TWO-LOOP Renormalization in the Making

Giampiero PASSARINO

Torino

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Outline of Part II

1. New coupling constant in the β_h scheme
2. New coupling constant in the β_t scheme
3. The Γ–β_t mixing
4. WSTI for two-loop gauge boson self-energies
5. WSTI at two loops: the role of reducible diagrams
6. The photon self-energy
7. The photon–Z mixing
8. The Z self-energy
9. The W self-energy
10. Dyson resummed propagators and their WSTI
11. The charged sector
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New coupling constant in the β_h scheme

The Z–γ transition in the SM does not vanish at zero squared momentum transfer. Although this fact does not pose any serious problem, not even for the renormalization of the electric charge, it is preferable to use an alternative strategy. Let's introduce the new $SU(2)$ coupling constant $\bar{g}$, the new mixing angle $\bar{\theta}$ and the new W mass $\bar{M}$ in the β_h scheme:

$$
\begin{aligned}
g &= \bar{g}\,(1+\Gamma) \qquad g' = -(\sin\bar{\theta}/\cos\bar{\theta})\,\bar{g} \\
v &= 2\bar{M}/\bar{g} \quad \lambda = \left(\bar{g}M_{_H}/2\bar{M}\right)^2 \quad \mu^2 = \beta_h - \tfrac{1}{2}M_{_H}^2
\end{aligned}
\tag{1}
$$

note: $g\sin\theta/\cos\theta = \bar{g}\sin\bar{\theta}/\cos\bar{\theta}$, where $\Gamma = \Gamma_1\,\bar{g}^2 + \Gamma_2\,\bar{g}^4 + \cdots$ is a new parameter yet to be specified. This change of parameters entails new $\bar{A}_\mu$ and $\bar{Z}_\mu$ fields related to B^3_μ and B^0_μ by

$$\begin{pmatrix} \bar{Z}^0_\mu \\ \bar{A}_\mu \end{pmatrix} = \begin{pmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{pmatrix} \begin{pmatrix} B^3_\mu \\ B^0_\mu \end{pmatrix}. \tag{2}$$

The replacement $g \to \bar{g}(1+\Gamma)$ introduces in the SM Lagrangian several terms containing the new parameter Γ. Let us take a close look at these 'Γ terms' in each sector of the SM.

- The pure Yang–Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^0_{\mu\nu}, \tag{3}$$

with $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g \epsilon^{abc} B^b_\mu B^c_\nu$ and $F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$, contains the following new Γ terms when we replace g by $\bar{g}(1+\Gamma)$:

$$\begin{aligned}
\Delta \mathcal{L}_{YM} = & -i \bar{g} \Gamma \bar{c} \Big[\partial_\nu \bar{Z}^0_\mu \left(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu \right) - \bar{Z}^0_\nu \left(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu \right) \\
& + \bar{Z}^0_\mu \left(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu \right) \Big] - i \bar{g} \Gamma \bar{s} \Big[\partial_\nu \bar{A}_\mu \left(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu \right) \\
& - \bar{A}_\nu \left(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu \right) + \bar{A}_\mu \left(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu \right) \Big] \\
& + \bar{g}^2 \Gamma (2+\Gamma) \Big[\frac{1}{2} \left(W^+_\mu W^-_\nu W^+_\mu W^-_\nu - W^+_\mu W^-_\mu W^+_\nu W^-_\nu \right) \\
& + \bar{c}^2 \left(\bar{Z}^0_\mu W^+_\mu \bar{Z}^0_\nu W^-_\nu - \bar{Z}^0_\mu \bar{Z}^0_\mu W^+_\nu W^-_\nu \right) + \bar{s}^2 \left(\bar{A}_\mu W^+_\mu \bar{A}_\nu W^-_\nu - \bar{A}_\mu \bar{A}_\mu W^+_\nu W^-_\nu \right) \\
& + \bar{s}\bar{c} \left(\bar{A}_\mu \bar{Z}^0_\nu (W^+_\mu W^-_\nu + W^+_\nu W^-_\mu) - 2 \bar{A}_\mu \bar{Z}^0_\mu W^+_\nu W^-_\nu \right) \Big],
\end{aligned} \tag{4}$$

where $\bar{s} = \sin\bar{\theta}$ and $\bar{c} = \cos\bar{\theta}$. As these terms are of $\mathcal{O}(\bar{g}^3)$ or $\mathcal{O}(\bar{g}^4)$, they do not contribute to the calculation of self-energies at the one-loop level, but they do beyond it.

• The scalar Lagrangian $\mathcal{L}_S$ contains several new Γ terms when we employ the relation $g = \bar{g}(1+\Gamma)$ and the β_h scheme of eqs. (1). Actually, the last two equations in (1) are not needed here, as the interaction part of the scalar Lagrangian does not induce Γ terms. They can be arranged in the following three classes

$$\Delta\mathcal{L}_{S,h} = \Delta\mathcal{L}_{S,h}^{(n_f=2)} + \Delta\mathcal{L}_{S,h}^{(n_f=3)} + \Delta\mathcal{L}_{S,h}^{(n_f=4)}, \tag{5}$$

according to the number of fields (n_f) appearing in each interaction term (indicated by the superscript in parentheses. Note that this superscript does not indicate, in general, the order in $\bar{g}$). The explicit expressions, up to terms of $\mathcal{O}(\bar{g}^4)$, are

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=2)}_{S,h} &= \bar{M}\Gamma\Big[-\frac{1}{2}\bar{M}\bar{s}^2\Gamma\bar{A}_\mu\bar{A}_\mu - \frac{1}{2}\bar{M}\left(2+\Gamma\bar{c}^2\right)\bar{Z}^0_\mu\bar{Z}^0_\mu \\
&\quad -\bar{M}\,\frac{\bar{s}}{\bar{c}}\left(1+\Gamma\bar{c}^2\right)\bar{A}_\mu\bar{Z}^0_\mu + \partial_\mu\phi_0\left(\bar{s}\bar{A}_\mu + \bar{c}\bar{Z}^0_\mu\right) \\
&\quad -\bar{M}\left(2+\Gamma\right)W^+_\mu W^-_\mu + W^-_\mu\partial_\mu\phi^+ + W^+_\mu\partial_\mu\phi^-\Big],
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=3)}_{S,h} &= \bar{g}\Gamma\Big[-\bar{M}H\left(\bar{Z}^0_\mu\bar{Z}^0_\mu + \frac{\bar{s}}{\bar{c}}\bar{A}_\mu\bar{Z}^0_\mu + 2W^+_\mu W^-_\mu\right) \\
&\quad +\frac{1}{2}\left(\bar{s}\bar{A}_\mu + \bar{c}\bar{Z}^0_\mu\right)\left(H\partial_\mu\phi^0 - \phi^0\partial_\mu H + i\phi^+\partial_\mu\phi^- - i\phi^-\partial_\mu\phi^+\right) \\
&\quad +i\left(\phi^- W^+_\mu - \phi^+ W^-_\mu\right)\left(\bar{s}\bar{M}\bar{A}_\mu - (\bar{s}^2/\bar{c})\bar{M}\bar{Z}^0_\mu + \frac{1}{2}\partial_\mu\phi^0\right) \\
&\quad +\frac{1}{2}W^-_\mu\partial_\mu\phi^+\left(H+i\phi^0\right) + \frac{1}{2}W^+_\mu\partial_\mu\phi^-\left(H-i\phi^0\right) - \frac{1}{2}\partial_\mu H\left(\phi^+W^-_\mu + \phi^-W^+_\mu\right)\Big],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=4)}_{S,h} &= \frac{\bar{g}^2}{2}\Gamma\Big\{-\frac{1}{2}\left(H^2+\phi_0^2\right)\left(\bar{Z}^0_\mu\bar{Z}^0_\mu + \frac{\bar{s}}{\bar{c}}\bar{A}_\mu\bar{Z}^0_\mu + 2W^+_\mu W^-_\mu\right) \\
&\quad +\phi^+\phi^-\left(-2\bar{s}^2\bar{A}_\mu\bar{A}_\mu + (1-2\bar{c}^2)\bar{Z}^0_\mu\bar{Z}^0_\mu + (\bar{s}/\bar{c} - 4\bar{s}\bar{c})\bar{A}_\mu\bar{Z}^0_\mu\right) \\
&\quad -2W^+_\mu W^-_\mu\phi^+\phi^- + \left(\bar{s}\bar{A}_\mu - (\bar{s}^2/\bar{c})\bar{Z}^0_\mu\right)\times \\
&\quad\quad \times\Big[\phi_0\left(\phi^+W^-_\mu + \phi^-W^+_\mu\right) - iH\left(\phi^+W^-_\mu - \phi^-W^+_\mu\right)\Big]\Big\}.
\end{aligned}
\tag{8}
$$

The interaction part of the scalar Lagrangian, $\mathcal{L}^I_S = -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2$, does not induce Γ terms; these are only originated by the term involving the covariant derivatives, $-(D_\mu K)^\dagger (D_\mu K)$. On the other hand, as $M/g = \bar{M}/\bar{g}$, the β_h terms induced by $\mathcal{L}^I_S$ are expressed in terms of the ratio of the barred parameters $\bar{M}/\bar{g}$.

