Evaluating multiloop Feynman integrals by Mellin–Barnes representation

V.A. Smirnov

Nuclear Physics Institute of Moscow State University

- Introduction. Evaluating Feynman integrals
- Mellin–Barnes representation. Simple one-loop examples
- General prescriptions. Multiple Mellin–Barnes integrals
- Two- and three-loop examples. A review of results
- Perspectives

details can be found in
V.A. Smirnov, *Evaluating Feynman integrals* (Springer 2004)

Introduction

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

$d = 4 - 2\epsilon$

The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0}\,, \qquad k^2 = k_0^2 - \vec{k}^2$$

UV, IR and collinear divergences

Regularization.

Dimensional regularization. Formally,

$$\mathrm{d}^4 k = \mathrm{d}k_0 \vec{k} \quad \rightarrow \quad \mathrm{d}^d k$$

where $d = 4 - 2\epsilon$

Informally, use alpha parameters

$$\frac{1}{(-k^2 + m^2 - i0)^a} = \frac{\mathrm{i}^a}{\Gamma(a)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{a-1} \mathrm{e}^{\mathrm{i}(p^2 - m^2)\alpha}$$

change the order of integration, take Gauss integrals over the loop momenta

$$\int \mathrm{d}^4 k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = -\mathrm{i}\pi^2 \alpha^{-2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

$\longrightarrow$

$$\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = \mathrm{e}^{\mathrm{i}\pi(1-d/2)/2} \pi^{d/2} \alpha^{-d/2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

Graph $\Gamma \rightarrow$

$$F_\Gamma(a_1 \ldots, a_L; d) \;=\; \frac{\mathrm{i}^{a+h(1-d/2)}\pi^{hd/2}}{\prod_l \Gamma(a_l)} \times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} \mathrm{e}^{\mathrm{i}\mathcal{V}/\mathcal{U} - \mathrm{i}\sum m_l^2 \alpha_l} \,,$$

where

$$\mathcal{U} \;=\; \sum_{\text{trees } T} \; \prod_{l \notin T} \alpha_l \,,$$

$$\mathcal{V} \;=\; \sum_{2-\text{trees } T} \; \prod_{l \notin T} \alpha_l \left(q^T\right)^2 \,.$$

One can deal with dimensionally regularized Feynman integrals as with usual integrals. They are even better ;-) Use integration by parts (IBP) and always neglect surface terms.

Methods to evaluate Feynman integrals: analytical, numerical, semianalytical . . .

A straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An advanced strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

A general integral of the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Methods to evaluate master integrals:

- Feynman/alpha parameters
- Mellin–Barnes representation
- method of differential equations

Mellin transformation, Mellin integrals as a tool for Feynman integrals: [M.C. Bergère & Y.-M.P. Lam'74]

Evaluating individual Feynman integrals:

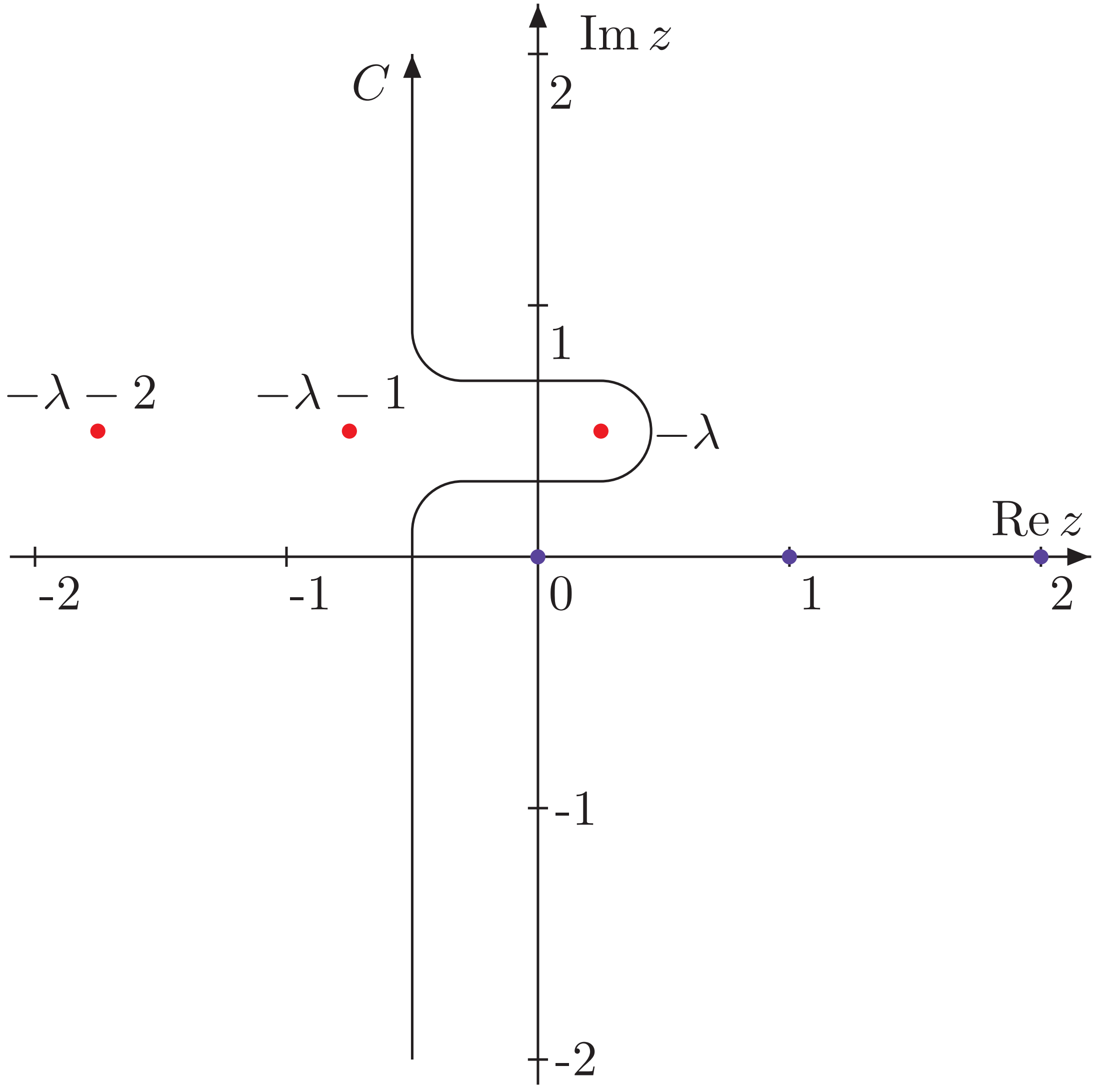
[N.I. Ussyukina'75. . . , A.I. Davydychev'89. . . ,]

Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ) [V.A. Smirnov'99, J.B. Tausk'99]

The basic formula:

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z) \; .$$

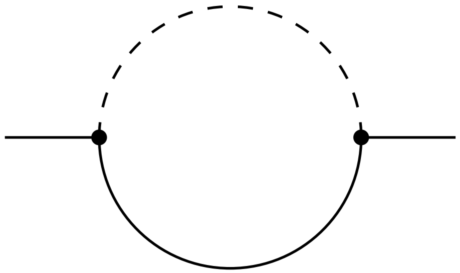
The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right


Im z
C
2
1
−λ − 2
−λ − 1
−λ
Re z
-2
-1
0
1
2
-1
-2

The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

Example 1



$$F_\Gamma(q^2,m^2;a_1,a_2,d) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1,a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} \; ,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

$$F_\Gamma(q^2,m^2;a_1,a_2,d) = \frac{\mathrm{i}\pi^{d/2}(-1)^{a_1+a_2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z) \times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)}$$

In particular,

$$F_\Gamma(2,1,4) = \frac{\mathrm{i}\pi^2}{q^2}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(1+z)\Gamma(-z)^2}{\Gamma(1-z)}$$

with $-1 < \mathrm{Re} z < 0$

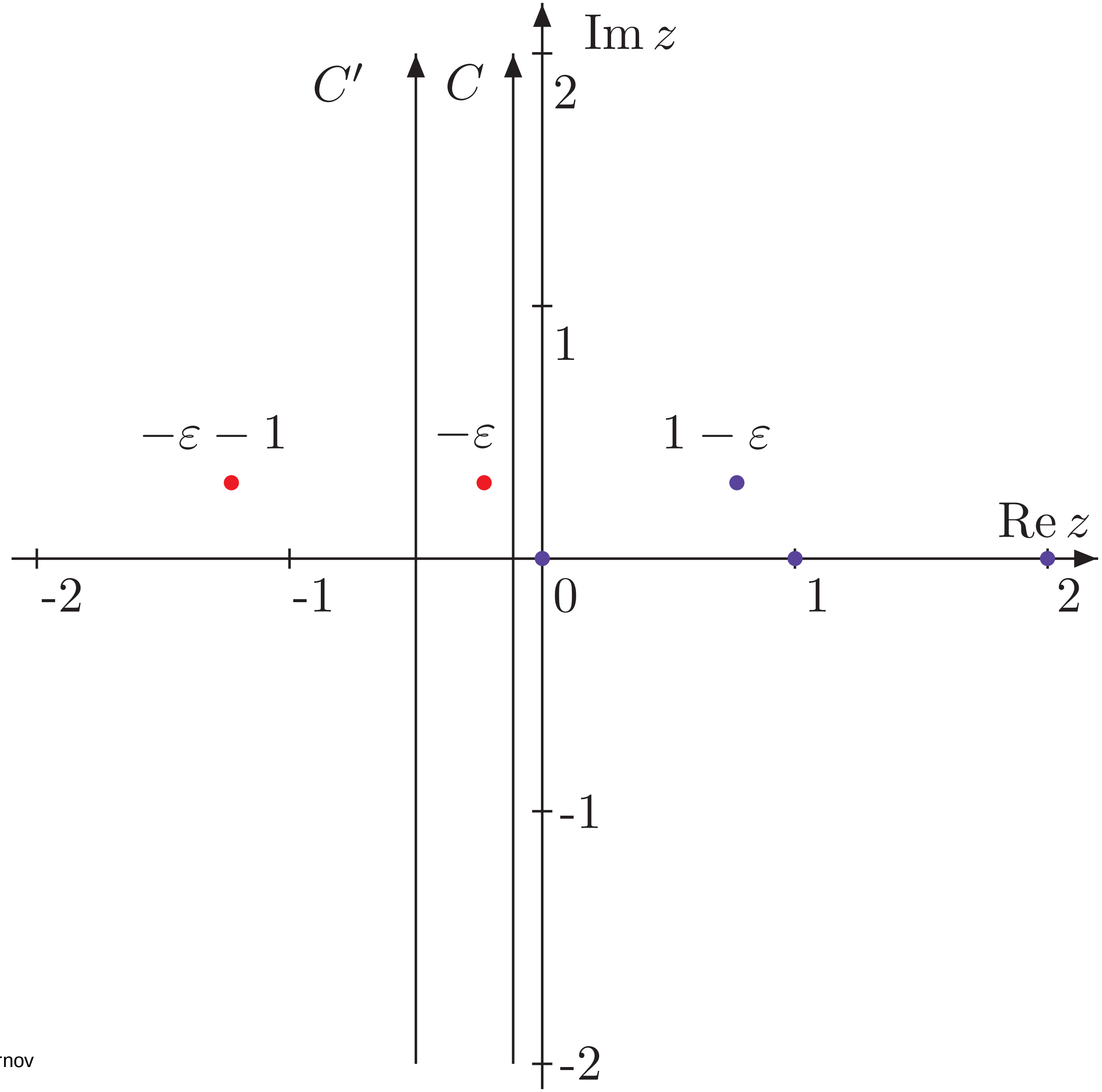
Closing the integration contour to the right and take a series of residues at the points $z = 0, 1, 2, \ldots \to$

