TWO-LOOP Renormalization in the Making

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Outline of Part IV

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1. Unstable particles
2. Solution of the renormalization equations
3. Some Input Parameter St
4. General structure of self-energies
5. Loop diagrams with dressed propagators
6. Unitarity, gauge parameter independence and WST identities
7. Unitarity
8. WST identities
9. Gauge parameter dependence
10. Complex poles

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Complex poles

To write additional renormalization equations we need *experimental* masses. For the W and Z bosons the IPS is defined in terms of pseudo-observables (PO); at first, OS quantities are derived by fitting the experimental lineshapes with

$$\Sigma_{VV}(s) = \frac{N}{(s - M^2_{OS})^2 + s^2\,\Gamma^2_{OS}/M^2_{OS}}, \qquad V = W, Z, \tag{1}$$

where N is an irrelevant (for our purposes) normalization constant. Secondly we define pseudo-observables (PO)

$$M_P = M_{OS}\,\cos\psi, \qquad \Gamma_P = \Gamma_{OS}\,\sin\psi, \qquad \psi = \arctan\frac{\Gamma_{OS}}{M_{OS}}, \tag{2}$$

which are inserted in the IPS.

Beyond one-loop

At one-loop level we can use directly the OS masses which are related to the zero of the real part of the inverse propagator. Beyond one-loop this would show a clash with gauge invariance since only the complex poles

$$s_V = \mu_V^2 - i\,\gamma_V\,\mu_V \qquad (3)$$

do not depend, to all orders, on gauge parameters. As a consequence, renormalization equations change their structure.

There is also a change of perspective with respect to old one-loop calculations.

- There one considers the cdb OS masses as input parameters independent of complex poles and *derive* the latter in terms of the former;
- Here the situation changes, renormalization equations are written for real, renormalized, parameters and solved in terms of (among other things) experimental complex poles.

When we constuct a propagator from an IPS that contains its complex pole, say s_V, we are left with a consistency relation between theoretical and experimental values of γ_V. If instead, we derive s_W from an IPS that contains s_Z, this is a prediction for the full W complex pole.

Furthermore, consistently with an order-by-order renormalization procedure, renormalized masses in loops and in vertices will be replaced with their real solutions of the renormalized equations, truncated to the requested order.
Alternatively, one could use Dyson resummed (*dressed*) propagators,

$$\bar{\Delta}_V = \frac{\Delta_V}{1 - i\,\Delta_V\,\Sigma_{VV}}, \tag{4}$$

also in loops, say two-loop resummed propagators in tree diagrams, one loop resummed in one-loop diagrams, tree in two-loop diagrams.

renormalization equations

Renormalization with complex poles

has more in it than the content of Eq.(3) and is not confined to prescribe a fixed width for unstable particles; it allows, al least in principle, for an elegant treatment of radiative corrections via effective, complex, couplings.
The corresponding formulation, however, cannot be extended naively beyond the fermion loop approximation; this is due, once again, to gauge parameter independence. We formulate the next renormalization equation in close resemblance with the language of effective couplings and will perform the proper expansions at the end.

We define residual functions according to

$$\Sigma_B(s) = \Sigma_{3Q}(s) + F_B(s), \qquad B = W, Z, \text{ and } H, \tag{5}$$

and discuss solutions of the renormalization equations for different IPS. As a consequence of introducing higher order corrections the coupling constant g will run according to

$$\frac{1}{g^2(s)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\Pi^{(1)}_{3Q}(s) - \frac{g^2}{(16\,\pi^2)^2}\,\Pi^{(2)}_{3Q}(s). \tag{6}$$

The running of $e^2 = g^2 s^2$ is controlled by

$$e^2(s) = \frac{4\,\pi\,\alpha}{1 - \frac{\alpha}{4\,\pi}\,\Pi_R(s)}, \tag{7}$$

while the running of the weak-mixing angle is defined according to

$$s^2(s) = \frac{e^2(s)}{g^2(s)}. \tag{8}$$

Eqs.(6)–(8) still contain bare parameters and in the following sections we will show how to replace bare quantities in terms of some IPS.

Input Parameter St

We use α, G_F and μ_W and predict, among other things, γ_W which, in turn, can be compared with the measured OS Γ_W. We begin with two equations

$$G\left[M^2 - \frac{g^2}{16\,\pi^2}F_W(0)\right] = \frac{g^2}{8}$$

$$\mu_W^2 = M^2 - \frac{g^2}{16\,\pi^2}\,\mathrm{Re}\left[\Sigma_{3Q}(s_W) + F_W(s_W)\right], \qquad (9)$$

where, to second order, we have

$$F_W = F_W^{(1)} + \frac{g^2}{16\,\pi^2}F_W^{(2)}, \qquad \Sigma_{3Q} = \Sigma_{3Q}^{(1)} + \frac{g^2}{16\,\pi^2}\Sigma_{3Q}^{(2)}. \qquad (10)$$

The (finite) mass counterterm of Eq.(9) is to be contrasted with the conventional mass renormalization where $\mathrm{Re}\,\Sigma_{WW}(M_W^2)$ is used.