- We choose the gauge-fixing Lagrangian $\mathcal{L}_{gf}$ with the following gauge functions:

$$\mathcal{C}_A = -\frac{1}{\xi_A}\partial_\mu \bar{A}_\mu, \quad \mathcal{C}_Z = -\frac{1}{\xi_Z}\partial_\mu \bar{Z}^0_\mu + \xi_Z \frac{\bar{M}}{\bar{c}}\phi_0, \quad \mathcal{C}_\pm = -\frac{1}{\xi_W}\partial_\mu W^\pm_\mu + \xi_W \bar{M}\phi_\pm. \tag{9}$$

gauge fixing

This R_ξ gauge Γ-independent $\mathcal{L}_{gf}$ cancels the zeroth order (in $\bar{g}$) gauge–scalar mixing terms introduced by $\mathcal{L}_S$, but not those proportional to Γ. Had one chosen gauge-fixing functions eqs. (9) with unbarred quantities, all the gauge–scalar mixing terms of $\mathcal{L}_S$ would be canceled, including those proportional to Γ, but additional new Γ vertices would also be introduced.

• New Γ terms are also originated in the Faddeev–Popov ghost sector. Studying the gauge transformations of the gauge-fixing functions $\mathcal{C}_A$, $\mathcal{C}_Z$ and $\mathcal{C}_\pm$ defined in eqs. (9), the additional new Γ terms of the FP Lagrangian in the β_h scheme are:

$$\Delta\mathcal{L}_{FP,h} = \Delta\mathcal{L}^{(n_f=2)}_{FP,h} + \Delta\mathcal{L}^{(n_f=3)}_{FP,h}, \tag{10}$$

where the two-field terms are,

$$\Delta\mathcal{L}^{(n_f=2)}_{FP,h} = -\Gamma\bar{M}^2\left[\xi_Z\bar{X}_Z\left(X_Z + \frac{\bar{s}}{\bar{c}}X_A\right) + \xi_W\left(\bar{X}_+X_+ + \bar{X}_-X_-\right)\right], \tag{11}$$

and the three-field terms are

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=3)}_{FP,h} \;=\;& \Gamma\bar{g}\,\Big\{ i\bar{c}W^{+}_{\mu}\Big((\partial_\mu\bar{X}_Z/\xi_Z)X_- - (\partial_\mu\bar{X}_+/\xi_W)X_Z\Big) \\
+\;& i\bar{s}W^{+}_{\mu}\Big((\partial_\mu\bar{X}_A/\xi_A)X_- - (\partial_\mu\bar{X}_+/\xi_W)X_A\Big) \\
+\;& i\bar{c}W^{-}_{\mu}\Big((\partial_\mu\bar{X}_-/\xi_W)X_Z - (\partial_\mu\bar{X}_Z/\xi_Z)X_+\Big) \\
+\;& i\bar{s}W^{-}_{\mu}\Big((\partial_\mu\bar{X}_-/\xi_W)X_A - (\partial_\mu\bar{X}_A/\xi_A)X_+\Big) \\
+\;& i\bar{c}\bar{Z}^{0}_{\mu}\;\Big((\partial_\mu\bar{X}_+/\xi_W)X_+ - (\partial_\mu\bar{X}_-/\xi_W)X_-\Big) \\
+\;& i\bar{s}\bar{A}_{\mu}\;\Big((\partial_\mu\bar{X}_+/\xi_W)X_+ - (\partial_\mu\bar{X}_-/\xi_W)X_-\Big) \\
+\;& \frac{1}{2}\xi_W\bar{M}\Big[i\phi_0\left(\bar{X}_+X_+ - \bar{X}_-X_-\right) - H\left(\bar{X}_+X_+ + \bar{X}_-X_-\right)\Big] \\
+\;& \frac{1}{2\bar{c}}\xi_Z\bar{M}\bar{X}_Z\Big[iX_-\phi_+ - iX_+\phi_- - \bar{s}HX_A - \bar{c}HX_Z\Big] \\
+\;& \frac{i}{2}\xi_W\bar{M}\Big[\bar{X}_-\phi_-\left(\bar{c}X_Z + \bar{s}X_A\right) - \bar{X}_+\phi_+\left(\bar{c}X_Z + \bar{s}X_A\right)\Big]\Big\}\,.
\end{aligned}
\tag{12}
$$

FP ghost fields

The bars over the FP ghost fields indicate conjugation. Obviously, the new FP fields X_A and X_Z should also be denoted with the bar for the field rediagonalization, just like the new fields $\bar{A}_\mu$ and $\bar{Z}_\mu$. However, this notation would be too messy and we will leave this point understood.

Note that the FP ghost – gauge boson vertices are simply the usual ones with g replaced by $\bar{g}\Gamma$. This is not the case, in general, for the FP ghost – scalar terms.

- Finally, the fermionic sector. The fermion – gauge boson Lagrangian,

$$
\begin{aligned}
\mathcal{L}_{fG} \;=\; & \frac{i}{2\sqrt{2}}\, g \left[W^{+}_{\mu} \bar{u} \gamma_{\mu} (1+\gamma_5)\, d \;+ W^{-}_{\mu} \bar{d} \gamma_{\mu} (1+\gamma_5)\, u \right] \\
& + \frac{i}{2c}\, g\, Z_{\mu} \bar{f} \gamma_{\mu} \Big(I_3 - 2 Q_f s^2 + I_3 \gamma_5 \Big) f \;+ i\, g s\, Q_f A_{\mu} \bar{f} \gamma_{\mu} f, \qquad (13)
\end{aligned}
$$

(where $I_3 = \pm 1/2$ is the weak isospin third component of the fermion f, and Q_f its charge in units of $|e|$) becomes, under the replacement $g \to \bar{g}(1+\Gamma)$ and the θ, A_μ and Z_μ redefinitions,

Fermions

$$
\begin{aligned}
\mathcal{L}_{fG} \;=\; & \frac{i}{2\sqrt{2}}\,\bar{g}\,(1+\Gamma)\left[W^{+}_{\mu}\bar{u}\gamma_{\mu}\,(1+\gamma_5)\,d \;+W^{-}_{\mu}\bar{d}\gamma_{\mu}\,(1+\gamma_5)\,u\,\right] \\
& + \frac{i}{2\bar{c}}\,\bar{g}\,\bar{Z}^{0}_{\mu}\,\bar{f}\gamma_{\mu}\left(I_3 - 2Q_f\bar{s}^2 + I_3\gamma_5\right)f \;+\; i\,\bar{g}\bar{s}\,Q_f\bar{A}_{\mu}\bar{f}\gamma_{\mu}f \\
& + \frac{i}{2}\,\bar{g}\,\Gamma\left(\bar{s}\bar{A}_{\mu} + \bar{c}\bar{Z}^{0}_{\mu}\right)I_3\,\bar{f}\gamma_{\mu}\,(1+\gamma_5)\,f\,. \qquad (14)
\end{aligned}
$$

The new neutral and charged current Γ vertices are immediately recognizable. The CKM matrix has been set to unity.

The fermion–scalar Lagrangian does not induce Γ terms. Indeed, the Yukawa couplings α and β in

$$\mathcal{L}_{fS} = -\alpha\bar{\psi}_L K u_R - \beta\bar{\psi}_L K^c d_R + \text{h.c.} \tag{15}$$

(where $K^c = i\tau_2 K^\star$ is the conjugate Higgs doublet) are set by $\alpha v/\sqrt{2} = m_u$ and $\beta v/\sqrt{2} = -m_d$. As $v = 2\bar{M}/\bar{g}$, it is $\alpha = \bar{g}m_u/\sqrt{2}\bar{M}$ and $\beta = -\bar{g}m_d/\sqrt{2}\bar{M}$, and no Γ appears in Eq.(15).

Yang–Mills

The Feynman rules for all these new Γ vertices are computed, up to terms of $\mathcal{O}(\bar{g}^4)$. Those corresponding to the pure Yang–Mills Lagrangian [Eq.(4)] are not listed, as they are identical to the usual Yang–Mills ones, except for the replacement $g \rightarrow \bar{g}\Gamma$ in the three-leg vertices, and $g^2 \rightarrow \bar{g}^2\Gamma(2+\Gamma)$ in the four-leg ones. In Appendix C, all bars over the various symbols (indicating rediagonalization) have been dropped, except over $\bar{g}$.

