C^{A_1A_2,B_1B_2} = E_C \left(C^{A_1A_2C,B_1B_2} + C^{A_1A_2\{B_1,C\,B_2\}} \right)$

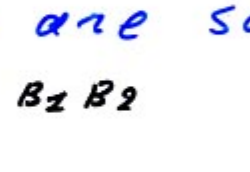
$C^{A_1A_2A_3,B_1B_2}$ - any traceless, V^A transversal tensor of the symmetry type 

Bianchi impose differential conditions on $C^{A_1A_2A_3,B_1B_2}$. These are solved in terms of $C^{A_1\ldots A_4,B_1B_2}$ 

••• $\mathcal{D}^L C^{A_1A_2A_3,B_1B_2} = E_C \big(C^{A_1A_2A_3C,B_1B_2} +$
$\qquad + \frac{2}{3} C^{A_1A_2A_3B_1,CB_2} \big) + \ldots$

Nonlinear corrections due to •

Continuation leads to an infinite system of equations on

$C^{A_1\ldots A_p,B_1B_2}(x)$ 

$\widetilde{\mathcal{D}}(C) = O(C^2)$

$C^{A_1\ldots A_p,B_1B_2}(x)$ forms some infinite-dimensional representation of $o(2,d-1)$ $o(1,d)$

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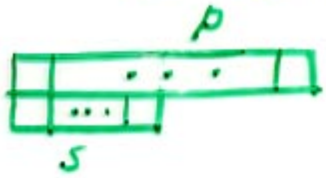
Any Spin

Weyl 0-forms

$C^{A_1\dots A_p,\, B_1\dots B_s}(x)$ $\qquad p \geq s \qquad$ s – spin

$C^{\{A_1\dots A_p,\, A_{p+1}\}B_2\dots B_s}(x) = 0$



$C^{A_1\dots A_{p-2}C}{}_{C,}{}^{B_1\dots B_s}(x) = 0$ $\qquad$ traceless

$V_{A_1} C^{A_1\dots A_p,\, B_1\dots B_s}(x) = 0$ $\qquad$ Lorentz tensors

Unfolded Field Equations

$$
\begin{cases}
R^{A_1\dots A_{s-1},\, B_1\dots B_{s-1}} = E_{A_s} \wedge E_{B_s}\, C^{A_1\dots A_s,\, B_1\dots B_s} \\
\qquad\qquad\qquad + \text{nonlinear corrections} \\
\widetilde{\mathcal{D}}(C^{A_1\dots A_p,\, B_1\dots B_s}) = \text{nonlinear corrections}
\end{cases}
$$

$$
\widetilde{\mathcal{D}}(C^{A_1\dots A_p,\, B_1\dots B_s}) = \mathcal{D}^L(C^{A_1\dots A_p,\, B_1\dots B_s}) +
$$

$$
+ E_C \Big(C^{A_1\dots A_p C,\, B_1\dots B_s} + \frac{s}{(p+2-s)} C^{A_1\dots A_p\{B_1,\, C B_2\dots B_s\}}
$$

$$
+ O(\Lambda) + \text{nonlinear corrections}
$$

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$s = 0$ – scalar field equation

$$
\partial_n C^{A_1\dots A_p}(x) + E_n{}^C C^{A_1\dots A_p}{}_C(x) = 0
$$

$\partial_n C(x) = C_n(x)$ $\qquad$ V-transversality

$\partial_n C_m(x) = C_{nm}(x) \Rightarrow \Box C = 0$ $\qquad$ tracelessness

Free differential algebras

Unfolded HS field equations have a form

$$d\omega + \omega * \omega = -F_2(\omega, C)$$

$$dC + (\omega \circ C) = -F_1(\omega, C)$$

$\omega = \omega_0 + \omega_1$ 1-form

C 0-form

From now on the wedge symbol $\wedge$ is omitted

It is a particular case of the system of the form

- $$R^{\Lambda} = 0$$

$$R^{\Lambda} \stackrel{d}{=} dW^{\Lambda} + F^{\Lambda}(W)$$

W^{Λ} - some set of p-forms ($p \geq 0$)

F^{Λ} contains wedge products of W^{Λ}

- is consistent if

$$F^{\Lambda} \frac{\delta F^{\Omega}}{\delta W^{\Lambda}} \equiv 0$$ generalized Jacobi identity

