Action Principle for Rotational Flows

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ABSTRACT The restriction of hydrodynamics to non-viscous, potential (gradient, irrotational) flows is a theory both simple and elegant; a favorite topic of introductory textbooks on hydrodynamics. It is known that this theory (under the stated limitations) can be formulated as an action principle. It finds its principle application to adiabatic systems and cannot aspire to account for viscosity or dissipation. However, there is no reason why it should not be possible to generalize it to include non-potential flows, including at least some aspects of turbulence. It is the purpose of this paper to provide this generalization.

The theory is a combination of Eulerian and Lagrangian hydrodynamics, with an extension to thermodynamics. It describes adiabatic phenomena and does not account for viscosity or dissipation. Nevertheless, it is an approach within which quasi static processes can be described. In the adiabatic context it appears to be an improvement of the Navier-Stokes equation; the principal advantage being a natural concept of energy in the form of a first integral of the motion, conserved by virtue of the Euler-Lagrange equations.

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I. Introduction

Many branches of theoretical physics have found their most powerful formulation as action principles. An action principle has been proposed for Eulerian hydrodynamics as well, but it has been limited to potential flows and its use was restricted. The purpose of this paper is to formulate an action principle for hydrodynamics, with an extension to thermodynamics, that can encompass general velocity fields.

The Navier-Stokes equation has maintained a dominant position in hydrodynamics for more than 100 years. It has know numerous successes and no real failures (Brenner 2013). It suffers, nevertheless, from an inherent incompatibility with the energy concept. This is of course inevitable, since it is designed to describe processes in which energy is dissipated, but it implies a lack of completeness for which most workers have felt a need to compensate. It is therefore usually supplemented by an equation referred to as an 'energy equation'. But the expression for this 'energy' is seldom canonical and cannot be justified. Dissatisfaction with this approach is occasionally expressed in the literature, as in this example (Khalatnikov (1965). After listing a "complete system of hydrodynamic equations" (Eq.s 2-5) he presents

one more equation, "the energy conservation law $\partial E/\partial t + \mathrm{div}Q = 0$", and then he says: "It is necessary to choose the unknown terms in Eq.s (2-5) in such a way that this last equation be automatically satisfied."

We aim of this work is not to provide an alternative to the Navier-Stokes equation. The proposed action principle describes only adiabatic processes. But a change of state that involves viscosity and other forms of dissipation is best described as a quasi-static evolution along a sequence of adiabatic equilibria; it is evident that such studies must be built on a coherent description of the adiabatic systems themselves, not just their equilibria (Prigogine 1965). The very concept of equilibrium only makes sense in the dynamical context within which it is an equilibrium configuration! It is evidently important to have a convenient formulation of general adiabatic systems and, at best, a formulation in the form of an action principle. The simplest non-adiabatic evolution is a sequence of equilibria of adiabatic systems, parameterized by the entropy.

The Navier-Stokes equation, when restricted to the case of negligible viscosity, provides a description of hydrodynamical systems with general velocity fields. * But perhaps it is not the best approach to adiabatic dynamics. It is in this narrow context of adiabatic dynamics that we shall formulate an action principle. In recognition of the fact that the Navier-Stokes equation is built on very solid physical principles we have tried to incorporate as many of its features as possible; that is, we have tried to retain the physics as well as the technical details that characterize this equation, in an attempt to discover a related variational principle.

The strategy that is pursued in this paper is to study the application of Navier-Stokes to a simple system, in order to learn how it describes physics that is difficult to understand within a variational context. Surprisingly, it turned out that, in this very limited context, the crucial difference is just a difference of sign. This simple fact inspired a resurrection of the 'Lagrangian' formulation of hydrodynamics. The solution is a variational principle that combines two classical formulations of hydrodynamics. It combines potential flows and 'solid-body' flows, just as Navier-Stokes does, but two independent vector fields are needed, one a gradient, the other not.

Hydrodynamics is known in two different formulations, sometimes said to be equivalent. Thus L.I. Sedov (1971), in his celebrated book on Continuous Mechanics, summarizes a detailed comparison between the two approaches by the the following paragraph,

"Clearly, specifications of a motion of a continuum from Lagrangean and Eulerian points of view are in a mechanical sense equivalent."

In the Eulerian version the principal variables are the density (a scalar field) and a vector field (denoted $\vec{v}$) associated with the velocity of flow. It is well known that this theory can be formulated as an action principle, but only under the restriction to potential flows, namely the condition that there be a scalar field Φ such that $\vec{v} = -\vec{\bigtriangledown}\Phi$. This is easy to understand since the important equation of continuity is a scalar equation and must be obtained from the action by variation of a scalar field. In the Lagrangian version the field $\vec{v}$ is represented as the time derivative of a more fundamental variable usually denoted $\vec{x}$, $\quad \vec{v} = d\vec{x}/dt$. There is an action principle associated with this theory, but it

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* In the case that the viscosity is zero the Navier-Stokes equation reduces to the Euler equation. This equation is less restrictive and it is not adequate for our purpose.

does not incorporate the important equation of continuity. So we must conclude that the equivalence of the two points of view may depend on the manner in which each one is implemented. In this paper we shall see that the two field theories are in a certain sense complimentary and that there are advantages to be gained by integrating them with each other.

The need to include additional variables, besides density and velocity) in the description of a rotating fluid is made especially clear in the treatments of Landau and Lifshitz (1958), and of Feynman (1954), of the excitations observed in superfluid Helium, but applicable to all liquids. A common and central feature is a differentiation between potential and solid-body flows. It is quite clear that any variational description of these excitations in the framework of macroscopic thermodynamics needs two independent vector fields.

The original aim of this work was to find appropriate matter sources for the dynamical metric field of General Relativity and the present results represent a step in that direction. (Fronsdal 2007, 2014c) But this report sticks to the non relativistic context, classical hydrodynamics and thermodynamics.

Summary

Section II presents the well known variational formulation of potential flows in hydrodynamics, and a short account of the connection to thermodynamics. The next section describes the strategy that was chosen to look for an action principle of wider generality: to focus on the special case of cylindrical Couette flow, with an unusual emphasis on laminar flow, in this simplest context in which non-potential flows are inevitable. Section IV reviews the classical application of the Navier-Stokes equation to this system and contrasts it with the failure of the potential theory, to discover the tight spot, and the remedy.

Section V is an outline of the proposed solution to the problem, a solution that is both simple and natural. It combines Eulerian and 'Lagrangian' hydrodynamics to describe both potential and rotational flows. Some features that are not yet fully understood are discussed in this section.

Section VI is a discussion of the relative merits of the variational principle and the Navier-Stokes equation in the context of adiabatic dynamics where both are applicable. The main issue is the role of energy. This section is followed by a study of plane Couette flow, where the relationship between the two theories turns out to be very different. Th variational approach has the greater predictive power.

Section VIII is a preliminary application of the variational approach to the problem of rotating heavenly bodies. The context is Newtonian gravity, for the theory has not yet been integrated with General Relativity.