New coupling constant in the β_t scheme

The β_t scheme equations corresponding to Eq.(1) are the following

$$
\begin{aligned}
g &= \bar{g}\,(1+\Gamma) \qquad g' = -(\sin\bar{\theta}/\cos\bar{\theta})\,\bar{g} \\
v &= 2\bar{M}'(1+\beta_t)/\bar{g} \quad \lambda = \left(\bar{g}M'_H/2\bar{M}'\right)^2 \quad \mu^2 = -\frac{1}{2}(M'_H)^2.
\end{aligned}
\tag{16}
$$

(Note: $g\sin\theta/\cos\theta = \bar{g}\sin\bar{\theta}/\cos\bar{\theta}$.) The analysis of the Γ terms presented in the previous section for the β_h scheme can be repeated for the β_t scheme using Eq.(16) instead of Eq.(1). The new fields $\bar{A}_\mu$ and $\bar{Z}_\mu$ are related to B^3_μ and B^0_μ by Eq.(2). Thus, we obtain the following results:

• The replacement $g \to \bar{g}(1+\Gamma)$ in the pure Yang–Mills sector introduces new Γ vertices collected in $\Delta\mathcal{L}_{YM}$, which does not depend on the parameters of the $\beta_{h,t}$ schemes. $\Delta\mathcal{L}_{YM}$ has already been given in Eq.(4).

• The new Γ terms introduced in $\mathcal{L}_S$ by eqs. (16) can be arranged once again in the three classes

$$\Delta\mathcal{L}_{S,t} = \Delta\mathcal{L}_{S,t}^{(n_f=2)} + \Delta\mathcal{L}_{S,t}^{(n_f=3)} + \Delta\mathcal{L}_{S,t}^{(n_f=4)}, \tag{17}$$

according to the number of fields appearing in the Γ terms. The explicit expression for $\Delta\mathcal{L}_{S,t}^{(2)}$ is, up to terms of $\mathcal{O}(\bar{g}^4)$,

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=2)}_{S,t} \;=\; & \bar{M}'\Gamma\left[-\frac{1}{2}\bar{M}'\bar{s}^2\Gamma\bar{A}_\mu\bar{A}_\mu - \frac{1}{2}\bar{M}'\left(2+\Gamma\bar{c}^2+4\beta_t\right)\bar{Z}^0_\mu\bar{Z}^0_\mu \quad (\right. \\
& -\bar{M}'\,\frac{\bar{s}}{\bar{c}}\left(1+\Gamma\bar{c}^2+2\beta_t\right)\bar{A}_\mu\bar{Z}^0_\mu + \partial_\mu\phi_0\left(\bar{s}\bar{A}_\mu+\bar{c}\bar{Z}^0_\mu\right)(1+\beta_t) \\
& -\bar{M}'\,(2+\Gamma+4\beta_t)\,W^+_\mu W^-_\mu + \left(W^-_\mu\partial_\mu\phi^+ + W^+_\mu\partial_\mu\phi^-\right)(1+
\end{aligned}
$$

with $\bar{s}=\sin\bar{\theta}$ and $\bar{c}=\cos\bar{\theta}$, while, up to the same $\mathcal{O}(\bar{g}^4)$,

more fields

$$\Delta\mathcal{L}_{S,t}^{(n_f=3,4)} = \Delta\mathcal{L}_{S,h}^{(n_f=3,4)}\left(\bar{M} \to \bar{M}'\right) \tag{19}$$

[$\Delta\mathcal{L}_{S,h}^{(n_f=3)}$ and $\Delta\mathcal{L}_{S,h}^{(n_f=4)}$ are given in eqs. (7) and (8)]. The subscripts t and h indicate the β_t and β_h schemes. Note the presence of β_t factors in the new Γ terms of Eq.(18). We will comment on this in sec. 23.

• Our recipe for gauge-fixing is the same as in the previous sections: we choose the R_ξ gauge $\mathcal{L}_{gf}$ to cancel the zeroth order (in $\bar{g}$) gauge–scalar mixing terms introduced by $\mathcal{L}_S$, but not those of higher orders (see discussions in 2). Here, this prescription is realized by $\mathcal{L}_{gf}$ with

$$\mathcal{C}_A = -\frac{1}{\xi_A}\partial_\mu \bar{A}_\mu, \quad \mathcal{C}_Z = -\frac{1}{\xi_Z}\partial_\mu \bar{Z}^0_\mu + \xi_Z \frac{\bar{M}'}{\bar{c}}\phi_0, \quad \mathcal{C}_\pm = -\frac{1}{\xi_W}\partial_\mu W^\pm_\mu + \xi_W \bar{M}'\phi_\pm, \tag{20}$$

clearly Γ-independent.

The new Γ terms of the FP ghost Lagrangian in the β_t scheme are:

$$\Delta\mathcal{L}_{FP,t} = \Delta\mathcal{L}_{FP,t}^{(n_f=2)} + \Delta\mathcal{L}_{FP,t}^{(n_f=3)} , \tag{21}$$

where the two-field terms are

$$\Delta\mathcal{L}_{FP,t}^{(n_f=2)} = -\left(1+\beta_t\right)\Gamma\bar{M}'^2 \left[\xi_Z \bar{X}_Z \left(X_Z + \frac{\bar{s}}{\bar{c}} X_A\right) + \xi_W \left(\bar{X}_+ X_+ + \bar{X}_- X_-\right)\right], \tag{22}$$

and the three-field terms are the same as in the β_h scheme, with $\bar{M}$ replaced by $\bar{M}'$: $\Delta\mathcal{L}_{FP,t}^{(n_f=3)} = \Delta\mathcal{L}_{FP,h}^{(n_f=3)}(\bar{M} \to \bar{M}')$ [Eq.(12)]. Like in the scalar sector, the Γ and β_t factors are entangled; see sec. 23 for a comment.

• We conclude this analysis with the fermionic sector. As in the Yang–Mills case, the fermion – gauge boson Lagrangian $\mathcal{L}_{fG}$ does not depend on the parameters of the β_h or β_t schemes. Its expression in terms of the new coupling constant $\bar{g}$ contains new Γ terms and is given in Eq.(14). The neutral sector rediagonalization induces no Γ terms in the fermion–scalar Lagrangian $\mathcal{L}_{fS}$ [Eq.(15)], which contains, however, the β_t vertices (the ratio M'/g is now replaced by the identical ratio $\bar{M}'/\bar{g}$).

The Feynman rules for all Γ vertices are listed in Appendix C, up to terms of $\mathcal{O}(\bar{g}^4)$. All primes and bars over A_μ, Z_μ, M, M_H and θ have been dropped (but not over $\bar{g}$). As we mentioned at the end of the previous section, the Γ vertices of the pure Yang–Mills sector need not be listed.

The Γ–β_t mixing

A comment on the presence of β_t factors in the new Γ vertices is now appropriate. Consider the scalar Lagrangian $\mathcal{L}_S$. As we already pointed out in sec. 2, the interaction part of $\mathcal{L}_S$, $\mathcal{L}_S^I = -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2$, does not induce Γ terms. On the other hand, $\mathcal{L}_S^I$ gives rise to β_t terms: as $M'/g = \bar{M}'/\bar{g}$, these β_t terms are simply expressed in terms of $\bar{M}'/\bar{g}$ instead of M'/g.

The derivative part of the scalar Lagrangian, $-(D_\mu K)^\dagger (D_\mu K)$, induces both Γ and β_t vertices, plus mixed ones which we still call Γ vertices (see the β_t factors in the two-leg Γ terms of $\Delta\mathcal{L}_{S,t}^{(n_f=2)}$).

It works like this: first, we replace $g \to \bar{g}(1+\Gamma)$ and $g' \to -\bar{g}(\bar{s}/\bar{c})$ in $-(D_\mu K)^\dagger(D_\mu K)$, splitting the result in two classes of terms, both written in terms of $\bar{g}$, with or without Γ.
Then we substitute in both classes $v \to 2\bar{M}'(1+\beta_t)/\bar{g}$: the class containing Γ is, up to terms of $\mathcal{O}(\bar{g}^4)$, $\Delta\mathcal{L}_{S,t}$ [Eq.(17)], and includes also β_t factors, while the class free of Γ has the same β_t vertices as Eq.(**??**) with g, θ, M', A_μ and Z_μ replaced by $\bar{g}$, $\bar{\theta}$, $\bar{M}'$, $\bar{A}_\mu$ and $\bar{Z}^0_\mu$. The upshot is that you need both the results for the new Γ vertices derived in the previous section 16 (containing β_t), *and* the expressions for the β_t terms.

The Γ and β_t terms of the Faddeev–Popov sector are intertwined just as in the case of the scalar Lagrangian.

Summary of the special vertices

The upshot of these first sections of the paper lies in the Appendices. There you find the full set of Standard Model Γ [up to $\mathcal{O}(\bar{g}^4)$] and $\beta_{h,t}$ special vertices in the R_ξ gauges. All primes and bars over A_μ, Z_μ, M, $M_{_H}$ and θ have been dropped, but not over $\bar{g}$, the $SU(2)$ coupling constant of the rediagonalized neutral sector. Just pick your tadpole scheme, β_h or β_t, and compute your Feynman diagrams including the $\beta_{h,t}$ vertices of Appendix A or B, respectively.

If you prefer to work with the rediagonalized neutral sector, you should simply replace g by $\bar{g}$ in the $\beta_{h,t}$ vertices, and add to them the Γ ones of Appendix C. There, Γ vertices are listed for the β_t scheme (note that Γ and β_t terms are intertwined — see sec. 23); just set $\beta_t = 0$ if you are using the β_h scheme instead.

