Elements of QFT in Curved Space-Time

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Contents of the mini-course

- **GR and its limits of applicability, Planck scale. Quantum gravity and semi-classical approach. Formulation of quantum field theory on curved background.**

- **Covariance and renormalizability in curved space-time. Renormalization group and conformal anomaly. Anomaly-induced effective action and Starobinsky model.**

- **Effective approach in curved space-time. The problem of cosmological constant and running in cosmology.**

Bibliography

I.L. Buchbinder, S.D. Odintsov, I.Sh., Effective Action in Quantum Gravity. (1992 - IOPP).

I.Sh., Class. Q. Grav. 25 (2008) 103001 (Topical review); 0801.0216.

L. Parker, D.J. Toms, Quantum Field Theory in Curved Spacetime, (2009 - Cambridge).

Lecture 1.

GR and its limits of applicability, Planck scale.
Quantum gravity and semi-classical approach.

GR and singularities.

Dimensional approach and Planck scale.

Quantum gravity and/or string theory.

Quantum Field Theory in curved space and its importance.

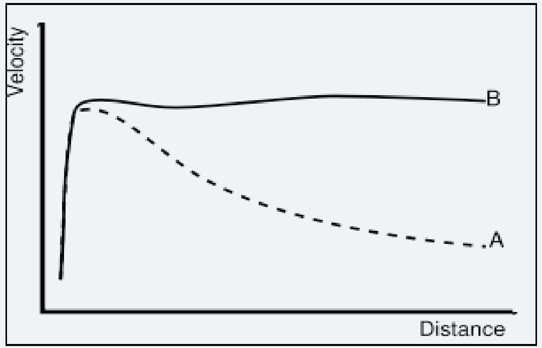
Formulation of classical fields in curved space.

Quantum theory with linearized parametrization of gravity.

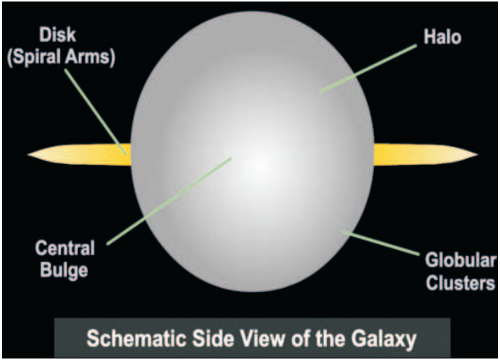
Classical Gravity – Newton's Law,

$$\vec{F}_{12} = -\frac{G\,M_1\,M_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} \quad \textbf{or} \quad U(r) = -G\,\frac{M_1\,M_2}{r}\,.$$

Newton's law work well from laboratory up to the galaxy scale.



For galaxies one needs, presumably, to introduce a HALO of Dark Matter, which consists from particles of unknown origin,



or modify the Newton's law - MOND,

$$F = F(\vec{r}, \vec{v})\,.$$

The real need to modify Newton gravity was because it is not relativistic while the electromagnetic theory is.

- Maxwell 1868 ... • Lorentz 1895 ... • **Einstein 1905**

Relativity: instead of **space + time**, **there is a unique space-time** M_{3+1} **(Minkowski space)**. Its coordinates are

$$x^\mu = (ct, x, y, z) .$$

The distances (intervals) are defined as

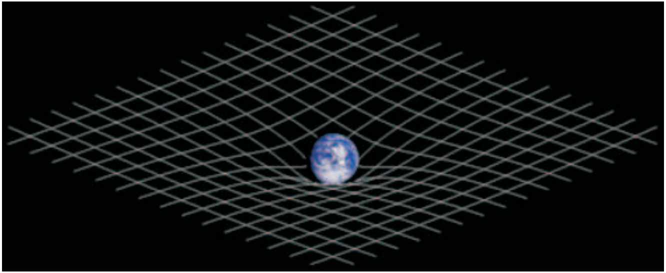
$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

How to incorporate gravity?

The Minkowski space is flat, as the surface of a table.

GR (A. Einstein, 1915): Gravitation = space-time metric.

- **Geometry shows matter how to move.**
- **Matter shows space how to curve.**



General Relativity and Quantum Theory

General Relativity (GR) is a complete theory of classical gravitational phenomena. It proved to be valid in the wide range of energies and distances.

The basis of the theory are the principles of equivalence and general covariance.
There are covariant equations for the matter (fields and particles, fluids etc) and Einstein equations for the gravitational field $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \, T_{\mu\nu} - \Lambda \, g_{\mu\nu} \,.$$

We have introduced Λ, cosmological constant (CC) for completeness.

The most important solutions of GR have specific symmetries.

1) Spherically-symmetric solution. Planets, Stars, Black holes.
2) Isotropic and homogeneous metric. Universe.

Spherically-symmetric solution of Schwarzschild.

This solution corresponds to the spherical symmetry in the static mass distribution and in the classical solution.
The metric may depend on the distance r and time t, but not on the angles φ and θ.

For the sake of simplicity we suppose that there is a point-like mass in the origin of the spherical coordinate system. The solution can be written in the standard Schwarzschild form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - r_g/r} - r^2 d\Omega\,.$$

where $r_g = 2GM$.

Performing a $1/r$ expansion we arrive at the Newton potential

$$\varphi(r) = -\frac{GM}{r} + \frac{G^2M^2}{2r^2} + \dots$$

Schwarzschild solution has two singularities:
At the gravitational radius $r_g = 2GM$ and at the origin $r = 0$.

The first singularity is coordinate-dependent, indicating the existence of the horizon.

Light or massive particles can not propagate from the interior of the black hole to an outside observer. The $r = r_g$ horizon looks as singularity only if it is observed from the “safe” distance.

An observer can change his coordinate system such that no singularity at $r = r_g$ will be observed.

On the contrary, $r = 0$ singularity is physical and indicates a serious problem.

Indeed, the Schwarzschild solution is valid only in the vacuum and we do not expect point-like masses to exist in the nature. The spherically symmetric solution inside the matter does not have singularity.

However, the object with horizon may be formed as a consequence of the gravitational collapse, leading to the formation of physical singularity at $r = 0$.

After all, assuming GR is valid at all scales, we arrive at the situation when the $r = 0$ singularity becomes real.

Then, the matter has infinitely high density of energy, and curvature invariants are also infinite. Our physical intuition tells that this is not a realistic situation.

Something must be modified.

Standard cosmological model

Another important solution of GR is the one for the homogeneous and isotropic metric (FLRW solution).

$$ds^2 = dt^2 - a^2(t) \cdot \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right) ,$$

Here r is the distance from some given point in the space (for homogeneous and isotropic space-time. The choice of this point is not important). $a(t)$ is the unique unknown function,

$k = (0, 1, -1)$ **defines the geometry of the space section M^3 of the 4-dimensional space-time manifold M^{3+1}.**

Consider only the case of the early universe, where the role of k and Λ is negligible and the radiation dominates over the matter.

Radiation-dominated epoch

is characterized by the dominating radiation with the relativistic relation between energy density and pressure $p = \rho/3$ and $T^\mu_\mu = 0$. Taking $k = \Lambda = 0$, we meet the Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3}\,\frac{\rho_0 a_0^4}{a^4}, .$$

Solving it, we arrive at the solution

$$a(t) = \Big(\frac{4}{3}\cdot 8\pi G\rho_0 a_0^4\Big)^{1/4} \times \sqrt{t}\,,$$

This expression becomes singular at $t \to 0$. Also, in this case the Hubble constant

$$H = \dot{a}/a = \frac{1}{2t}$$

also becomes singular, along with ρ_r and with components of the curvature tensor.