II. The Fetter-Walecka action principle.

A variational formulation of simple hydrodynamics may be found in a book by Fetter and Walecka (1980). The action is

$$A = \int dt \int d^3x \, \mathcal{L}, \quad \mathcal{L} = \rho(\dot{\Phi} - \vec{v}^{\,2}/2) - W(\rho), \tag{2.1}$$

with the definition

$$\vec{v} = -\text{grad} \, \Phi. \tag{2.2}$$

That is a strong restriction on the velocity field. The potential W is related to the pressure, as will be shown.

The equations of motion are the Euler-Lagrange equations derived by variation of the two scalar fields. Variation of Φ gives the equation of continuity,

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{2.3}$$

and variation of ρ,

$$\dot{\Phi} - \vec{v}^{\,2}/2 = \frac{\partial W}{\partial \rho}, \tag{2.4}$$

or

$$\rho \dot{\vec{v}} + \rho \vec{\nabla} \vec{v}^{\,2}/2 = -\text{grad} \, p, \quad p := \rho \frac{\partial W}{\partial \rho} - W, \tag{2.5}$$

where p is the pressure. The two terms on the left side can be combined,

$$\dot{\vec{v}} + \vec{\nabla} \vec{v}^{\,2}/2 = \frac{D\vec{v}}{Dt} := \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla}) \, \vec{v}.$$

That the substantial derivative $D\vec{v}/Dt$ appears here, as it should, comes about because of the identity

$$\text{grad} \, \vec{v}^{\,2}/2 = (\vec{v} \cdot \vec{\nabla}) \, \vec{v}, \tag{2.6}$$

which is true by virtue of (2.2). Eq. (2.5) is the Bernoulli equation, written in the form in which it is most easily compared to the Navier-Stokes equation. It is the gradient of the 'integrated Bernoulli equation' (2.4).

Thermodynamics

This action principle can be combined with Gibbs thermodynamic principle of minimum energy (Gibbs, 1878). Applied to a one component thermodynamic system the lagrangian density takes the form

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^{\,2}/2) - f(\rho, T) - sT. \tag{2.7}$$

Here f is the free energy density and $s = \rho S$ is the entropy density. Variation of T (the temperature) gives the adiabatic equation that permits the elimination of the temperature

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in favor of the specific entropy density S, assumed uniform. Variation of Φ gives the equation of continuity and variation of the density gives

$$\dot{\Phi} - \vec{v}^2/2 = q,$$

where q is the chemical potential density. Note that the chemical potential depends on the entropy. In the case of an ideal gas it can expressed in terms of the temperature and we obtain in that case

$$\dot{\Phi} - \vec{v}^2/2 = C_V T.$$

This form of the equation is, in our opinion, much to be preferred, for it requires no knowledge of the entropy. The comparison of theory and experiments and, consequently, the interpretation, would be greatly facilitated if it were possible to measure the temperature profile instead of the pressure.

Up to this point in our work (Fronsdal 2014a, 2014b) the attention has been focused, deliberately, on the special case when the flow velocity is a gradient, the vorticity or curl being zero,

$$\vec{v} = -\vec{\bigtriangledown}\Phi, \quad \rightarrow \quad \vec{\bigtriangledown} \wedge \vec{v} = 0.$$

Although it is unduly restrictive, this limitation was accepted as an apparently necessary condition to formulate hydrodynamics (and thermodynamiccs) as a lagrangian field theory. Many problems in thermodynamics and hydrodynamics involve no flow and in many others the flow is irrotational. But a relativistic lagrangian is needed for General Relativity and here the restriction to potential flows is hard to bear. To discover how to achieve greater generality we shall now turn our attention to a simple systems where a generalization is obviously and urgently needed.
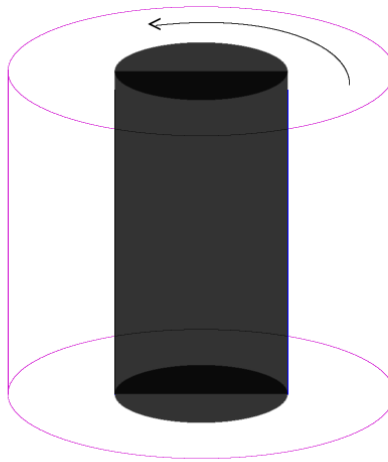

Fig.1. Cylindrical Couette flow is the horizontal, rotational flow between two concentric cylinders.

III. Couette flow, potential flow

This simple example is well suited to serve as a first introduction to vorticity.

We shall consider a situation that is effectively 2-dimensional because of translational symmetry, when nothing depends on a vertical coordinate z and the flow is confined to the horizontal x, y plane. Couette flow is the flow of a fluid between of a pair of concentric cylinders that can be rotated around a common (vertical) axis. With both cylinders at rest we postulate a state in which the space bounded by the two cylinders is filled with a fluid at rest, with all variables time independent, and uniform. The effect of gravity will be neglected. The cylinders are long enough that end effects can be neglected as well. *

We begin to rotate the inner cylinder. For this to have any effect on the fluid we need to postulate a degree of adherence of the liquid to the surface of the inner cylinder. This is summed up by the no-slip boundary condition

$$\vec{v}|_{r=r_0} = \vec{v}|_{\text{inner boundary}} = \omega_0(-y, x, 0).$$

The angular velocity ω_0 is that of the cylinder.

The no-skip boundary condition has been widely applied, and because we wish to compare our approach to the traditional one it serves our purpose to do the same. See Brenner (2011), Priezjev and Troian (2005), Dukowicz, Goldstein, Handler and Sirovich (1993).

The rotational axis is the z axis. The coordinates are inertial and cartesian. No boundary conditions are imposed on the velocity at the outer wall, so far.

We propose to treat this system, with fixed boundary conditions, as an adiabatic system, and the stationary solution as its equilibrium configuration. We begin with the familiar lagrangian density

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2) - f - sT. \tag{3.1}$$

The problem is essentially 2-dimensional and r is the cylindrical, or polar radial coordinate. When all the variables are time independent the Euler-Lagrange equations reduce to

$$\text{div}(\rho\vec{v}) = 0, \quad \dot{\Phi} - \vec{v}^2/2 = q, \tag{3.2}$$

where q is the chemical potential and $\dot{\Phi}$ is a constant. In the case of an ideal gas ,

$$\dot{\Phi} - \vec{v}^2/2 = C_V T. \tag{3.3}$$

If we assume that the specific entropy is uniform, then this last equation can be transformed to the more familiar Bernoulli equation,

$$\dot{\vec{v}} + \rho\vec{\nabla}(\vec{v}^2/2) = -\vec{\nabla}p. \tag{3.4}$$

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* According to a Merriam Webster web page, the term Couette flow is derived from "French couette machine bearing, literally, feather bed, from Old French coute, cuilte quilt, mattress." To make up for this slight we cite several of Couette's pioneering papers (Couette 1887,1888,1889,1890).