Finally, the following table graphically summarizes which of the SM sectors provide each type of special vertex. Note the overlap of Γ and β_t terms in the scalar and Faddeev–Popov sectors.

SECTOR	β_h	β_t	Γ
Scalar: $(D_\mu K)^\dagger (D_\mu K)$		$\bullet$	$\bullet$
Scalar: $\mu^2 K^\dagger K + (\lambda/2)(K^\dagger K)^2$	$\bullet$	$\bullet$	
Yang–Mills			$\bullet$
Gauge-Fixing			
Faddeev–Popov		$\bullet$	$\bullet$
Fermion – gauge boson			$\bullet$
Fermion – Higgs		$\bullet$	

WSTI for two-loop gauge boson self-energies

WSTI

The purpose of this section is to discuss in detail the structure of the (doubly-contracted) Ward-Slavnov-Taylor identities (WSTI) for the two-loop gauge boson self-energies in the Standard Model, focusing in particular on the role played by the reducible diagrams. This analysis is performed in the 't Hooft–Feynman gauge.

Definitions and WST identities

Let Π_{ij} be the sum of all diagrams (both one-particle reducible and irreducible) with two external boson fields, i and j, to all orders in perturbation theory (as usual, the external Born propagators are not to be included in the expression for Π_{ij})

$$\Pi_{ij} = \sum_{n=1}^{\infty} \frac{g^{2n}}{(16\pi^2)^n}\, \Pi^{(n)}_{ij}\,. \tag{23}$$

In the subscripts of the quantities $\Pi^{(n)}_{ij}$ we will also explicitly indicate, when necessary, the appropriate Lorentz indices with Greek letters. At each order in the perturbative expansion it is convenient to make explicit the tensor structure of these functions by employing the following definitions:

$$\Pi^{(n)}_{\mu\nu,VV} = D^{(n)}_{VV}\,\delta_{\mu\nu} + P^{(n)}_{VV}\,p_\mu\,p_\nu \quad \Pi^{(n)}_{\mu,VS} = -ip_\mu\,M_S\,G^{(n)}_{VS} \quad \Pi^{(n)}_{SS} = R^{(n)}_{SS}\,, \tag{24}$$

where the subscripts V and S indicate vector and scalar fields, M_S is the mass of the Nambu–Goldstone scalar S, and p is the incoming momentum of the vector boson (note: $\Pi^{(n)}_{\mu,SV} = -\Pi^{(n)}_{\mu,VS}$). The quantities D_{ij}, P_{ij}, G_{ij}, and R_{ij} depend only on the squared four-momentum and are symmetric in i and j. Furthermore, D and R have the dimensions of a mass squared, while G and P are dimensionless.

The WST identities require that, at each perturbative order, the gauge-boson self-energies

satisfy the equations

$$
\begin{aligned}
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,AA} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,AZ} + i p_\mu\, M_0\, \Pi^{(n)}_{\mu,A\phi_o} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,ZZ} + M_0^2\, \Pi^{(n)}_{\phi_o\phi_o} + 2\, i p_\mu\, M_0\, \Pi^{(n)}_{\mu,Z\phi_o} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,WW} + M^2\, \Pi^{(n)}_{\phi\phi} + 2\, i p_\mu\, M\, \Pi^{(n)}_{\mu,W\phi} &= 0\,, \qquad (25)
\end{aligned}
$$

which imply the following relations among the form factors D, P, G, and R

$$
\begin{aligned}
D^{(n)}_{AA} + p^2\, P^{(n)}_{AA} &= 0 && (26)\\
D^{(n)}_{AZ} + p^2\, P^{(n)}_{AZ} + M_0^2\, G^{(n)}_{A\phi_0} &= 0 && (27)\\
p^2\, D^{(n)}_{ZZ} + p^4\, P^{(n)}_{ZZ} + M_0^2\, R^{(n)}_{\phi_0\phi_0} &= -2\, M_0^2\, p^2 G^{(n)}_{Z\phi_0} && (28)\\
p^2\, D^{(n)}_{WW} + p^4\, P^{(n)}_{WW} + M^2\, R^{(n)}_{\phi\phi} &= -2\, M^2\, p^2 G^{(n)}_{W\phi}\,. && (29)
\end{aligned}
$$

The subscripts A, Z, W, ϕ and ϕ_0 clearly indicate the SM fields. We have verified these WST Identities at the two-loop level (i.e. $n = 2$) with our code `GraphShot`.

WSTI at two loops: the role of reducible diagrams

At any given order in the coupling constant expansion, the SM gauge boson self-energies satisfy the WSTI (25). For $n \geq 2$, the quantities $\Pi^{(n)}_{ij}$ contain both one-particle irreducible (1PI) and reducible (1PR) contributions. At $\mathcal{O}(g^4)$, the SM $\Pi^{(n)}_{ij}$ functions contain the following *irreducible* topologies:

- eight two-loop topologies,
- three one-loop topologies with a β_{t_1} vertex,
- four one-loop topologies with a Γ_1 vertex,
- and one tree-level diagram with a two-leg $\mathcal{O}(g^4)$ β_t or Γ vertex .

Reducible $\mathcal{O}(g^4)$ graphs involve the product of two $\mathcal{O}(g^2)$ ones:

two one-loop diagrams,

one one-loop diagram and a tree-level diagram with a $\mathcal{O}(g^2)$ two-leg vertex insertion,

or two tree-level diagrams, each with a $\mathcal{O}(g^2)$ two-leg vertex insertion.

There are also $\mathcal{O}(g^4)$ topologies containing tadpoles but, as we discussed in previous sections, their contributions add up to zero as a consequence of our choice for β_t.
In the following we analyze the structure of the $\mathcal{O}(g^4)$ WSTI for photon, Z, and W self-energies, as well as for the photon–Z mixing, emphasizing the role played by the reducible diagrams.

The photon self-energy

The contribution of the 1PR diagrams to the photon self-energy at $\mathcal{O}(g^4)$ is given, in the 't Hooft–Feynman gauge, by (with obvious notation)

$$\Pi^{(2)R}_{\mu\nu,AA} = \frac{1}{(2\pi)^4 i} \left[\frac{1}{p^2}\, \tilde{\Pi}^{(2)R}_{\mu\nu,AA} + \frac{1}{p^2 + M_0^2}\, \hat{\Pi}^{(2)R}_{\mu\nu,AA} \right], \qquad (30)$$

where

$$\tilde{\Pi}^{(2)R}_{\mu\nu,AA} = \Pi^{(1)}_{\mu\alpha,AA}\, \Pi^{(1)}_{\alpha\nu,AA} \qquad \hat{\Pi}^{(2)R}_{\mu\nu,AA} = \Pi^{(1)}_{\mu\alpha,AZ}\, \Pi^{(1)}_{\alpha\nu,ZA} + \Pi^{(1)}_{\mu,A\phi_0}\, \Pi^{(1)}_{\nu,\phi_0 A}\,.$$

It is interesting to consider separately the reducible diagrams that involve an intermediate photon propagator ($\tilde{\Pi}^{(2)R}_{\mu\nu,AA}$) and those including an intermediate Z or ϕ_0 propagator ($\hat{\Pi}^{(2)R}_{\mu\nu,AA}$). By employing the definitions given in the previous subsection and eq. (26) with $n = 1$, one verifies that $\tilde{\Pi}^{2R}_{\mu\nu,AA}$ obeys the photon WSTI by itself,

Theorem

$$p_\mu \, p_\nu \, \tilde{\Pi}^{(2)R}_{\mu\nu,AA} = p^2 \left[D^{(1)}_{AA} + p^2 \, P^{(1)}_{AA} \right]^2 = 0 \, . \tag{31}$$

This is not the case for $\hat{\Pi}^{(2)R}_{\mu\nu,AA}$, although most of its contributions cancel when contracted by $p_\mu p_\nu$ as a consequence of eq. (27) ($n = 1$),

$$p_\mu\, p_\nu \hat{\Pi}^{(2)R}_{\mu\nu,AA} = p^2\, M_0^2\, \left(p^2 + M_0^2\right) \left[G^{(1)}_{A\phi_o}\right]^2 . \tag{32}$$

The only diagrams contributing to the A–ϕ_0 mixing up to $\mathcal{O}(g^2)$ are those with a W–ϕ or FP ghosts loop, and the tree-level diagram with a Γ insertion. Their contribution, in the 'tHooft–Feynman gauge, is

$$G^{(1)}_{A\phi_0} = (2\pi)^4 i\, sc\, \left[2B_0(p^2, M, M) + 16\pi^2\Gamma_1\right] . \tag{33}$$

A direct calculation (e.g. with `GraphShot`) shows that this residual contribution of the reducible diagrams to the $\mathcal{O}(g^4)$ photon WSTI, eq. (32), is exactly canceled by the contribution of the $\mathcal{O}(g^4)$ irreducible diagrams, which include two-loop diagrams as well as one-loop graphs with a two-leg vertex insertion.