The situation is qualitatively similar to the black hole singularity.

Applicability of GR

The singularities are significant, because they emerge in the most important solutions, in the main areas of application of GR.

Extrapolating backward in time we find that the use of GR leads to a problem, while at the late Universe GR provides a consistent basis for cosmology and astrophysics. The most natural resolution of the problem of singularities is to assume that

- **GR is not valid at all scales**.

At the very short distances and/or when the curvature becomes very large, the gravitational phenomena must be described by some other theory, more general than the GR.

But, due to success of GR, we expect that this unknown theory coincides with GR at the large distance & weak field limit.

The most probable origin of the deviation from the GR are quantum effects.

Need for quantum field theory in curved space-time.

Let us use the dimensional arguments.

The expected scale of the quantum gravity effects is associated to the Planck units of length, time and mass. The idea of Planck units is based on the existence of the 3 fundamental constants:

$$c = 3 \cdot 10^{10}\, cm/s\,,$$

$$\hbar = 1.054 \cdot 10^{-27}\, erg \cdot sec\,;$$

$$G = 6.67 \cdot 10^{-8}\, cm^3/sec^2\, g\,.$$

One can use them uniquely to construct the dimensions of

length $\quad l_P = G^{1/2}\,\hbar^{1/2}\,c^{-3/2} \,\approx\, 1.4 \cdot 10^{-33}\, cm;$

time $\quad t_P = G^{1/2}\,\hbar^{1/2}\,c^{-5/2} \,\approx\, 0.7 \cdot 10^{-43}\, sec;$

mass $\quad M_P = G^{-1/2}\,\hbar^{1/2}\,c^{1/2} \,\approx\, 0.2 \cdot 10^{-5} g \,\approx\, 10^{19}\, GeV\,.$

One can use these fundamental units in different ways.

In particle physics people use to set $c = \hbar = 1$ and measure everything in GeV. Indeed, for everyday life it may not be nice.

E.g., you have to schedule the meeting "just $10^{27}\, GeV^{-1}$ from now", but " 15 minutes" may be appreciated better.

However, in the specific area, when all quantities are (more or less) of the same order of magnitude, GeV units are useful.

One can measure Newton constant G in GeV.
Then $G = 1/M_P^2$ and $t_P = l_P = 1/M_P$.

Now, why do not we take M_P as a universal measure for everything? Fix $M_P = 1$, such that $G = 1$. Then everything is measured in the powers of the Planck mass M_P.

"20 grams of butter" $\equiv$ "10^6 of butter"

Warning: sometimes you risk to be misunderstood !!

Status of QFT in curved space

One may suppose that the existence of the fundamental units indicates fundamental physics at the Planck scale.

It may be Quantum Gravity, String Theory ...
We do not know what it really is.

So, which concepts are certain?

Quantum Field Theory and Curved space-time definitely are.

Therefore, our first step should be to consider QFT of matter fields in curved space.

Different from quantum theory of gravity, QFT of matter fields in curved space is renormalizable and free of conceptual problems.

However, deriving many of the most relevant observables is yet an unsolved problem.

Formulation of classical fields on curved background

• We impose the principles of locality and general covariance.

• Furthermore, we require the symmetries of a given theory (specially gauge invariance) in flat space-time to hold for the theory in curved space-time.

• It is also natural to forbid the introduction of new parameters with the inverse-mass dimension.

These set of conditions leads to a simplest consistent quantum theory of matter fields on the classical gravitational background.

• The form of the action of a matter field is fixed except the values of a few parameters which remain arbitrary.

• The procedure which we have described above, leads to the so-called non-minimal actions.

Along with the nonminimal scheme, there is a more simple, minimal one. According to it one has to replace

$$\partial_\mu \to \nabla_\mu\,, \quad \eta_{\mu\nu} \to g_{\mu\nu}\,, \quad d^4x \to d^4x\sqrt{-g}\,.$$

Below we consider the fields with spin zero (scalar), spin $1/2$ (Dirac spinor) and spin 1 (massless vector).

The actions for other possible types of fields (say, massive vectors or antisymmetric $b_{\mu\nu}$, spin $3/2$, etc), can be constructed using the same approach.

Scalar field

The minimal action for a real scalar field is

$$S_0 = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \, \partial_\nu\varphi - V_{min}(\varphi) \right\} ,$$

where $$V_{min}(\varphi) = -\frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4$$

is a minimal potential term.

The possible nonminimal structure is

$$S_{non-min} = \frac{1}{2} \int d^4x \sqrt{-g} \, \xi \, \varphi^2 R \, .$$

The new quantity ξ is called nonminimal parameter.

Since the non-minimal term does not have derivatives of the scalar field, it should be included into the potential term, and thus we arrive at the new definition of the classical potential.

$$V(\varphi) = -\frac{1}{2} \left(m^2 + \xi R\right) \varphi^2 + \frac{f}{4!} \varphi^4 \, .$$

In case of the multi-scalar theory the nonminimal term is

$$\int d^4x\sqrt{-g}\,\xi_{ij}\;\varphi^i\varphi^j\,R\,.$$

Further non-minimal structures involving scalar are indeed possible, for example

$$\int R^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\,.$$

However, these structures include constants of inverse mass dimension, therefore do not fit the principles declared above.

In fact, these terms are not necessary for the construction of consistent quantum theory.

Along with the non-minimal term, our principles admit some terms which involve only metric. These terms are conventionally called "vacuum action" and their general form is the following

$$S_{vac} \;=\; S_{EH} \;+\; S_{HD}$$

where $$S_{EH} \;=\; \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R+2\Lambda\}\,.$$

is the Einstein-Hilbert action with the CC

S_{HD} includes higher derivative terms. The most useful form is

$$S_{HD} = \int d^4x\sqrt{-g}\,\big\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\big\}\,,$$

where $$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2$$

is the square of the Weyl tensor in $n=4$,

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is the integrand of the n=4 Gauss-Bonnet topological invariant.

In $n = 4$ case some terms in the action

$$S_{vac} = S_{EH} + S_{HD}$$

gain very special properties.

S_{HD} **includes a conformal invariant $\int C^2$, topological and surface terms, $\int E$ and $\int \Box R$.**

The last two terms do not contribute to the classical equations of motion for the metric.

Moreover, in the FRW case $\int C^2 = const$ and only $\int R^2$ is relevant!

However, as we shall see later on, all these terms are important, for they contribute to the dynamics at the quantum level, e.g., through the conformal anomaly.

The basis E, C^2, R^2 is, in many respects, more useful than $R^2_{\mu\nu\alpha\beta}$, $R^2_{\alpha\beta}$, R^2, and that is why we are going to use it here.

For the Dirac spinor the minimal procedure leads to the expression

$$S_{1/2} = i \int d^4x \sqrt{-g} \,\left(\bar{\psi}\, \gamma^\alpha \, \nabla_\alpha \psi - im\, \bar{\psi}\psi \right) ,$$

where γ^μ and ∇_μ are γ-matrices and covariant derivatives of the spinor in curved space-time.

Let us define both these objects.