We assume that the flow lines are circles, $r^2 = x^2 + y^2 = $ constant,

$$\vec{v} = \omega(r)(-y, x, 0).$$

This is a field with vanishing curl only if

$$\omega(r) = ar^{-2}, \quad a = \text{constant},$$

and even then it is not, in the strict sense, a gradient. Although

$$\vec{v} = a \, \text{grad} \, \theta, \quad \theta := \arctan \frac{y}{x}, \tag{3.5}$$

the scalar θ is not one-valued, though the vector field is. By allowing the velocity potential Φ to be multivalued,

$$\vec{v} = a \frac{(-y, x, 0)}{r^2} = -\vec{\bigtriangledown}\Phi, \quad \Phi = a\theta, \quad a = \text{constant},$$

one extends the approach to include special rotations, with $\vec{\bigtriangledown} \wedge \vec{v} \neq 0$ at the origin only (though ill defined at that one point). In the present case the origin is outside the vessel and the curl is zero at all points of the fluid. It is customary and convenient to encompass this situation in the concept "potential flow".

This is the only possible horizontal, circulating, potential flow. The angular velocity is greater at smaller radius; it is therefore likely that it is driven by rotating the inner cylinder and that it settles according to the no slip condition. There is no opportunity to adjust the outer boundary conditions; either the fluid slips there or else the angular speed of the outer cylinder must be adjusted in accordance with $\omega(r)r^2 = a$. We assume that this is done.

The divergence of $\vec{v}$ (for any choice of $\omega(r)$) is zero, so the equation of continuity reduces to $\vec{v} \cdot \vec{\bigtriangledown}\rho = 0$, requiring ρ to depend on r and z only. From now on it is taken for granted that ρ is a function of r only.

This type of stationary motion has been observed, see for example Joseph and Renardy (1985), but the principal goal of most experiments has been to measure the onset of turbulence as, with increasing rotational speeds, the laminar flow breaks down (Couette 1887). Our interest shall be focused, instead, on the laminar flow at low angular speeds.

The principal equation of motion, Eq. (3.3) reduces, in the case that the velocity takes the form (3.5), to

$$\frac{a^2}{2r^2} = C - C_V T, \quad C = \text{constant} \tag{3.6}$$

It appears that this temperature lapse has not been measured. The experiment would be difficult to interpret because of the suspected heat flow caused by the friction, within the liquid and between the liquid and the wall. The equation is consistent with the interpretation of the fluid as a collection of classical particles, and in full agreement with the Navier-Stokes equation, except that the latter simply balances the pressure gradient

against the centripetal force and a contribution associated with the viscosity, usually without relating it to an equation of state or to the temperature, and very rarely connecting it to entropy. * As is seen from Eq. (3.4), the kinetic energy acts as an effective potential and gives rise to a repulsive, radial force, balanced by the negative of the pressure gradient. We shall have more to say about this interpretation later.
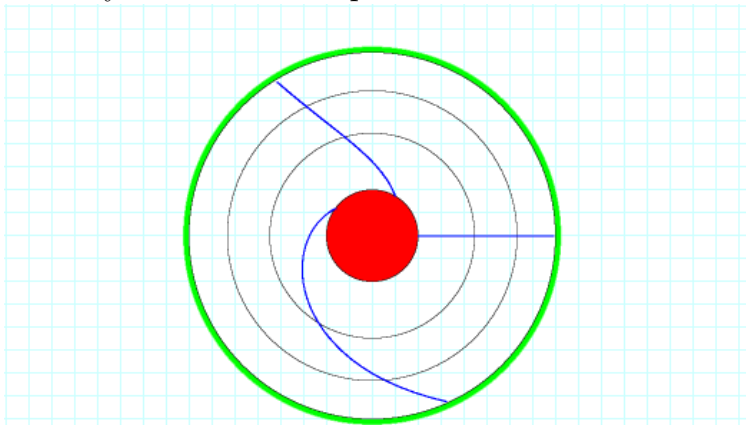

Fig.2. The gradient flow that results from rotating the inner cylinder. The curved lines show the positions of particles initially lined up radially. The inner cylinder is the driver of the motion and it is turning anti-clockwise.

IV. 'Solid-body' flow

Consider next the complimentary experiment in which there is friction at the outer cylinder only, making it the driver of the motion. The problem is the same, except for different boundary conditions,

$$\vec{v}|_{r=r_1} = \vec{v}|_{\text{outer boundary}} = \omega_1(-y, x).$$

Our initial study of potential motion revealed only one solution, and here its application is anti-intuitive since it has the angular velocity increasing towards the center, away from the driving wall. For the general problem, in which the two cylinders move independently, we have two independent boundary conditions but a one dimensional space of potential flows. Clearly we have come to a situation where we cannot limit our attention to gradient velocity fields. Still, we are not willing to concede that this is the limit of the action principle approach to hydrodynamics.

It is reported that the motion of a liquid, even a rarified gas, driven by the rotation of the outer cylinder, tends towards the motion of a solid body. See for example Andereck, Liu and Swinney (1986), de Socio, Ianiro and Marino (2000). As the speed is increased, various

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* It is common, in the case of liquids, to treat them as incompressible. In this case it is justified to treat the pressure gradient as unknown, to be determined by the need to get reasonable solutions to the equations. But of course this puts severe limits on the predictive power of the theory.

instabilities set in, but our only concern is to understand the laminar motion observed at low speeds.

The motion of a rigid body is characterized by the velocity field

$$\vec{v} = b(-y, x, 0), \quad b \text{ constant,}$$

and this is not a gradient. If we analyze this flow in the same manner that we did potential flow; that is, if we simply use the above velocity field in the field equations, then we obtain similar results; in particular, instead of (3.6),

$$\frac{b^2}{2} r^2 = C - \mathcal{R}T.$$

Note that to get this unphysical result we applied the equations of the potential theory beyond their domain of validity. Here the left side has the wrong sign, the force is attractive. This error in sign is a problem that frustrates all efforts, until the only possible solution pops into view.

We have to escape from under the restriction to gradient velocities and a generalization of the action principle is required. To gather further clues to help us proceed it is natural to ask how the Navier Stokes equation handles this situation.

Navier Stokes

The standard treatment of non-potential flows is based on the continuity equation and the Navier-Stokes equation (Navier 1827, Stokes 1843, Navier 1882),

$$\dot{\rho} + \text{div}(\rho\vec{v}) = 0,$$

$$\rho\left(\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = -\vec{\nabla}p - \nu\Delta\vec{v}. \tag{4.1}$$

This allows for flows of both kinds, potential flow and solid body flow. The new elements are three. First of course, the nature of the velocity field is not constrained to be a gradient. In the second place the term $\rho\vec{\nabla}\vec{v}^2/2$ in (3.3) has been replaced by the term $\rho(\vec{v} \cdot \vec{\nabla})\vec{v}$ in (4.1). Finally, there is a new term, the viscosity term $\nu\Delta\vec{v}$.