We look for a solution with the following form:

$$g^2 = 8\,G\,\mu_W^2\left[1+\sum_{n=1}C_g(n)\,\left(\frac{G}{\pi^2}\right)^n\right],$$
$$M^2 = \mu_W^2\left[1+\sum_{n=1}C_M(n)\,\left(\frac{G}{\pi^2}\right)^n\right]. \qquad (11)$$

The solution is

$$C_g(1) = \frac{1}{2}\left[\mathrm{Re}\,\Sigma^{(1)}_{WW}(s_W) - F^{(1)}_W(0)\right], \qquad C_M(1) = \frac{1}{2}\,\mathrm{Re}\,\Sigma^{(1)}_{WW}(s_W),$$
$$C_g(2) = C_g^2(1) + \frac{1}{4}\,\mu_W^2\left[\mathrm{Re}\,\Sigma^{(2)}_{WW}(s_W) - F^{(2)}_W(0)\right],$$
$$C_M(2) = C_M^2(1) + \frac{1}{4}\,\mathrm{Re}\left[\mu_W^2\,\Sigma^{(2)}_{WW}(s_W) - F^{(1)}_W(0)\,\Sigma^{(1)}_{WW}(s_W)\right]. \qquad (12)$$

In particular we obtain

$$\frac{\boldsymbol{M}^2}{g^2} = \frac{1}{8\,G}\left[1 + \frac{G}{2\,\pi^2}\,F_W^{(1)}(0) + \frac{G^2}{4\,\pi^4}\,\mu_W^2\,F_W^{(2)}(0)\right]. \tag{13}$$

For this input parameter set renormalization of g is obtained after inserting Eq.(12) into Eq.(6),

$$\frac{1}{g^2(s)} = \frac{1}{8\,G\,\mu_W^2} - \frac{1}{16\,\pi^2\,\mu_W^2}\,\delta\,g^{(1)} - \frac{G}{32\,\pi^4}\,\delta\,g^{(2)},$$
$$\delta\,g^{(n)} = \mu_W^2\,\Pi_{3Q}^{(n)}(s) + \mathrm{Re}\,\Sigma_{WW}^{(n)}(s_W) - F_W^{(n)}(0). \tag{14}$$

The renormalization equation for s^2 is

$$g^2 s^2 = 4\pi\alpha \left[1 - \frac{g^2 s^2}{16\pi^2} \Pi_{QQ}(0)\right]. \tag{15}$$

with a solution given by

$$s^2 = \frac{1}{2} A \left[1 + \sum_{n=1} C_s(n) \left(\frac{G}{\pi^2}\right)^n\right], \qquad A = \frac{\pi\alpha}{G\,\mu_W^2},$$
$$C_s(1) = -\frac{1}{2}\,\delta s^{(1)}, \qquad C_s(2) = -\frac{1}{4}\left[\delta s^{(2)} - \mu_W^2 A\, \Pi^{(n)}_{QQ}(0)\,\delta s^{(1)}\right],$$
$$\delta s^{(n)} = \mathrm{Re}\,\Sigma^{(n)}_{WW}(s_W) - F^{(n)}_W(0) + \mu_W^2 A\, \Pi^{(n)}_{QQ\,;\,\mathrm{ext}}(0). \tag{16}$$

In $\delta s^{(2)}$ we have a residual dependence on s^2 which must be set to its lowest order value,

$$\bar{s}^2 = \frac{1}{2} A. \tag{17}$$

For the W propagator we factorize a g^2, insert the solution and write its inverse as

$$\left[g^2\,\Delta_W(s)\right]^{-1} = \frac{s}{g^2(s)} - \frac{1}{8\,G} + \frac{1}{16\,\pi^2}\left[F_W^{(1)}(s) - F_W^{(1)}(0)\right] + \frac{G\,\mu_W^2}{32\,\pi^4}\left[F_W^{(2)}(s) - F_W^{(2)}(0)\right]. \tag{18}$$

Using Eq.(14) the same expression can be rewritten as

$$\left[g^2\,\Delta_W(s)\right]^{-1} = \frac{s}{g^2(s)} - \frac{\mu_W^2}{g^2(s_W)} + \frac{i}{16\,\pi^2}\,R_W^{(1)}(s_W) + \frac{i\,G\,\mu_W^2}{32\,\pi^4}\,R_W^{(2)}(s_W), \tag{19}$$

where the remainders are:

$$R_W^{(n)}(s_W) = \mathrm{Im}\,\Sigma_{WW}^{(n)}(s_W) - \mu_W\,\gamma_W\,\Pi_{3Q\,;\,\mathrm{ext}}^{(n)}(s_W). \tag{20}$$