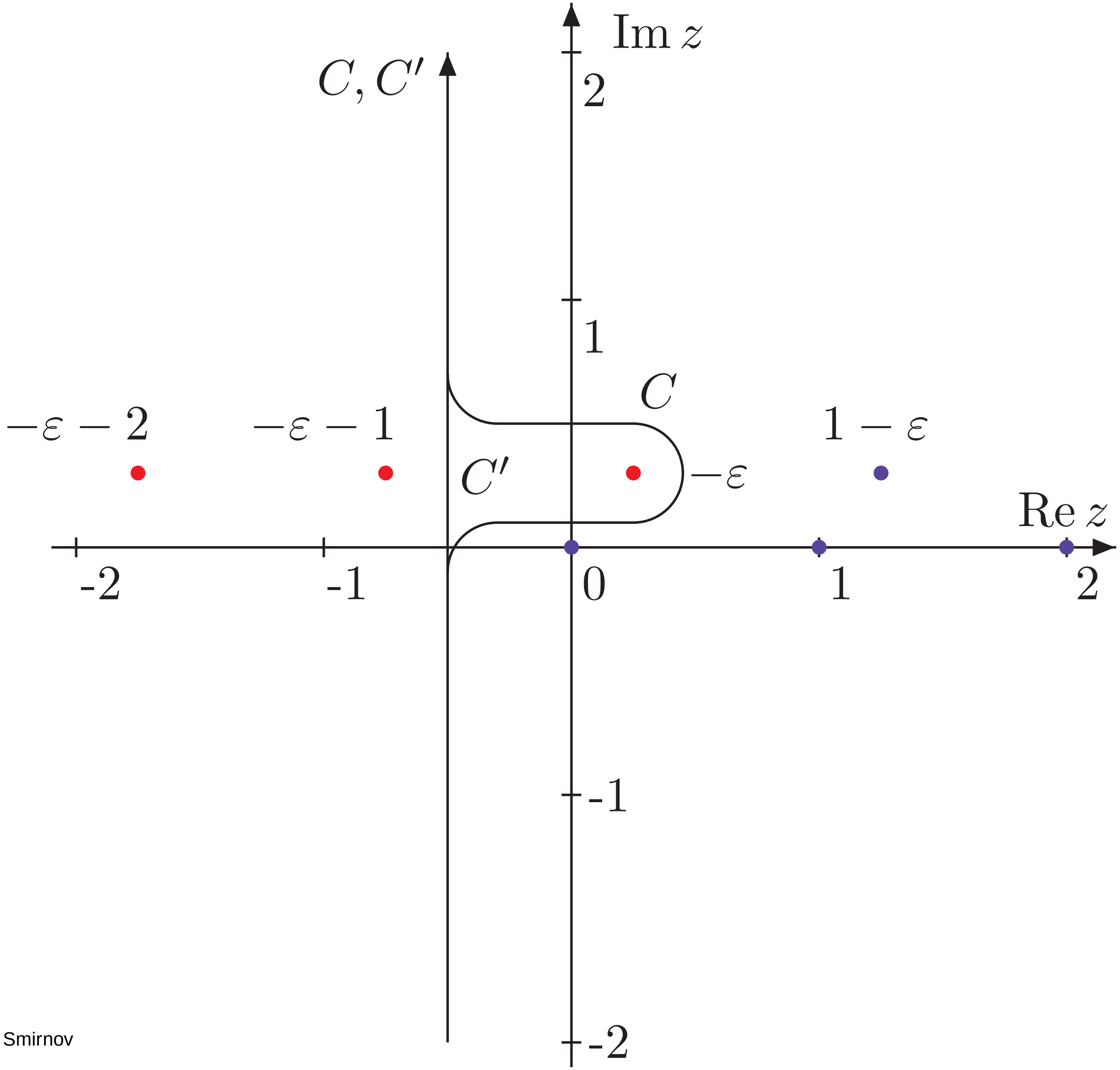
$$F_\Gamma(2,1,4) = \mathrm{i}\pi^2 \frac{\ln\left(1 - q^2/m^2\right)}{q^2}$$

$$F_\Gamma(q^2, m^2; 1, 1, d) \;=\; \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}$$

$$\times \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

$\Gamma(\epsilon+z)\Gamma(-z) \to$ a singularity in ϵ


Im z
C'
C
2
1
$-\varepsilon-1$
$-\varepsilon$
$1-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2


Im z
C, C'
2
1
C
$-\varepsilon - 2$
$-\varepsilon - 1$
$1 - \varepsilon$
C'
$-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2

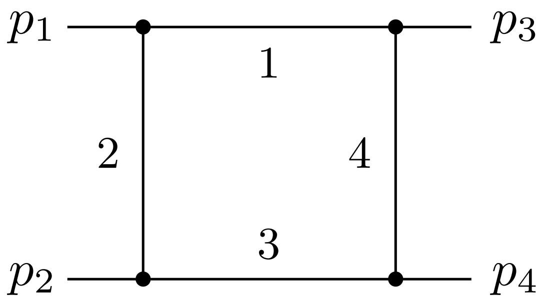
Take a residue at $z = -\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)}$$

and shift the contour:

$$\mathrm{i}\pi^2 \frac{1}{2\pi \mathrm{i}} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z}$$

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0,\ i = 1, 2, 3, 4$



$$
F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)
$$

$$
= \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_3)^2]^{a_4}} ,
$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)$$

$$= (-1)^a \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}$$

$$\times \int_0^1 \int_0^1 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\epsilon-2}} ,$$

where $a = a_1 + a_2 + a_3 + a_4$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

Change the order of integration over z and ξ-parameters, evaluate parametric integrals in terms of gamma functions

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

$$\times\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$

$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

```
(*   The integrand of the MB integral for the one-
   loop massless box diagram with  p1^2=p2^2=p3^2=p4^2=0                  *)
```

In[1]:= `Box1[a1_, a2_, a3_, a4_] :=`

$(S^{2-a1-a2-a3-a4-ep-z}\ T^{z}\ \text{Gamma}[a1+a2+a3+a4-2+ep+z]\ \text{Gamma}[a2+z]$
$\text{Gamma}[a4+z]\ \text{Gamma}[2-a1-a2-a4-ep-z]\ \text{Gamma}[2-a2-a3-a4-ep-z]\ \text{Gamma}[-z])\ /$
$(\text{Gamma}[a1]\ \text{Gamma}[a2]\ \text{Gamma}[a3]\ \text{Gamma}[a4]\ \text{Gamma}[4-a1-a2-a3-a4-2\,ep]);$

```
(*  Notation:
   s=(p1+p2)^2=-S, t=(p1+p2)^2=-T;
 I Pi^(d/2) is pulled out, as always  *)
```

```
(*   The box with the powers of the propagators equal to one *)
```

In[2]:= `Box1[1, 1, 1, 1]`

Out[2]= $\frac{1}{\text{Gamma}[-2\,ep]}(S^{-2-ep-z}\ T^{z}\ \text{Gamma}[-1-ep-z]^{2}\ \text{Gamma}[-z]\ \text{Gamma}[1+z]^{2}\ \text{Gamma}[2+ep+z])$

In[3]:= `% /. {S → 1, T → x}`

Out[3]= $\frac{x^{z}\ \text{Gamma}[-1-ep-z]^{2}\ \text{Gamma}[-z]\ \text{Gamma}[1+z]^{2}\ \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}$

(* The initial quantity is obtained from the last one by the change x→ T/S and multiplying by S^{-2-ep} *)

(* The product $\text{Gamma}[1+z]^{2}\ \text{Gamma}[-1-ep-z]^{2}$ generates a singularity in ep when ep→0.
The first pole of Gamma[1+z] and the first pole of Gamma[-1-ep-z] glue together at ep=0.
(There is no place for a contour between these poles.)
We take minus residue
at z=-1-ep
and shift the contour so that the nature of
the first pole of Gamma[-1-ep-z] in the new integral changes.

Let us evaluate the box in expansion in ep up to ep^1.
*)

In[4]:= `-Residue[%3, {z, -1 - ep}]`

Out[4]= $-\frac{1}{\text{Gamma}[-2\,ep]}(x^{-1-ep}\ \text{Gamma}[-ep]^{2}\ \text{Gamma}[1+ep]$
$(\text{EulerGamma}+\text{Log}[x]+2\,\text{PolyGamma}[0,-ep]-\text{PolyGamma}[0,1+ep]))$

In[5]:= `FullSimplify[Normal[Series[% E^(EulerGamma ep), {ep, 0, 1}]]]`

Out[5]= $\frac{1}{6\,ep^{2}\,x}(24-8\,ep^{2}\,\pi^{2}+ep\,(-12+7\,ep^{2}\,\pi^{2})\ \text{Log}[x]+2\,ep^{3}\ \text{Log}[x]^{3}-68\,ep^{3}\ \text{Zeta}[3])$

```
(* Expanding in ep in the integral over the new contour *)
```

(*   The integration is at $-1<\text{Re}(z)<0$ *)

In[6]:= `Simplify[Normal[Series[%3 E^(EulerGamma ep), {ep, 0, 1}]]]`

Out[6]= $-2\,ep\,x^{z}\ \text{Gamma}[-1-z]^{2}\ \text{Gamma}[-z]\ \text{Gamma}[1+z]^{2}\ \text{Gamma}[2+z]$

```
(*  The MB integral can be evaluated by closing the
   integration contour to the right in the complex z-plane.      *)
```

```
In[7]:= Simplify[% //. {Gamma[-1 - z] → Gamma[-z] / (-1 - z), Gamma[2 + z] → Gamma[1 + z] (1 + z) }]
```

Out[7]= $-\frac{2\,ep\,x^z\,Gamma[-z]^3\,Gamma[1+z]^3}{1+z}$

```
In[8]:= % /. {Gamma[-z]^3 Gamma[1 + z]^3 → -π^3 Csc[π z]^3}
```

Out[8]= $\frac{2\,ep\,\pi^3\,x^z\,Csc[\pi z]^3}{1+z}$

```
(*  Now we take residues at z=0,1,2,.,..    *)

In[9]:= % /. z → z + n
```

Out[9]= $\frac{2\,ep\,\pi^3\,x^{n+z}\,Csc[\pi (n+z)]^3}{1+n+z}$

```
In[10]:= % /. {Csc[π (n + z)] → (-1)^n Csc[π z] };

In[11]:= % /. { (-1)^(3 n) → (-1)^n}
```

Out[11]= $\frac{2\,(-1)^n\,ep\,\pi^3\,x^{n+z}\,Csc[\pi z]^3}{1+n+z}$

```
In[12]:= -Residue[%, {z, 0}]
```

Out[12]= $-\frac{1}{(1+n)^3}\left((-1)^n\,ep\,x^n\,(2+\pi^2+2\,n\,\pi^2+n^2\,\pi^2-2\,Log[x]-2\,n\,Log[x]+Log[x]^2+2\,n\,Log[x]^2+n^2\,Log[x]^2)\right)$

```
In[13]:= Apart[% /. n → n - 1, n]
```

Out[13]= $\frac{2\,(-1)^n\,ep\,x^{-1+n}}{n^3}-\frac{2\,(-1)^n\,ep\,x^{-1+n}\,Log[x]}{n^2}+\frac{(-1)^n\,ep\,x^{-1+n}\,(\pi^2+Log[x]^2)}{n}$

```
In[14]:= Sum[%, {n, 1, Infinity}]
```

Out[14]= $-\frac{1}{x}(ep\,(\pi^2\,Log[1+x]+Log[x]^2\,Log[1+x]+2\,Log[x]\,PolyLog[2,-x]-2\,PolyLog[3,-x]))$

```
(*  Numerical check    *)

In[33]:= %14 /. {x → 0.76, ep → 0.3}

Out[33]= -2.91293

In[34]:= NIntegrate[%6 /. {x → 0.76, ep → 0.3, z → -0.5 + I * y1}, {y1, -5, 5}] / 2 / Pi

Out[34]= -2.91293 + 0. i

In[16]:= %14 + %5
```

Out[16]= $-\frac{1}{x}(ep\,(\pi^2\,Log[1+x]+Log[x]^2\,Log[1+x]+2\,Log[x]\,PolyLog[2,-x]-2\,PolyLog[3,-x]))+\frac{1}{6\,ep^2\,x}(24-8\,ep^2\,\pi^2+ep\,(-12+7\,ep^2\,\pi^2)\,Log[x]+2\,ep^3\,Log[x]^3-68\,ep^3\,Zeta[3])$

```
(*  This is our result (up to I Pi^(d/2) ) *)
```