Any F^{Λ} - some FDA

Properties of FDA's

(i) invariance under diffeomorphisms because of exterior algebra formalism

(ii) gauge invariance

W^Λ - p-form
ε^Λ - p-1-form

$$\delta W^\Lambda = d\varepsilon^\Lambda - \varepsilon^\Omega \frac{\delta F^\Lambda}{\delta W^\Omega}$$

is a consequence of the generalized Jacobi identity

(iii) the equations

$$R^\Lambda = 0$$

are formally consistent as a consequence of the generalized Jacobi identities and gauge invariant because

$$\delta R^\Lambda = -R^\Omega \frac{\delta}{\delta W^\Omega}\left(\varepsilon^\Phi \frac{\delta F^\Lambda}{\delta W^\Phi}\right)$$

As a result, once free field dynamics of massless fields is reformulated in the "unfolded form" of some linearized FDA, any its consistent nonlinear deformation will describe consistent interactio

Particular examples

(i) if W^Λ consists of only 1-forms ω^i, then

$$F^i(\omega) = f^i_{jk}\, \omega^j \wedge \omega^k$$

and the generalized Jacobi identity amounts to the usual Jacobi identity for a Lie algebra g with structure coefficients $f^i_{jk} = -f^i_{kj}$

$$f_{[ij}{}^{e} f_{k]e}{}^{n} = 0$$

The equation

$\circ$ $0 = R^i \stackrel{d}{=} d\omega^i + f^i_{jk}\, \omega^j \wedge \omega^k$

is the zero-curvature condition for the Lie algebra g. In applications g is the (A)dS space-time symmetry algebra $o(d-1,2)$ or $o(d,1)$ or some its extension: superextension, higher spin extension, etc.

A solution of $\circ$ describes some vacuum of the theory

(ii) if W^1 contains in addition some p-forms C^α (for example Weyl 0-forms in the HS case) and the function F^α is linear in C^α

$$F^\alpha = 2 t_i{}^\alpha{}_\beta \, \omega^i C^\beta$$

the generalized Jacoby identities demand $t_i{}^\alpha{}_\beta$ to form some representation of g

$$[t_i, t_j]^\alpha{}_\beta = f_{ij}^k \, t_k{}^\alpha{}_\beta$$

As a result, the equations

$$\mathcal{D} C^\alpha = d C^\alpha + 2 \, t_i{}^\alpha{}_\beta \, \omega^i C^\beta = 0$$

are covariant constancy condition for C^α taking values in an appropriate module over g.

$\mathcal{D}$ is covariant derivative in the representation t.

$\mathcal{D}^2 = 0 \iff d\omega^i + f_{jk}^i \, \omega^j \wedge \omega^k = 0$

What is a representation t for

 massless Weyl 0-forms $C_{A_1 \ldots A_n, B_1 \ldots B_s}$

Twisted Adjoint Representation

τ - some automorphism

$$\tau(ab) = \tau(a)\,\tau(b)$$

τ - twisted representation covariant derivative:

$$\widehat{D}(C) = dC + \omega C - C\,\tau(\omega)$$

In HS theories

$$\tau(P^{a}) = -P^{a} \qquad \tau(L^{ab}) = L^{ab}$$

for the $(A)dS_d$ subalgebra

$$P^{a} = M^{aB} V_B \qquad L^{ab} = M^{ab}$$

V^A transversal part of M^{AB}

τ reflection with respect to V^A

$$\tau(f(Y)) = \tilde{f}(Y) \equiv f(\widehat{Y})$$

$$\tilde{A}^{A} = A^{A} - \frac{2}{V^2} V^A V_B A^B$$

$\mathcal{V}$ reflection is a symmetry of the metric η^{AB}

$$\tilde{\eta}^{AB} = \eta^{AB}$$

As a result $\mathcal{T}$ preserves the defining commutation relations

$$[Y^A_i, Y^B_j]_* = \epsilon_{ij}\eta^{AB}$$

as well as the $sp(2)$ generator

$$t_{ij} = Y^A_i Y_{jA} \qquad \mathcal{T}(t_{ij}) = t_{ij}$$

Thus $\mathcal{T}$ is automorphism of the star product

$$f * g(Y) = \frac{1}{(2\pi)^{2(d+2)}} \int dS\, dT \exp(-2 S^A_i T^B_j \epsilon^{ij} \eta_{AB})$$
$$f(Y+S)\, g(Y+T)$$

as well as of the HS subalgebra formed by $sp(2)$ singlets (two row rectangular Young tableaux of the (A)dS symmetry) modulo the ideal I formed by the elements proportional to t_{ij}

Covariant derivatives in the adjoint and twisted adjoint representations

$$W^{AB} = \omega^{L\,AB} - \Lambda\,(E^A V^B - E^B V^A)$$

Lorents connection $\omega^{L\,AB}$ is V-transversal in A and B

E-dependent term is V-transversal in one index and V-parallel in another ($E^A V_A = 0$).