The complex zero of this expression is the theoretical prediction for the complex pole of the W boson. The real part has been fixed to μ_W^2; the solution for the imaginary part is

$$\gamma_W^{\rm th} = \frac{G\,\mu_W}{2\,\pi^2}\left(\gamma_1 + \frac{G}{2\,\pi^2}\,\gamma_2\right),$$

$$\gamma_1 = \mathrm{Im}\,\Sigma^{(1)}_{WW}(\mu_W^2),$$

$$\gamma_2 = \mathrm{Im}\,\Sigma^{(1)}_{WW}(\mu_W^2)\left[\mathrm{Re}\,F^{(1)}_W(\mu_W^2) - F^{(1)}_W(0)\right] + \mu_W^2\left[\mathrm{Im}\,F^{(2)}_W(\mu_W^2) - \mathrm{Im}\,F^{(1)}_W(\mu_W^2)\,\mathrm{Re}\,\Sigma^{(1)}_{WW\,;\,p}(\mu_W^2)\right], \qquad (21)$$

where the suffix p denotes derivation.

We have one consistency condition obtained by comparing the derived width of Eq.(21) with the experimental input γ_W. The goodness of the comparison is a precision test of the standard model.
Furthermore, the parameter controlling perturbative (non-resummed) expansion is $G_F\,\mu_W^2$ and we derive,

$$G = G_F\left\{1 - \delta_G^{(1)}\,\frac{G_F\,\mu_W^2}{2\,\pi^2} + \left[2\,(\delta_G^{(1)})^2 - \frac{2}{\mu_W^2}\,\delta_G^{(1)}\,C_g(1) - \delta_G^{(2)}\right]\left(\frac{G_F\,\mu_W^2}{2\,\pi^2}\right)^2\right\}. \tag{22}$$

In other words, we can go from the G option to the G_F option by replacing in the previous results

$$\begin{aligned} F_W^{(1)}(0) \quad &\to \quad \overline{F}_W^{(1)} = F_W^{(1)}(0) + \mu_W^2\,\delta_G^{(1)}, \\ F_W^{(2)}(0) \quad &\to \quad \overline{F}_W^{(2)} = F_W^{(2)}(0) + \mu_W^2\,\delta_G^{(2)} + \delta_G^{(1)}\Big[\mu_W^2\,\delta_G^{(1)} + \mathrm{Re}\,F_W^{(1)}(s_W) \\ &\qquad + \mathrm{Re}\,\Sigma^{(1)}_{3Q\,;\,\mathrm{ext}}(s_W) - 2\,\overline{F}_W^{(1)}\Big], \end{aligned} \tag{23}$$

and $G \to G_F$.

All function appearing in the results depend also on internal masses, M etc. Therefore we always use, for and arbitrary f

$$f^{(1)}(s\,;\,M^2\,,\,\dots) = f^{(1)}(s\,;\,\mu_W^2\,,\,\dots) + \frac{G\,\mu_W^2}{2\,\pi^2}\,\mathrm{Re}\,\Sigma_{WW}^{(1)}(s_W\,;\,\mu_W^2\,,\,\dots) \times \frac{\partial}{\partial M^2}\,f^{(1)}(s\,;\,M^2\,,\,\dots)\Big|_{M^2=\mu_W^2}. \tag{24}$$

A last subtlety in Eq.(18) is represented by the residual s^2 dependence of the W self-energy and of δ_G; we use

$$s^2 = \bar{s}^2\left[1 - \frac{G_F}{2\,\pi^2}\,\delta s^{(1)}\right] \quad \text{in} \quad F_W^{(1)},\ \delta_G^{(1)}$$
$$s^2 = \bar{s}^2 \quad \text{in} \quad F_W^{(2)},\ \delta_G^{(2)}. \tag{25}$$

Self-energies

Consider a two-point function to all orders in perturbation theory,

$$\Sigma_{VV}(s,\xi) = \sum_{n=2}^{\infty} \Sigma^{(n)}_{VV}(s,\xi)\, g^{2n}. \tag{26}$$

All one-loop self-energies corresponding to physical particles are gauge-parameter independent when put on their, bare or renormalized, mass-shell and coincide with the corresponding $\xi = 1$ expression, i.e.

$$\Sigma^{(1)}_{VV}(s,\xi) = \Sigma^{(1)}_{VV\,;\,I}(s) + (s - M_V^2)\, \Phi_{VV}(s,\xi). \tag{27}$$