In[17]:= `(% /. x → T / S ) S^(-2-ep)`

Out[17]=
$$S^{-2-ep}\left(-\frac{1}{T}\left(ep\,S\left(\pi^2\,Log\left[1+\frac{T}{S}\right]+Log\left[\frac{T}{S}\right]^2 Log\left[1+\frac{T}{S}\right]+2\,Log\left[\frac{T}{S}\right]PolyLog\left[2,-\frac{T}{S}\right]-2\,PolyLog\left[3,-\frac{T}{S}\right]\right)\right)+\frac{1}{6\,ep^2\,T}\left(S\left(24-8\,ep^2\,\pi^2+ep\,(-12+7\,ep^2\,\pi^2)\,Log\left[\frac{T}{S}\right]+2\,ep^3\,Log\left[\frac{T}{S}\right]^3-68\,ep^3\,Zeta[3]\right)\right)\right)$$

```
(*  Suppose now that we want to evaluate the box
    in expansion in ep up to ep^2.
 *)
```

In[4]:= `-Residue[%3, {z, -1 - ep}]`

Out[4]=
$$-\frac{1}{Gamma[-2\,ep]}\left(x^{-1-ep}\,Gamma[-ep]^2\,Gamma[1+ep]\,(EulerGamma+Log[x]+2\,PolyGamma[0,-ep]-PolyGamma[0,1+ep])\right)$$

In[36]:= `FullSimplify[Normal[Series[%4 E^(EulerGamma ep), {ep, 0, 2}]]]`

Out[36]=
$$\frac{1}{360\,ep^2\,x}\left(1440-480\,ep^2\,\pi^2-41\,ep^4\,\pi^4-60\,ep\,Log[x]\,(12-7\,ep^2\,\pi^2+ep^2\,Log[x]\,(3\,ep\,\pi^2+Log[x]\,(-2+ep\,Log[x])))+240\,ep^3\,(-17+10\,ep\,Log[x])\,Zeta[3]\right)$$

```
(* Expanding in ep in the integral over the new contour *)

(*   The integration is at -1<Re (z)<0    *)
```

In[37]:= `Simplify[Normal[Series[%3 E^(EulerGamma ep), {ep, 0, 2}]]]`

Out[37]=
$$2\,ep\,x^z\,Gamma[-1-z]^2\,Gamma[-z]\,Gamma[1+z]^2\,Gamma[2+z]\,(-1+ep\,EulerGamma+2\,ep\,PolyGamma[0,-1-z]-ep\,PolyGamma[0,2+z])$$

In[38]:= `Simplify[% //. { Gamma[-1 - z] → Gamma[-z] / (-1 - z), Gamma[2 + z] → Gamma[1 + z] (1 + z) }]`

Out[38]=
$$\frac{1}{1+z}\left(2\,ep\,x^z\,Gamma[-z]^3\,Gamma[1+z]^3\,(-1+ep\,EulerGamma+2\,ep\,PolyGamma[0,-1-z]-ep\,PolyGamma[0,2+z])\right)$$

In[39]:= `Simplify[% //. {PolyGamma[0, -1 - z] → PolyGamma[0, -z] - (1 / (-1 - z)),`
`PolyGamma[0, 2 + z] → PolyGamma[0, 1 + z] + 1 / (1 + z)}]`

Out[39]=
$$\frac{1}{(1+z)^2}\left(2\,ep\,x^z\,Gamma[-z]^3\,Gamma[1+z]^3\,(-1+ep+ep\,EulerGamma-z+ep\,EulerGamma\,z+2\,ep\,(1+z)\,PolyGamma[0,-z]-ep\,(1+z)\,PolyGamma[0,1+z])\right)$$

In[40]:= `% /. {`
`PolyGamma[0, -z] → PolyGamma[0, 1 + z] + π Cot[π z]}`

Out[40]=
$$\frac{1}{(1+z)^2}\left(2\,ep\,x^z\,Gamma[-z]^3\,Gamma[1+z]^3\,(-1+ep+ep\,EulerGamma-z+ep\,EulerGamma\,z-ep\,(1+z)\,PolyGamma[0,1+z]+2\,ep\,(1+z)\,(\pi\,Cot[\pi\,z]+PolyGamma[0,1+z]))\right)$$

```
In[41]:= % /. {Gamma[-z]^3 Gamma[1 + z]^3 → -π^3 Csc[π z]^3}

Out[41]= -1/(1 + z)^2 (2 ep π^3 x^z Csc[π z]^3 (-1 + ep + ep EulerGamma - z + ep EulerGamma z -
      ep (1 + z) PolyGamma[0, 1 + z] + 2 ep (1 + z) (π Cot[π z] + PolyGamma[0, 1 + z])))

        (*  Now we take residues at z=0,1,2,.,..   *)

In[42]:= % /. z → z + n

Out[42]= -1/(1 + n + z)^2 (2 ep π^3 x^(n+z) Csc[π (n + z)]^3
      (-1 + ep + ep EulerGamma - n - z + ep EulerGamma (n + z) - ep (1 + n + z) PolyGamma[0, 1 + n + z] +
        2 ep (1 + n + z) (π Cot[π (n + z)] + PolyGamma[0, 1 + n + z])))

In[43]:= % /. {Csc[π (n + z)] → (-1)^n Csc[π z] , Cot[π (n + z)] →  Cot[π z] };

In[44]:= % /. { (-1)^(3 n) → (-1)^n}

Out[44]= -1/(1 + n + z)^2
    (2 (-1)^n ep π^3 x^(n+z) Csc[π z]^3 (-1 + ep + ep EulerGamma - n - z + ep EulerGamma (n + z) - ep
          (1 + n + z) PolyGamma[0, 1 + n + z] + 2 ep (1 + n + z) (π Cot[π z] + PolyGamma[0, 1 + n + z])))

In[45]:=  -Residue[%, {z, 0}]

Out[45]= 1/(3 (1 + n)^4) ((-1)^n ep x^n
    (-6 + 6 ep + 6 ep EulerGamma - 6 n + 6 ep EulerGamma n - 3 π^2 + ep π^2 + 3 ep EulerGamma π^2 -
      9 n π^2 + 2 ep n π^2 + 9 ep EulerGamma n π^2 - 9 n^2 π^2 + ep n^2 π^2 + 9 ep EulerGamma n^2 π^2 -
      3 n^3 π^2 + 3 ep EulerGamma n^3 π^2 + 6 Log[x] - 6 ep EulerGamma Log[x] + 12 n Log[x] -
      12 ep EulerGamma n Log[x] + 6 n^2 Log[x] - 6 ep EulerGamma n^2 Log[x] +
      2 ep π^2 Log[x] + 6 ep n π^2 Log[x] + 6 ep n^2 π^2 Log[x] + 2 ep n^3 π^2 Log[x] -
      3 Log[x]^2 - 3 ep Log[x]^2 + 3 ep EulerGamma Log[x]^2 - 9 n Log[x]^2 -
      6 ep n Log[x]^2 + 9 ep EulerGamma n Log[x]^2 - 9 n^2 Log[x]^2 - 3 ep n^2 Log[x]^2 +
      9 ep EulerGamma n^2 Log[x]^2 - 3 n^3 Log[x]^2 + 3 ep EulerGamma n^3 Log[x]^2 + 2 ep Log[x]^3 +
      6 ep n Log[x]^3 + 6 ep n^2 Log[x]^3 + 2 ep n^3 Log[x]^3 + 6 ep PolyGamma[0, 1 + n] +
      6 ep n PolyGamma[0, 1 + n] + 3 ep π^2 PolyGamma[0, 1 + n] + 9 ep n π^2 PolyGamma[0, 1 + n] +
      9 ep n^2 π^2 PolyGamma[0, 1 + n] + 3 ep n^3 π^2 PolyGamma[0, 1 + n] -
      6 ep Log[x] PolyGamma[0, 1 + n] - 12 ep n Log[x] PolyGamma[0, 1 + n] -
      6 ep n^2 Log[x] PolyGamma[0, 1 + n] + 3 ep Log[x]^2 PolyGamma[0, 1 + n] +
      9 ep n Log[x]^2 PolyGamma[0, 1 + n] + 9 ep n^2 Log[x]^2 PolyGamma[0, 1 + n] +
      3 ep n^3 Log[x]^2 PolyGamma[0, 1 + n] - 6 ep PolyGamma[1, 1 + n] -
      12 ep n PolyGamma[1, 1 + n] - 6 ep n^2 PolyGamma[1, 1 + n] + 6 ep Log[x] PolyGamma[1, 1 + n] +
      18 ep n Log[x] PolyGamma[1, 1 + n] + 18 ep n^2 Log[x] PolyGamma[1, 1 + n] +
      6 ep n^3 Log[x] PolyGamma[1, 1 + n] + 3 ep PolyGamma[2, 1 + n] +
      9 ep n PolyGamma[2, 1 + n] + 9 ep n^2 PolyGamma[2, 1 + n] + 3 ep n^3 PolyGamma[2, 1 + n]))
```

```
In[46]:= Apart[% /. n → n - 1, n]

Out[46]= - 2 (-1)^n ep^2 x^(-1+n) / n^4 - 2 (-1)^n ep x^(-1+n) (-1 + ep EulerGamma + ep PolyGamma[0, n]) / n^3 -
   1/(3 n^2) ((-1)^n ep x^(-1+n) (ep π^2 + 6 Log[x] - 6 ep EulerGamma Log[x] -
      3 ep Log[x]^2 - 6 ep Log[x] PolyGamma[0, n] - 6 ep PolyGamma[1, n])) -
   1/(3 n) ((-1)^n ep x^(-1+n) (-3 π^2 + 3 ep EulerGamma π^2 + 2 ep π^2 Log[x] - 3 Log[x]^2 +
      3 ep EulerGamma Log[x]^2 + 2 ep Log[x]^3 + 3 ep π^2 PolyGamma[0, n] +
      3 ep Log[x]^2 PolyGamma[0, n] + 6 ep Log[x] PolyGamma[1, n] + 3 ep PolyGamma[2, n]))

(*  Mathematica does not work here  ;-(      *)

In[47]:= Sum[%, {n, 1, Infinity}]

(* OK,     let us help it.
   See the file
  1 lB0MBaux.nb
       *)

In[90]:= %87 + %36

Out[90]= 1/(360 ep^2 x) (1440 - 480 ep^2 π^2 - 41 ep^4 π^4 -
    60 ep Log[x] (12 - 7 ep^2 π^2 + ep^2 Log[x] (3 ep π^2 + Log[x] (-2 + ep Log[x]))) +
    240 ep^3 (-17 + 10 ep Log[x]) Zeta[3]) - 1/(6 x)
   (ep (6 π^2 Log[1 + x] - 6 ep π^2 Log[x] Log[1 + x] + 6 Log[x]^2 Log[1 + x] -
      4 ep Log[x]^3 Log[1 + x] + 3 ep π^2 Log[1 + x]^2 + 6 ep Log[-x] Log[x] Log[1 + x]^2 +
      3 ep Log[x]^2 Log[1 + x]^2 - 6 Log[x] (-2 + ep Log[x] - 2 ep Log[1 + x]) PolyLog[2, -x] +
      12 ep Log[x] Log[1 + x] PolyLog[2, 1 + x] - 12 PolyLog[3, -x] -
      12 ep Log[1 + x] PolyLog[3, -x] - 12 ep Log[x] PolyLog[3, 1 + x] + 12 ep PolyLog[4, -x] -
      12 ep PolyLog[2, 2, -x] + 12 ep Log[x] Zeta[3] + 12 ep Log[1 + x] Zeta[3]))

(*  This is our result (up to I Pi^(d/2) ) *)

In[91]:= (% /. x → T / S) S^(-2-ep)

Out[91]= S^(-2-ep) (1/(360 ep^2 T) (S (1440 - 480 ep^2 π^2 - 41 ep^4 π^4 -
      60 ep Log[T/S] (12 - 7 ep^2 π^2 + ep^2 Log[T/S] (3 ep π^2 + Log[T/S] (-2 + ep Log[T/S])))) +
      240 ep^3 (-17 + 10 ep Log[T/S]) Zeta[3])) - 1/(6 T)
   (ep S (6 π^2 Log[1 + T/S] - 6 ep π^2 Log[T/S] Log[1 + T/S] + 6 Log[T/S]^2 Log[1 + T/S] -
      4 ep Log[T/S]^3 Log[1 + T/S] + 3 ep π^2 Log[1 + T/S]^2 + 6 ep Log[-T/S] Log[T/S] Log[1 + T/S]^2 +
      3 ep Log[T/S]^2 Log[1 + T/S]^2 - 6 Log[T/S] (-2 + ep Log[T/S] - 2 ep Log[1 + T/S])
       PolyLog[2, -T/S] + 12 ep Log[T/S] Log[1 + T/S] PolyLog[2, 1 + T/S] -
      12 PolyLog[3, -T/S] - 12 ep Log[1 + T/S] PolyLog[3, -T/S] -
      12 ep Log[T/S] PolyLog[3, 1 + T/S] + 12 ep PolyLog[4, -T/S] -
      12 ep PolyLog[2, 2, -T/S] + 12 ep Log[T/S] Zeta[3] + 12 ep Log[1 + T/S] Zeta[3])))
```

General recipes

- Derive a (multiple) MB representation for general powers of the propagators. (The number of MB integrations can be large (more than 10)).
  - Use it for checks. Reducing a line to a point $\to$ tending a_i to zero $\to$ (usually) taking some residues. A typical situation:
    $$\frac{\Gamma(a_2+z)\Gamma(-z)}{\Gamma(a_2)}, \quad a_2 \to 0$$
    Gluing of poles of different nature. Take a (minus) residue at $z_2 = 0$, then set $a_2 = 0$.
  - Unambiguous prescriptions for choosing integration contours
  - Try to have a minimal number of MB integrations.