$$\mathcal{D}_0 = \mathcal{D}_0^L + 2\,[{}^{\shortparallel}Y_i^{\;A} V_A\, {}^{\perp}Y_j^{\;B} \underset{\circ}{E}_B,\ \ ]_*$$

$$\widetilde{\mathcal{D}}_0 = \mathcal{D}_0{}^L + 2\,\{{}^{\shortparallel}Y_i^{\;A\perp}Y_j^{\;B}\, V_A^{\perp} \underset{\circ}{E}_B,\ \ \}_*$$

where $\mathcal{D}_0{}^L$ is the local Lorentz derivative

$$\mathcal{D}_0{}^L = d + [\,\omega^{L\,AB\,\perp}Y_{Ai}\,{}^{\perp}Y_{Bj}\,\epsilon^{ij},\ \ ]_*$$

and we use notations

$$A^A = {}^{\shortparallel}A^A + {}^{\perp}A^A$$

$${}^{\shortparallel}A^A = \frac{1}{V^2} V^A V_B A^B, \qquad {}^{\perp}A^A = A^A - \frac{1}{V^2} V^A V_B A^B$$

V-reflection acts as follows

$$\widetilde{{}^{\shortparallel}A}^A = -\,{}^{\shortparallel}A^A \qquad \widetilde{{}^{\perp}A}^A = {}^{\perp}A$$

Using simple formulas

$$Y_i^A * = Y_i^A + \frac{1}{2}\frac{\partial}{\partial Y_A{}^i}$$

$$Y_A{}^i = \eta_{AB}\epsilon^{ij}Y_j^B$$

$$* Y_i^A = Y_i^A - \frac{1}{2}\frac{\partial}{\partial Y_A{}^i}$$

one gets

$$\mathcal{D}_0 = \mathcal{D}_0{}^L - \Lambda\, E_0{}^A V^B \left({}^{\perp}Y_{Ai}\frac{\partial}{\partial {}^{\parallel}Y_i^B} - {}^{\parallel}Y_{Bi}\frac{\partial}{\partial {}^{\perp}Y_i^A}\right)$$

$$\widetilde{\mathcal{D}}_0 = \mathcal{D}_0{}^L - 2\Lambda E_0{}^A V^B \left({}^{\perp}Y_A^i\, {}^{\parallel}Y_{Bi} - \frac{1}{4}\epsilon^{ij}\frac{\partial^2}{\partial {}^{\perp}Y^{Ai}\partial {}^{\parallel}Y^{Bj}}\right)$$

$\mathcal{D}_0$ acts on finite-dimensional subspaces spanned by homogeneous polynomials

Invariant subspaces of $\widetilde{\mathcal{D}}_0$ are infinite-dimensional because they involve polynomials of arbitrary high degrees

$$[N^{ad}, \mathcal{D}_0] = 0 \qquad N^{ad} = Y_i^A\frac{\partial}{\partial Y_i^A}$$

$$[N^{tw}, \widetilde{\mathcal{D}}_0] = 0 \qquad N^{tw} = {}^{\perp}Y_i^A\frac{\partial}{\partial {}^{\perp}Y_i^A} - {}^{\parallel}Y_i^A\frac{\partial}{\partial {}^{\parallel}Y_i^A}$$

Unfolded Form of Free Equations

$$\begin{cases} R_1(Y) = E_0{}^A \wedge E_0{}^B \dfrac{\partial^2}{\partial^{\perp} Y_i{}^A \partial^{\perp} Y_j{}^B} \epsilon_{ij} C(Y)\Big|_{{}^{\parallel}Y=0} \\ \widetilde{\mathcal{D}}_0 C(Y) = 0 \end{cases}$$

Free equations for all totally symmetric massless fields

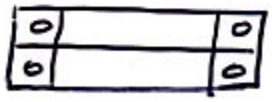
$$R_1(Y) = \mathcal{D}_0 \omega_1$$

The field equations describe a set of subsystems

$$(N^{\perp} + N^{\parallel})\omega = N^{ad}\omega = 2(s-1)\omega$$

$$(N^{\perp} - N^{\parallel})C = N^{tw} C = 2s\, C$$

ω and C are $sp(2)$ singlets



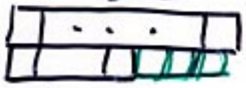
⊡ : $A = 0 \dots d$

$N^{\perp} \geq N^{\parallel} \Rightarrow s \geq 0$ since otherwise the tensor is zero

Sets of Lorentz tensors

ω :