If the coefficient ν (the kinematical viscosity) vanishes, of the second equation only the radial component remains. It is an ordinary differential equation for the functions ω, ρ and p and the system is under-determined. If $\nu \neq 0$, the tangential projection imposes the additional requirement that

$$\Delta\vec{v} = 0. \tag{4.2}$$

We shall see that this leads to unique solutions for reasonable boundary conditions, either or both cylinders driving. The fact that uniqueness is obtained only when the viscosity is taken into account is a little odd, in our opinion, since there is no lower limit on the value of the coefficient ν. But this is the standard treatment. Given the form

$$\vec{v} = \omega(r)(-y, x, 0)$$

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of the velocity field, but with $\omega(r)$ now arbitrary since it is not required to be a gradient, within the class of flows under consideration the general solution of Eq.(4.2) is

$$\vec{v} = \omega(r)(-y, x, 0), \quad \omega(r) = \frac{a}{r^2} + b, \tag{4.3}$$

a and b constants; the two types of flow already considered are the only ones allowed by Eq. (4.2). The boundary conditions at $= r_0, r_1$ give us

$$\omega(r_0) = a r_0^{-2} + b = \omega_0, \quad \omega(r_1) = a r_1^{-2} + b = \omega_1,$$

When the a term dominates we have the highest angular velocity at the inner surface; this is as expected when the inner cylinder is driving. If the b term dominates we have nearly constant angular velocity, as expected for a solid body. If both cylinders are rotating, in opposite directions, and both are driving, then $\omega(r)$ will have a change of sign. Explicitly,

$$a = \frac{\omega_0 - \omega_1}{r_0^{-2} - r_1^{-2}},$$

and

$$b = \frac{1}{r_1^2 r_2^2} \frac{r_1^2 \omega_1 - r_0^2 \omega_0}{r_0^{-2} - r_1^{-2}}.$$

The result is that Navier-Stokes, with non zero viscosity, has just one extra solution besides the gradient, allowing it to satisfy no slip boundary conditions for all values of the angular velocities of the two cylinders. It is important that this new solution, with $\vec{v} \propto (-y, x, 0)$, is the same as the static state observed from a rotating reference frame; its existence is required by the relativistic equivalence theorem. And the fact that the Navier-Stokes singles out just two, radically different kinds of flow, and nothing interpolating between them, is highly significant.

The salient differences between Navier-Stokes and the theory attempted in the preceding subsection are the following. As we said, the term $\vec{\nabla} v^2/2$ in Eq.(3.3) is replaced by the term $\vec{v} \cdot \vec{\nabla} \vec{v}$ in (4.1). One verifies that, when the velocity field is a gradient, both expressions are equal, but in the case of solid body rotation there is a change of sign,

$$\vec{\nabla} \vec{v}^2/2 = b^2 \vec{r} \text{ but } \vec{v} \cdot \vec{\nabla} \vec{v} = -b^2 \vec{r}.$$

The force produced by the first expression is attractive, while the Navier-Stokes force is repulsive, as it must be. (A rapidly rotating rigid body breaks apart, it does not collapse towards the center!) Evidently, we must understand what lies behind this difficulty.

The context is cylindrical Couette flow and our guide is the Navier-Stokes equation.

The centrifugal force

In particle physics the dynamical variable is the position of the particle.

The 'fictitious' centrifugal force in particle physics can be seen as coming from an effective 'kinematic' potential. The force is $\omega^2 \vec{r}$ directed outwards, and this is $-\vec{\nabla}[-\omega^2 r^2/2]$,

so the effective potential is the negative of the kinetic energy. The unexpected sign comes from the fact that the the origin of this potential is in the term $m\dot{\vec{x}}^2/2$ in the Lagrangian; this term appears with the same sign in the Hamiltonian. Confusing? Let us say it in another way, the term in question appears with the positive sign in the Lagrangian and with the positive sign in the Hamiltonian; but its contribution to the equation of motion is opposite in sign from that of a normal potential. We have seen that our potential model when used outside its domain of validity, gives the wrong sign in the equation of motion.
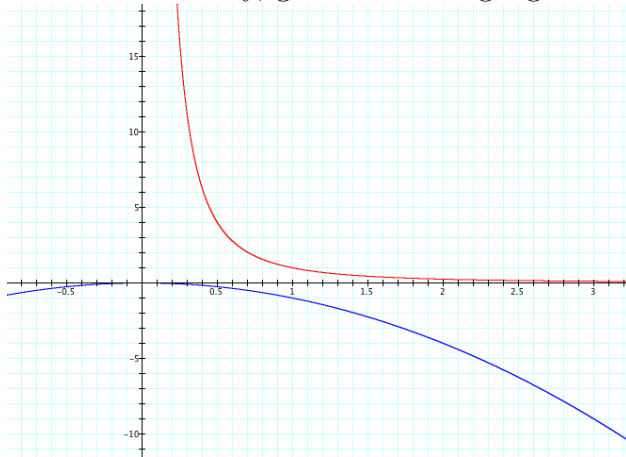

Fig. 3. Centrifugal potentials. The abscissa is the distance from the axis of rotation. The upper curve is the potential associated with a gradient velocity, a/r^2; it appears as a positive contribution to the Hamiltonian and with negative sign in the Lagrangian. The force is the negative of the gradient of the potential, pointing outwards. The lower curve is the negative of the potential associated with solid body rotation. This potential, $br^2/2$, appears with positive sign in the Hamiltonian, and with the same sign in the Lagrangian; but the force is in the direction of the gradient, likewise outwards.

Fluid mechanics can be understood as a field theory, and there are two versions, some times said to be equivalent (Stanyukovich 1960, Sedov 1971), a Eulerian formulation and an alternative 'Lagrangian' formulation. The principal distinction is the following.

In the Eulerian formulation of fluid mechanics the dynamical variables are the scalar density field and a velocity vector field. This theory becomes a Lagrangian field theory, in the sense of being based on a dynamical action principle, only in the case that the velocity field is the gradient of a velocity potential. Thus formulated, this theory cannot describe solid body rotation because the potential $\rho\vec{v}^2/2$ appears with the negative sign in the Lagrangian. Let us be clear about this: the velocity associated with rotational motion in the x, y plane is $\omega(-y, x, 0)$ and it is not a gradient, but there is an additional obstacle. If we introduce a term $-\vec{v}^2/2(\text{or}+\vec{v}^2/2)$ in the Lagrangian we get the right sign in the Hamiltonian but the wrong sign in the equations of motion (or *vice versa*).

The 'Lagrangian' version of fluid mechanics handles the centrifugal force correctly, just as particle physics does, but the kinetic energy is in the kinetic part of the Lagrangian. This way one gets the right sign for solid-body motion. This is because the dynamical variables are not ρ and $\vec{v}$ but ρ and a vector field usually denoted $\vec{x}$, satisfying $d\vec{x}/dt = \vec{v}$.

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To avoid confusion we shall call it $\vec{X}$. The books explain that this vector field may be interpreted as the position of a particle in the fluid; one chooses an initial value and calls on the equations of motion to predict the future trajectory. This explanation tends to obscure the fact that this version of the theory is also a field theory, with field variables ρ and $\vec{X}$, and

$$\vec{v} := \dot{\vec{X}}.$$

The velocity is the time derivative of the basic field variable and the term $\rho\vec{v}^2/2$ is now in the kinetic part of the lagrangian.