- Resolve the singularity structure in ϵ. The goal: to represent a given MB integral as a sum of integrals where a Laurent expansion in ϵ becomes possible.
  The basic procedure:
  take residues and shift contours
  Two strategies:
  - #1 [V.A. Smirnov'99]
    E.g., the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$ generates a pole of the type $\Gamma(-\epsilon)$.
    The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$. 'Key' gamma functions

- #2 [J.B. Tausk'99, Anastasiou'05, Czakon'05].

  Lecture by T. Riemann.

  Two algorithmic descriptions [C. Anastasiou'05, M. Czakon'05]

  The Czakon's version is already implemented in `Mathematica`!

- Evaluate MB integrals after expansion in ϵ

In the last step:
Apply the first and the second Barnes lemmas

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$

$$= \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_6 + z)}$$

$$= \frac{\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_4)\Gamma(\lambda_3 + \lambda_4)\Gamma(\lambda_1 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5)\Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)}$$

$$\times \frac{\Gamma(\lambda_2 + \lambda_5)\Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)} , \qquad \lambda_6 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$$

multiple corollaries, e.g.,

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma^*(\lambda_2 + z)\Gamma(-\lambda_2 - z)\Gamma(\lambda_3 - z)$$
$$= \Gamma(\lambda_1 - \lambda_2)\Gamma(\lambda_2 + \lambda_3) \left[\psi(\lambda_1 - \lambda_2) - \psi(\lambda_1 + \lambda_3)\right]$$

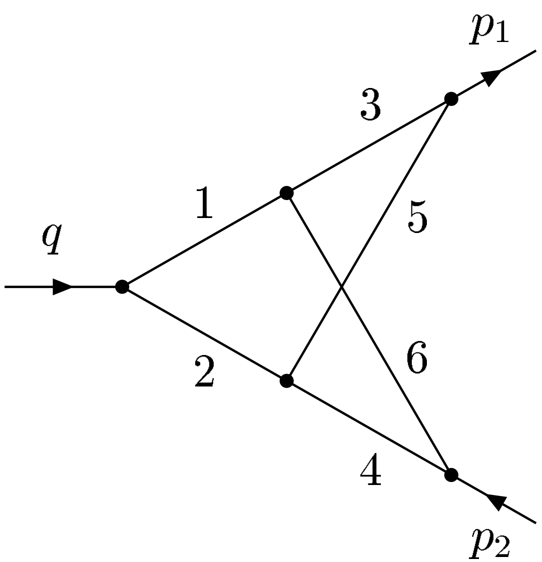
Use SUMMER [J.A.M. Vermaseren'00]

IBP is also possible, e.g.

$$\int_C \mathrm{d}z \frac{f(z)}{z^2} = \int_C \mathrm{d}z \frac{f'(z)}{z}$$

Example 3. Non-planar two-loop massless vertex diagram

with $p_1^2 = p_2^2 = 0$, $Q^2 = -(p_1 - p_2)^2 = 2 p_1 \cdot p_2$



$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) \;=\; \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{[(k+l)^2 - 2p_1\cdot(k+l)]^{a_1}}$$

$$\times \frac{1}{[(k+l)^2 - 2p_2\cdot(k+l)]^{a_2}(k^2 - 2p_1\cdot k)^{a_3}(l^2 - 2p_2\cdot l)^{a_4}(k^2)^{a_5}(l^2)^{a_6}}$$

Gonsalves'83:

$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) \;=\; \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-\epsilon-a_{35})\Gamma(2-\epsilon-a_{46})}{(Q^2)^{a+2\epsilon-4} \prod \Gamma(a_l)\; \Gamma(4-2\epsilon-a_{3456})}$$

$$\times\Gamma(a+2\epsilon-4)\int_0^1 \mathrm{d}\xi_1 \ldots \int_0^1 \mathrm{d}\xi_4\, \xi_1^{a_3-1}(1-\xi_1)^{a_5-1}\xi_2^{a_4-1}(1-\xi_2)^{a_6-1}$$

$$\times\xi_3^{a_1-1}\xi_4^{a_2-1}(1-\xi_3-\xi_4)_+^{a_{3456}+\epsilon-3} A(\xi_1,\xi_2,\xi_3,\xi_4)^{4-2\epsilon-a}\;,$$

where

$$A(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_3\xi_4 + (1-\xi_3-\xi_4)[\xi_2\xi_3(1-\xi_1)+\xi_1\xi_4(1-\xi_2)]$$

$$\frac{\Gamma(a+2\epsilon-4)}{\left[\eta\xi(1-\xi)+(1-\eta)(\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2))\right]^{a+2\epsilon-4}}$$

$$= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\Gamma(-z_1)\eta^{z_1}\xi^{z_1}(1-\xi)^{z_1}}{(1-\eta)^{a+2\epsilon-4+z_1}}$$

$$\times \frac{\Gamma(a+2\epsilon-4+z_1)}{\left[\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2)\right]^{a+2\epsilon-4+z_1}}$$

The last line $\rightarrow$

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_2\,\Gamma(a+2\epsilon-4+z_1+z_2)\Gamma(-z_2)\xi^{z_2}\xi_2^{z_2}(1-\xi_1)^{z_2}}{(1-\xi)^{a+2\epsilon-4+z_1+z_2}\xi_1^{a+2\epsilon-4+z_1+z_2}(1-\xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-\epsilon-a_{35})}{(Q^2)^{a+2\epsilon-4}\Gamma(6-3\epsilon-a)\prod\Gamma(a_l)}$$

$$\times \frac{\Gamma(2-\epsilon-a_{46})}{\Gamma(4-2\epsilon-a_{3456})} \frac{1}{(2\pi\mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_1 \mathrm{d}z_2 \Gamma(a+2\epsilon-4+z_1+z_2)$$

$$\times \Gamma(-z_1)\Gamma(-z_2)\Gamma(a_4+z_2)\Gamma(a_5+z_2)\Gamma(a_1+z_1+z_2)$$

$$\times \frac{\Gamma(2-\epsilon-a_{12}-z_1)\Gamma(4-2\epsilon+a_2-a-z_2)}{\Gamma(4-2\epsilon-a_{1235}-z_1)\Gamma(4-2\epsilon-a_{1246}-z_1)}$$

$$\times \Gamma(4-2\epsilon+a_3-a-z_1-z_2)\Gamma(4-2\epsilon+a_6-a-z_1-z_2) \; ,$$

where $a_{3456} = a_3 + a_4 + a_5 + a_6$, etc.

```
(*   2 fold MB representation for the non-planar vertex massless diagram.
     The factor QQ^(4-a1-a2-a3-a4-a5-a6-2 ep) is omitted.
     QQ=-(p1-p2)^2.
   The factor (I Pi^(d/2))^2 is also omitted as usually.  *)

In[1]:= NPMB[a1_, a2_, a3_, a4_, a5_, a6_] := ((-1)^(a1 + a2 + a3 + a4 + a5 + a6) /
          (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[a5] Gamma[a6]) Gamma[2 - ep - a3 - a5]
        Gamma[2 - ep - a4 - a6] / Gamma[4 - 2 ep - a3 - a4 - a5 - a6] / Gamma[6 - 3 ep - a1 - a2 - a3 - a4 - a5
        Gamma[a1 + a2 + a3 + a4 + a5 + a6 + 2 ep - 4 + z1 + z2] Gamma[-z1] Gamma[-z2]
        Gamma[2 - ep - a1 - a2 - z1] Gamma[a4 + z2] Gamma[a1 + z1 + z2]
        Gamma[4 - 2 ep - a1 - a3 - a4 - a5 - a6 - z2] Gamma[4 - 2 ep - a1 - a2 - a4 - a5 - a6 - z1 - z2]
        Gamma[a5 + z2] Gamma[4 - 2 ep - a1 - a2 - a3 - a4 - a5 - z1 - z2] / Gamma[4 - 2 ep - a1 - a2 - a4 - a6
         Gamma[4 - 2 ep - a1 - a2 - a3 - a5 - z1]);

(*   The diagram with all powers of the propagators equal to
  one. We shall evaluate it in expansion in ep up to ep^0. When expanding
  in ep we shall pull out  E^(-2 EulerGamma ep) as usually      *)

In[2]:= NPMB[1, 1, 1, 1, 1, 1]

Out[2]= (Gamma[-ep]^2 Gamma[-ep - z1] Gamma[-z1] Gamma[-1 - 2 ep - z2] Gamma[-1 - 2 ep - z1 - z2]^2
           Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 ep + z1 + z2]) /
         (Gamma[-3 ep] Gamma[-2 ep] Gamma[-2 ep - z1]^2)

(*  The factor QQ^(-2-2 ep) is pulled out     *)

(*  A useful change of variables     *)

In[3]:= % /. z1 -> -1 - z1 - z2

Out[3]= (Gamma[-ep]^2 Gamma[1 + 2 ep - z1] Gamma[-z1] Gamma[-2 ep + z1]^2 Gamma[-1 - 2 ep - z2]
           Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[1 - ep + z1 + z2]) /
         (Gamma[-3 ep] Gamma[-2 ep] Gamma[1 - 2 ep + z1 + z2]^2)

(*    Notation:
  We call poles of Gamma[a-z] ultriviolet (UV) and
   poles of Gamma[a+z] infrared (IR).
    If we want to change the nature of the first
   pole of some gamma function it is denoted by green colour.
    If the first pole of a gamma function Gamma[a-z] has changed its nature,
 i.e. has become IR we denote
  such gamma function by red colour, i.e. Gamma[a-z].

   If the first pole of a gamma function Gamma[a+z] has changed its nature,
 i.e. has become UV we denote
  such gamma function by violet colour, i.e. Gamma[a+z].
*)

Gamma[-2 ep + z1]^2 Gamma[-z1] Gamma[1 + 2 ep - z1]

Gamma[1 + z2]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2]

Gamma[1 + z1 + z2] Gamma[1 - ep + z1 + z2]

(*  There is gluing when ep->0
   in the following pairs of products:
     Gamma[-2 ep+z1]^2 Gamma[-z1] and
     Gamma[1+z2]^2 Gamma[-1-2 ep-z2].

      There is no other way to generate a pole in ep.
                                                     *)

(*  Take care of Gamma[-2 ep+z1]^2 and Gamma[-1-2 ep-z2]  *)
```

```
In[4]:= -Residue[%, {z2, -1 - 2 ep}]

Out[4]= (Gamma[-2 ep] Gamma[-ep]^2 Gamma[1 + 2 ep] Gamma[1 + 2 ep - z1] Gamma[-z1]
          Gamma[-3 ep + z1] Gamma[-2 ep + z1]^3) / (Gamma[-3 ep] Gamma[-4 ep + z1]^2)

        (*  Gamma[-3 ep+z1] Gamma[-2 ep+z1]^3 Gamma[1+2 ep-z1] Gamma[-z1]            *)

        (*  There is gluing of UV and IR poles due to the
          product of Gamma[-3 ep+z1] Gamma[-2 ep+z1]^3 with Gamma[-z1].
           So we take care of  the first poles of Gamma[-3 ep+z1] and Gamma[-2 ep+z1] *)

In[5]:= Residue[%4, {z1, 2 ep}]
```