$s-1$



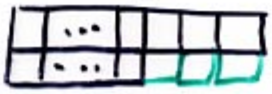
t

$0 \leq t \leq s-1$

□ - V - parallel

t

C :



s

$s \leq t < \infty$

$t = s$: Weyl tensors

Examples

$s=0$

$0 \le t < \infty$

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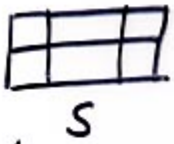
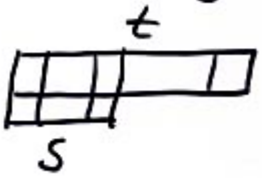
$s=2$

$2 \le t < \infty$

Lowest components



Weyl tensors



$t > s$: $(t-s)^{th}$ derivatives of the Weyl tensors

Towards interactions

$$R(Y) = E^A \wedge E^B \frac{\partial^2}{\partial Y^A_i \, \partial Y^B_j} \epsilon_{ij} \, C(Y)\Big|_{Y=0} + \dots$$

$$\tilde{\mathcal{D}} C(Y) = 0 + \dots$$

where

$$R(Y) = d\omega(Y) + \omega(Y) \wedge * \, \omega(Y)$$

$$\tilde{\mathcal{D}} C(Y) = dC(Y) + \omega(Y) * C(Y) - C(Y) * \omega(\tilde{Y})$$

$+\dots$ denote non-linear corrections

According to general properties of FDA's, once formal consistency is achieved, the system describes gauge invariant HS interactions

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Deformed oscillator algebra

The key element of the construction is the $sp(2)$ invariance condition which singles out appropriate tensor representations for higher spin fields.

To keep $sp(2)$ invariance at the interaction level it is useful to use the properties of the following algebra

$$[q_i, q_j] = \epsilon_{ij}(1 + \nu k)$$

$$k q_i = - q_i k \qquad k^2 = I$$

ν - some parameter

(More generally $\nu k \Rightarrow Q$, $Q q_i = - q_i Q$)

Introduce

$$t_{ij} = \frac{1}{2}\{q_i, q_j\}$$

- $[t_{ij}, q_k] = \epsilon_{jk} q_i + \epsilon_{ik} q_j$

for any ν

$$[t_{ij}, q_k] = \frac{1}{2}\{q_i, 1+\nu K\}\epsilon_{jk} + \frac{1}{2}\{q_j, 1+\nu K\}\epsilon_{ik}$$

the ν-dependent terms cancel out because $\{q_i, K\} = 0$

As a result, for any ν,

q_i transforms as $sp(2)$ vector and, therefore, t_{ij} form $sp(2)$ algebra

$$\begin{cases} t_{ij} = \frac{1}{2}\{q_i, q_j\} \\ [t_{ij}, q_k] = \epsilon_{jk} q_i + \epsilon_{ik} q_j \\ [t_{ij}, t_{kl}] = \epsilon_{jk} t_{il} + \epsilon_{ik} t_{jl} + \epsilon_{jl} t_{ik} + \epsilon_{il} t_{jk} \end{cases}$$

Algebra $osp(1,2)$

q_i – supercharges

A particular representation was found by Wigner in 1950 who was looking for most general commutation relations for oscillators $a^\pm$ such that

$$[H, a^\pm] = \pm a^\pm, \quad H = \frac{1}{2}\{a, a^+\}$$

$a^\pm : q_{1,2}$ $\qquad H = t_{12}$ $\qquad$ 2-body Calodgero model: ν-coupling constant

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Parameter ν is related to the quadratic casimir operator of $osp(1,2)$

$$C_2 = \frac{1}{2}(t_{ij}t^{ij} - q_i q^i) = \frac{1}{4}(1-\nu^2)$$

Deformed oscillator algebra allows one to describe a large class of representations of $osp(1,2)$ $(sp(2))$ including all lowest (highest) weight representations

$$q_2|0\rangle = 0, \qquad K|0\rangle = |0\rangle$$

$$H|0\rangle = \frac{1}{2}(1+\nu)|0\rangle \qquad H = t_{12} = \frac{1}{2}\{q_1, q_2\}$$

Deformed oscillator algebra is a usuful realization of the fuzzy hyperboloid

$sp(2)$ $\qquad\qquad$ $O(1,2)$

t_{ij}

$$X_\alpha = \gamma_\alpha{}^{ij} t_{ij}$$

$$[X_\alpha, X_\beta] = \epsilon_{\alpha\beta\gamma} X^\gamma$$

$$C_2 = \frac{1}{2} X_\alpha X^\alpha = \frac{1}{2}R^2$$

$R = R(\nu)$ $\qquad$ $R \xrightarrow[\nu\to\infty]{} \infty$

Nonlinear HS dynamics hep-th/0304049

Idea to describe complicated nonlinear corrections as a solution of some simply formulated "differential equations" with respect to additional variables Z^A_i:

$$W = dx^n W_n(Z, Y|x)$$ gauge fields

$$B = B(Z, Y|x)$$ HS Weyl tensors + $s=0$ matter field

$$S = dZ^A_i S^i_A(Z, Y|x)$$ Z-connection

Dynamical HS fields used to formulate free field dynamics: "initial data" for the evolution in Z-directions

$$W(0, Y|x) = \omega(Y|x)$$

$$B(0, Y|x) = C(Y|x)$$

$$S = dZ^A_i Z^i_A + \ldots$$

Star-product of the form

$$(f*g)(Z,Y)=\frac{1}{\pi^{2(d+1)}}\int dS\,dT\,\exp -2S_i{}^A T_A{}^i$$

$$\times f(Z+S,Y+S)\,g(Z-T,Y+T)$$

Oscillator algebra

$$[Y_i{}^A, Y_j{}^B]_* = \epsilon_{ij}\eta^{AB}$$

$$[Z_i{}^A, Z_j{}^B]_* = -\epsilon_{ij}\eta^{AB}$$

$$[Y_i{}^A, Z_j{}^B]_* = 0$$

$\overleftarrow{(Z-Y)}\ \overrightarrow{(Z+Y)}$ normal ordered

$$f(Z-Y)*g(Z,Y) = f(Z-Y)\,g(Z,Y)$$

$$f(Z,Y)*g(Z+Y) = f(Z,Y)\,g(Z+Y)$$

This star product admits inner Klein operator

$$k=\exp\left(-\frac{2}{V^2}V_A Z^A_i V_B Y^{Bi}\right)$$

$$k*f(Z,Y)=\widetilde{f}(Z,Y)*k$$

$$\widetilde{f}(Z,Y)=f(\widetilde{Z},\widetilde{Y})$$

$$\widetilde{({}^{\parallel}A^A)} = -\,{}^{\parallel}A^A \qquad \widetilde{({}^{\perp}A^A)} = {}^{\perp}A^A$$

Full Nonlinear System of HS equations in any dimension

$$
* \begin{cases} dW + W * W = 0 \\ dS + W * S + S * W = 0 \\ dB + W * B - B * \widetilde{W} = 0 \end{cases}
$$

zero-curvature + covariant constancy conditions

$$d = dx^n \frac{\partial}{\partial x^n}$$

$$
** \begin{cases} S * B = B * \widetilde{S} \\ S * S = -\frac{1}{2} dZ_i^A \, dZ^{Bi} (\eta_{AB} - 4 V_A V_B \, B * k) \end{cases}
$$

$\widetilde{S}(dZ, Z, Y) = S(\widetilde{dZ}, \widetilde{Z}, \widetilde{Y})$

Λ^{-1}

The differentials dZ_i^A, dx_n anticommute with themselves but commute with $f(Z, Y|x)$

k is inner Klein operator

$$k * f(Z, Y) = \widetilde{f}(Z, Y) * k$$

The system is invariant under HS gauge transformations

$$\delta W = d\varepsilon + [W, \varepsilon]_*$$ adjoint

$$\delta S = [S, \varepsilon]_*$$ adjoint

D57 $$\delta B = B * \widetilde{\varepsilon} - \varepsilon * B$$ twisted adjoint

Formal Consistency

Usual gauge fields

$$\Omega = d + W$$

$$R = \Omega^2 = \{d, W\} + W^2 \qquad d^2 = 0$$

Bianchi identities

$$0 = [\Omega, R] \equiv \Omega(\Omega\Omega) - (\Omega\Omega)\Omega$$

$\doteq$ associativity

Gauge transformations are similarity transformations

$$\Omega' = U\,\Omega\,U^{-1}$$

If some conditions on the curvature R are imposed, they should respect Bianchi identities and be gauge invariant

Non commutative connection

$$S = dZ^A_i \, S^i_A(Z, Y | x)$$

can be decomposed

$$S = S_0 + S_1$$

$$S_0 = dZ^A_i Z^i_A \qquad \text{differential}$$

$$S_1 \qquad \text{gauge field}$$

$$[S_0, \ ] = -dZ^A_i \frac{\partial}{\partial Z^A_i}$$

analogue of de Rahm differential

$$S_0 * S_0 = -\tfrac{1}{2} dZ^A_i \, dZ^i_A$$

Bianchi identities = associativity

$$(S * S) * S = S * (S * S)$$

+ analogous conditions between X - sector and Z - sector

In terms of the universal connection

$$\mathcal{W} = d + W + S$$

$d = dx^n \frac{\partial}{\partial x^n}$

$W = dx^n W_n$

$S = dZ^A_i \, S^i_A$

the full system of HS equations takes the following simple form

$$\begin{cases} \mathcal{W} * \mathcal{W} = -\frac{1}{2} dZ^A_i dZ^i_A + 2\, dz_i dz^i \, B * K & * \\ \mathcal{W} * B = B * \widetilde{\mathcal{W}} & ** \end{cases}$$