The definition of γ^μ requires the tetrad (vierbein)

$$e^\mu_a \cdot e^{\nu a} = g^{\mu\nu} \,, \quad e^a_\mu \cdot e^{\mu b} = \eta^{ab} \,.$$

Now, we set $\gamma^\mu = e^\mu_a \, \gamma^a$, where γ^a is usual (flat-space) γ-matrix.

The new γ -matrices satisfy Clifford algebra in curved space-time

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} .$$

The covariant derivative of a Dirac spinor $\nabla_\alpha \psi$ should be consistent with the covariant derivative of tensors. We suppose

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{2}\, w_\mu^{ab}\, \sigma_{ab}\, \psi \,,$$

w_μ^{ab} **is usually called spinor connection and**

$$\sigma_{ab} = \frac{i}{2}\, \left(\gamma_a \gamma_b - \gamma_b \gamma_a\right).$$

The conjugated expression is

$$\nabla_\mu\, \bar{\psi} = \partial_\mu\, \bar{\psi} - \frac{i}{2}\, \bar{\psi}\; w_\mu^{ab}\; \sigma_{ab}\,.$$

In order to establish the form of the spinor connection, consider the covariant derivative acting on the vector $\bar{\psi}\gamma^\alpha\psi$.

$$\nabla_\mu(\bar{\psi}\gamma^\alpha\psi) = \partial_\mu(\bar{\psi}\gamma^\alpha\psi) + \Gamma^\alpha_{\mu\lambda}\bar{\psi}\gamma^\lambda\psi\,.$$

The solution has the form

$$w_{\mu ab} \;=\; \frac{1}{2}\, \Big(e_{\alpha[b}\partial_\mu\, e^\alpha_{a]} \;+\; \Gamma^\alpha_{\lambda\mu}\, e_{\alpha[b}\, e^\lambda_{a]}\Big)\;.$$

The minimal generalization for massless Abelian vector field A_μ is straightforward

$$S_1 = \frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu}\,,$$

where $\quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$

In the non-Abelian case we have very similar structure.

$$A_\mu \to A^a_\mu\,,$$

$$F_{\mu\nu} \to G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\,.$$

In both Abelian and non-Abelian cases the minimal action keeps the gauge symmetry. The non-minimal covariant terms for spins 1/2 **and** 1 **have inverse mass dimension and the vacuum terms are the same as before.**

Interaction with external gravity does not spoil gauge invariance of a fermion or charged scalar coupled to a gauge field. Also, the Yukawa interaction can be obtained via the minimal procedure, $\int d^4x\sqrt{-g}\varphi\bar{\psi}\psi$.

The quantization in curved space can be performed by means of the path integral approach.

The generating functional of the connected Green functions $W[J, g_{\mu\nu}]$ is defined as

$$e^{iW[J,g_{\mu\nu}]} = \int d\Phi\, e^{iS[\Phi,g]+i\Phi J}\,,$$

$d\Phi$ is the invariant measure of the functional integral and $J(x)$ are independent sources for the fields $\Phi(x)$.

- **The classical action is replaced by the Effective Action (EA)**

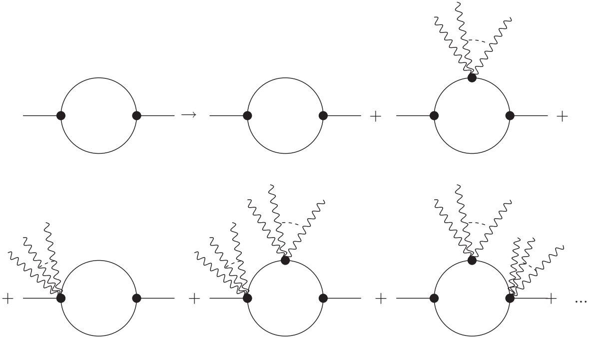
$$\Gamma[\Phi, g_{\mu\nu}] = W[J(\Phi), g_{\mu\nu}] - J(\Phi)\cdot\Phi\,, \qquad \Phi = \frac{\delta W}{\delta J}\,,$$

which depends on the mean fields Φ and on $g_{\mu\nu}$.

The QFT in curved space, as it is formulated above, is renormalizable and consistent.

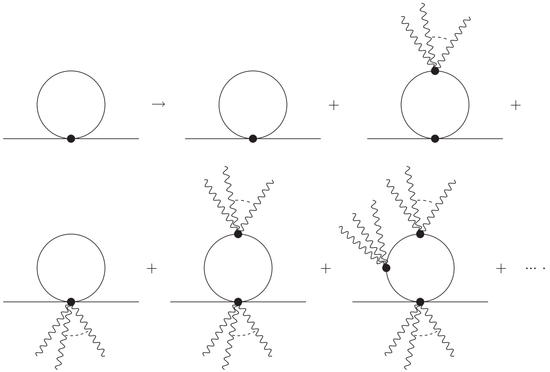
The main difference with QFT in flat space is that in curved space EA depends on the background metric, $\Gamma[\Phi, g_{\mu\nu}]$

In terms of Feynman diagrams, one has to consider graphs with internal lines of matter fields & external lines of both matter and metric. In practice, one can consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.



An important observation is that
all those “new” diagrams with $h_{\mu\nu}$ legs have superficial degree of divergence equal or lower that the “old” flat-space diagrams.

Consider the case of scalar field which shows why the nonminimal term is necessary



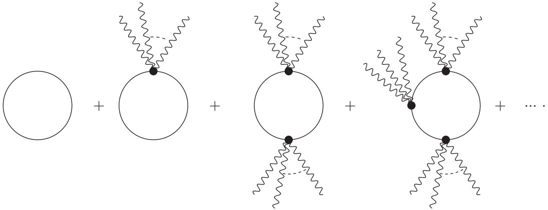
In general, the theory in curved space can be formulated as renormalizable. One has to follow the prescription

$$S_t = S_{min} + S_{non.min} + S_{vac} \,.$$

Renormalization involves fields and parameters like couplings and masses, ξ and vacuum action parameters.

Introduction: *Buchbinder, Odintsov & I.Sh. (1992).*

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the same structure as

$$S_{vac} = S_{EH} + S_{HD}$$

More observations about higher derivatives

The consistent theory can be achieved only if we include

$$S_{HD} = \int d^4x \sqrt{-g}\,\left\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right\},$$

$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2$ **is the square of the Weyl tensor.**

In quantum gravity such a HD term means massive ghost, the gravitational spin-two particle with negative kinetic energy. This leads to the problem with unitarity, at least at the tree level.

One can achive unitary and superrenormalizable quantum gravity by constructing a theory with complex poles only.

L. Modesto, I.Sh, arXiv:1512.07600.

On another side, real or complex ghosts provide removal of $r = 0$ Newtonian singularity, e.g., in the four-derivative gravity,

$$\varphi(r) = -GM\left(\frac{1}{r} - \frac{4}{3}\frac{e^{-m_{(2)}r}}{r} + \frac{1}{3}\frac{e^{-m_{(0)}r}}{r}\right)$$

K. Stelle, Phys. Rev. D 16 (1977) 953.

In the $4+$ derivative gravity there is similar cancelation of Newtonian singularity.

L. Modesto, Tiberio P. Netto, I.Sh. arXiv:1412.0740, JHEP.

In the non-local theory

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\left\{R + G_{\mu\nu}\,\frac{a(\Box)-1}{\Box}\,R^{\mu\nu}\right\}, \quad a(\Box) = e^{-\Box/m^2}\,.$$

there is also a non-singular Newtonian limit

$$\varphi(r) = -\frac{GM}{r}\,\mathrm{erf}\left(\frac{mr}{2}\right)\,.$$

A. Tseytlin, hep-th/9509050, PLB **also** *W. Siegel, hep-th/0309093.*

In the framework of semiclassical theory, gravity is external and unitarity of the gravitational S-matrix is not really important.