Out[5]= $\frac{1}{6\,\text{Gamma}[-3\,\text{ep}]}$ (Gamma[-ep]^2 Gamma[1 + 2 ep] (12 EulerGamma^2 Gamma[-ep] +
2 π^2 Gamma[-ep] + 36 EulerGamma Gamma[-ep] PolyGamma[0, -2 ep] +
27 Gamma[-ep] PolyGamma[0, -2 ep]^2 - 12 EulerGamma Gamma[-ep] PolyGamma[0, -ep] -
18 Gamma[-ep] PolyGamma[0, -2 ep] PolyGamma[0, -ep] + 3 Gamma[-ep] PolyGamma[0, -ep]^2 -
3 Gamma[-ep] PolyGamma[1, -2 ep] + 3 Gamma[-ep] PolyGamma[1, -ep]))

```
        (*  We expand all parts of our result in ep up to ep^0  *)

In[6]:= FullSimplify[Normal[Series[% E^(2 EulerGamma ep), {ep, 0, 0}]]]
```

Out[6]= $\frac{24 - 8\,\text{ep}^2 \pi^2 + 5\,\text{ep}^4 \pi^4 - 568\,\text{ep}^3\,\text{Zeta}[3]}{16\,\text{ep}^4}$

```
In[7]:= Residue[%4, {z1, 3 ep}]

Out[7]= Gamma[1 - ep] Gamma[-2 ep] Gamma[ep]^3 Gamma[1 + 2 ep]

In[8]:= FullSimplify[Normal[Series[% E^(2 EulerGamma ep), {ep, 0, 0}]]]
```

Out[8]= $-\frac{15 + 15\,\text{ep}^2 \pi^2 + 9\,\text{ep}^4 \pi^4 + 5\,\text{ep}^3\,\text{PolyGamma}[2, 1]}{30\,\text{ep}^4}$

```
In[10]:= FullSimplify[%6 + %8 /. PolyGamma[2, 1] → -2 Zeta[3]]
```

Out[10]= $\frac{1}{\text{ep}^4} - \frac{\pi^2}{\text{ep}^2} + \frac{\pi^4}{80} - \frac{211\,\text{Zeta}[3]}{6\,\text{ep}}$

```
        (* Now the first poles of Gamma[-3 ep+z1] and
          Gamma[-2 ep+z1]^3 have changed their nature and we have:

           Gamma[-3 ep+z1] Gamma[-2 ep+z1]^3 Gamma[-z1] Gamma[1+2 ep-z1]

            Now there is no gluing of UV and
            IR poles and we can safely expand the integarnd in ep.
                      *)
```

```
In[11]:= Simplify[Normal[Series[%4 E^(2 EulerGamma ep), {ep, 0, 0}]]]
```

Out[11]= $\frac{1}{8\,ep^2}$ (Gamma[1 - z1] Gamma[-z1] Gamma[z1]^2
(12 + 12 ep EulerGamma + 6 ep^2 EulerGamma^2 + ep^2 π^2 + 24 ep^2 PolyGamma[0, 1 - z1]^2 -
12 ep (1 + ep EulerGamma) PolyGamma[0, z1] + 6 ep^2 PolyGamma[0, z1]^2 +
24 ep PolyGamma[0, 1 - z1] (1 + ep EulerGamma - ep PolyGamma[0, z1]) +
24 ep^2 PolyGamma[1, 1 - z1] - 66 ep^2 PolyGamma[1, z1]))

```
(* The calculation of the integral of this expression
   over z2 at -1<z2<0 is presented in the
  file 2 lNPMBaux.nb. This is the corresponding result:  *)
```

In[12]:= $\frac{\pi^2}{4\,ep^2} - \frac{41\,\pi^4}{48} + \frac{3\,Zeta[3]}{ep}$;

```
(* Numerical check     *)

In[13]:= % /. {ep → 0.3}

Out[13]= -43.7675

In[14]:= NIntegrate[%11 /. {ep → 0.3, z1 → -0.5 + I * y1}, {y1, -3, 3}] / 2 / Pi

Out[14]= -43.7675 + 0. i

(* Now we have a 2 fold MB integral where the first pole of
 Gamma[-1-2 ep-z2] has changed its nature, i.e. has become IR  *)

(* Gamma[-2 ep+z1]^2 Gamma[-z1] Gamma[1+2 ep-z1]

 Gamma[1+z2]^2 Gamma[-z2] Gamma[-1-2 ep-z2]

 Gamma[1+z1+z2] Gamma[1-ep+z1+z2]      *)

(* Now we take care of Gamma[-2 ep+z1]^2 in a similar way. *)

In[15]:= Residue[%3, {z1, 2 ep}]
```

Out[15]= $\frac{1}{Gamma[-3\,ep]}$ (Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2]
(-EulerGamma Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] - Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2]
PolyGamma[0, -2 ep] - 2 Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] PolyGamma[0, 1 + z2] +
Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] PolyGamma[0, 1 + ep + z2] +
Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] PolyGamma[0, 1 + 2 ep + z2]))

```
In[16]:= Factor[%15]
```

Out[16]= $-\frac{1}{Gamma[-3\,ep]}$ (Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2]
Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] (EulerGamma + PolyGamma[0, -2 ep] +
2 PolyGamma[0, 1 + z2] - PolyGamma[0, 1 + ep + z2] - PolyGamma[0, 1 + 2 ep + z2]))

```
(* There is no gluing here because the first pole of
 Gamma[-1-2 ep-z2] is IR and its the product with Gamma[1+ep+z2]
 Gamma[1+2 ep+z2]  is unambiguously integrated. So we may expand in ep:  *)
```

```
In[17]:= Simplify[Normal[Series[% E^(2 EulerGamma ep), {ep, 0, 0}]]]
```

$$\text{Out[17]=}\ \frac{1}{8\,\text{ep}^2}\,(3\,\text{Gamma}[-1-\text{z2}]\,\text{Gamma}[-\text{z2}]\,\text{Gamma}[1+\text{z2}]^2\,(4+4\,\text{ep EulerGamma}+2\,\text{ep}^2\,\text{EulerGamma}^2-5\,\text{ep}^2\,\pi^2+8\,\text{ep}^2\,\text{PolyGamma}[0,-1-\text{z2}]^2+12\,\text{ep}\,(1+\text{ep EulerGamma})\,\text{PolyGamma}[0,1+\text{z2}]+18\,\text{ep}^2\,\text{PolyGamma}[0,1+\text{z2}]^2-8\,\text{ep PolyGamma}[0,-1-\text{z2}]\,(1+\text{ep EulerGamma}+3\,\text{ep PolyGamma}[0,1+\text{z2}])+8\,\text{ep}^2\,\text{PolyGamma}[1,-1-\text{z2}]-14\,\text{ep}^2\,\text{PolyGamma}[1,1+\text{z2}]))$$

```
(*  The calculation of the integral of this expression
    over z2 at -1<z2<0 is presented in the
   file 2 lNPMBaux.nb. This is the corresponding result:  *)
```

$$\text{In[18]:=}\ -\frac{\pi^2}{4\,\text{ep}^2}+\frac{31\,\pi^4}{60}+\frac{9\,\text{Zeta}[3]}{2\,\text{ep}};$$

```
(*  Numerical check       *)

In[19]:= % /. {ep → 0.3}

Out[19]= 40.9433

In[20]:= NIntegrate[%17 /. {ep → 0.3, z2 → -0.5 + I * y1}, {y1, -3, 3}] / 2 / Pi

Out[20]= 40.9433 + 0. i

(*  Now we have a 2 fold MB integral where the first poles of
  Gamma[-1-2 ep-z2] and Gamma[-2 ep+z1] have changed their nature,
 i.e. have become IR and UV, respectively  *)

(*  Gamma[-2 ep+z1]^2  Gamma[-z1] Gamma[1+2 ep-z1]

  Gamma[1+z2]^2  Gamma[-z2]  Gamma[-1-2 ep-z2]

  Gamma[1+z1+z2]  Gamma[1-ep+z1+z2]       *)

(*  There is no gluing in this integral
  so that we can safely expand the integrand in ep  *)

In[21]:= Simplify[Normal[Series[%3 E^(2 EulerGamma ep), {ep, 0, 0}]]]

Out[21]= 6 Gamma[1 - z1] Gamma[-z1] Gamma[z1]^2 Gamma[-1 - z2] Gamma[-z2] Gamma[1 + z2]^2

(*   The integral is factorized  *)

In[22]:= 6 Gamma[1 - z1] Gamma[-z1] Gamma[z1]^2 ;

In[23]:= Gamma[-1 - z2] Gamma[-z2] Gamma[1 + z2]^2;
```

```
In[24]:= Simplify[%22 %23 - %21]

Out[24]= 0

        (*  The calculation of these two integrals is presented
          in the file 2 lNPMBaux.nb. These are the corresponding results  *)
```

In[70]:= π^2;

In[90]:= $-\frac{\pi^2}{6}$;

```
        (*  Numerical checks       *)

In[92]:= %70 /. {ep → 0.3} // N

Out[92]= 9.8696

In[93]:= NIntegrate[%22 /. {ep → 0.3, z1 → -0.5 + I * y1}, {y1, -3, 3}] / 2 / Pi

Out[93]= 9.8696 + 0. i

In[94]:= %90 /. {ep → 0.3} // N

Out[94]= -1.64493

In[96]:= NIntegrate[%23 /. {ep → 0.3, z2 → -0.5 + I * y1}, {y1, -3, 3}] / 2 / Pi

Out[96]= -1.64493 + 0. i

        (*  So this is the result for the above 2 fold MB integral   *)

In[99]:= %70 %90
```

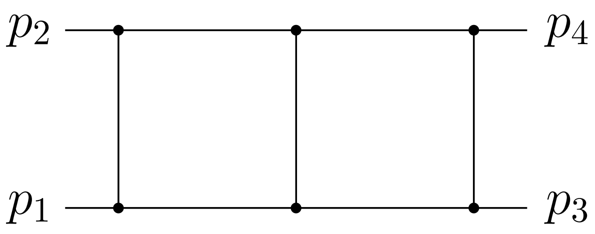
Out[99]= $-\frac{\pi^4}{6}$

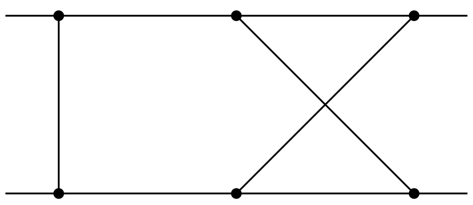
```
        (*  Collecting all the contributions   *)

In[100]:= Expand[% + %10 + %12 + %18]
```

Out[100]= $\frac{1}{ep^4} - \frac{\pi^2}{ep^2} - \frac{59\,\pi^4}{120} - \frac{83\,Zeta[3]}{3\,ep}$

```
        (*   This is our result     *)
```

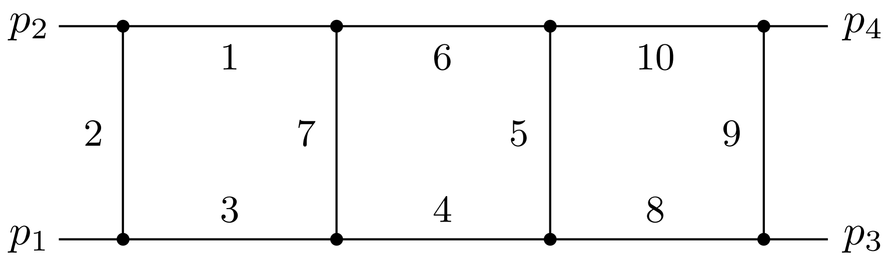
$$\frac{1}{ep^4} - \frac{\pi^2}{ep^2} - \frac{83\,Zeta[3]}{3\,ep} - \frac{59\,\pi^4}{120};$$