The Navier-Stokes equation is based directly on particle mechanics. The main equation does not involve the gradient of a hamiltonian and no hamiltonian exists. The dynamical field is the velocity but the gradient of no kinetic potential enters the calculations. Instead the substantive derivative appears,

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}.$$

Let us look once more at the example of rotary motion. The gradient theory describes a motion of the type $\vec{v} = (a/r^2)(-y, x, 0)$, and in this case

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}\vec{v}^2/2.$$

But in the case of solid body motion, when $\vec{v} = b(-x, y, 0)$ (a and b are constants), we find that

$$(\vec{v} \cdot \vec{\nabla})v = -\vec{\nabla}\vec{v}^2/2.$$

The Navier-Stokes equation is based on particle mechanics and both kinds of motion are treated correctly, but there is no hamiltonian; this is because the dynamical variable is the velocity. To get beyond this stage, to set up a lagrangian theory of non potential flow, it is necessary to introduce the field $\vec{X}$, as in the 'Lagrangian' version of fluid mechanics.

V. Action principle for general flow

With the Lagrangian density that we have used for potential flow as a starting point, we begin by adding a new term,

$$\mathcal{L} = \rho\left(\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 - \frac{1}{2}(\vec{\nabla}\Phi)^2\right) - f - sT. \tag{5.1}$$

The second term appears with the sign that is appropriate for solid-body rotational motion. The potential terms $-(\vec{\nabla}\Phi)^2/2$ may be considered as a part of the free energy density; the new term $(\rho/2)\dot{\vec{X}}^2$ plays exactly the same role as the second term in

$$F' = F - M\Omega. \tag{5.2}$$

This equation appears in the work of Hall and Vinen (1956) (and in Landau and Lifshitz 19555, 1958); F is the free energy, M is the angular momentum and Ω is the angular

velocity. This quantity F' is required to be a minimum with respect to variation of Ω, with M fixed, and the resulting equation is the same as when $C\Omega^2/2$ is varied with $C = M/\Omega$ held fixed. The context is superfluid Helium but the theory applies just as well to all fluids. The physical interpretation as well as the mathematical structure is thus precisely the same as that of the above tentative Lagrangian (5.1). This explains the negative sign: F' is not a modified free energy but a Lagrangian!

The Hamiltonian density is now

$$\mathcal{H} = \frac{\rho}{2}\dot{\vec{X}}^2 + \frac{\rho}{2}(\vec{\nabla}\Phi)^2 + f + sT. \tag{5.3}$$

This is correct for all flows, most notably for solid body rotation, $\dot{\vec{X}} = b(-y, x, 0)$.

We shall compare this theory with the Navier-Stokes equation, but we first need to add another term to the Lagrangian, the final form being

$$\mathcal{L} = \rho\left(\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 + \dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2\right) - f - sT. \tag{5.4}$$

Variation of Φ will give the equation of continuity and the term $\rho\dot{\vec{X}} \cdot \vec{\nabla}\Phi$ is included in order to get the correct expression for the flow. The expression (5.3) for the Hamiltonian is not affected.

Let us list all the Euler - Lagrange equations.

1. Variation of T: the adiabatic relation, as before.
2. Variation of Φ: the continuity equation, now

$$\dot{\rho} + \vec{\nabla} \cdot \left(\rho\dot{\vec{X}} - \rho\vec{\nabla}\Phi\right) = 0. \tag{5.5}$$

The unique flow velocity is therefore

$$\vec{v} := \dot{\vec{X}} - \vec{\nabla}\Phi. \tag{5.6}$$

This is what corresponds to the velocity that appears in the Navier-Stokes equation.

3. Variation of the vector field:

$$\frac{\partial}{\partial t}\left(\rho\dot{\vec{X}} + \rho\vec{\nabla}\Phi\right) = 0. \tag{5.7}$$

4. Variation of the density:

$$\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 + \dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2 - \frac{\partial f}{\partial \rho} - ST = 0. \tag{5.8}$$

Note that two independent vector fields, $\dot{\vec{X}}$ and $-\vec{\nabla}\Phi$, are needed to get the correct signs for both squared velocity terms; opposite signs in (5.8), both positive in the Hamiltonian

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density. The term $\dot{\vec{X}}\cdot\vec{\bigtriangledown}\Phi$ does not affect the Hamiltonian, but it is crucial for the equation of continuity.

Numbers 2 and 3 are solved by taking ρ to depend on r only, $\dot{\Phi}$ constant, and

$$\Phi = t\dot{\Phi} + a\theta, \quad \vec{\bigtriangledown}\Phi = \frac{a}{r^2}(-y, x, 0), \quad \vec{X} = bt(-y, x, 0). \tag{5.9}$$

When these solutions are inserted into (5.8) we get, in the case of an ideal gas, for stationary flow,

$$\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + ab = C - (n+1)\mathcal{R}T. \tag{5.10}$$

This is the equation that tries to agree with the Navier-Stokes equation and it very nearly does, but to compare we must take the gradient.

$$\rho\vec{\bigtriangledown}\left(\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + ab\right) = -\vec{\bigtriangledown}p.$$

With the same flow, the Navier-Stokes equation is

$$\rho\vec{\bigtriangledown}\left(\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + 2ab\ln r\right) = -\vec{\bigtriangledown}p - \nu\vec{\bigtriangledown}\vec{v}.$$

Of course, we cannot reproduce the viscosity term. The cross term ab is a constant, while in the Navier-Stokes equation it is replaced by the logarithm.

Taking stock

With the lagrangian density (5.1) a significant forward step in our program has been achieved. We now know that the idea of a viable lagrangian formulation of thermodynamics is not absolutely limited to potential flows. This leaves many things yet to be done, for example, the complete interpretation of the two vector fields that make up the total flow, the integration with electromagnetism, and with General Relativity. But first we need to apply the new, tentative insight to a range of phenonomena; all that we know at present is that our new lagrangian density is suitable for cylindrical Couette flow in the laminar regime. A closely related phenomenon, the flow along plane, moving walls or linear Couette flow will be examined below, in Section VII.

The appearance, in the action principle for general flows, of an extra variable should not be surprising. The work of Feynman (1954), and of Landau and Lipshitz (1955, 1958) on the application of statistical mechanics to liquid Helium is based on a separate and different treatment of two types of excitations, the 'phonons' being accounted for by a gradient vector field and the 'rotons' understood as a different degree of freedom. These papers, and especially a much quoted paper by Hall and Vinen (1956) that they inspired, briefly refer to an action principle. Interestingly, in the papers that followed, the strong hint of a Lagrangian structure is minimalized is a function $F' = F - \Omega M$, where the second term can be identified with the term $\rho\dot{\vec{X}}^2$ in our Lagrangian. (The rest of our the kinetic

part of our Lagrangian are included in the free energy F.) Interestingly, in the papers that followed, the strong hint of a Lagrangian structure was systematically suppressed. We read words to this effect "The equations of Hall and Vinen can be written as follows ... ". But what is written is the Euler-Lagrange equations, without any mention of their relation to an action principle.