The photon–Z mixing

We now consider the second of eqs. (25) for $n = 2$. Reducible diagrams contribute to both A–Z and A–ϕ_0 transitions. Following the example of Eq.(30), we divide these contributions in two classes: the diagrams that include an intermediate photon propagator and those mediated by a Z or a ϕ_0, namely, for the photon–Z transition in the 't Hooft–Feynman gauge,

$$
\begin{aligned}
\Pi^{(2)R}_{\mu\nu,AZ} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu\nu,AZ} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu\nu,AZ}\right] \\
\tilde{\Pi}^{(2)R}_{\mu\nu,AZ} &= \Pi^{(1)}_{\mu\alpha,AA}\,\Pi^{(1)}_{\alpha\nu,AZ} \\
\hat{\Pi}^{(2)R}_{\mu\nu,AZ} &= \Pi^{(1)}_{\mu\alpha,AZ}\,\Pi^{(1)}_{\alpha\nu,ZZ} + \Pi^{(1)}_{\mu,A\phi_0}\,\Pi^{(1)}_{\nu,\phi_0 Z}\,,
\end{aligned}
\tag{34}
$$

and, for the photon–ϕ_0 transition in the same gauge,

$$
\begin{aligned}
\Pi^{(2)R}_{\mu,A\phi_o} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu,A\phi_o} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu,A\phi_o}\right] \\
\tilde{\Pi}^{(2)R}_{\mu,A\phi_o} &= \Pi^{(1)}_{\mu\alpha,AA}\,\Pi^{(1)}_{\alpha,A\phi_o} \\
\hat{\Pi}^{(2)R}_{\mu,A\phi_o} &= \Pi^{(1)}_{\mu\alpha,AZ}\,\Pi^{(1)}_{\alpha,Z\phi_o} + \Pi^{(1)}_{\mu,A\phi_o}\,\Pi^{(1)}_{\phi_o\phi_o}\,.
\end{aligned}
\tag{35}
$$

The reducible diagrams with an intermediate photon propagator satisfy the WSTI by themselves. Indeed,

$$p_\mu\, p_\nu\, \tilde{\Pi}^{(2)R}_{\mu\nu,AZ} + iM_0 p_\mu \tilde{\Pi}^{(2)R}_{\mu,A\phi_o} = 0\,, \tag{36}$$

as it can be easily checked using eq. (26) with $n = 1$. On the contrary, the remaining reducible diagrams must be added to the irreducible $\mathcal{O}(g^4)$ contributions in order to satisfy the WSTI for the photon–Z mixing:

Theorem

$$\begin{aligned} & p_\mu p_\nu \left[\frac{\hat{\Pi}^{(2)R}_{\mu\nu,AZ}}{(2\pi)^4 i (p^2 + M_0^2)} + \Pi^{(2)I}_{\mu\nu,AZ} \right] \\ & + iM_0 p_\mu \left[\frac{\hat{\Pi}^{(2)R}_{\mu,A\phi_0}}{(2\pi)^4 i (p^2 + M_0^2)} + \Pi^{(2)I}_{\mu,A\phi_0} \right] \\ & = 0. \end{aligned} \tag{37}$$

The Z self-energy

Also in the case of the WSTI for the $\mathcal{O}(g^4)$ Z self-energy it is convenient to separate the reducible contributions mediated by a photon propagator from the rest of the reducible diagrams. In the 't Hooft–Feynman gauge it is

$$
\begin{aligned}
\Pi^{(2)R}_{\mu\nu,ZZ} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu\nu,ZZ}\right] \\
\tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} &= \Pi^{(1)}_{\mu\alpha,ZA}\,\Pi^{(1)}_{\alpha\nu,AZ} \\
\hat{\Pi}^{(2)R}_{\mu\nu,ZZ} &= \Pi^{(1)}_{\mu\alpha,ZZ}\,\Pi^{(1)}_{\alpha\nu,ZZ} + \Pi^{(1)}_{\mu,Z\phi_0}\,\Pi^{(1)}_{\nu,\phi_0 Z}\,,
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\Pi^{(2)R}_{\mu,Z\phi_0} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu,Z\phi_0}\right] \\
\tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} &= \Pi^{(1)}_{\mu\alpha,ZA}\,\Pi^{(1)}_{\alpha,A\phi_0} \\
\hat{\Pi}^{(2)R}_{\mu,Z\phi_0} &= \Pi^{(1)}_{\mu\alpha,ZZ}\,\Pi^{(1)}_{\alpha,Z\phi_0} + \Pi^{(1)}_{\mu,Z\phi_0}\,\Pi^{(1)}_{\phi_0\phi_0}\,,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\Pi^{(2)R}_{\phi_0\phi_0} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\phi_0\phi_0} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\phi_0\phi_0}\right] \\
\tilde{\Pi}^{(2)R}_{\phi_0\phi_0} &= \Pi^{(1)}_{\alpha,\phi_0 A}\,\Pi^{(1)}_{\alpha,A\phi_0} \\
\hat{\Pi}^{(2)R}_{\phi_0\phi_0} &= \Pi^{(1)}_{\alpha,\phi_0 Z}\,\Pi^{(1)}_{\alpha,Z\phi_0} + \Pi^{(1)}_{\phi_0\phi_0}\,\Pi^{(1)}_{\phi_0\phi_0},
\end{aligned}
\qquad (40)
$$

and, once again, the reducible diagrams mediated by a photon propagator satisfy the WSTI by themselves, i.e.

$$
p_\mu\, p_\nu\, \tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} + M_0^2\, \tilde{\Pi}^{(2)R}_{\phi_0\phi_0} + 2\, i\, p_\mu\, M_0\, \tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} = 0\,, \qquad (41)
$$

as it can be easily checked using the one-loop WSTI for the photon–Z mixing [eq. (27) with $n = 1$].

The W self-energy

All the $\mathcal{O}(g^4)$ 1PR contributions to the WSTI for the W self-energy are mediated, in the 't Hooft–Feynman gauge, by a charged particle of mass M. A separate analysis of their contribution does not lead, in this case, to particularly significant simplifications of the structure of the WSTI. However, some cancellations among the reducible terms occur, allowing to obtain a relation that will be useful in the discussion of the Dyson resummation of the W propagator. The 1PR quantities that contribute to the $\mathcal{O}(g^4)$ WSTI for the W self-energy have the following form:

$$\Pi^{(2)R}_{\mu\nu,WW} = \frac{1}{(2\pi)^4 i\ (p^2+M^2)} \Big\{ \Big(D^{(1)}_{WW}\Big)^2 \delta_{\mu\nu} + p_\mu\, p_\nu \Big[2\, D^{(1)}_{WW}\, P^{(1)}_{WW} + p^2\ \Big(P^{(1)}_{WW}\Big)^2 + M^2\ \Big(G^{(1)}_{W\phi}\Big)^2 \Big] \Big\} \quad (42)$$

$$\Pi^{(2)R}_{\mu,W\phi} = \frac{-i\,p_\mu\,M}{(2\pi)^4 i\,(p^2+M^2)}\,G^{(1)}_{W\phi}\left[D^{(1)}_{WW} + p^2\,P^{(1)}_{WW} + R^{(1)}_{\phi\phi}\right]$$

$$\Pi^{(2)R}_{\phi\phi} = \frac{1}{(2\pi)^4 i\,(p^2+M^2)}\left[p^2\,M^2\,\left(G^{(1)}_{W\phi}\right)^2 + \left(R^{(1)}_{\phi\phi}\right)^2\right]. \qquad (43)$$

Contracting the free indices with the corresponding external momenta, summing the three contributions and employing eq. (29) with $n=1$, we obtain

$$(2\pi)^4 i\left[p_\mu\,p_\nu\,\Pi^{(2)R}_{\mu\nu,WW} + M^2\,\Pi^{(2)R}_{\phi\phi} + 2\,i\,p_\mu\,M\,\Pi^{(2)R}_{\mu,W\phi}\right] = p^2\,M^2\,\left(G^{(1)}_{W\phi}\right)^2$$
$$-\,R^{(1)}_{\phi\phi}\,\left[D^{(1)}_{WW} + p^2\,P^{(1)}_{WW}\right]. \qquad (44)$$

Dyson resummed propagators and their WSTI

Dyson resummed propagators

We will now present the Dyson resummed propagators for the electroweak gauge bosons. We will then employ the results of sec. 27 to show explicitly, up to terms of $\mathcal{O}(g^4)$, that the resummed propagators satisfy the WST identities.
Following definition (23) for Π_{ij}, the function Π'_{ij} represents the sum of all 1PI diagrams with two external boson fields, i and j, to all orders in perturbation theory (as usual, the external Born propagators are not to be included in the expression for Π'_{ij}).