Massless on-shell ($p_i^2 = 0,\ i = 1,2,3,4$) double boxes:
done in 1999-2000, with multiple subsequent applications.
Master integrals calculated with the help of MB representation [V.A. Smirnov'99, J.B Tausk'99, V.A. Smirnov & O.L. Veretin'99]

more loops, more legs, more parameters…
triple boxes, $\#$loops $+$ $\#$legs $= 3 + 4 = 7 \gg 1$



The general planar triple box Feynman integral

$$T(a_1,\ldots,a_{10};s,t;\epsilon) \;=\; \int\int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l\, \mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}}$$

$$\times \frac{1}{[(k+p_1+p_2)^2]^{a_3}[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}}$$

$$\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}}$$

General 7fold MB representation:

$$T(a_1,\ldots,a_8;s,t,m^2;\epsilon) \quad = \quad \frac{\left(i\pi^{d/2}\right)^3 (-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$

$$\times \frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times \frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$

$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$

$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$

$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$

$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$

$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

The master triple box:

$$
\begin{aligned}
& T(1,1,\ldots,1;s,t;\epsilon) \\
= & \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
\times & \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
\times & \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
\times & \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
\times & \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
\times & \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)
\end{aligned}
$$

Result

[V.A. Smirnov'03]

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}e^{-\gamma_{\rm E}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j} \,,$$

where $x = -t/s$, $L = \ln(s/t)$, and

$$c_6 = \frac{16}{9}\,, \quad c_5 = -\frac{5}{3}L\,, \quad c_4 = -\frac{3}{2}\pi^2\,,$$

$$c_3 = 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3\,,$$

$$c_2 = -3\left(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\right)$$

$$-L\left(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x)$$

$$-\left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4\,,$$

$$
\begin{aligned}
c_1 =& \, 3\left(81 H_{0,0,0,0,1}(x) + 41 H_{0,0,0,1,1}(x) + 37 H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right. \\
&+33 H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29 H_{1,0,0,0,1}(x) \\
&\left.+H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177 H_{0,0,0,1}(x) + 85 H_{0,0,1,1}(x)\right. \\
&\left.+73 H_{0,1,0,1}(x) + 3 H_{0,1,1,1}(x) + 61 H_{1,0,0,1}(x) + 3 H_{1,0,1,1}(x) + 3 H_{1,1,0,1}(x)\right) \\
&+\left(\frac{119}{2} L^2 + \frac{139}{12}\pi^2\right) H_{0,0,1}(x) + \left(\frac{47}{2} L^2 + 20\pi^2\right) H_{0,1,1}(x) \\
&+\left(\frac{35}{2} L^2 + 14\pi^2\right) H_{1,0,1}(x) + \frac{3}{2}\left(L^2 + \pi^2\right) H_{1,1,1}(x) \\
&+\left(\frac{23}{2} L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right) H_{0,1}(x) + \left(\frac{3}{2} L^3 + \pi^2 L - 3\zeta_3\right) H_{1,1}(x) \\
&+\left(\frac{9}{8} L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right) H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5 \,,
\end{aligned}
$$

$$
\begin{aligned}
c_0 = &-\left(951H_{0,0,0,0,0,1}(x) + 819H_{0,0,0,0,1,1}(x) + 699H_{0,0,0,1,0,1}(x) + 195H_{0,0,0,1,1,1}(x)\right.\\
&+547H_{0,0,1,0,0,1}(x) + 231H_{0,0,1,0,1,1}(x) + 159H_{0,0,1,1,0,1}(x) + 3H_{0,0,1,1,1,1}(x)\\
&+363H_{0,1,0,0,0,1}(x) + 267H_{0,1,0,0,1,1}(x) + 195H_{0,1,0,1,0,1}(x) + 3H_{0,1,0,1,1,1}(x)\\
&+123H_{0,1,1,0,0,1}(x) + 3H_{0,1,1,0,1,1}(x) + 3H_{0,1,1,1,0,1}(x) + 147H_{1,0,0,0,0,1}(x)\\
&+303H_{1,0,0,0,1,1}(x) + 231H_{1,0,0,1,0,1}(x) + 3H_{1,0,0,1,1,1}(x) + 159H_{1,0,1,0,0,1}(x)\\
&+3H_{1,0,1,0,1,1}(x) + 3H_{1,0,1,1,0,1}(x) + 87H_{1,1,0,0,0,1}(x) + 3H_{1,1,0,0,1,1}(x)\\
&\left.+3H_{1,1,0,1,0,1}(x) + 3H_{1,1,1,0,0,1}(x)\right)\\
&-L\left(729H_{0,0,0,0,1}(x) + 537H_{0,0,0,1,1}(x) + 445H_{0,0,1,0,1}(x) + 133H_{0,0,1,1,1}(x)\right.\\
&+321H_{0,1,0,0,1}(x) + 169H_{0,1,0,1,1}(x) + 97H_{0,1,1,0,1}(x) + 3H_{0,1,1,1,1}(x)\\
&+165H_{1,0,0,0,1}(x) + 205H_{1,0,0,1,1}(x) + 133H_{1,0,1,0,1}(x) + 3H_{1,0,1,1,1}(x)\\
&\left.+61H_{1,1,0,0,1}(x) + 3H_{1,1,0,1,1}(x) + 3H_{1,1,1,0,1}(x)\right)\\
&-\left(\frac{531}{2}L^2 + \frac{89}{4}\pi^2\right)H_{0,0,0,1}(x) - \left(\frac{311}{2}L^2 + \frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)\\
&-\left(\frac{247}{2}L^2 + \frac{307}{12}\pi^2\right)H_{0,1,0,1}(x) - \left(\frac{71}{2}L^2 + 32\pi^2\right)H_{0,1,1,1}(x)
\end{aligned}
$$

$$
-\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x)
$$

$$
-\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x)
$$

$$
-\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L\right.
$$

$$
\left.-120\zeta_3\right)H_{0,1,1}(x)-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L\right.
$$

$$
\left.-3\zeta_3\right)H_{1,1,1}(x)-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x)
$$

$$
-\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x)
$$

$$
-\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x)
$$

$$
+\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6\,.
$$

$\zeta_3 = \zeta(3), \zeta_5 = \zeta(5)$ and $\zeta(z)$ is the Riemann zeta function. The functions $H_{a_1,a_2,\ldots,a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x)$, with $a_i = 1, 0, -1$, are HPL [E. Remiddi and J.A.M. Vermaseren'00]

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t)\,,$$

where $f(\pm 1; t) = 1/(1 \mp t), \quad f(0; t) = 1/t,$

$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x\,,$$

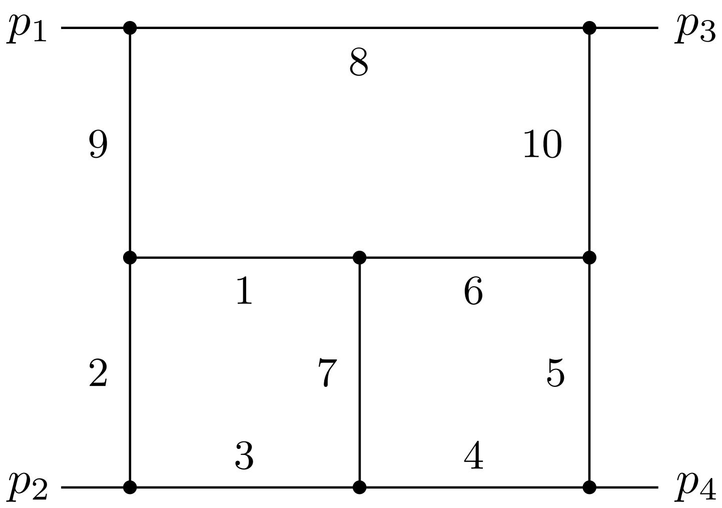
with $a_i = 1, 0, -1$.

NB All the terms of the result have the same degree of transcendentality!

Studying cross order relations in $N = 4$ supersymmetric gauge theories [C. Anastasiou, L.J. Dixon, Z. Bern & D.A. Kosower'03,04]

One more triple box is needed

The 'tennis court' graph with numerator $(l_1 + l_3)^2$



[Z. Bern, L.J. Dixon & V.A. Smirnov'05]

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{1+3\epsilon}t^2}$$

$$\times \frac{1}{(2\pi \mathrm{i})^8} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \ldots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}w\, \mathrm{d}z_1 \prod_{j=2}^{7} \mathrm{d}z_j \Gamma(-z_j) \left(\frac{t}{s}\right)^w \Gamma(1+3\epsilon+w)$$

$$\times \frac{\Gamma(-3\epsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\epsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_1+z_2+z_3)}$$

$$\times \frac{\Gamma(-1-\epsilon-z_1-z_2-z_4)\Gamma(2+\epsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\epsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\epsilon+z_4+z_5+z_6+z_7)}$$

$$\times \Gamma(-\epsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)\Gamma(-1+w-z_5-z_6)$$

$$\times \Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)$$

$$\times \Gamma(-\epsilon+z_1+z_2-z_5-z_6-z_7)\Gamma(1-\epsilon-w+z_4+z_5+z_6+z_7)$$

$$\times \Gamma(1+\epsilon-z_1-z_2-z_3+z_5+z_6+z_7)$$

Result:

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{(-s)^{1+3\epsilon}\, t^2} \sum_{i=0}^{6} \frac{c_j}{\epsilon^j} \;,$$

where

$$\begin{aligned}
c_6 &= \frac{16}{9}\,, \quad c_5 = -\frac{13}{6}\ln x\,, \quad c_4 = -\frac{19}{12}\pi^2 + \frac{1}{2}\ln^2 x \\
c_3 &= \frac{5}{2}\left[\mathsf{Li}_3(-x) - \ln x\,\mathsf{Li}_2(-x)\right] + \frac{7}{12}\ln^3 x - \frac{5}{4}\ln^2 x\ln(1+x) \\
&\quad + \frac{157}{72}\pi^2\ln x - \frac{5}{4}\pi^2\ln(1+x) - \frac{241}{18}\zeta(3)\;\ldots
\end{aligned}$$

[C. Anastasiou, Z. Bern, L.J. Dixon & D.A. Kosower'03; Z. Bern, L.J. Dixon & D.A. Kosower'04]:

for the planar MHV four-point amplitude in $N = 4$ SUSY YM in two loops, one has

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_4^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon)\,,$$

where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)\,, \quad C^{(2)} = -\frac{1}{2}\zeta_2^2$$

[Z. Bern, L.J. Dixon & V.A. Smirnov'05]:

taking into account the results for the ladder triple box and the tennis court diagram up to ϵ^0, for planar double box up to ϵ^2, and for the box up to ϵ^4, we obtain, in three loops,

$$
\begin{aligned}
M_4^{(3)}(\epsilon) &= -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) \\
&\quad + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + O(\epsilon)\,,
\end{aligned}
$$

where

$$
f^{(3)}(\epsilon) = \frac{11}{2}\,\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,,
$$

$$
C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,.
$$

An exponentiation of the planar MHV n-point amplitudes in $N=4$ SUSY YM at L loops:

$$\begin{aligned} \mathcal{M}_n &\equiv 1+\sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \\ &= \exp\left[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\Big)\right] . \end{aligned}$$

where

$$a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon} ,$$

$M_n^{(1)}(l\epsilon)$ is the all-orders-in-ϵ one-loop amplitude (with $\epsilon \to l\epsilon$), and

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \,.$$

The constants $f_k^{(l)}$ and $C^{(l)}$ are independent of the number of legs n.

The $E_n^{(l)}(\epsilon)$ are non-iterating $O(\epsilon)$ contributions to the l-loop amplitudes (with $E_n^{(l)}(0) = 0$).