Theorem

from arguments based on Nielsen identities we know that

$$\frac{\partial}{\partial \xi} \Sigma_{VV}(s_P, \xi) = 0, \tag{28}$$

where

$$s_P - M_V^2 + \Sigma_{VV}(s_P) = 0. \tag{29}$$

We write

$$\Sigma^{(n)}_{VV}(s, \xi) = \Sigma^{(n)}_{VV\,;\,I}(s) + \Sigma^{(n)}_{VV\,;\,\xi}(s, \xi), \tag{30}$$

use

$$
\begin{aligned}
M_V^2 &= s_P + g^2\,\Sigma^{(1)}_{VV}(s_P) \\
&+ \boldsymbol{g}^4 \left[\Sigma^{(1)}_{VV\,;\,I}(s_P)\,\Sigma^{(1)}_{VV\,;\,\xi}(s_P,\xi) - \Sigma^{(2)}_{VV\,;\,I}(s_P) - \Sigma^{(2)}_{VV\,;\,\xi}(s_P,\xi)\right] \\
&+ \mathcal{O}\left(g^6\right), \qquad (31)
\end{aligned}
$$

to derive, as a consequence of Eq.(28),

$$
\Sigma^{(n)}_{VV\,;\,\xi}(s_P,\xi) = \Sigma^{(n-1)}_{VV\,;\,I}(s_P)\,\Phi_{VV}(s_P,\xi), \qquad (32)
$$

etc. As a consequence we obtain

$$
\Sigma_{VV}(s_P) = \sum_{n=2}^{\infty} \Sigma^{(n)}_{VV\,;\,I}(s_P)\,g^{2n}. \qquad (33)
$$

Dressed propagators

Suppose that we have a simple model with an interaction Lagrangian

$$L = \frac{g}{2}\,\Phi(x)\,\phi^2(x). \tag{34}$$

The mass M of the Φ -field and m of the ϕ -field be such that the Φ -field be unstable. Let Δ_i be the lowest order propagators and $\overline{\Delta}_i$ the one-loop dressed propagators, i.e.

$$\overline{\Delta}_\Phi = \frac{\Delta_\Phi}{1 - \Delta_\Phi\,\Sigma_{\Phi\Phi}}, \qquad \overline{\Delta}_\phi = \frac{\Delta_\phi}{1 - \Delta_\phi\,\Sigma_{\phi\phi}}, \tag{35}$$

etc. In fixed order perturbation theory, the ϕ self-energy is given in Fig. 1.

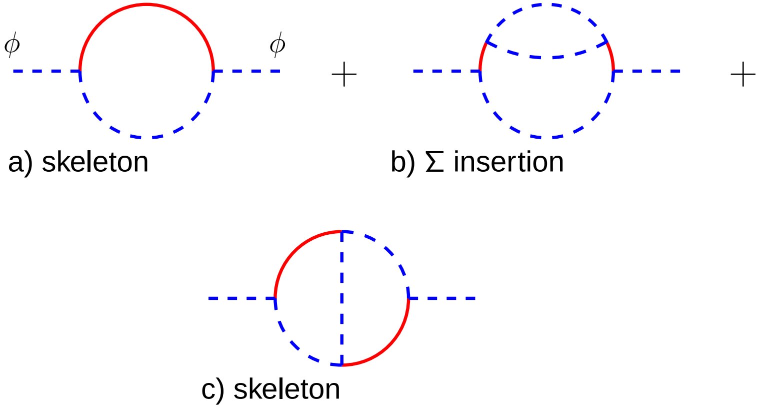

Figure: The ϕ self-energy with skeleton expansion, diagrams a) and c), and insertion of a sub-loop $\Sigma_{\Phi\Phi}$, diagram b).

ϕ imaginary part

Note that the imaginary part of $\Sigma_{\phi\phi}$ is non-zero only for

$$-p^2 > 9\,m^2, \qquad \text{(the three-particle cut of diagram b) in Fig. 1)}, \\ \text{if} \quad m \ll M. \tag{36}$$

When we use dressed propagators only diagrams a) and c) are retained in Fig. 1 (for two-loop accuracy) but in a) we use $\overline{\Delta}_{\Phi}$ with one-loop accuracy:

$$\Sigma^{(a)}_{\phi\phi} = \int \frac{d^n q_2}{\left(q_2^2 + M^2 - \frac{g^2}{16\,\pi^2}\,\Sigma_{\Phi\Phi}(q_2^2)\right)\,\left((q_2+p)^2 + m^2\right)},$$

$$\Sigma_{\Phi\Phi}(q_2^2) = B_0(q_2^2\,;\,m\,,\,m), \tag{37}$$

where we assume $p^2 < 0$.