$*$ means that non-zero curvature components appear only in $dZ\,dZ$ - sector

$dz_i = V_A dZ^A_i$ - V parallel components of $dZ^A{}_i$

$\widetilde{(dz_i)} = -(dz_i)$

Because inner Klein operator K acts only on $Z^A{}_i$, Y^A_i but not on $dZ^A{}_i$

$**$ implies

$$\mathcal{W} * (B * K) = (B * K) * \mathcal{W}'$$

$$\mathcal{W}'(dZ, Z, Y | x) = \mathcal{W}(\widetilde{dZ}, Z, Y | x)$$

Consistent because $(dz_i)^3 \equiv 0 \quad (i = 1, 2)$

HS equations and deformed oscillator

The nontrivial curvature proportional to the field B (to be proportional to HS Weyl tensors) is in the sector of V-parallel components of S_A^i

Let

$$S^i = \frac{1}{\sqrt{V^2}} V^A S_A^i$$

The nontrivial part of equations

$$\begin{cases} [S^i, S^j]_* = -\epsilon^{ij}(1+4\Lambda^{-1} B*k) \\ S^i * (B*k) = -(B*k) * S^i \end{cases}$$

Deformed oscillator algebr

$$T_{ij} = -\frac{1}{2}\{S_i^A, S_{jA}\}_* = -\frac{1}{2}\left(\{S_i, S_j\}_* + \{\overset{\perp}{S}{}^A_i, \overset{\perp}{S}_{jA}\}\right)$$

T_{ij} form $sp(2)$ for any B — ordinary oscillators

Howe dual $sp(2)$ generators for non-linear theory

$$t^{int}_{ij} = Y^A_i Y_{Aj} - Z^A_i Z_{Aj} - T_{ij} \qquad W t^{int}_{ij} = t^{int}_{ij} W$$

HS equations : fuzzy hyperboloid of radius $\quad r = \Lambda^{-1} B*k$

Λ^{-1}: R^2_{AdS} $\qquad r \to \infty$ in the flat space-time limit

Perturbative Analysis

Simplest vacuum solution

$$B_0 = 0 \,, \qquad S_0 = dZ^A{}_i \, Z^i{}_A$$

$$W_0 = \frac{1}{4}\, \omega_0{}^{AB}(x) \{Y^i{}_A, Y_{iB}\}_*$$

where $\omega_0{}^{AB}(x)$ describes $(A)dS_d$

to satisfy $R(W_0) = 0$

Expanding

$$W = W_0 + W_1 \,, \quad S = S + S_1 \,, \quad B = B_1$$

one solves the equations which contain $\frac{\partial}{\partial Z}$ (i.e. S)

Modulo gauge ambiguity, all fields are expressed in terms of "initial data"

$$C(Y|x) = B(0, Y|x)$$

$$\omega(Y|x) = W(0, Y|x)$$

The S-independent equations

$$dW + W * W = 0 \,, \qquad dB + W * B - B * \tilde{W} = 0$$

reproduce the free massless equations with all nonlinear corrections

$e^2 g$ - Kristiansson, Rajan (2003)

D62

Linearized equations

(i) Solve

$$S_0 * B_1 = B_1 * S_0 \qquad \tilde{S}_0 = S_0$$

$$\Downarrow$$

$$dZ^A_i \frac{\partial}{\partial Z^A_i} B_1 = 0$$

$$\Downarrow$$

$$B_1(Z, Y|x) = C(Y|x)$$

(ii) Since all components of the curvature in ${}^{\perp}Z^A_i$ directions are zero, the components ${}^{\perp}S^i_A$ can be gauge fixed to ${}^{\perp}S^i_{0A}$. The leftover symmetry parameters $\varepsilon(Z, Y|x)$ satisfy $[{}^{\perp}S^i_{0A}, \varepsilon]_* = 0$, i.e. ε is independent of ${}^{\perp}Z^A_i$:

$$\varepsilon = \varepsilon(z, Y|x)$$

Analogously, from

$$dS + \{W, S\}_* = 0$$

in the sector of $d{}^{\perp}Z^A_i$ it follows

$$W = W(z, Y|x)$$

As a result, Z^A_i dependence enters only via $z_i = \frac{1}{\sqrt{V^2}} V_A Z^A_i$