By definition, the all-orders-in-ϵ one-loop amplitude is absorbed into $M_n^{(1)}(\epsilon)$:

$$f^{(1)}(\epsilon) = 1 \,, \qquad C^{(1)} = 0 \,, \qquad E_n^{(1)}(\epsilon) = 0 \,.$$

$1/\epsilon^2$ pole of the four-point amplitude $\to$
soft anomalous dimension at 3 loops in $N = 4$ SUSY YM

[A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko & V.N. Velizhanin'04]

$\leftrightarrow$

leading-transcendentality part of three-loop soft anomalous dimension in QCD [S. Moch, J.A.M. Vermaseren & A. Vogt'04]

$\leftrightarrow$

$j \to \infty$ formulae [M. Staudacher]

J. Drummond, J. Henn, V.A. Smirnov & E. Sokatchev:

For off-shell Feynman diagrams ($p_i^2 \neq 0$) exactly at $d = 4$, we have

Tennis court $=$ s $\times$ Triple box

```
(*     Triple box     *)
```

In[1]:=
```
B3off[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_, a9_, a10_] :=
  (P11^z12 P22^z13 P33^(z14+z4+z9) P44^(z10+z15+z5)
    S^(6-a1-a10-a2-a3-a4-a5-a6-a7-a8-a9-3 ep-z10-z11-z12-z13-z14-z15-z4-z5-z9) T^z11 Gamma[-z10] Gamma[-z11]
    Gamma[-z12] Gamma[-z13] Gamma[a9 + z11 + z12 + z13] Gamma[-z14] Gamma[-z15]
    Gamma[-z2] Gamma[-z3] Gamma[a7 + z1 + z2 + z3] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2 - z4]
    Gamma[-z4] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3 - z5] Gamma[-z5]
    Gamma[a5 + z1 + z4 + z5] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 + z2 + z3 + z4 + z5]
    Gamma[z11 + z14 + z15 - z6] Gamma[-z7] Gamma[2 - a8 - a9 - ep - z11 - z12 - z14 + z6 + z7]
    Gamma[2 - a2 - a3 - ep + z1 - z10 + z3 - z6 - z8]
    Gamma[-2 + a10 + a8 + a9 + ep + z11 + z12 + z13 + z14 + z15 - z6 - z7 - z8] Gamma[-z8]
    Gamma[2 - a10 - a9 - ep - z11 - z13 - z15 + z6 + z8] Gamma[a2 + z6 + z7 + z8]
    Gamma[2 - a1 - a2 - ep + z1 + z2 - z6 - z7 - z9] Gamma[-z9] Gamma[-z1 + z10 + z6 + z9]
    Gamma[-2 + a1 + a2 + a3 + ep - z1 + z10 - z2 - z3 + z6 + z7 + z8 + z9]) /
  (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[a9]
    Gamma[4 - a4 - a5 - a6 - a7 - 2 ep] Gamma[a1 - z2] Gamma[a3 - z3]
    Gamma[4 - a1 - a2 - a3 - 2 ep + z1 + z2 + z3] Gamma[a8 - z7]
    Gamma[a10 - z8] Gamma[4 - a10 - a8 - a9 - 2 ep + z6 + z7 + z8])
```

In[2]:=
```
B3off[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

Out[2]=
```
(P11^z12 P22^z13 P33^(z14+z4+z9) P44^(z10+z15+z5) S^(-4-3 ep-z10-z11-z12-z13-z14-z15-z4-z5-z9) T^z11
   Gamma[-z10] Gamma[-z11] Gamma[-z12] Gamma[-z13] Gamma[1 + z11 + z12 + z13]
   Gamma[-z14] Gamma[-z15] Gamma[-z2] Gamma[-z3] Gamma[1 + z1 + z2 + z3]
   Gamma[-1 - ep - z1 - z2 - z4] Gamma[-z4] Gamma[-1 - ep - z1 - z3 - z5]
   Gamma[-z5] Gamma[1 + z1 + z4 + z5] Gamma[2 + ep + z1 + z2 + z3 + z4 + z5]
   Gamma[z11 + z14 + z15 - z6] Gamma[-z7] Gamma[-ep - z11 - z12 - z14 + z6 + z7]
   Gamma[-ep + z1 - z10 + z3 - z6 - z8] Gamma[1 + ep + z11 + z12 + z13 + z14 + z15 - z6 - z7 - z8]
   Gamma[-z8] Gamma[-ep - z11 - z13 - z15 + z6 + z8] Gamma[1 + z6 + z7 + z8]
   Gamma[-ep + z1 + z2 - z6 - z7 - z9] Gamma[-z9] Gamma[-z1 + z10 + z6 + z9]
   Gamma[1 + ep - z1 + z10 - z2 - z3 + z6 + z7 + z8 + z9]) /
 (Gamma[-2 ep] Gamma[1 - z2] Gamma[1 - z3] Gamma[1 - 2 ep + z1 + z2 + z3]
   Gamma[1 - z7] Gamma[1 - z8] Gamma[1 - 2 ep + z6 + z7 + z8])
```

In[3]:=
```
<< e:/job2006/czakon/MB.m
```

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 24 Jan 06
```

In[5]:=
```
B3 = B3off[1, 1, 1, 1, 1, 1, 1, 1, 1, 1] ;
```

In[6]:=
```
B3rules = MBoptimizedRules[B3, ep → 0, {}, {ep}]
```

Out[6]=

$$\left\{\left\{ep \to -\frac{1}{2}\right\}, \left\{z1 \to -\frac{5895}{8192}, z10 \to -\frac{1}{16}, z11 \to -\frac{1}{2}, z12 \to -\frac{5}{16}, z13 \to -\frac{1}{8}, z14 \to -\frac{1}{32}, z15 \to -\frac{1}{8}, z2 \to -\frac{25}{128}, z3 \to -\frac{1}{128}, z4 \to -\frac{729}{8192}, z5 \to -\frac{97}{512}, z6 \to -\frac{5377}{8192}, z7 \to -\frac{67}{256}, z8 \to -\frac{663}{8192}, z9 \to -\frac{1}{2048}\right\}\right\}$$

In[7]:=
```
B3cont = MBcontinue[B3, ep → 0, B3rules]
```

In[8]:=
```
B3select = MBpreselect[MBmerge[B3cont], {ep, 0, 0}]
```

In[9]:=
```
B3exp = MBexpand[B3select, Exp[3 ep EulerGamma], {ep, 0, 0}]
```

Out[9]=
```
{MBint[
  (P11^z12 P22^(-1-z11-z12) P33^(-1-z11-z12) P44^z12 S^(-2+z11) T^z11 Gamma[-z1] Gamma[-z10] Gamma[-z11]
     Gamma[-z12] Gamma[1 + z11 + z12] Gamma[-z14] Gamma[1 + z11 + z12 + z14]
     Gamma[1 + z1 - z10 + z12 - z15]^2 Gamma[1 + z1 - z11 - z14 - z15]
     Gamma[-z15] Gamma[-z12 + z15] Gamma[z10 - z12 + z15]^2
     Gamma[1 + z14 + z15] Gamma[-z1 + z10 + z11 + z14 + z15]) /
   (Gamma[2 + z11 + z12 + z14] Gamma[2 + z1 - z10 + z12 - z15]
     Gamma[1 - z12 + z15] Gamma[1 + z10 - z12 + z15]),
  {{ep -> 0}, {z1 -> -5895/8192, z10 -> -1/16, z11 -> -1/2, z12 -> -5/16, z14 -> -1/32, z15 -> -1/8}}]}
```