The consistency criteria include: physically reasonable solutions and their stability under small perturbations.

Fabris, Pelinson and I.Sh., NPB, hep-th/0009197;
Fabris, Pelinson, Salles and I.Sh., JCAP, arXiv:1112.5202;
F. Salles and I.Sh., PRD, arXiv:1401.4583.

The stability does not actually depend on quantum corrections. It is completely defined by the sign of the classical coefficient a_1 of the Weyl-squared term.

The sign of the Weyl-squared term defines whether graviton or ghost has positive kinetic energy, also whether ghost is also a tachyon etc.

Conclusions

- **QFT of matter fields in curved space-time is definitely a very important object of study, because it concerns real and not completely well-understood physics.**

- **QFT of matter fields in curved space-time can be always formulated as renormalizable theory if the corresponding theory in flat space-time is renormalizable.**

- **The action of QFT of matter fields in curved space-time includes additional non-minimal term in the scalar sector and also higher derivative terms in the vacuum (gravity) sector.**

- **Different from QG, the higher derivative terms do not necessary pose a problem, because we do not need physical interpretation for the gravitational propagator. The issue of stability of classical solutions remains important, of course.**

Elements of QFT in Curved Space-Time

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Lecture 2.

Methods for evaluating quantum corrections: divergent part.

Local momentum representation. Covariance.

Schwinger-DeWitt method. Examples of Renormalization.

Renormalization group.

In curved space the Effective Action (EA) depends on metric

$$\Gamma[\Phi] \to \Gamma[\Phi, g_{\mu\nu}]\,.$$

Feynman diagrams: one has to consider grafs with internal lines of matter fields and external limes of both matter and metric. In practice, one can consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Is it possible to get EA for an arbitrary background in this way? Perhaps not. But it is sufficient to explore renormalization!

An important aspect is that the general covariance in the non-covariant gauges can be shown in the framework of mathematically rigid Batalin-Vilkovisky quantization scheme:

- *P. Lavrov and I.Sh., Phys. Rev. D81 (2010).*

Strong arguments supporting locality of the counterterms follow from the “quantum gravity completion” consideration.

Still, it would be very nice to have an explicitly covariant method of deriving counterterms at all loop orders.

Riemann normal coordinates.

A.Z.Petrov, Einstein Spaces. (Pergamon Press, 1969).
T.S. Bunch & L. Parker, Phys. Rev. D20 (1979) 2499. + InSpire.

Consider manifold $M_{3,1}$ and choose a point with coordinates x'^{μ}. The normal coordinates $y^{\mu} = x^{\mu} - x'^{\mu}$ satisfy several conditions.

The lines of constant coordinates are geodesics which are completely defined by the tangent vectors

$$\xi^{\mu} = \frac{dx^{\mu}}{d\tau}\Big|_{x'}, \qquad \tau(x') = 0$$

and τ is natural parameter along the geodesic. Moreover, we request that metric at the point x' be the Minkowski one $\eta_{\mu\nu}$. For an arbitrary function $A(x)$

$$A(x' + y) = A' + \frac{\partial A}{\partial y^{\alpha}}\Big| \, y^{\alpha} + \frac{1}{2}\,\frac{\partial^2 A}{\partial y^{\alpha}\partial y^{\beta}}\Big| \, y^{\alpha} y^{\beta} + \dots,$$

where the line indicates $y^{\mu} = 0$.

Direct calculations show that

$$\Gamma^{\lambda}_{\alpha\beta}(x) \,=\, \frac{2}{3}\,{R'}^{\lambda}{}_{(\alpha\beta)\nu}\,y^{\nu} - \frac{1}{2}\,{R'}^{\lambda}{}_{\nu\alpha(\mu\,;\,\beta)}\,y^{\mu}\,y^{\nu} \,+\, ...\,,$$

r.h.s. depends only on the tensor quantities at the point $y = 0$.
From this follows the expansion for the metric

$$g_{\alpha\beta}(y) = \eta_{\alpha\beta} - \frac{1}{3}\,R'_{\alpha\mu\beta\nu}y^{\mu}y^{\nu} - \frac{1}{6}R'_{\alpha\nu\beta\lambda;\mu}y^{\mu}y^{\nu}y^{\lambda} \,+\, ...\,.$$

and $\quad R_{\mu\rho\nu\sigma}(y) = R'_{\mu\rho\nu\sigma} + R'_{\mu\rho\nu\sigma;\lambda}y^{\lambda} + ...$

The most fortunate feature of these series is that coefficients are curvature tensor and its covariant derivatives at one point $y = 0$.

We gain a tool for deriving local quantities, e.g., counterterms. The covariance is guaranteed by construction!

The procedure is as follows:

- **Introduce local momentum representation at the point** $y = 0$.
- **Develop Feynman technique in the momentum space.**
- **Calculate diagrams with the new propagators and vertices.**

Everything is manifestly covariant with respect to the transformations in the point x'.

Example. Scalar field propagator. The bilinear operator

$$\hat{H} = -\frac{1}{\sqrt{-g}}\,\frac{\delta^2 S_0}{\delta\varphi(x)\,\delta\varphi(x')}\,.$$

It has the form $\quad \hat{H} = \left(\Box - m^2 - \xi R\right)_x\,.$

Expanding $\hat{H}$ in normal coordinates in $\mathcal{O}(R)$

$$\hat{H} = \partial^2 - m^2 - \xi R + \frac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}y^{\alpha}y^{\beta}\partial_{\mu}\partial_{\nu} - \frac{2}{3}R^{\alpha}_{\beta}y^{\beta}\partial_{\alpha} + \,\dots\,.$$

The equation for the propagator is

$$\hat{H}\,G(x,x') = -\,\delta(x,x')\,.$$

which leads to the following expression:

$$G(k) = \frac{1}{k^2+m^2} + \frac{1}{3}\,\frac{(1-3\xi)R}{(k^2+m^2)^2} - \frac{2}{3}\,\frac{R^{\mu\nu}k_{\mu}k_{\nu}}{(k^2+m^2)^3} + \mathcal{O}\left(\frac{1}{k^3}\right)\,.$$

One can continue this expansion to further orders in curvature.

It is clear that higher orders in an expansion

$$G(k) = \frac{1}{k^2+m^2} + \frac{1}{3}\,\frac{(1-3\xi)R}{(k^2+m^2)^2} - \frac{2}{3}\,\frac{R^{\mu\nu}k_\mu k_\nu}{(k^2+m^2)^3} + \mathcal{O}\left(\frac{1}{k^3}\right)$$

will always produce less divergences when replaced into internal line of the loop Feynman diagram.

The same effect occurs in the expansion in y^α for vertices.

For instance, any divergent diagram in renormalizable flat-space QFT has

$$d + D \leq 4\,,$$

where D is superficial degree of divergence and d is number of derivatives on external lines.

Clearly, the terms with background curvatures will have smaller $d+D$ and the maximal number of metric derivatives in vacuum diagrams is four.

The described method is explicitly covariant.

Combining the information from the two methods

- **Usual Feynman technique with external $h_{\mu\nu}$;**
- **Local momentum representation.**

The necessary counterterms in curved space are covariant local expressions constructed from matter fields and metric.