As we did in eqs. (24), we write explicitly its ,

Lorentz structure

$$\Pi^{l}_{\mu\nu,vv} = D^{l}_{vv}\,\delta_{\mu\nu} + P^{l}_{vv}\,p_{\mu}\,p_{\nu} \qquad (45)$$

$$\Pi^{l}_{\mu,vs} = -ip_{\mu}\,M_{s}\,G^{l}_{vs} \qquad \Pi^{l}_{ss} = R^{l}_{ss}\,, \qquad (46)$$

where V and S indicate SM vector and scalar fields, and p_{μ} is the incoming momentum of the vector boson [note: $\Pi^{l}_{\mu,sv} = -\Pi^{l}_{\mu,vs}$].

We also introduce the

transverse and longitudinal projectors

$$
t^{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,, \qquad l^{\mu\nu} = \frac{p_\mu p_\nu}{p^2}\,,
$$
$$
t^{\mu\alpha}\,t^{\alpha\nu} = t^{\mu\nu}\,, \quad l^{\mu\alpha}\,l^{\alpha\nu} = l^{\mu\nu}\,, \quad t^{\mu\alpha}\,l^{\alpha\nu} = 0\,,
$$
$$
\Pi^{l}_{\mu\nu,VV} = D^{l}_{VV}\,t_{\mu\nu} + L^{l}_{VV}\,l_{\mu\nu}, \quad L^{l}_{VV} = D^{l}_{VV} + p^2\,P^{l}_{VV}\,. \tag{47}
$$

The full propagator for a field i which mixes with a field j via the function Π^{l}_{ij} is given by the perturbative series

$$\bar{\Delta}_{ii} = \Delta_{ii} + \Delta_{ii} \sum_{n=0}^{\infty} \prod_{l=1}^{n+1} \sum_{k_l} \Pi^{l}_{k_{l-1}k_l} \Delta_{k_l k_l} \tag{48}$$

$$= \Delta_{ii} + \Delta_{ii} \Pi^{l}_{ii} \Delta_{ii} + \Delta_{ii} \sum_{k_1=i,j} \Pi^{l}_{ik_1} \Delta_{k_1 k_1} \Pi^{l}_{k_1 i} \Delta_{ii} + \cdots ,$$

where $k_0 = k_{n+1} = i$, while for $l \neq n+1$, k_l can be i or j. Δ_{ii} is the Born propagator of the field i.

We rewrite Eq.(48) as

$$\bar{\Delta}_{ii} = \Delta_{ii}\,\left[1 - (\Pi\,\Delta)_{ii}\right]^{-1}, \tag{49}$$

and refer to $\bar{\Delta}_{ii}$ as the *resummed* propagator. The quantity $(\Pi\,\Delta)_{ii}$ is the sum of all the possible products of Born propagators and self-energies, starting with a 1PI self-energy Π^{I}_{ii}, or transition Π^{I}_{ij}, and ending with a propagator Δ_{ii}, such that each element of the sum cannot be obtained as a product of other elements in the sum.

A diagrammatic representation of $(\Pi\,\Delta)_{ii}$ is the following,

$(\Pi\,\Delta)_{ii} =$ 

where the Born propagator of the field i (j) is represented by a dotted (solid) line, the white blob is the i 1PI self-energy, and the dots at the end indicate a sum running over an infinite number of 1PI j self-energies (black blobs) inserted between two 1PI i–j transitions (gray blobs).

It is also useful to define, as an auxiliary quantity, the *partially resummed* propagator for the field i, $\hat{\Delta}_{ii}$, in which we resum only the proper 1PI self-energy insertions Π^{I}_{ii}, namely,

$$\hat{\Delta}_{ii} = \Delta_{ii}\,\left[1 - \Pi^{I}_{ii}\,\Delta_{ii}\right]^{-1}. \tag{50}$$

If the particle i were not mixing with j through loops or two-leg vertex insertions, $\hat{\Delta}_{ii}$ would coincide with the resummed propagator $\bar{\Delta}_{ii}$.

$\hat{\Delta}_{ii}$ can be graphically depicted as

$\hat{\Delta}_{ii} =$  $+ \cdots.$

Partially resummed propagators allow for a compact expression for $(\Pi\,\Delta)_{ii}$,

$$(\Pi\,\Delta)_{ii} = \Pi^{I}_{ii}\Delta_{ii} + \Pi^{I}_{ij}\hat{\Delta}_{jj}\Pi^{I}_{ji}\Delta_{ii}, \tag{51}$$

so that the resummed propagator of the field i can be cast in the form

$$\bar{\Delta}_{ii} = \Delta_{ii}\,\left[1 - \left(\Pi^{I}_{ii} + \Pi^{I}_{ij}\hat{\Delta}_{jj}\Pi^{I}_{ji}\right)\Delta_{ii}\right]^{-1}. \tag{52}$$

We can also define a resummed propagator for the i–j transition. In this case there is no corresponding Born propagator, and the resummed one is given by the sum of all possible products of 1PI i and j self-energies, transitions, and Born propagators starting with Δ_{ii} and ending with Δ_{jj}. This sum can be simply expressed in the following compact form,

$$\bar{\Delta}_{ij} = \bar{\Delta}_{ii}\,\Pi^{I}_{ij}\,\hat{\Delta}_{jj}. \tag{53}$$

The charged sector

We now apply Eq.(50), Eq.(52), Eq.(53)) to W and charged Goldstone boson fields. The *partially* resummed propagator of the charged Goldstone scalar follows immediately from Eq.(50). The Born W and ϕ propagators in the 't Hooft–Feynman gauge are

$$\Delta^{\mu\nu}_{WW} = \frac{\delta_{\mu\nu}}{p^2 + M^2}, \quad \Delta_{\phi\phi} = \frac{1}{p^2 + M^2}, \tag{54}$$

where, for simplicity of notation, we have dropped the coefficients $(2\pi)^4 i$.

In the same gauge, the partially resummed ϕ and W propagators are

$$\hat{\Delta}_{\phi\phi} = \Delta_{\phi\phi}\left[1 - \Pi'_{\phi\phi}\,\Delta_{\phi\phi}\right]^{-1} = \left[p^2 + M^2 - R'_{\phi\phi}\right]^{-1} \tag{55}$$

$$\hat{\Delta}^{\mu\nu}_{WW} = \frac{1}{p^2 + M^2 - D'_{WW}}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu P'_{WW}}{p^2 + M^2 - D'_{WW} - p^2 P'_{WW}}\right). \tag{56}$$

Equation (56) assumes a more compact form when expressed in terms of the transverse and longitudinal projectors $t_{\mu\nu}$ and $l_{\mu\nu}$,

$$\hat{\Delta}^{\mu\nu}_{WW} = \frac{t^{\mu\nu}}{p^2 + M^2 - D^l_{WW}} + \frac{l^{\mu\nu}}{p^2 + M^2 - L^l_{WW}}. \tag{57}$$

The resummed W and ϕ propagators can be then derived from Eq.(52),

$$\bar{\Delta}_{\phi\phi} = \left[p^2 + M^2 - R^l_{\phi\phi} - \frac{p^2\, M^2\, (G^l_{W\phi})^2}{p^2 + M^2 - L^l_{WW}} \right]^{-1} \tag{58}$$

$$\bar{\Delta}^{\mu\nu}_{WW} = \frac{t^{\mu\nu}}{p^2 + M^2 - D^l_{WW}} + l^{\mu\nu} \left[p^2 + M^2 - L^l_{WW} - \frac{p^2 M^2 (G^l_{W\phi})^2}{p^2 + M^2 - R^l_{\phi\phi}} \right]^{-1}. \tag{59}$$

The resummed propagator for the W–ϕ transition is provided by Eq.(53),

$$\bar{\Delta}^{\mu}_{W\phi} = \frac{-ip_{\mu} M G^{I}_{\phi W}}{p^2 + M^2 - R^{I}_{\phi\phi}} \left[p^2 + M^2 - L^{I}_{WW} - \frac{p^2 M^2 (G^{I}_{W\phi})^2}{p^2 + M^2 - R^{I}_{\phi\phi}} \right]^{-1} . \qquad (60)$$

We will now show explicitly, up to terms of $\mathcal{O}(g^4)$, that the resummed propagators defined above satisfy the following WST identity:

Theorem

$$p_{\mu}\, p_{\nu}\, \bar{\Delta}^{\mu\nu}_{WW} + i\, p_{\mu}\, M\, \bar{\Delta}^{\mu}_{W\phi} - i\, p_{\nu}\, M\, \bar{\Delta}^{\nu}_{\phi W} + M^2\, \bar{\Delta}_{\phi\phi} = 1\,, \qquad (61)$$

which, in turn, is satisfied if

$$p^2M^2\left(G^{l}_{w\phi}\right)^2 + M^2R^{l}_{\phi\phi} + p^2L^{l}_{ww} - R^{l}_{\phi\phi}L^{l}_{ww} + 2p^2M^2G^{l}_{w\phi} = 0\,. \qquad (62)$$

This equation can be verified explicitly, up to terms of $\mathcal{O}(g^4)$, using the WSTI for the W self-energy: at $\mathcal{O}(g^2)$ Eq.(62) becomes simply

$$M^2R^{(1)}_{\phi\phi} + p^2L^{(1)}_{ww} + 2p^2M^2G^{(1)}_{w\phi} = 0\,, \qquad (63)$$

which coincides with eq. (29) for $n = 1$.