In[10]:=
```
(P11^z12 P22^(-1-z11-z12) P33^(-1-z11-z12) P44^z12 S^(-2+z11) T^z11
    Gamma[-z1] Gamma[-z10] Gamma[-z11] Gamma[-z12] Gamma[1 + z11 + z12]
    Gamma[-z14] Gamma[1 + z11 + z12 + z14] Gamma[1 + z1 - z10 + z12 - z15]^2
    Gamma[1 + z1 - z11 - z14 - z15] Gamma[-z15] Gamma[-z12 + z15]
    Gamma[z10 - z12 + z15]^2 Gamma[1 + z14 + z15] Gamma[-z1 + z10 + z11 + z14 + z15]) /
  (Gamma[2 + z11 + z12 + z14] Gamma[2 + z1 - z10 + z12 - z15]
    Gamma[1 - z12 + z15] Gamma[1 + z10 - z12 + z15]);
```

```
In[11]:= % /. {z10 → z2, z14 → z3, z15 → z4, z11 → z5, z12 → z6}

Out[11]= (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-2+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3]
    Gamma[-z4] Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5]
    Gamma[-z1 + z2 + z3 + z4 + z5] Gamma[z4 - z6] Gamma[z2 + z4 - z6]^2 Gamma[-z6]
    Gamma[1 + z1 - z2 - z4 + z6]^2 Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6]) /
  (Gamma[1 + z4 - z6] Gamma[1 + z2 + z4 - z6] Gamma[2 + z1 - z2 - z4 + z6] Gamma[2 + z3 + z5 + z6])

In[12]:= TB =
   (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-2+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3] Gamma[-z4]
      Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5] Gamma[-z1 + z2 + z3 + z4 + z5]
      Gamma[z4 - z6] Gamma[z2 + z4 - z6]^2 Gamma[-z6] Gamma[1 + z1 - z2 - z4 + z6]^2
      Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6]) / (Gamma[1 + z4 - z6]
      Gamma[1 + z2 + z4 - z6] Gamma[2 + z1 - z2 - z4 + z6] Gamma[2 + z3 + z5 + z6]);


(*       Tennis court   *)

In[13]:= TCoff[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_, a9_, a10_, a11_] :=
  (P11^z12 P22^z13 P33^(z10+z14+z5) P44^(z15+z8)
    S^(6-a1-a10-a11-a2-a3-a4-a5-a6-a7-a8-a9-3 ep-z10-z11-z12-z13-z14-z15-z5-z8)
    T^z11 Gamma[-z10] Gamma[-z11] Gamma[-z12] Gamma[-z13]
    Gamma[a9 + z11 + z12 + z13] Gamma[-z14] Gamma[-z15] Gamma[-z2]
    Gamma[-z3] Gamma[a7 + z1 + z2 + z3] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2 - z4]
    Gamma[-z4] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3 - z5] Gamma[-z5]
    Gamma[a5 + z1 + z4 + z5] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 + z2 + z3 + z4 + z5]
    Gamma[-z6] Gamma[-z7] Gamma[2 - a2 - a3 - ep + z1 - z10 + z3 - z6 - z8]
    Gamma[6 - a1 - a11 - a2 - a3 - a4 - a5 - a6 - a7 - a8 - a9 - 3 ep - z10 - z11 - z12 -
      z14 - z5 - z6 - z7 - z8] Gamma[-z8] Gamma[-6 + a1 + a10 + a11 + a2 + a3 + a4 +
      a5 + a6 + a7 + a8 + a9 + 3 ep + z10 + z11 + z12 + z13 + z14 + z15 + z5 + z6 + z8]
    Gamma[a2 + z6 + z7 + z8] Gamma[2 - a1 - a2 - ep + z1 + z2 - z6 - z7 - z9]
    Gamma[6 - a1 - a10 - a11 - a2 - a3 - a4 - a5 - a6 - a7 - a9 - 3 ep - z10 - z11 -
      z13 - z15 - z4 - z5 - z6 - z8 - z9] Gamma[-z9] Gamma[-z1 + z10 + z6 + z9]
    Gamma[-2 + a1 + a2 + a3 + ep - z1 + z10 - z2 - z3 + z6 + z7 + z8 + z9] Gamma[-4 + a1 +
      a11 + a2 + a3 + a4 + a5 + a6 + a7 + 2 ep + z10 + z11 + z14 + z15 + z4 + z5 + z6 + z7 + z8 + z9]) /
   (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[a9]
    Gamma[4 - a4 - a5 - a6 - a7 - 2 ep] Gamma[a1 - z2] Gamma[a3 - z3]
    Gamma[4 - a1 - a2 - a3 - 2 ep + z1 + z2 + z3] Gamma[a10 - z7]
    Gamma[8 - a1 - a10 - a11 - a2 - a3 - a4 - a5 - a6 - a7 - a8 - a9 - 4 ep - z10 - z5 - z6 - z8]
    Gamma[a8 - z4 - z9]
    Gamma[-4 + a1 + a11 + a2 + a3 + a4 + a5 + a6 + a7 + 2 ep + z10 + z4 + z5 + z6 + z7 + z8 + z9])
```

```
In[14]:= TC3 = TCoff[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1]

Out[14]= (P11^z12 P22^z13 P33^(z10+z14+z5) P44^(z15+z8) S^(-3-3 ep-z10-z11-z12-z13-z14-z15-z5-z8) T^z11
   Gamma[-z10] Gamma[-z11] Gamma[-z12] Gamma[-z13] Gamma[1 + z11 + z12 + z13]
   Gamma[-z14] Gamma[-z15] Gamma[-z2] Gamma[-z3] Gamma[1 + z1 + z2 + z3]
   Gamma[-1 - ep - z1 - z2 - z4] Gamma[-z4] Gamma[-1 - ep - z1 - z3 - z5]
   Gamma[-z5] Gamma[1 + z1 + z4 + z5] Gamma[2 + ep + z1 + z2 + z3 + z4 + z5]
   Gamma[-z6] Gamma[-z7] Gamma[-ep + z1 - z10 + z3 - z6 - z8]
   Gamma[-2 - 3 ep - z10 - z11 - z12 - z14 - z5 - z6 - z7 - z8] Gamma[-z8]
   Gamma[3 + 3 ep + z10 + z11 + z12 + z13 + z14 + z15 + z5 + z6 + z8]
   Gamma[1 + z6 + z7 + z8] Gamma[-ep + z1 + z2 - z6 - z7 - z9]
   Gamma[-2 - 3 ep - z10 - z11 - z13 - z15 - z4 - z5 - z6 - z8 - z9] Gamma[-z9]
   Gamma[-z1 + z10 + z6 + z9] Gamma[1 + ep - z1 + z10 - z2 - z3 + z6 + z7 + z8 + z9]
   Gamma[2 + 2 ep + z10 + z11 + z14 + z15 + z4 + z5 + z6 + z7 + z8 + z9]) /
  (Gamma[-2 ep] Gamma[1 - z2] Gamma[1 - z3] Gamma[1 - 2 ep + z1 + z2 + z3]
   Gamma[1 - z7] Gamma[-1 - 4 ep - z10 - z5 - z6 - z8]
   Gamma[1 - z4 - z9] Gamma[2 + 2 ep + z10 + z4 + z5 + z6 + z7 + z8 + z9])

In[15]:= TCrules = MBoptimizedRules[TC3, ep → 0, {}, {ep}]

Out[15]= {{ep → -1/8}, {z1 → -531259/983040, z10 → -1/256, z11 → -1/8, z12 → -194967/327680, z13 → -265019/983040,
  z14 → -1601/98304, z15 → -1/32, z2 → -256571/983040, z3 → -32407/163840, z4 → -19553/98304,
  z5 → -256091/983040, z6 → -1658441/3932160, z7 → -260897/983040, z8 → -409857/1310720, z9 → -149787/1310720}}

In[16]:= TCcont = MBcontinue[TC3, ep → 0, TCrules]

In[17]:= TCselect = MBpreselect[MBmerge[TCcont], {ep, 0, 0}]

In[18]:= TCexp = MBexpand[TCselect, Exp[3 ep EulerGamma], {ep, 0, 0}]

Out[18]= {MBint[
   (P11^z12 P22^(-1-z11-z12) P33^(-1-z11-z12) P44^z12 S^(-1+z11) T^z11 Gamma[-z1] Gamma[-z10] Gamma[-z11]
     Gamma[-z12] Gamma[1 + z11 + z12] Gamma[z1 - z10 - z11 - z12 - z14]^2
     Gamma[-z14] Gamma[1 + z11 + z12 + z14] Gamma[1 + z10 + z11 + z12 + z14]^2
     Gamma[1 + z1 - z11 - z14 - z15] Gamma[-z15] Gamma[-z12 + z15]
     Gamma[1 + z14 + z15] Gamma[-z1 + z10 + z11 + z14 + z15]) /
    (Gamma[1 + z1 - z10 - z11 - z12 - z14] Gamma[2 + z11 + z12 + z14]
     Gamma[2 + z10 + z11 + z12 + z14] Gamma[1 - z12 + z15]), {{ep → 0},
   {z1 → -531259/983040, z10 → -1/256, z11 → -1/8, z12 → -194967/327680, z14 → -1601/98304, z15 → -1/32}}]}

In[19]:= (P11^z12 P22^(-1-z11-z12) P33^(-1-z11-z12) P44^z12 S^(-1+z11) T^z11 Gamma[-z1] Gamma[-z10] Gamma[-z11]
     Gamma[-z12] Gamma[1 + z11 + z12] Gamma[z1 - z10 - z11 - z12 - z14]^2 Gamma[-z14]
     Gamma[1 + z11 + z12 + z14] Gamma[1 + z10 + z11 + z12 + z14]^2 Gamma[1 + z1 - z11 - z14 - z15]
     Gamma[-z15] Gamma[-z12 + z15] Gamma[1 + z14 + z15] Gamma[-z1 + z10 + z11 + z14 + z15]) /
    (Gamma[1 + z1 - z10 - z11 - z12 - z14] Gamma[2 + z11 + z12 + z14]
     Gamma[2 + z10 + z11 + z12 + z14] Gamma[1 - z12 + z15]);
```

```
In[20]:= % /. {z10 → z2, z14 → z3, z15 → z4, z11 → z5, z12 → z6}

Out[20]= (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-1+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3]
           Gamma[-z4] Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5]
           Gamma[-z1 + z2 + z3 + z4 + z5] Gamma[z4 - z6] Gamma[z1 - z2 - z3 - z5 - z6]^2
           Gamma[-z6] Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6] Gamma[1 + z2 + z3 + z5 + z6]^2) /
         (Gamma[1 + z4 - z6] Gamma[1 + z1 - z2 - z3 - z5 - z6]
           Gamma[2 + z3 + z5 + z6] Gamma[2 + z2 + z3 + z5 + z6])

In[21]:= (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-1+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3]
            Gamma[-z4] Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5]
            Gamma[-z1 + z2 + z3 + z4 + z5] Gamma[z4 - z6] Gamma[z1 - z2 - z3 - z5 - z6]^2
            Gamma[-z6] Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6] Gamma[1 + z2 + z3 + z5 + z6]^2) /
          (Gamma[1 + z4 - z6] Gamma[1 + z1 - z2 - z3 - z5 - z6]
            Gamma[2 + z3 + z5 + z6] Gamma[2 + z2 + z3 + z5 + z6]);

In[22]:= % /. z2 → -z2 + z1 - z3 - z4 - z5

Out[22]= (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-1+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3]
           Gamma[-z4] Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5]
           Gamma[-z1 + z2 + z3 + z4 + z5] Gamma[z4 - z6] Gamma[z2 + z4 - z6]^2 Gamma[-z6]
           Gamma[1 + z1 - z2 - z4 + z6]^2 Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6]) /
         (Gamma[1 + z4 - z6] Gamma[1 + z2 + z4 - z6] Gamma[2 + z1 - z2 - z4 + z6] Gamma[2 + z3 + z5 + z6])

In[23]:= TC =
           (P11^z6 P22^(-1-z5-z6) P33^(-1-z5-z6) P44^z6 S^(-1+z5) T^z5 Gamma[-z1] Gamma[-z2] Gamma[-z3] Gamma[-z4]
              Gamma[1 + z3 + z4] Gamma[1 + z1 - z3 - z4 - z5] Gamma[-z5] Gamma[-z1 + z2 + z3 + z4 + z5]
              Gamma[z4 - z6] Gamma[z2 + z4 - z6]^2 Gamma[-z6] Gamma[1 + z1 - z2 - z4 + z6]^2
              Gamma[1 + z5 + z6] Gamma[1 + z3 + z5 + z6]) / (Gamma[1 + z4 - z6]
              Gamma[1 + z2 + z4 - z6] Gamma[2 + z1 - z2 - z4 + z6] Gamma[2 + z3 + z5 + z6]);

In[24]:= TC / TB

Out[24]= S
```

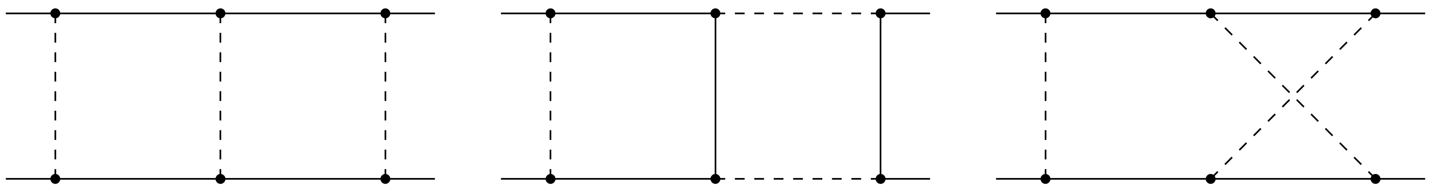
Other applications of the method of MB representation: Massless double boxes with one leg off-shell, $p_1^2 = q^2 \neq 0$, $p_i^2 = 0, \; i = 2,3,4$:

- Reduction to master integrals [T. Gehrmann & E. Remiddi'01]
- Master integrals:
  - first results obtained by MB [V.A. Smirnov'01,02]
  - systematic evaluation by differential equations [T. Gehrmann & E. Remiddi'01]

All results are expressed in terms of two-dimensional harmonic polylogarithms which generalize harmonic polylogarithms [E. Remiddi & J.A.M. Vermaseren'00]

Applications to the process $e^+e^- \to 3$jets

Massive on-shell 2-boxes, $p_i^2 = m^2, \; i = 1,2,3,4$



- first results obtained by MB

  [V.A. Smirnov'02,04; G. Heinrich & V.A. Smirnov'04]

- Reduction to master integrals by Laporta's algorithm

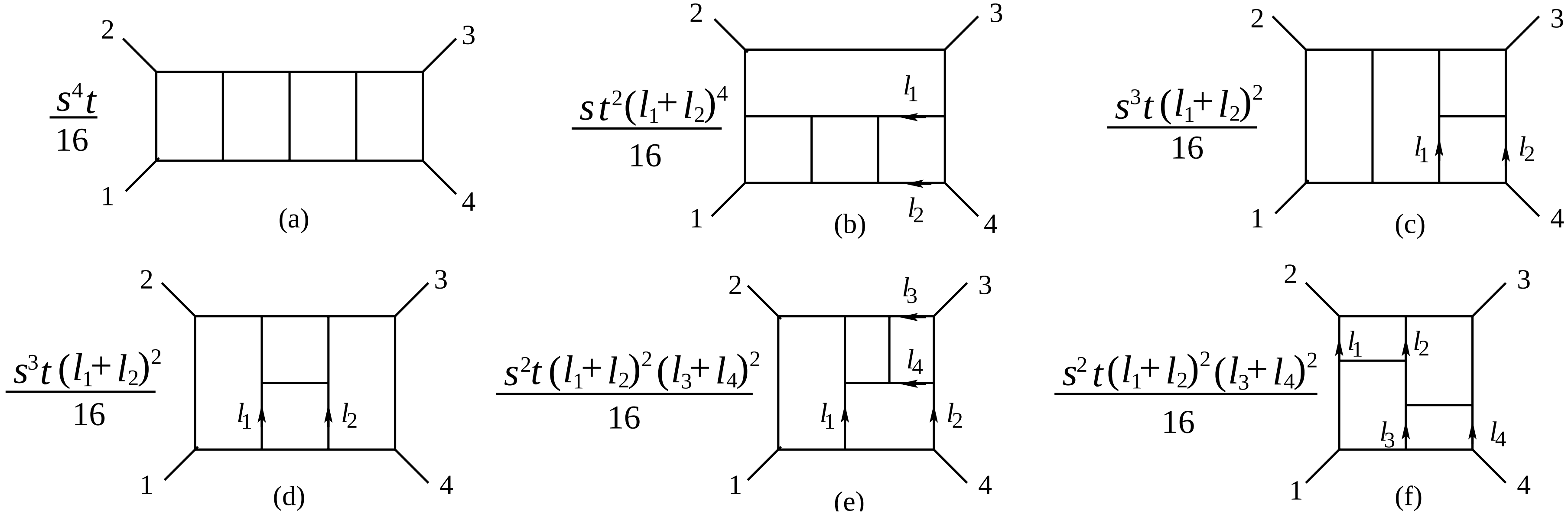
  [M. Czakon, J. Gluza & T. Riemann'04]

- Evaluating all the master integrals by differential equations and MB [M. Czakon, J. Gluza & T. Riemann'05]

Work in progress. What functions shall we have in results?

Perspectives

- The method of MB representation is a powerful method. In particular, it is very flexible in resolving the singularity structure in ϵ.
- Studying iteration relations in $N = 4$ SYSY YM in four loops (work in progress)



- Automation of Strategy #1 is also possible.
- Evaluating every Feynman integral of a given family, without reduction to master integrals, appears to be a possible alternative.