Since the complex Φ pole is defined by

$$M^2 - s_M - \frac{g^2}{16\,\pi^2}\,\Sigma_{\Phi\Phi}(-s_M) = 0, \tag{38}$$

we write the inverse (dressed) propagator as

$$\Big[1 - \frac{g^2}{16\,\pi^2}\,\frac{\Sigma_{\Phi\Phi}(q_2^2) - \Sigma_{\Phi\Phi}(-s_M)}{q_2^2 + s_M}\Big]\,\Big(q_2^2 + s_M\Big)\,, \tag{39}$$

expand in g as if we were in a gauge theory with problems of gauge parameter dependence and obtain

$$\Sigma^{(a)}_{\phi\phi} = g^2 \int \frac{d^n q}{(q^2 + s_M)\,\Big((q+p)^2 + m^2\Big)} \times \Big[1 + \frac{g^2}{16\,\pi^2}\,\frac{\Sigma_{\Phi\Phi}(q^2) - \Sigma_{\Phi\Phi}(-s_M)}{q^2 + s_M}\Big] \tag{40}$$

$$= \frac{i}{2}\, g^2\, \pi^2\, B_0\left(1, 1\, ;\, p^2\, ;\, s_{_M}\, ,\, m^2\right) + i\, \frac{g^4}{16}\, S^{E}\left(p^2\, ;\, m^2, m^2, s_{_M}, m^2, s_{_M}\right)$$

$$+\, i\, \frac{g^4}{16}\, B_0\left(2, 1\, ;\, p^2\, ;\, s_{_M}\, ,\, m^2\right)\, \left[\Delta_{UV} - \ln\frac{m^2}{\mu^2} + 2 - \beta\, \ln\frac{\beta+1}{\beta-1}\right], \qquad (41)$$

where

$$\beta^2 = 1 - 4\, \frac{m^2}{s_{_M}}. \qquad (42)$$

More on dressed propagators

Note that there is an interply between using dressed propagators for all internal lines of a diagram and combinatorial factors and number of diagrams with and without dressed propagators.

Note that the poles in the q^0 complex plane remain in the same quadrants as in the Feynman prescription and Wick rotation can be carried out, as usual.

Evaluation of diagrams with complex masses does not pose a serious problem; in the analytical approach one should, hovever, pay the due attention to splitting of logarithms.

Consider a B_0 function,

$$B_0(p^2\,;\,M_1\,,\,M_2) = \Delta_{UV} - \int_0^1 dx\,\frac{\chi(x)}{\mu^2},$$
$$\chi(x) = -p^2\,x^2 + (p^2 + M_2^2 - M_1^2)\,x + M_1^2, \tag{43}$$

where one usually writes

$$\ln\frac{\chi(x)}{\mu^2} = \ln(-\frac{p^2}{\mu^2} - i\,\delta) + \ln(x - x_-) + \ln(x - x_+). \tag{44}$$

Since $\mathrm{Im}\,\chi(x)$ does not change sign with in $[0,1]$ the correct recipe for $M^2 = m^2 - i\,m\,\gamma$ is

$$\ln\frac{\chi(x)}{\mu^2} = \ln|\,p^2\,| + \ln(x - x_-) + \theta(-p^2)\,\Big[\ln(x - x_+) + \eta(-x_-\,,\,-x_+)\Big]$$
$$+\,\theta(p^2)\,\Big[\ln(x_+ - x) + \eta(-x_-\,,\,x_+)\Big]. \tag{45}$$

In the numerical treatent, instead, no splitting is performed and no special care is needed.
A t -channel propagator deserves some additional comment: one should not confuse the position of the pole which is always at $\mu^2 - i\,\mu\,\gamma$ with the fact that a dressed propagator function is real in the t -channel.

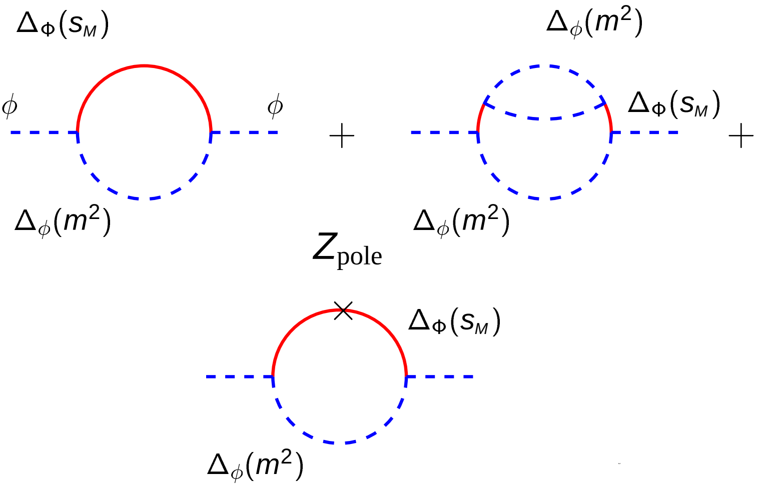

Figure: Diagram b) of Fig. 1 with one-loop dressed Φ propagators is equivalent, up to $\mathcal{O}\left(g^4\right)$, to the sum of three diagrams with lowest order