Consider a theory power-counting renormalizable in flat space.

Using Feynman technique with external $h_{\mu\nu}$ tails we observe an increase of the number of propagators and vertices
$\Longrightarrow$ **superficial degree of divergence decrease.**

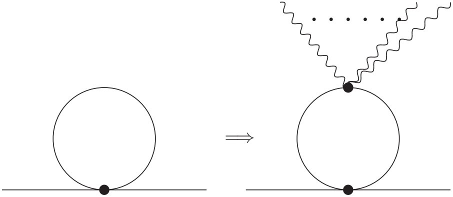
Using local momentum representation: the new terms always have some extra negative powers of momenta k, compensated by the background curvatures and their derivatives
$\Longrightarrow$ **superficial degree of divergence decrease.**

Therefore, independent of the approach, the new counterterms do not have $O(1/mass)$ -factors and the theory remains power-counting renormalizable in curved space.

Types of the counterterms:

- **Minimal, e.g.,** $m^2\varphi^2, \;\; (\nabla\varphi)^2, \;\; i\bar{\psi}\gamma^\mu\nabla_\mu\psi$.
- **Non-minimal in the scalar sector,** $R\varphi^2$.

E.g., the quadratically divergent diagram



in the $\lambda\varphi^4$ **theory produces log. divergences corresponding to** $\int d^4\sqrt{-g}R\varphi^2$ **counterterm.**

- **Vacuum terms** $\Lambda, R, R^2, C^2,$ **etc.**

Renormalization doesn't depend on the choice of the metric!

Renormalization in matter fields sector

It is possible to perform renormalization in curved space in a way similar to the one in flat space.

Counterterms are controlled by symmetries & power counting.

In the simple case of the scalar $\lambda\varphi^4$-theory,

$$S_0 = \int d^4x \sqrt{-g}\left\{ \frac{1}{2}\, g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + \frac{1}{2}\,\left(m^2+\xi R\right)\varphi^2 - \frac{f}{4!}\,\varphi^4 \right\},$$

we meet, in dim. regularization, the following counterterms:

$$\Delta S_{scal} = \int d^nx\,\sqrt{-g}\;\mu^{n-4}\,\sum_{k=1}^{10}\alpha_k\left(\frac{1}{n-4}\right)\times\mathcal{L}_k\,,$$

where $\mathcal{L}_7 = (\nabla\varphi)^2\,,\quad \mathcal{L}_8 = m^2\varphi^2\,,$

$$\underline{\mathcal{L}_9 = R\varphi^2}\,,\quad \mathcal{L}_{10} = \varphi^4\,,\quad \mathcal{L}_{1,\dots,6} = \mathcal{L}_{1,\dots,6}(g_{\mu\nu})\,.$$

$\alpha_k(x)$ **are polynomials of the order equal to the loop order.**

The situation is similar for any theory which is renormalizable in flat space: only $\xi R\varphi^2$ counterterms represent a new element in the matter sector.

Moreover, due to covariance, multiplicative renormalization factors, e.g., Z_1, in

$$\varphi_0 = \mu^{\frac{n-4}{2}}\, Z_1^{1/2}\,\varphi\,,$$

are exactly the same as in the flat space.

The renormalization relations for the scalar mass m and nonminimal parameter ξ have the form

$$m_0^2 = Z_2\, m^2\,, \qquad \xi_0 - \frac{1}{6} = \tilde{Z}_2\, \left(\xi - \frac{1}{6}\right) + Z_3\,.$$

At one loop we have, also,

$$\tilde{Z}_2 = Z_2\,, \qquad Z_3 = 0\,.$$

So, in principle, we even do not need to perform a special calculation of renormalization for ξ at the 1-loop order.

Renormalization in the vacuum sector

Remember the action of vacuum is $S_{vac} = S_{EH} + S_{HD}$,

where
$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, \{R + 2\Lambda\}\,.$$

and
$$S_{HD} = \int d^4x \sqrt{-g}\, \{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

The possible counterterms are:

$$\Delta S_{vac} = \int d^n x\, \sqrt{-g}\, \mu^{n-4} \sum_{k=1}^{6} \hat{\alpha}_k \left(\frac{1}{n-4}\right) \times \hat{\mathcal{L}}_k\,,$$

where $\hat{\mathcal{L}}_\Lambda = 1\,, \quad \hat{\mathcal{L}}_G = R\,, \quad \hat{\mathcal{L}}_1 = C^2\,,$

$$\hat{\mathcal{L}}_2 = E\,, \quad \hat{\mathcal{L}}_3 = \Box R\,, \quad \hat{\mathcal{L}}_4 = R^2\,.$$

$\alpha_k(x)$ **are polynomials of the order equal to the loop order.**

General situation

- The theory such as SM, GUT etc,
which is renormalizable in flat space, can be formulated as renormalizable in curved space

- The action of the theory can be divided into following three sectors:
1. Minimal matter sector;
2. Non-minimal matter sector;
3. Vacuum (metric-dependent) sector.

- The renormalization satisfies the hierarchy

$$1. \Longrightarrow 2. \Longrightarrow 3.$$

In the minimal sector it is identical to the one in flat space.

- The conformal invariance is supposed to hold in the one-loop counterterms, $\xi = 1/6, \quad a_4 = 0.$

Renormalization group equations

Renormalization group (RG) is one of the most efficient methods of Quantum Field Theory, also in Stat. Mechanics.

In QFT there are many versions of RG

- Perturbative RG based on the minimal subtraction scheme of renormalization ($\overline{\mathrm{MS}}$).

- Perturbative RG which is based on a more physical, e.g., momentum subtraction scheme of renormalization.

- Non-Perturbative RG based on the path integral integration over momenta beyond some cut-off (Wilson approach).

- Intermediate approach with the cut-off dependence for the Green functions by Polchinsky.

- The same in the EA formalism, by Wetterich, Morris, Percacci, Reuter et al.

● Consider the standard $\overline{\text{MS}}$-based formalism of RG in curved space. Let us denote Φ the full set of matter fields

$$\Phi = \varphi,\, \psi,\, A$$

and P the full set of parameters: couplings, masses, ξ and vacuum parameters.

The bare action $S_0[\Phi_0, P_0]$ depends on bare quantities, $S[\Phi, P]$ is the renormalized action.

Multiplicative renormalizability:

$$S_0[\Phi_0, P_0] \,=\, S[\Phi, P]\,,$$

(Φ_0, P_0) and (Φ, P) are related by proper renormalization transformation. The generating functionals of the bare and renormalized Green functions are

$$e^{iW_0[J_0]} \,=\, \int d\Phi_0\, e^{i(S_0[\Phi_0,P_0]+\Phi_0\cdot J_0)}\,,$$

$$e^{iW[J]} \,=\, \int d\Phi\, e^{i(S[\Phi,P]+\Phi\cdot J)}\,.$$

The transformation for matter fields is

$$\Phi_0 = \mu^{\frac{n-4}{2}} Z_1^{1/2} \Phi \,.$$

Make this change of variables and denote

$$J_0 = \mu^{\frac{4-n}{2}} Z_1^{-1/2} J \,.$$

Then

$$W_0[J_0] \,=\, W[J] \,.$$

Consequently, for the mean field we meet

$$\bar{\Phi}_0 \,=\, \frac{\delta W[J_0]}{\delta J_0} \,=\, \frac{\delta W[J]}{\delta J} \frac{\delta J}{\delta J_0} \,=\, \mu^{\frac{n-4}{2}} Z_1^{1/2} \bar{\Phi} \,.$$

Finally, for the effective action we find

$$\Gamma_0[\Phi_0, P_0] \,=\, W_0[J_0] - \bar{\Phi}_0 \cdot J_0 \,=\, W[J] - \bar{\Phi} \cdot J \,=\, \Gamma[\Phi, P] \,.$$

S_0 and Γ_0 are 4 - dimensional integrals,
while S and Γ are n - dimensional integrals.