(iii) To solve the fuzzy hyperboloid equation one first uses

$$(f * \kappa)(Z;Y) = \exp -2z_i y^i f(-{}^{\parallel}Y, Z^{\perp}; {}^{\parallel}Z, {}^{\perp}Y)$$

$$\kappa = \exp -2z_i y^i$$

i.e. κ interchanges ${}^{\parallel}Y$ and ${}^{\parallel}Z$

(elementary to check using the $*$ product

Then, the fuzzy hyperboloid equation

$$\frac{\partial}{\partial z_i} S_1^j - \frac{\partial}{\partial z_j} S_1^i = -4\Lambda^{-1}\epsilon^{ij} C(-{}^{\parallel}Z, {}^{\perp}Y)\exp(-2z_k y^k)$$

solves to

$$S^j = \frac{\partial}{\partial z_j}\varepsilon_1 + 2\Lambda^{-1} z^j \int_0^1 dt\, t\, C(-t\,{}^{\parallel}Z, {}^{\perp}Y)\times$$
$$\times \exp(-2t z_k y^k)$$

(General formula: if $\partial^i S^j - \partial^j S^i = -2\epsilon^{ij} f(z)$
then $S^j = \partial^j \varepsilon + z^j \int_0^1 dt\, t\, f(tz)$

The ambiguity in ε_1 manifests the gauge ambiguity $\delta S_1^j = [S_0^j, \varepsilon_1]$

Convenient gauge: $\frac{\partial}{\partial z_i}\varepsilon_1 = 0$

$$\varepsilon_1 = \varepsilon_1(Y|x) \text{ -- HS gauge}$$

parameters of dynamical degrees of freedom.

As a result S is expressed in terms of $B_1 = C(Y)$: non-commutative connection is expressed via non-comm. curvature modulo gauge ambiguity.

It remains to analyse the equations with space-time differential

$\circ\ dW + W * W = 0$

$\circ\circ\ dS + W * S + S * W = 0$

$\circ\circ\circ\ dB + W * B - B * \widetilde{W} = 0$

(iv) The nontrivial part (i.e. V-parallel) reads

$$\frac{\partial}{\partial z_i} W_1 = dS_1^i + W_0 * S_1^i - S_1^i * W_0$$

Using that the equation $\frac{\partial}{\partial z_i}\varphi = \chi^i$ solves as $\varphi(z) = const + \int_0^1 dt\, z_i \chi^i(tz)$ provided that $\frac{\partial}{\partial z_i}\chi^i = 0$ $(i=1,2)$ one gets

$$W_1(Z,Y|x) = \omega(Y|x) - Z_A^i V^A \int_0^1 dt\,(1-t) \times$$
$$\times e^{-2t z_i y^i} E^B \frac{\partial}{\partial Y^{jB}} C(-t''Z, Y^\perp)$$

$\omega(Y|x)$ is the dynamical HS gauge field.

It remains to plug this expression into $\circ$ and $B = C(Y|x)$ into $\circ\circ\circ$

Because the dependence on Z is fixed by the equations consistent with $\circ$, $\circ\circ$, $\circ\circ\circ$ equations, it is enough to consider $\circ$, $\circ\circ\circ$ say at

265 $Z = 0$

As a result

$$dW + W * W = 0$$

$$dB + W * B - B * \widetilde{W} = 0$$

give at $Z_i = 0$

$$\begin{cases} R_1(Y|x) = E_0{}^A \wedge E_0{}^B \dfrac{\partial^2}{\partial^\perp Y_i^A \, \partial^\perp Y_j^B} \epsilon_{ij} \, C(Y|x)\Big|_{\text{``}Y=0} + \ldots \\ \widetilde{\mathcal{D}}_0 C(Y|x) = 0 + \ldots \end{cases}$$

because "$Z=0$

$$R_1 = \mathcal{D}_0 \, \omega(Y|x)$$

where ... denote all non-linear corrections which can be systematically reconstructed using the same procedure.

Comment: Klein operator $\kappa = \exp -2 z_i y^i$ is non-polynomial and it is necessary to prove that all star products are well-defined (finite). This is indeed true within the star-product used. All higher-order corrections are well-defined.