Theorem

Therefore, using one-loop diagrams with one-loop dressed Φ propagators is equivalent, to $\mathcal{O}\left(g^4\right)$, to use the sum of the three diagrams of Fig. 2 where Φ propagators are at lowest order but with complex mass s_M and where the vertex $Z_{\rm pole}$ is defined by

$$Z_{\rm pole} = \frac{g^2}{16\,\pi^2}\,B_0\left(-s_M\,;\,m\,,\,m\right). \tag{46}$$

Unitarity and gauge invariance

When dealing with the calculation of physical processes, with one and two loops, that include unstable particles, one should construct a scheme that

a) respects the unitarity of the $S-$matrix;

b) gives results that are gauge-parameter independent;

c) satisfies the whole set of WST identities.

Resummation will be part of any scheme, a fact that indroduces additional subtleties if $a - c$) are to be respected. Consider in more details the definition of dressed propagator: we consider a skeleton expansion of the self-energy Σ with progators that are resummed up to $\mathcal{O}(n)$ and define

Recursion relation

$$\Delta^{(n+1)}(p^2) = \Delta^{(0)}(p^2)\left[\Delta^{(0)}(p^2) - \Sigma^{(n+1)}\left(p^2\,,\;\Delta^{(n)}(p^2)\right)\right]^{-1}, \tag{47}$$

where

$$\Delta^{(0)}(p^2) = \frac{1}{p^2 + m^2}. \tag{48}$$

If it exists, we define a dressed propagator as

$$\overline{\Delta}(p^2) = \lim_{n \to \infty} \Sigma^{(n)}(p^2),$$
$$\overline{\Delta}(p^2) = \Delta^{(0)}(p^2)\left[\Delta^{(0)}(p^2) - \Sigma\left(p^2\,,\,\overline{\Delta}(p^2)\right)\right], \qquad (49)$$

which is not equivalent to a *rainbow* approximation and coincides with the Schwinger - Dyson solution for the propagator.

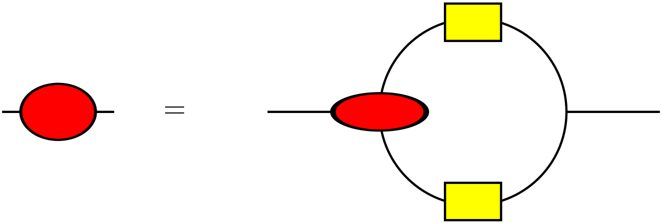

Figure: Schwinger - Dyson equation for the self-energy

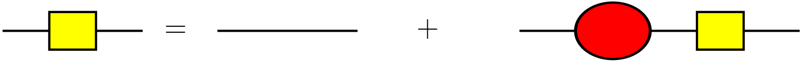

Figure: Dressed propagator

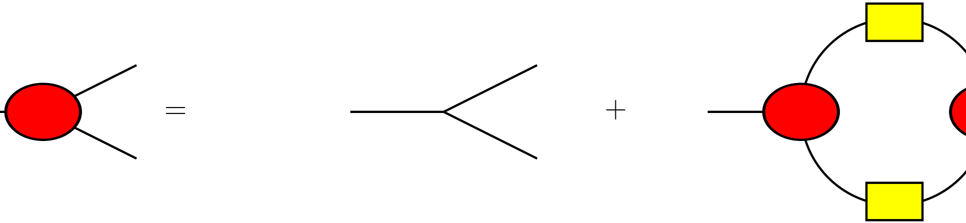

Figure: Dressed vertex

Cutting rules

– Cutting rules

We assume that Eq.(49) has a solution that obeys Källen - Lehmann representation,

$$\mathrm{Re}\,\overline{\Delta}(p^2) = \mathrm{Im}\,\Sigma(p^2)\left[\Big(p^2+m^2-\mathrm{Re}\,\Sigma(p^2)\Big)^2+\Big(\mathrm{Im}\,\Sigma(p^2)\Big)^2\right]^{-1} = \pi\,\rho(-p^2). \tag{50}$$

A dressed propagator, being the result of an infinite number of iterations,

$$\mathrm{Re}\,\overline{\Delta}(p^2) = \int_0^\infty ds\,\frac{\rho(s)}{p^2+s-i\,\delta}, \tag{51}$$

is a formal object which is difficult to handle for all practical pourposes.