The specific model for cylindrical Couette flow has some features that warn of difficulties expected to arise. So far, the field $\vec{X}$ appears through its time derivative only, but for various reasons space derivatives are expected to be needed as well. From at least one perspective such terms are disquieting. One of the most tempting interpretations of the field X relates it to the electromagnetic vector potential; this is because of the existence of various effects, including the magnetic Barnett effect (Barnett 1915) that signals a strong link between magnetization and rotation, and the electric Seebeck effect (Seebeck 1825). * But the solution (5.9) gives a specific value for the 'magnetic' field $\vec{\nabla} \wedge \vec{X}$, a strong field aligned with the axis of rotation. We know that the steady rotation of the earth is accompanied by a magnetic field that is relatively weak and strongly erratic in direction and magnitude. For this reason, while we continue to ponder the possibility of an interpretation of the vector field that is strongly tied to electromagnetism, we are also looking for alternatives.

VI. Comparing two approaches

Viscosity. An important feature of reality that is encompassed by the Navier-Stokes equation is viscosity. In the Lagrangian approach one concedes from the start that viscosity is going to be neglected, the context being adiabatic dynamics. Our point of view, as explained in the introduction, is that viscosity and dissipation are phenomena that can be discussed only after a complete description of adiabatic phenomena is at hand. The new action principle, in the general, thermodynamic form, is a family of adiabatic systems explicitly parameterized by the entropy. Quasi static evolution is a sequence of equilibria of this family, parameterized by the entropies of the adiabatic systems to which they belong.

Predicting the flows. A major success of the application of the Navier-Stokes equation to laminar, cylindrical Couette flow is the prediction, on the basis of the tangential component (4.2): $\nu \Delta \vec{v} = 0$, of just two kinds of flow, in this case potential flow and solid-body flow. Intuitively, the latter is a little unexpected; this flow is attained, or nearly attained, when the outer cylinder is driving and the inner cylinder is slipping. The Euler Lagrange equations associated with the lagrangian density (4.2) place no restrictions on the factor b in the solution $\vec{X} = b(-y, x, 0)$, but the anticipated inclusion of spatial derivatives of the field $\vec{X}$ will lead to a modification of Eq.(5.7) and restrictions on the flow.

The density profile. The Euler Lagrange equation obtained by variation of the density, Eq,(5.6), reduces in the case that the flows are as in (5.9) to Eq. (5.10). This is in agreement with the Navier-Stokes equation except that the cross term ab is constant,

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* Feynman, in a similar context, also comments on this analogy (Feynman 1954).

while in the Navier-Stokes equation it depends logarithmically on the radius. If the angular velocity b were to vary with r, the term $\dot{\vec{X}} \cdot \vec{\nabla}\Phi$ in the lagrangian density would no longer be a constant. If b is replaced by $br^{-\tau}$ we obtain

$$\frac{1}{2}\dot{\vec{X}}^2 + \dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2 = \frac{a^2}{2r^2} - \frac{b^2}{2}r^{2(1-\tau)} + abr^{-\tau} = C - (n+1)\mathcal{R}T. \qquad (6.1)$$

This is in substantial agreement with Navier-Stokes in the limit of small τ. And it offers an extra free parameter for a small amount of fudging.

Fig. 4 shows a sample temperature profile. The small dip for small radius is produced by the ab term on the left hand side of (6.1). A similar feature has been observed.

Fig.4. A temperature profile for cylindrical Couette flow, plotted against the horizontal radius, based on Eq.(6.1). The position of the walls is not indicated.

The expression $(n+1)\mathcal{R}T$ on the right hand side is not valid at very low temperatures, but in the case of an ideal gas, with uniform entropy, it is proportional to a power of the density and it is perhaps reasonable to expect it to go to zero with the density. In that case one may expect the onset of turbulence to occur for a fixed value of the parameter

$$\frac{a^2}{2r^2} - \frac{b^2}{2}r^{(1-\beta)} + abr^{\beta} = C_0. \qquad (6.2)$$

In the a, b plane this is a hyperbola open to the positive b axis. This too is in qualitative accord with experiments.

The energy. The principal reason for bringing out an alternative to the approach that relies heavily on the Navier-Stokes equation is of course the improved position of the energy concept. The equations of motion do not differ greatly from the equations used in the traditional method but, instead of an hoc formula for "energy" that is required to

be conserved as an additional postulate (See for example Müller 2007) , we have a first integral of the equations of motion. We note that some authors require that the "energy equation" hold in consequence of the other conservation equations, thus expressing a point of view similar to ours. (Khalatnikov 1956.)

VII. Couette flows between plane walls

This example serves to show that one cannot expect the new action principle to agree with the Navier-Stokes equation in all cases, even in the limit of negligible viscosity. It is an example where the Navier-Stokes equation has very limited predictive power beyond the identification of the two principal modes of flow.

Here again is the Navier-Stokes equation,

$$\rho\left(\dot{\vec{v}} + (\vec{v} \cdot \vec{\bigtriangledown})\vec{v}\right) = -\vec{\bigtriangledown}p - \mu\Delta\vec{v}. \tag{7.1}$$

Consider the problem of flow in the space bounded by plane walls parallel to the x, z-plane, $y = y_0$ and $y = y_1$. We limit our attention to stationary flows parallel to the x axis,

$$\vec{v} = (v, 0, 0),$$

with the function v depending only on y.

What the Navier-Stokes equation has to say is this. Projected on the direction of the flow, the velocity and the pressure are homogeneous (constant), and the equation reduces to

$$\Delta v_x = 0.$$

with the general solution

$$\vec{v} = (a + by)(1, 0, 0), \quad a, b \quad \text{constant}. \tag{7.2}$$

This is highly significant, for it shows that the distinction between potential and solid-body flow is less than fundamental; in this case we have one type of flow (with $b = 0$) that is both potential type and solid-body type, and another (with $a = 0$) that is neither.

We are studying stationary flows, with $\dot{\vec{v}} = 0$; and since the velocity is constant along the direction of motion,

$$\frac{D\vec{v}}{Dt} := \dot{\vec{v}} + (\vec{v} \cdot \vec{\bigtriangledown})\vec{v} = 0. \tag{6.3}$$

Consequently, all that remains is the equation

$$0 = -\vec{\bigtriangledown}p - \mu\Delta\vec{v}, \tag{6.4}$$

projected on the y-axis; the pressure gradient is balanced against the viscosity. Although the kinetic energy - if it is relevant to invoke energy in the context of the Navier-Stokes equation - varies with the position, there is no force associated with this variation of the kinetic energy. It is true that the kinetic energy, in the special case that the flow is potential,

is uniform, so that no force is associated with it, but the absence of a kinetic force when $b \neq 0$ tells us that this other type of flow is of a very different character, since a non zero gradient of the kinetic energy in any flow was expected to generate a force. Perhaps this is behind some imprecise remarks found in the literature, to the effect that, in the context of the Navier-Stokes equation, the relation between linear flow and cylindrical flow is not perfectly understood (Faisst and Eckhardt 2000).