To prove Eq.(62) at $\mathcal{O}(g^4)$ we can combine the last of Eq.(25) with $n = 2$ and Eq.(44) to get [1]

$$p^2 M^2 \left(G^{(1)}_{W\phi}\right)^2 + M^2 R^{(2)\prime}_{\phi\phi} + p^2 L^{(2)\prime}_{WW} - R^{(1)}_{\phi\phi} L^{(1)}_{WW} + 2p^2 M^2 G^{(2)\prime}_{W\phi} = 0\,. \quad (64)$$

[1]For simplicity of notation, in this section we dropped the coefficients $(2\pi)^4 i$.

The neutral sector

neutral sector

The SM neutral sector involves the mixing of three boson fields, A_μ, Z_μ and ϕ_0. As the definitions for the resummed propagators presented at the beginning of sec. 44 refer to the mixing of only two boson fields, we will now discuss their generalization to the three-field case.

Consider three boson fields i, j and k mixing up through radiative corrections. For each of them we can define a *partially resummed* propagator $\hat{\Delta}_{ll}$ $(l = i, j, \text{ or } k)$ according to Eq.(50). For each pair of the three fields, say (j, k), we can also define the following *intermediate* propagators

$$
\begin{aligned}
\tilde{\Delta}_{jj}(j,k) &= \Delta_{jj}\left[1 - \left(\Pi^{l}_{jj} + \Pi^{l}_{jk}\hat{\Delta}_{kk}\Pi^{l}_{kj}\right)\Delta_{jj}\right]^{-1} && (65)\\
\tilde{\Delta}_{jk}(j,k) &= \tilde{\Delta}_{jj}(j,k)\,\Pi^{l}_{jk}\,\hat{\Delta}_{kk}\,, && (66)
\end{aligned}
$$

where the parentheses on the l.h.s. indicate the chosen pair of fields. [$\tilde{\Delta}_{kk}(j,k)$ and $\tilde{\Delta}_{kj}(j,k)$ can be simply derived from the above definitions by exchanging $j \leftrightarrow k$.] The reader will immediately note that the r.h.s. of the above eqs. (65, 66) are almost identical to those of eqs. (52, 53), with the appropriate renaming of the fields. Equations (65, 66), introduced in the context of three-field mixing, define however only *intermediate* propagators (denoted by the tilde), while eqs. (52, 53), presented in the analysis of the two-field mixing case, define the complete resummed propagators (denoted by the bar).

Indeed, the definition of full resummed propagator in the three-field mixing scenario requires one further step: the resummed propagator for a field i mixing with the fields j and k via the functions Π'_{ij}, Π'_{ik} and Π'_{jk} can be cast in the following form

$$\bar{\Delta}_{ii} = \Delta_{ii}\left[1 - \left(\Pi'_{ii} + \sum_{l,m} \Pi'_{il}\,\tilde{\Delta}_{lm}(j,k)\,\Pi'_{mi}\right)\Delta_{ii}\right]^{-1}, \qquad (67)$$

where l and m can be j or k, while the resummed propagator for the transition between the fields i and k is

$$\bar{\Delta}_{ik} = \bar{\Delta}_{ii} \sum_{l=j,k} \Pi^{l}_{il} \tilde{\Delta}_{lk}(j, k)\,. \tag{68}$$

Armed with eqs. (65)–(68), we can now present the A_μ, Z_μ and A_μ–Z_μ propagators. First of all, the Born A_μ, Z_μ and ϕ_0 propagators in the 't Hooft–Feynman gauge are

$$\Delta^{\mu\nu}_{AA} = \frac{\delta_{\mu\nu}}{p^2}, \qquad \Delta^{\mu\nu}_{ZZ} = \frac{\delta_{\mu\nu}}{p^2 + M_0^2}, \qquad \Delta_{\phi_0\phi_0} = \frac{1}{p^2 + M_0^2}, \tag{69}$$

where, for simplicity of notation, we have dropped once again the coefficients $(2\pi)^4 i$. The *partially resummed* propagators (three) can be immediately computed via Eq.(50) and the *intermediate* ones (twelve) via eqs. (65) and (66). Finally, after some algebra, eqs. (67) and (68) provide us with the fully resummed propagators:
$\bar{\Delta}_{VV} = t_{\mu\nu}\bar{\Delta}^T_{VV} + l_{\mu\nu}\bar{\Delta}^L_{VV}$, with $V = A, Z$ and

$$\bar{\Delta}^T_{AA} = \left[p^2 - D^l_{AA} - \frac{(D^l_{AZ})^2}{p^2 + M_0^2 - D^l_{ZZ}}\right]^{-1} \tag{70}$$

$$\bar{\Delta}^T_{ZZ} = \left[p^2 + M_0^2 - D^l_{ZZ} - \frac{(D^l_{AZ})^2}{p^2 - D^l_{AA}}\right]^{-1} \tag{71}$$

$$\bar{\Delta}^T_{AZ} = D^l_{AZ}\left[\left(p^2 - D^l_{AA}\right)\left(p^2 + M_0^2 - D^l_{ZZ}\right) - (D^l_{AZ})^2\right]^{-1}. \tag{72}$$

The expressions of the longitudinal components of these propagators are more lengthy and we will only present them up to terms of $\mathcal{O}(g^4)$:

$$\bar{\Delta}^L_{AA} = \left[p^2 + \mathcal{O}(g^6)\right]^{-1} \tag{73}$$

$$\bar{\Delta}^L_{ZZ} = \left[p^2 + M_0^2 - L^I_{ZZ} - \frac{(L^I_{AZ})^2}{p^2} - \frac{p^2 M_0^2 (G^I_{Z\phi_0})^2}{p^2 + M_0^2} + \mathcal{O}(g^6)\right]^{-1} \tag{74}$$

$$\bar{\Delta}^L_{AZ} = \frac{L^I_{AZ}}{p^2\left(p^2 + M_0^2 - L^I_{ZZ}\right)} + \frac{M_0^2}{\left(p^2 + M_0^2\right)^2} G^I_{A\phi_0} G^I_{Z\phi_0} + \mathcal{O}(g^6). \tag{75}$$

Equation (73) achieves its compact form due to the use of the WSTI (26) and (27) with $n = 1, 2$. Also eq. (75) has been simplified using $L^{(1)}_{AA} = 0$ [i.e. eq. (26) with $n = 1$]. We point out that if we use the one-loop WSTI for the photon self-energy, eq. (26), the transverse part of the resummed A–Z propagator becomes, up to terms of $\mathcal{O}(g^4)$,

$$\bar{\Delta}^T_{AZ} = D^I_{AZ}\left[p^2\left(1 + P^I_{AA}\right)\left(p^2 + M_0^2 - D^I_{ZZ}\right)\right]^{-1} + \mathcal{O}(g^6), \qquad (76)$$

thus showing a pole at $p^2 = 0$ if $D^I_{AZ}(p^2 = 0)$ were not vanishing because of the rediagonalization of the neutral sector.

In order to show explicitly, up to terms of $\mathcal{O}(g^4)$, that the above resummed propagators satisfy their WSTI, we also present the resummed propagators involving the neutral scalar ϕ_0:

$$\bar{\Delta}^{\mu}_{A\phi_o} = -ip_\mu \frac{M_0}{p^2}\left[\frac{G^I_{Z\phi_o} L^I_{AZ}}{\left(p^2+M_0^2\right)^2} + \frac{G^I_{A\phi_o}}{p^2+M_0^2-R^I_{\phi_o\phi_o}}\right] + \mathcal{O}(g^6) \tag{77}$$

$$\bar{\Delta}^{\mu}_{Z\phi_o} = \frac{-ip_\mu M_0}{p^2+M_0^2-L^I_{ZZ}}\left[\frac{G^I_{A\phi_o} L^I_{AZ}}{p^2\left(p^2+M_0^2\right)} + \frac{G^I_{Z\phi_o}}{p^2+M_0^2-R^I_{\phi_o\phi_o}}\right] + \mathcal{O}(g^6) \tag{78}$$

$$\bar{\Delta}_{\phi_o\phi_o} = \left[p^2+M_0^2-R^I_{\phi_o\phi_o} - M_0^2\left(G^I_{A\phi_o}\right)^2 - \frac{p^2M_0^2}{p^2+M_0^2}\left(G^I_{Z\phi_o}\right)^2\right]^{-1} + \mathcal{O}(g^6). \tag{79}$$

With these results, and with the WSTI (Eq.(26))–(Eq.(28)), (Eq.(37)) and (Eq.(41)), we can finally prove, up to $\mathcal{O}(g^4)$, the following WSTI for the resummed A, Z and A–Z propagators,

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{AA} = 1 \tag{80}$$

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{AZ} + i p_\mu M_0\, \bar{\Delta}^{\mu}_{A\phi_0} = 0 \tag{81}$$

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{ZZ} + M_0^2\, \bar{\Delta}_{\phi_0\phi_0} + 2 i p_\mu M_0\, \bar{\Delta}^{\mu}_{Z\phi_0} = 1\,. \tag{82}$$