Γ depends on the dimensional parameter μ, while Γ_0 does not depend on μ by construction.

Therefore,

$$\Gamma_0[g_{\alpha\beta},\Phi_0,P_0,4]\,=\,\Gamma[g_{\alpha\beta},\Phi,P,n,\mu]\,,$$

and we arrive at the differential equation

$$\mu\frac{d}{d\mu}\,\Gamma[g_{\alpha\beta},\Phi,P,n,\mu]\,=\,0\,.$$

Taking into account the possible μ- dependence of P and Φ we recast this equation into

$$\Big\{\,\mu\frac{\partial}{\partial\mu}+\mu\frac{dP}{d\mu}\,\frac{\partial}{\partial P}+\int d^nx\,\mu\frac{d\Phi(x)}{d\mu}\,\frac{\delta}{\delta\Phi(x)}\,\Big\}\,\Gamma[g_{\alpha\beta},\Phi,P,n,\mu]\,=\,0\,.$$

We define, as in flat space-time

$$\beta_P(n) \,=\, \mu\frac{dP}{d\mu}\,, \qquad\qquad \beta_P(4) \,=\, \beta_P$$

$$\gamma_\Phi(n) \,=\, \mu\frac{d\Phi}{d\mu}\,, \qquad\qquad \gamma_\Phi(4) \,=\, \gamma_\Phi\,.$$

Then, the RG equation is cast in the form

$$\left\{\mu\frac{\partial}{\partial\mu} + \int_{x,n}\gamma_\Phi(n)\,\frac{\delta}{\delta\Phi} \,+\, \beta_P(n)\,\frac{\partial}{\partial P}\,\right\}\Gamma[g_{\alpha\beta},\Phi,P,n,\mu] \,=\, 0\,.$$

This is the general RG equation which can be used for different purposes, depending on the physical interpretation of μ.

Here $\displaystyle\int_{x,n} \,=\, \int d^nx\sqrt{-g}$ **and** $\displaystyle\int_x \,=\, \int_{x,4}\,.$

Short distance limit.

Perform a global rescaling of quantities according to their dimension

$$\Phi \to \Phi k^{-d_\Phi}, \quad P \to P k^{-d_P}, \quad \mu \to k\mu, \quad l \to k^{-1} l.$$

The effective action Γ does not change.

Since Γ does not depend on x^μ explicitly, one can replace $l \to l \times k^{-1}$ by the transformation of the metric $g_{\mu\nu} \to k^2 g_{\mu\nu}$.

Then, in addition to RG, we meet an identity

$$\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] \,=\, \Gamma[k^2 g_{\alpha\beta}, k^{-d_\Phi}\Phi, k^{-d_P} P, n, k^{-1}\mu]\,,$$

whereas the curvatures transform as

$$R^2_{\mu\nu\alpha\beta} \sim k^{-4}, \quad R^2_{\alpha\beta} \sim k^{-4}, \quad R \sim k^{-4}\,.$$

Replace $k = e^{-t}$,

$$\frac{d}{dt}\,\Gamma[e^{2t}g_{\alpha\beta}, e^{-d_\Phi t}\Phi, e^{d_P t}P, n, e^{-t}\mu] = 0\,.$$

For $t = 0$ **we meet**

$$\Big\{ \int d^n x \,\Big(2g_{\alpha\beta}\,\frac{\delta}{\delta\, g_{\alpha\beta}} - d_\Phi\,\frac{\delta}{\delta\Phi}\Big) - d_P\,\frac{\partial}{\partial P} - \mu\frac{\partial}{\partial\mu}\Big\}\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] = 0.$$

Together with the RG equation it gives the solution

$$\Gamma[g_{\alpha\beta}e^{-2t}, \Phi, P, n, \mu] \;=\; \Gamma[g_{\alpha\beta}, \Phi(t), P(t), n, \mu]\,,$$

where $P(t)$ **and** $\Phi(t)$ **satisfy RG equations for "effective charges"**

$$\frac{d\Phi}{dt} \;=\; (\gamma_\Phi - d_\Phi)\Phi\,, \qquad\qquad \frac{dP}{dt} \;=\; \beta_P - Pd_P\,.$$

The limit $t \to \infty$ means, the limit of short distances and great curvatures.

It is equivalent to the standard rescaling of momenta in the flat-space QFT.

However, one has to be careful!

The time-dependence of the metric is very similar to the rescaling (we denote time as τ in order to avoid confusion)

$$g_{\alpha\beta} \,\to\, g_{\alpha\beta} \cdot e^{H\tau}\,, \qquad \text{where} \qquad H = const\,.$$

However, this situation does not correspond to the RG, because scalar curvature remains constant $R = -12H^2$.

For the most interesting physical applications we need some special scale-setting procedure, to associate μ with some physically relevant quantity (lecture IV - seminar).

What are the terms in the EA which are behind the RG?

An example of finite (nonlocal) corrections (factor $1/64\pi^2$,)

$$\mathcal{L}_{eff} = C_{\mu\nu\alpha\beta}\left[\frac{1}{60\epsilon} + \frac{8Y}{15a^4} + \frac{2}{45a^2} + \frac{1}{150}\right] C^{\mu\nu\alpha\beta}$$

$$+\lambda\phi^2\left[\frac{Y(a^2-4)}{12a^2} - \frac{1}{36} - \left(\frac{1}{2\epsilon} - Y\right)\left(\xi - \frac{1}{6}\right)\right] R + \dots,$$

where $$\frac{1}{\epsilon} = \frac{1}{2-\omega} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - \gamma\,,$$

$$Y = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2}\,.$$

One can get a full form of the Appelquist and Carazzone theorem for gravity out of these expressions.

Gorbar & Sh. JHEP, 2003, hep-ph/0210388, 0303124; 0311190; Gorbar, Berredo-Peixoto & Sh. 2005, and others.

Schwinger-DeWitt technique

is the most useful method for practical 1-loop calculations.

Consider the typical form of the operator

$$\hat{\mathbf{H}} \,=\, \hat{1}\Box + \hat{\Pi} + \hat{1}m^2\,.$$

It depends on the metric and maybe other external parameters (via $\hat{\Pi}$). The one-loop EA is given by the expression

$$\frac{i}{2}\,\mathrm{Tr}\,\ln\hat{\mathbf{H}}\,.$$

Let us perform variation with respect to the external parameters.

$$\frac{i}{2}\,\delta\,\mathrm{Tr}\,\ln\hat{\mathbf{H}} \,=\, \frac{i}{2}\,\mathrm{Tr}\,\hat{\mathbf{H}}^{-1}\,\delta\,\hat{\mathbf{H}}\,.$$

The Schwinger proper-time representation for the propagator

$$\hat{\mathbf{H}}^{-1} \,=\, \int_0^\infty ids\, e^{-is\,\hat{\mathbf{H}}}\,.$$

Then, we transform $$\delta\,\hat{\mathbf{H}}\cdot\int_0^\infty ids\, e^{-is\,\hat{\mathbf{H}}} \,=\, \delta\,\int_0^\infty \frac{ds}{i\,s}\;e^{-is\,\hat{\mathbf{H}}}\,.$$

After all, we arrive at

$$\frac{i}{2}\,\mathrm{Tr}\,\log\hat{\mathbf{H}} = const - \frac{i}{2}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}\,e^{-is\,\hat{\mathbf{H}}}\,,$$

where constant term can be disregarded.