What is the meaning of (6.4)? One may, for example, assume that the pressure is that of an ideal gas,

$$p = \mathcal{R}\rho T,$$

but since the temperature is very rarely measured, perhaps the polytropic relation will be used instead,

$$p \propto \rho^{\gamma}.$$

Unfortunately the pressure also is rarely measured with the required precision. As far as the viscosity term is concerned the situation is less satisfactory, for usually, when any measurements were made, the purpose was to measure the viscosity. Consequently, it is very rare that an analysis is made in the interest of verifying a theory and all that can be said is that experimental results are not known to contradict the Navier-Stokes equation. Indeed, how could they?

It is unfortunate that few experiments on Couette flow are made with the idea of studying the laminar flow; for in fact the aim is usually to determine the conditions under which it breaks down to be replaced by more complicated flows including turbulence. Our purpose is more modest, since we aim to understand laminar flow only, but more ambitious since we are looking for an application of a general theory with definite precepts including as far as possible a lagrangian variational principle. The most urgent question is always the simplest: can we understand the observed laminar flow in the limiting case in which the role of viscosity is negligible?

Our theory, in the present state of development, includes an equation of continuity that, when the flow has the form (6.2),

$$\vec{v} = \dot{\vec{X}} - \vec{\nabla}\Phi = (a + by)(1, 0, 0), \tag{6.5}$$

demands that the density depend on the coordinate y only. Solutions for the vector and scalar fields include the following

$$-\vec{\nabla}\Phi = a(1, 0, 0), \quad \vec{X} = bt(y, 0, 0).$$

The field equation is satisfied,

$$\partial_t(\rho\dot{\vec{X}} + \vec{\nabla}\Phi) = 0,$$

and finally there is the equation that comes from variation of the density,

$$\frac{a^2}{2} - aby - \frac{b^2 y^2}{2} = C - (n+1)\mathcal{R}T. \tag{6.6}$$

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or

$$(n+1)\mathcal{R}T = \frac{1}{2}(by+a)^2 + \text{constant}$$

This contrasts with the Navier-Stokes equation (6.4). The fluid is contained between moving walls located at two values of the coordinate y. If this interval includes the plane with coordinate $y = -a/b$, then the temperature has a minimum there, see Fig. (6.2).

The order of magnitude of the variation of temperature is that of $v^2/\mathcal{R}$. If the velocity is 1 m/sec this will imply a temperature variation of order $10^4/C_p$, a very small variation indeed. So it is unlikely that this temperature variation has been observed.

It is nevertheless significant that, in remarkable contrast with the Navier-Stokes approach, our approach leads to a definite, quantitative prediction without the intervention of viscosity, presumably valid in the limit when the viscosity can be neglected. Admittedly, this prediction rests on the assumption that the flow profile is as in (6.2). This restriction cannot yet be derived within the new approach but it is not illogical to attribute it to viscosity.

There is no conflict between Eq.s (6.4) (Navier-Stokes) and (6.6); the latter applies under the assumption that the viscosity is negligible in this particular situation.
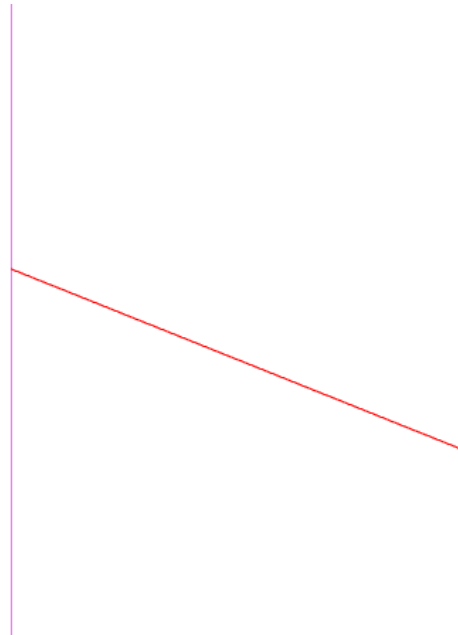

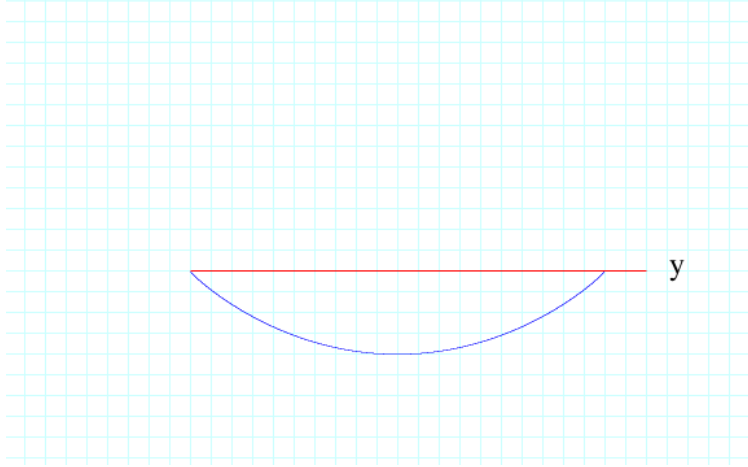Fig.5. The velocity profile. The two walls are moving in opposite directions. Improve this.

Fig.6. An example of the temperature profile. The two walls are moving in opposite direction with the same speed.

VIII. A rotating galaxy in Newtonian gravity

In analogy with Emden's "Gas Kugeln" (Emden 2007), consider an isolated concentration of mass, held together by Newtonial gravity, in uniform, Couette-like motion around a fixed axis, described in terms of inertial coordinates. The central axis of rotation is the z coordinate axis and the motion follows circles centered on this axis and confined to planes of fixed z. The Lagrangian density (5.2) must be supplemented by the Newtonian potential energy $\rho\phi$,

$$\mathcal{L} = \rho\left(\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 + \dot{\vec{X}}\cdot\vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2 - \phi\right) - f - sT,$$

where ϕ is the appropriate solution of Poisson's equation

$$\Delta\phi = -4\pi G\rho.$$

Solid-body flow

Consider the case of uniform, solid-body rotation about a fixed axis. The stationary, solid-body solution is $\vec{v} = \dot{\vec{X}} = \omega(-y, x, 0)$, $\omega = $ constant. To satisfy the equation of continuity, the density, the pressure and the temperature depend on $r = \sqrt{x^2 + y^2}$ and z only. Then the only remaining matter equation is,

$$\frac{\omega^2}{2}r^2 - \phi = q, \tag{8.1}$$

wherer q is the chemical potential, reducing to $C_V T$ in the case of an ideal gas, and R is the distance from the center. The gradient of this equation is the universally used hydrostatic equation. It coincides with the Navier-Stokes equation in the case of vanishing viscosity (the Euler equation).

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Besides Eq.(8.1), and the equation of state, the only other equation of Newtonian gravity is Poisson's equation.