Unitarity

Theorem

Unitarity follows if

- *we add all possible ways in which a diagram with given topology can be cut in two;*
- *the shaded line separates S from $S^\dagger$. F*

For a stable particle the cut line, proportional to $\overline{\Delta}^+$, contains a pole term

$$\overline{\Delta}^+ = 2\,i\,\pi\,\theta(p_0)\,\delta(p^2+m^2), \tag{52}$$

whereas there is no such contribution for an unstable particle. We express $\mathrm{Im}\,\Sigma$ in terms of cut self-energy diagrams and repeat the procedure ad libidum and prove that cut unstable lines are left with no contribution, i.e. unstable particles contribute to the unitarity of the $S-$matrix via their stable decay products.

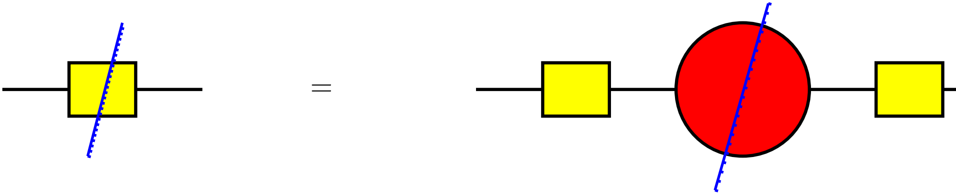

Figure: Cutting equation for dressed propagator.

Unitarity

The consistent use of dressed propagators gives a general scheme where unitarity is satisfied which is essentially a statement on the imaginary parts of the diagrams.

Approximated, or truncated, schemes (e.g. resummation of one-loop self energies, or *rainbow* approximation without further resummation of the vertex functions) usually lead to gauge dependent results.

WST identities

WST identities

We assume that WST identities hold at any fixed order in perturbation theory for diagrams that contain bare propagators and vertices; they again form dressed propagators and vertices when summed.

We expect that an arbitrary truncation that preferentially resums specific topologies will lead to violations of WST identities. Of course such violations are absent if exact calculations were possible.

Approximations

Gauge parameter dependence

A truncated approximation, e.g. simple resummation of two-point functions, necessarily leads to gauge dependent results. A convenient tool is to analyze the gauge invariance of the effective action where one can show that on-shell gauge dependence always occurs at higher order than the order of truncation.

Introducing complex poles

Complex pole

A property of the $S-$matrix is the complex pole

$$\overline{\Delta}^{-1}(p^2 = -s_P) = 0, \tag{53}$$

which is gauge parameter independent as shown by a study of Nielsen identities. An approximate solution of the unitarity constraint is as follows:

$$2\,\mathrm{Im}\,T_{ii} = \sum_n \left|T_{ni}\right|^2, \qquad \sum_n \left|T_{ni}\right|^2 = \left|D(p^2)\right|^2 \sum_n \int dPS_n \left|M_{1\to n}\right|^2, \tag{54}$$

where, $S = 1 + i\,T$ and where $D(p^2)$ is the unknown form of the propagator.

Making the approximation,

$$\sum_n \int dPS_n \left| M^{1\to n} \right|^2 \equiv m\,\Gamma_{\rm tot}, \tag{55}$$

we derive

$$\mathrm{Im}\, D(p^2) = m\,\Gamma_{\rm tot}. \tag{56}$$

A simple but, once again, approximate solution is

$$D(p^2) = \left(p^2 + m^2 - i\, m\Gamma_{\rm tot} \right)^{-1}, \tag{57}$$

which is valid far from the mass shell and where the invariant mass at which the decay is evaluated is identified with m^2.

We can improve upon this solution by writing instead

$$D(p^2) = \Big(p^2 - s_P\Big)^{-1}, \tag{58}$$

which is equivalent to resum only the self-energy (up to some fixed order), and to use $m^2 = s_P + \Sigma(s_P)$

$$\begin{aligned} D(p^2) &= -\Big[s - s_P - \Sigma(s) + \Sigma(s_P)\Big]^{-1} \\ &= -\Big(p^2 - s_P\Big)^{-1} + \text{h.o.}, \end{aligned} \tag{59}$$

where higher order terms are neglected. Another way to see that Eq.(58) is an improvement of Eq.(57) is to observe that

$$p^2 + m^2 + i\,\frac{\Gamma_{\text{tot}}}{m}\,p^2 = \Big(1 + i\,\frac{\Gamma_{\text{tot}}}{m}\Big)\,(p^2 + s_P) + \text{h.o.} \approx p^2 + s_P. \tag{60}$$

A propagator with the correct analytical structure, $p^2 - s_P$, will be represented with a thick dot. The approximation of Eq.(58) allows us to write the cutting equation of Fig. 7.