Internal Symmetries

$$W_\nu{}^\mu(Z,Y|x)\,,\quad S_\nu{}^\mu(Z,Y|x),\ B_\nu{}^\mu(Z,Y|x)$$

$$\nu,\mu = 1\dots p$$

Equations: associative structure

Reality conditions

$$W^\dagger(Z,Y|x) = -W(-iZ,iY|x)$$

HS algebra $hu(p|sp(2)[2,d-1])$

AdS

All fields in $u(p)$ - YM algebra

Orthogonal and symplectic reductions

$$W_\nu{}^\mu(Z,Y|x) = -\rho^{\mu\mu'}W_{\mu'}{}^{\nu'}(-iZ,iY|x)\rho_{\nu'\nu}$$

$\rho^{\nu\mu}=\rho^{\mu\nu}$: $ho(p|sp(2)[2,d-1])$

YM : $o(p)$

$\rho^{\nu\mu}=-\rho^{\mu\nu}$: $husp(p|sp(2)[2,d-1])$

YM : $usp(p)$

Odd spins: in YM algebra

YM sector

Even spins: in opposite symmetry

graviton second rank representation which contains YM singlet

D67 Minimal model - only even spins $ho(1|sp(2)[2,d-1])$

C2.6

Interaction Ambiguity

No ambiguity modulo nonlinear field redefinitions!

$$g^2 = |\Lambda|^{\frac{d-2}{2}} \varkappa^2$$

YM constant can be rescaled away for the classical HS model

(Different from 4d model that has a freedom in one function distinguishing between self-dual and anti self-dual sectors. Lagrangian!)

Nontrivial part of the system is fixed by the condition

$$t_{ij} = \tfrac{1}{2}\{Y^A{}_i, Y_{jA}\}_* \to t_{ij}^{int} \text{ forms } sp(2)$$

and allows

$$D(t_{ij}^{int}) = 0 \qquad [S, t_{ij}^{int}]_* = 0$$

$$B * \tilde{t}_{ij}^{int} - t_{ij}^{int} * B = 0$$

reduction of the space of fields to  and factorization $\mathcal{I}^{int}_j$

Discussion

Quantum mechanical nonlocality of the star-product gives rise to higher derivatives in higher spin interactions via field equations

For example

$$\widetilde{\mathcal{D}}_0 C = 0$$

$$\widetilde{\mathcal{D}}_0 = \mathcal{D}_0^L - 2\Lambda E_0^A V^B \left({}^{I}Y_A{}^{i}\,{}^{II}Y_{Bi} - \frac{1}{4}\epsilon^{ij}\frac{\partial^2}{\partial^I Y^{Ai}\,\partial^I Y^{Bj}}\right)$$

links $\frac{\partial}{\partial X}$ with $\frac{\partial^2}{\partial Y \partial Y}$

HS * product algebras

$\Updownarrow$ AdS

$\Downarrow$ infinite towers of spins

$\Downarrow$ space-time "non-locality" of HS models

The part of the nonlinear system that contains space-time derivatives

$$\begin{cases} dW + W * W = 0 \\ dS + W * S + S * W = 0 \\ dB + W * B - B * \widetilde{W} = 0 \end{cases}$$

admits a solution (locally)

$$W = g^{-1} * dg$$

$$S = g^{-1} * s * g$$

$$B = g^{-1} * b * \widetilde{g}$$

where

$g = g(Z, Y|x)$ is an arbitrary gauge function

$$\left.\begin{array}{l} s = s(Z,Y) = dZ^A_i\, s^i_A(Z,Y) \\ b = b(Z,Y) \end{array}\right\} \text{arbitrary } x\text{-independent elements}$$

Because of gauge invariance, one is left with the equations

$$\begin{cases} s * b = b * \tilde{s} \\ s * s = -\frac{1}{2} dZ^A_i\, dZ^{Bi} (\eta_{AB} - 4 V_A V_B\, b * k) \end{cases}$$

which do not contain x

HS nonlinear dynamics is reformulated as the fuzzy hyperboloid equation in a non-commutative space-time!

Local events!!

Conclusions

- A class of non-linear HS gauge theories in (A)dSd exists.

  So far - totally symmetric HS gauge fields.

  SUSY in AdS_4

  Different Chan-Paton structures = = Yang-Mills algebras

- HS equations describe a noncommutative hyperboloid in the "auxiliary" noncommutative space

  $$R_{H2}(\Lambda^{-1} B(Z,Y|x))$$

  HS curvaturees

  Minkowski coordinates x allow visualization (local events)

- Higher derivatives in x ~~allow~~ result from Moyal non-locality in the "fuzzy" fiber.

Open Problems

- Extension to mixed symmetry massless fields  Labastida (1989)

Free mixed symmetry fields in AdS_d are different from (irreducible) fields in Minkowski

Brink, Metsaev, M.V. (2000)

- Action in all orders
- To find nontrivial solutions with massive modules m to define a low-energy expansion in $\left(\frac{\partial}{m}\right)^p$

Λ : $\quad (\Lambda^{-\frac{1}{2}} D)^p \sim 1 \qquad [D, D] \sim \Lambda$

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strings

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