The earth

The rotation of the earth is very nearly of the solid-body type. The gradient of $C_V T$ relates to the pressure; it is normal to the surface; therefore the formula for the surface takes the form

$$\frac{\omega^2}{2} r^2 - \phi = \text{constant}.$$

If the rotation is relatively slow the following is a good approximation to the potential,

$$\phi \approx -\frac{MG}{R}. \tag{8.2}$$

In this approximation the formula for the surface takes the form

$$\frac{\omega^2}{2} r^2 + \frac{MG}{R} \approx \text{constant}. \tag{8.3}$$

As the first term increases from the poles to the equator, R must increase also; the distance from the center is greater at the equator (the equatorial bulge). Since the variation of R from an average R_0 is small we can further approximate this as

$$z^2 + r^2 (1 - \frac{\omega^2}{2} \frac{R_0^3}{MG}) = \text{constant}.$$

This is the shape of the earth as given in the text books. It is an oblate spheroid, flattened at the poles.

Recently it has been reported that the central core of the earth rotates at a slower speed than the crust. This can be accommodated by allowing a small amount of potential flow,

$$-\vec{\bigtriangledown} \Phi = a(-y, x, 0), \quad \dot{\vec{X}} = b(-y, x, 0).$$

But a more general treatment is required to take into account the changes of phase between the solid central core, the liquid magma and the solid outer crust. This is possible since the thermodynamic action principle has been generalized to describe such transitions (Fronsdal 2014a).

Galaxies

The problem of rapidly rotating galaxies cannot be treated by Newtonian gravity. The action principle can, in principle at least, be understood as the restriction of a fully relativistic field theory to a particular frame of reference. So far, the lagrangian for rotational motion has been determined in one reference frame, in the case that space derivatives of the field $vecX$ can be neglected. Next, we must calculate the energy momentum tensor and solve Einstein's equations with this rotating source.

References

Andereck, C.D.; Liu, S.S. and Swinney, H.L. (1986). "Flow regimes in a circular Couette system with independently rotating cylinders". Journal of Fluid Mechanics **164** 155-183.

Barnett, S. J., "Magnetization by Rotation". Physical Review 6 (4): 239-270 (1915).

Brenner, H., "Proposal of a critical test of the Navier-Stokes-Fourier paradigm for compressible fluid continua", Phys.Rev. E 87, 013014 (2013).

Brenner, H., "Beyond the no-slip boundary condition", Physical Review E 84, 046309 (2011).

Chandrasekhar, S., *An Introduction to Stellar Structure*, U. Chicago Press 1938.

Couette, M., "Oscillations tournantes d'un solide de rvolution en contact avec un fluide visqueux," Compt. Rend. Acad. Sci. Paris **105**, 1064-1067 (1887).

Couette, M., "Sur un nouvel appareil pour l'étude du frottement des fluides" Compt.Rend. Acad. Sci. Paris **107**, 388-390 (1888).

Couette M., "La viscosit des liquides," Bulletin des Sciences Physiques **4**, 40-62, 123-133, 262-278 (1888).

Couette, M., "Distinction de deux rgimes dans le mouvement des fluides," Journal de Physique [Ser. 2] IX, 414-424 (1890)

M. Couette, M., "Etudes sur le frottement des liquides," Ann. de Chim. et Phys. [Ser. 6] 21, 433?510 (1890).

de Socio, L.M., Ianiro, N. and Marino,L., "Effects of the Centrifugal Forces on a Gas Between Rotating Cylinders", J. Thermophysics and Heat Transfer, **14** No. 2, 2000.

Dukowicz,J.K., Stephen F. Price,S,F and Lipscomb, W.H., "Consistent approximations and boundary conditions for ice-sheet dynamics from a principle of least action", Climate, Ocean and Sea-Ice Modeling (COSIM) Project, Group T-3, MS B216, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Emden, *Gaskugeln*, Teubner 1907.

Faisst, H. and Eckhardt, B., "Transition from the Couette-Taylor system to the plane Couette system", arXiv:physics/0003102v1 [physics.flu-dyn] 30 Mar 2000.

Fetter, A.L. and Walecka, J.D., *Theoretical Mechanics of Particles and Continua*, MacGraw-Hill NY 1980.

Feynman, R.P., "Atomic Theory of the Two-Fluid Model of Liquid Helium", Phys. Rev. **94**, 262-277 (1954).

Fronsdal, C. , Thermodynamics of fluids, a monograph, in progress 2014. Available at fronsdal.physics.ucla.edu.

Fronsdal, C., "Ideal Stars and General Relativity", Gen.Rel.Grav. 39 1971-2000 (2007).

Fronsdal, C., "Heat and Gravitation. The Action Principle", Entropy, 16, 1515-1546 (2014).

(2014). doi:10.3390/c16031515. arXiv 0812.4990v2, revised 2010.

Fronsdal, C. "Relativistic Thermodynamics", 2014; to be published.

Gibbs, J.W., "On the equilibrium of heterogeneous substances" Trans.Conn.Acad. 1878.

Goldstein, D., Handler, R. and Sirovich L., "Modeling a No-Slip Flow Boundary with an external Force Field", J. Comp. Phys. **105** 354(1993).

Hall, H.E. and Vinen, W.F., "The Rotation of Liquid Helium II. II. The Theory of Mutual Friction in Uniformly Rotating Helium II", Proc. R. Soc. Lond. A 1956 238, doi: 10.1098/rspa.1956.0215, 1956.

Joseph, D.D. and Renardy, Y., "Couette flow of two fluids between concentric cylinders", J . Fluid Mech. (1985). 150, 381-394 (1985).

Khalatnikov, I.M., *An Introduction to the Theory of Superfluidity*, Westview Press 2000. Original published as part of Frontiers in Physics SEries, D.Pines (ed.) 1965.

Landau, L. D., and Lifschitz, E. M., Doklady Akad. NAUK **100** 669 (1955).

Landau, L. D., and Lifschitz, E. M., *Statistical Physics*, Pergamon Press, (1958).

Mass maintaining its Volume by its internal Heat, and depending on the laws of gases as known to terrestrial Experiment, Amer.J.Sci.Arts, Series 2, **4**, 57- (1870).

Müller, I., *A History of Thermodynamics*, Springer, Berlin 2007.

Navier, L Mm. Acad. Sci. **7** (1827) pp. 375?394 (1827).

Navier,C.L.M.H., "Memoire sur les lois du mouvement des fluides", Mem. Acad. Sci. Inst. France, **6** 389-440 (1882).

Priezjev, N.V., Darhuber, A.A. and Troian, S.M., "Slip length in sheared liquid films subject to mixed boundary conditions," Phys. Rev. E 71, 041608 (2005).

Prigogine, I., "Evolution criteria, variational properties and fluctuations", in "Thermodynamics, variational techniques and stability,"Donnelly et al, U. Chicago Press 1965.

Sedov, L.I., "A course in continuous mechanics", Woolters-Noordhoff, Groeningen, 1971.

Seebeck, T.J., "Magnetische Polarisation der Metalle und Erze durch Temperatur-Differenz" (Magnetic polarization of metals and minerals by temperature differences), Abhandlungen der Königlichen Akad. Wissenschaften, Berlin, pp. 265-373 (1825).

Stanyukovich, K.P., *Unsteady motion of continuous media*, Pergamon Press N. Y. (1960).

G.G. Stokes, G.G., Trans. Cambridge Phil. Soc. , 8 (1843) pp. 287-319.