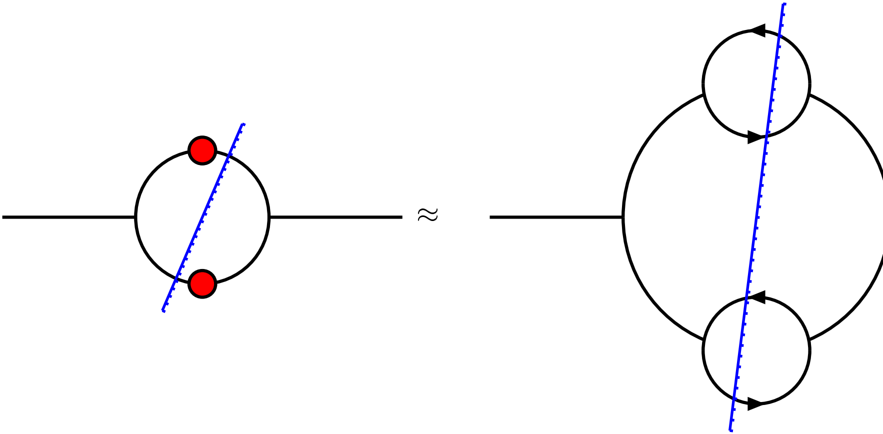

Figure: Cutting equation for a contribution to the Z self-energy using W propagators of Eq.(58).

truncated propagators

One can see that using truncated propagators with complex poles (at the one-loop level of accuracy) is still respecting unitarity of the $S-$matrix within the approximation of Eq.(55) if the complex pole is computed from fermions only; however, this scheme violates gauge invariance since vertices are not included.

There is a solution to this problem, namely replacing everywhere the (real) masses with the complex poles, couplings included; this is known in the literature as complex mass scheme.

CM scheme

The complex mass scheme

Since WST identities are algebraic relations satisfied separately by the real and the imaginary part one starts from WST identities with real masses, satisfied at any given order, replaces everywhere $m^2 \to s_P$ without violating the invariance.

In turns, this scheme violates unitarity, i.e. we cannot identify the two sides of any cut diagram with T and $T^\dagger$ respectively.

To summarize, the analytical structure of the $S-$matrix is correctly reproduced when we use propagator factors $p^2 - s_P$ but unitarity of S requires more, a dressed propagator

$p^2 + s_P$	$p^2 + s_P - \Sigma(p^2) + \Sigma(-s_P)$
analyticity	unitarity

Another drawback of the scheme is that all propagators for unstable particles will have the same functional form both in the time-like and in the space-like region while, for a dressed propagator the presence of a pole on the second Riemann sheet does not change the real character of the function if we are in a $t-$channel.
In some sense the scheme becomes more appealing when we go beyond one loop. WST identities are satisfied with bare (i.e. non-dressed) propagators and vertices up to two-loops; we may assume that they are verified order by order to all orders,

$$W^{(1)}(\{\Gamma\}) = W^{(2)}(\{\Gamma\}) = \cdots = 0, \tag{61}$$

where $\{\Gamma\}$ is a set of (off-shell) Green function and cdr $W = 0$ is the WST identity.

Next we write the same set of WST identities but using a skeleton expansion with one-loop dressed propagators. Calling the scheme *complex mass scheme* is somehow misleading; to the requested order we replace everywhere m^2 with $s_P + \Sigma(s_P)$ which is real by construction. If only one-loop is needed then $m^2 \to s_P$ everywhere (therefore justifying the name *complex mass*) and

$$W^{(1)}\left(\{\Gamma\}\right)\Big|_{m^2 = s_P} = 0, \tag{62}$$

is trivially true. Also,

$$W^{(2)}\left(\{\Gamma\}\right)\Big|_{m^2 = s_P} = 0. \tag{63}$$

At the two-loop level we have two-loop diagrams with no self-energy insertions where $m^2 = s_P$ and one-loop diagrams where $m^2 = s_P + \Sigma(s_P)$ and the factor

$$\frac{\Sigma(p^2) - \Sigma(s_P)}{p^2 + s_P}, \tag{64}$$

expanded to first order with $\Sigma = \Sigma^{(1)}$.

Furthermore, in vertices we use $m^2 = s_P$ in two-loop diagrams and $m^2 = s_P + \Sigma(s_P)$ in one-loop diagrams. Expanding the factor of Eq.(64) generates two-loop diagrams with insertion of one-loop self-energies plus one-loop diagrams with one more propagator and a vertex proportional to $\Sigma(s_P)$; furthermore one-loop diagrams with m^2 dependent vertices get multiplied by $\Sigma(s_P)$; it follows that

Theorem

$$\begin{aligned} W^{(1+2)}\left(\{\Gamma\}_{\rm skeleton}\right)\Big|_{m^2=s_P+\Sigma(s_P)} &= W^{(1+2)}\left(\{\Gamma\}\right)\Big|_{m^2=s_P} \\ &+ \Sigma(s_P)\,\frac{d}{dm^2}\,W^{(1)}\left(\{\Gamma\}\right)\Big|_{m^2=s_P} \\ &= 0, \end{aligned} \tag{65}$$

as a consequence of Eqs.(62)–(63).