The next step is to introduce

$$\hat{U}(x,x';s) = e^{-is\,\hat{\mathbf{H}}}$$

$\hat{\mathbf{H}}$ **acts on the covariant** δ **-function and it proves useful to define**

$$\hat{U}_0(x,x';s) = \frac{\mathcal{D}^{1/2}(x,x')}{(4\pi i\, s)^{n/2}}\,\exp\left\{\frac{i\sigma(x,x')}{2s} - m^2 s\right\}.$$

$\sigma(x,x')$ **- geodesic distance between** x **and** x'. **It satisfies an identity** $2\sigma = (\nabla\sigma)^2 = \sigma^\mu\sigma_\mu$ **.**

$\mathcal{D}$ **is the Van Vleck-Morett determinant**

$$\mathcal{D}(x,x') = \det\left[-\frac{\partial^2\sigma(x,x')}{\partial x^\mu\,\partial x'^\nu}\right]\,,$$

which is a double tensor density, with respect to x **and** x'.

A useful representation for the evolution operator $\hat{U}(x,x';s)$ is

$$\hat{U}(x,x';s) = \hat{U}_0(x,x';s)\sum_{k=0}^{\infty}(is)^k\,\hat{a}_k(x,x'),$$

$\hat{a}_k(x,x')$ **are Schwinger-DeWitt coefficients.**

The evolution operator satisfies the equation

$$i\,\frac{\partial\hat{U}(x,x';s)}{\partial s} = -\hat{\mathbf{H}}\hat{U}(x,x';s), \qquad U(x,x';0) = \delta(x,x').$$

Using these relations one can construct the equation for the coefficients $\hat{a}_k(x,x')$:

$$\sigma^\mu\nabla_\mu\hat{a}_0 = 0,$$

$$(k+1)\hat{a}_{k+1}+\sigma^\mu\nabla_\mu\hat{a}_{k+1} = \Delta^{-1/2}\Box(\Delta^{1/2}\hat{a}_k)+\hat{\Pi}\hat{a}_k, \qquad k = 1,2,3,\dots.$$

It is sufficient to know the coincidence limits

$$\lim_{x\to x'}\hat{a}_k(x,x').$$

If we consider more general operator

$$S_2 \;=\; \hat{H} \;=\; \hat{1}\Box + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi}\,,$$

the linear term can be indeed absorbed into the covariant derivative $\nabla_\mu \to \mathcal{D}_\mu = \nabla_\mu + \hat{h}_\mu$.

The commutator of the new covariant derivatives will be

$$\hat{S}_{\mu\nu} \;=\; \hat{R}_{\mu\nu} - (\nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu) - (\hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu)$$

and we arrive at

$$\hat{a}_1\Big| \;=\; \hat{a}_1(x,x) \;=\; \hat{P} \;=\; \hat{\Pi} + \frac{\hat{1}}{6}\,R - \nabla_\mu \hat{h}^\mu - \hat{h}_\mu \hat{h}^\mu\,.$$

and

$$\hat{a}_2\Big| \;=\; \hat{a}_2(x,x) \;=\; \frac{\hat{1}}{180}\,(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R)$$

$$+\,\frac{1}{2}\,\hat{P}^2 + \frac{1}{6}\,(\Box \hat{P}) + \frac{1}{12}\,\hat{S}^2_{\mu\nu}\,.$$

The great advantage of these expressions is their universality. They enable to analyze EA in various QFT models.

In 4-dimensional space-time $\hat{a}_2 \Big|$ **logarithmic divergences, while** $\hat{a}_1 \Big|$ **defines quadratic divergences.**

The derivation of the "magic" coefficient

$$a_2 \quad \equiv \quad \mathrm{Tr}\; \hat{a}_2 \Big|$$

is, in many cases, the most important thing.
The divergent part of EA, in the dimensional regularization, is

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{-g}\; \mathrm{tr}\, \hat{a}_2(x,x)\,, \quad \textbf{where} \quad \epsilon = (4\pi)^2(n-4)\,.$$

The last formula is a very powerful tool for deriving the divergences in the models of field theory in flat and curved space-times or even in Quantum Gravity.

Sometimes it has to be modified, for example the sign gets changed for a fermionic case.

In complicated cases we need the generalized Schwinger-DeWitt technique (Barvinsky & Vilkovisky, 1985).

Further coefficients $\hat{a}_k$, $k \geq 3$ correspond to the finite part.

They are given by the expressions like

$$\frac{1}{m^2}\,\mathcal{O}(R^3)\,, \qquad \frac{1}{m^2}\,R_{\mu\nu}\Box R^{\mu\nu}\,,\ldots \qquad (a_3 \text{ case})$$

and therefore contribute only to the finite part of EA.

Practical calculation of the coefficients $\hat{a}_k$, $k \geq 3$ is possible, despite rather difficult.

The $\hat{a}_3$ coefficient has been derived by Gilkey (1979) and by Avramidy (1986), who also derived $\hat{a}_4$ coefficient.
In 1989-1990 *I. Avramidy* **and** *A. Barvinsky & G.V. Vilkovisky* **derived important resummation of the Schwinger-DeWitt series.**
As an important application one can obtain, for massive theories, the exact one-loop form factors of the terms

$$R^2\,, \quad C^2\,, \quad F_{\mu\nu}^2\,, \quad (\nabla\phi)^2\,, \quad \phi^4\,.$$

E.Gorbar, I.Sh., G.de Berredo-Peixoto, B.Gonçalves,
JHEP (2003); CQG (2005); PRD (2009).

In the EA $\Gamma[\Phi, g_{\mu\nu}]$ one can separate the part $\Gamma[g_{\mu\nu}]$ which doesn't depend on matter fields.

It corresponds to the Feynman diagrams, with the internal lines of matter fields and the external lines of the metric only.

$\Gamma[g_{\mu\nu}]$ is called the EA of vacuum. It is the most important part of EA, as far as gravitational applications are concerned.

Path integral representation of the vacuum EA

$$e^{i\Gamma_{vac}[g_{\mu\nu}]} = \int d\Phi \; e^{iS[\Phi;\, g_{\mu\nu}]}\,.$$

Here Φ is the set of all matter fields and gauge ghosts.
Γ_{vac} admits a loop expansion, at the tree level it is equal to S_{vac}.
Already at this level one can make some strong statements about possible and impossible form of quantum corrections.

Consider one-loop divergences for the free fields, scalars, spinors and massless vectors in curved space-time.

Scalar field. N_s**-component case.**

$$\hat{H} \,=\, \delta^i_j \,\left(\Box - m_s^2 - \xi R\,\right)_x\,, \quad \textbf{where} \quad i,j = 1,2,...,N_s.$$

The identification with the general expression

$$\hat{H} \,=\, \hat{1}\Box + 2\hat{h}^\mu\nabla_\mu + \hat{\Pi} \quad \textbf{gives} \quad \hat{h}^\mu = 0\,, \quad \hat{\Pi} = -\delta^i_j\,(m_s^2 + \xi R).$$

$$\textbf{Then,} \quad \hat{S}_{\mu\nu} = 0 \quad \textbf{and} \quad \hat{P} \,=\, \delta^i_j\,\left[\left(\xi - \frac{1}{6}\right)R - m_s^2\right]\,.$$

Finally,

$$\bar{\Gamma}^{(1)}_{div} = -\frac{N_s\;\mu^{n-4}}{\epsilon}\int d^nx\sqrt{-g}\Big\{\frac{1}{2}\,m_s^4\,+\,m_s^2\Big(\xi - \frac{1}{6}\Big)R$$

$$+\,\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2 R^2 + \frac{1}{180}\Big(R^2_{\mu\nu\alpha\beta}-R^2_{\alpha\beta}\Big)-\frac{1}{6}\Big(\xi-\frac{1}{5}\Big)\Box R\Big\}\,.$$

For a complex scalar field, the divergent part of the EA is twice of the previous result. This is nothing but the overall factor N_s.

In general, free fields give additional and independent contributions to the vacuum divergences.

In the $n = 4$ conformal case $m_s = 0,\ \xi = 1/6$

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{360\epsilon}\int d^n x\sqrt{-g}\,\left\{3C^2 - E + 2\Box R\right\}.$$

Both classical action

$$S_0^c = \int d^4 x\,\sqrt{-g}\left\{\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + \frac{1}{12}\,R\varphi^2\right\}$$

and the log. divergence are conformal invariant

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}\,e^{2\sigma(x)}\,, \qquad \varphi \longrightarrow \varphi\,e^{-\sigma(x)}\,.$$

In the conformal scalar case the pole terms are conformal invariant or surface structures.

!! This result holds only in certain regularizations and may be violated in others.

Spinor field. **We meet another operator**

$$\hat{H} = i\,\big(\,\gamma^\alpha \nabla_\alpha - i\,m_f\,\big)\,.$$

The 1-loop EA is

$$\bar{\Gamma}^{(1)} = -\frac{i}{2}\,\mathrm{Tr}\,\log \hat{H}\,.$$

The sign change is due to the odd Grassmann parity of the fermion field, while Tr **is taken in the usual "bosonic" way.**

After some algebra we arrive at the following expression

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{\epsilon}\int d^n x\sqrt{-g}\,\Big\{\,\frac{m_f^2}{3}R - 2m_f^4 + \frac{1}{20}C^2(4) - \frac{11}{180}E + \frac{1}{30}\Box R\Big\}\,.$$

Again, in the conformal case $m_f = 0$ **we meet only the conformal-invariant counterterms.**

Vector field

In the massless case we do not need to distinguish Abelian and non-Abelian vectors, since only the free parts are important.

Consider a single Abelian vector. The action must be supplemented by the gauge fixing and ghost terms.

The 1-loop contribution to the vacuum EA

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\mathrm{Tr}\,\log\hat{H} - i\,\mathrm{Tr}\,\log\hat{H}_{gh}\,,$$

$\hat{H}$ **and** $\hat{H}_{gh}$ **are bilinear forms of the field and ghost actions.**

The divergent part is

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{180\epsilon}\int d^4x\sqrt{-g}\,\big\{18(C^2 - \Box R) - 31\,E\big\}\,.$$

The divergences include conformal-invariant and surface terms.

An example of RG equation.

The divergent part of the EA of vacuum for the theory with N_s scalars, N_f spinors and N_v vectors

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^n x\sqrt{-g}\,\{\beta_{EH}R + \beta_{CC} + \beta_W C^2 + \beta_E E + \beta_{R2}R^2 + \beta_d \Box R\}\,,$$

where $\beta_i = \dfrac{k_i}{(4\pi)^2}$ **and** $k_{CC} = \dfrac{1}{2}m_s^4 - 4m_f^4\,,$

$$k_{EH} = N_s m_s^2\Big(\xi - \frac{1}{6}\Big) + \frac{2N_f m_f^2}{3}\,, \qquad k_{R2} = \frac{N_s}{2}\Big(\xi - \frac{1}{6}\Big)^2\,,$$

$$w = k_W = \frac{N_s}{120} + \frac{N_f}{20} + \frac{N_v}{10}\,,$$

$$b = k_E = -\frac{N_s}{360} - \frac{11\,N_f}{360} - \frac{31\,N_v}{180}\,,$$

$$c = k_\Box = \frac{N_s}{180} + \frac{N_f}{30} - \frac{N_v}{10}\,.$$

Consider the Weyl-squared term.

$$\Delta S_W = \frac{\mu^{n-4}}{\epsilon}\int d^nx\sqrt{-g}\,wC^2\,, \quad w = \frac{N_s}{120} + \frac{N_f}{20} + \frac{N_v}{10}\,.$$

Renormalized action = to the bare one, $S_W(n) + \Delta S_W = S_W^0\,.$

Obviously, this means $a_1^0 = \mu^{n-4}\left(a_1 + \frac{w}{\epsilon}\right)\,.$ Taking

$$0 = \mu\,\frac{da_1^0}{d\mu} = \mu^{n-4}\left[(n-4)\left(a_1 + \frac{w}{\epsilon}\right) + \mu\,\frac{da_1}{d\mu}\right]$$

In this way we arrive at $\mu\,\frac{da_1}{d\mu} = -(n-4)a_1 - \frac{w}{(4\pi)^2}\,.$

or $\beta_W \,=\, \mu\,\frac{da_1}{d\mu}\Big|_{n=4} \,=\, -\,\frac{w}{(4\pi)^2}\,.$

For the coupling parameter $\lambda = -(2a_1)^{-1}$ **we have**

$$\mu\,\frac{d\lambda}{d\mu} \,=\, -\,\frac{w}{2\,(4\pi)^2}\,\lambda^2\,,$$

indicating asymptotic freedom, since in all cases $w > 0$.

In a similar way one can derive RG equations for $a_{2,3,4}$ and also for Λ and G , namely

$$\frac{da_3}{dt} = \mu\,\frac{da_3}{d\mu} = \frac{N_s}{2\,(4\pi)^2}\,\left(\xi - \frac{1}{6}\right)^2\,,$$

$$(4\pi)^2\,\frac{d}{dt}\,\left(\frac{\Lambda}{8\pi G}\right) = \frac{N_s m_s^4}{2} - 2N_f m_f^4\,.$$

$$(4\pi)^2\,\mu\,\frac{d}{d\mu}\,\left(\frac{1}{16\pi G}\right) = \frac{N_s m_s^2}{2}\,\left(\xi - \frac{1}{6}\right) + \frac{N_f m_f^2}{3}\,.$$

These equations describe the short distance behavior of the corresponding effective charges.

However, it is not really clear how to apply them, e.g., to cosmology or to the black hole physics.

Conclusions

- **Plane diagrams and local momentum representation, together, tell us the full story of renormalization in curved space.**

- **The renormalization program is a full success of we are interesting in getting free of divergences.**

- **Perturbative Renormalization Group is formulated without difficulties within Minimal Subtraction scheme.**

- **Unfortunately the problems start right at the point when we need to calculate finite part of EA. For, example, there is no unique interpretation of μ or $t = \ln(\mu/\mu_0)$ for the case of inflation and, in fact, in many other cases.**