The LQ basis

For the purpose of the renormalization, it is more convenient to extract from the quantities defined in the previous sections the factors involving the weak mixing angle θ. To achieve this goal, we employ the LQ basis, which relates the photon and Z fields to a new pair of fields, L and Q:

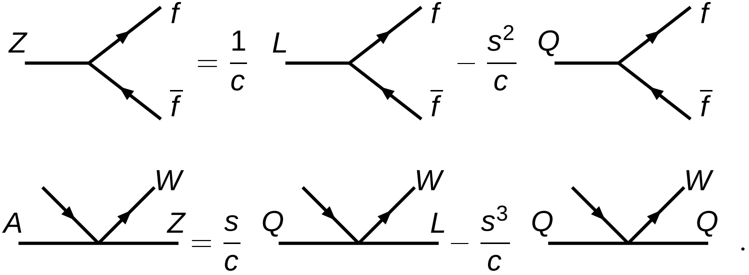
$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & 0 \\ s & 1/s \end{pmatrix} \begin{pmatrix} L_\mu \\ Q_\mu \end{pmatrix}. \tag{83}$$

Consider the fermion currents j_A^μ and j_Z^μ coupling to the photon and to the Z. As the Lagrangian must be left unchanged under this transformation, namely $j_Z^\mu Z_\mu + j_A^\mu A_\mu = j_L^\mu L_\mu + j_Q^\mu Q_\mu$, the currents transform as

$$\begin{pmatrix} j_Z^\mu \\ j_A^\mu \end{pmatrix} = \begin{pmatrix} 1/c & -s^2/c \\ 0 & s \end{pmatrix} \begin{pmatrix} j_L^\mu \\ j_Q^\mu \end{pmatrix}. \tag{84}$$

If we rewrite the SM Lagrangian in terms of the fields L and Q, and perform the same transformation (83) on the FP ghosts fields [from (X_A,X_Z) to (X_L,X_Q)], then all the interaction terms of the SM Lagrangian are independent of θ. Note that this is true only if the relation $M/c = M_0$ is employed, wherever necessary, to remove the remaining dependence on θ. In this way the dependence on the weak mixing angle is moved to the kinetic terms of the L and Q fields which, clearly, are not mass eigenstates.

The relevant fact for our discussion is that the couplings of Z, photon, X_Z and X_A are related to those of the fields L and Q, X_L and X_Q by identities like the one described, in a diagrammatic way, in the following figure:


Z
f
f̄
=
1/c
L
f
f̄
−
s²/c
Q
f
f̄
A
W
Z
=
s/c
Q
W
L
−
s³/c
Q
W
Q
.

As the couplings of the fields L, Q, X_L and X_Q do not depend on θ, all the dependence on this parameter is factored out in the coefficients in the r.h.s. of these identities.
Since θ appears only in the couplings of the fields A, Z, X_A and X_Z (once again, the relation $M/c = M_0$ must also be employed, wherever necessary), it is possible to single out this parameter in the two-loop self-energies of the vector bosons. Consider, for example, the transverse part of the photon two-loop self-energy $D^{(2)}_{AA}$ (which includes the contribution of both *irreducible* and *reducible* diagrams). All diagrams contributing to $D^{(2)}_{AA}$ can be classified in two classes: those including (i) one internal A, Z, X_A or X_Z field, and (ii) those not containing any of these fields. The complete dependence on θ can be factored out by expressing the external photon couplings and the internal A, Z X_A or X_Z couplings of the diagrams of class (i) in terms of the couplings of the fields L, Q, X_L and X_Q, namely

$$D^{(2)}_{AA} = s^2 \left[\frac{1}{c^2} f^{AA}_1 + f^{AA}_2 + s^2 f^{AA}_3 \right], \tag{85}$$

where the functions f^{AA}_i ($i = 1, 2, 3$) are θ-independent. Similarly, we can factor out the θ dependence of the transverse part of the two-loop photon–Z mixing and Z self-energy,

$$D^{(2)}_{AZ} = \frac{s}{c} \left[\frac{1}{c^2} f^{AZ}_1 + f^{AZ}_2 + s^2 f^{AZ}_3 + s^4 f^{AZ}_4 \right], \tag{86}$$

$$D^{(2)}_{ZZ} = \frac{1}{c^2} \left[\frac{1}{c^2} f^{ZZ}_1 + f^{ZZ}_2 + s^2 f^{ZZ}_3 + s^4 f^{ZZ}_4 + s^6 f^{ZZ}_5 \right], \tag{87}$$

where, once again, the functions f_i^{AZ} and f_i^{ZZ} $(i = 1, \dots, 5)$ do not depend on θ. Analogous relations hold for the longitudinal components of the two-loop self-energies.

We note that $D^{(2)}_{AZ}$ and $D^{(2)}_{ZZ}$ also contain a third class of diagrams containing more than one internal Z (or X_z) field (up to three, in $D^{(2)}_{ZZ}$). However, the diagrams of this class involve the trilinear vertex ZHZ (or $\bar{X}_z H X_z$), which does not induce any new θ dependence.

However, from the point of view of renormalization it is more convenient to distinguish between the θ dependence originating from external legs and the one introduced by external legs. We define, to all orders,

$$
\begin{aligned}
D_{AA} &= s^2\, \Pi_{QQ\,;\,\mathrm{ext}}\, p^2 = s^2 \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\, p^2, \\
D_{AZ} &= \frac{s}{c}\, \Sigma_{AZ\,;\,\mathrm{ext}} = \frac{s}{c} \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Sigma^{(n)}_{AZ\,;\,\mathrm{ext}}, \\
D_{ZZ} &= \frac{1}{c^2}\, \Sigma_{ZZ\,;\,\mathrm{ext}} = \frac{1}{c^2} \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Sigma^{(n)}_{ZZ\,;\,\mathrm{ext}}, \\
\Sigma^{(n)}_{AZ\,;\,\mathrm{ext}} &= \Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} - s^2\, \Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\, p^2, \\
\Sigma^{(n)}_{ZZ\,;\,\mathrm{ext}} &= \Sigma^{(n)}_{33\,;\,\mathrm{ext}} - 2\, s^2\, \Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} + s^4\, \Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\, p^2.
\end{aligned}
\tag{88}
$$

Furthermore, our procedure is such that

$$\Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} = \Pi^{(n)}_{3Q\,;\,\mathrm{ext}}\, p^2, \tag{89}$$

with $\Pi^{(n)}_{3Q\,;\,\mathrm{ext}}$ regular at $p^2 = 0$. At $\mathcal{O}\left(g^2\right)$ the *external* quantities are θ-independent while, at $\mathcal{O}\left(g^4\right)$ the relation with the coefficients of Eqs.(85)–(87) is

$$\begin{aligned}
\Pi^{(2)}_{QQ\,;\,\mathrm{ext}}\, p^2 &= \frac{1}{c^2}\, f^{AA}_1 + f^{AA}_2 + f^{AA}_3\, s^2, \\
\Sigma^{(2)}_{3Q\,;\,\mathrm{ext}} &= \frac{1}{c^2}\,(f^{AA}_1 + f^{AZ}_1) - f^{AA}_1 + f^{AZ}_2 + s^2\,(f^{AA}_2 + f^{AZ}_3) + s^4\,(f^{AA}_3 + f^{AZ}_4) \\
\Sigma^{(2)}_{33\,;\,\mathrm{ext}} &= \frac{1}{c^2}\,(f^{AA}_1 + 2\, f^{AZ}_1 + f^{ZZ}_1) - f^{AA}_1 - 2\, f^{AZ}_1 + f^{ZZ}_2 \\
&+ s^2\,(-f^{AA}_1 + 2\, f^{AZ}_2 + f^{ZZ}_3) + s^4\,(f^{AA}_2 + 2\, f^{AZ}_3 + f^{ZZ}_4) \\
&+ s^6\,(f^{AA}_3 + 2\, f^{AZ}_4 + f^{ZZ}_5),
\end{aligned} \tag{90}$$

and s, c in Eq.(90) should be evaluated at $\mathcal{O}\left(g^0\right)$.

Consider the process $\overline{f}f \to \overline{h}h$; taking into account Dyson re-summed propagators and neglecting, for the moment, vertices and boxes we write

$$
\begin{aligned}
\mathcal{M}(\overline{f}f \to \overline{h}h) = (2\pi)^4\, i \Big[& -e^2\, Q_f Q_h\, \gamma^\mu \otimes \gamma^\mu\, \bar{\Delta}^T_{AA} \\
& - \frac{eg}{2\,c}\, Q_f\, \gamma^\mu \otimes \gamma^\mu\, (v_h + a_h\, \gamma_5)\, \bar{\Delta}^T_{ZA} \\
& - \frac{eg}{2\,c}\, Q_h\, \gamma^\mu\, (v_f + a_f\, \gamma_5) \otimes \gamma^\mu\, \bar{\Delta}^T_{ZA} \\
& - \frac{g^2}{4\,c^2}\, \gamma^\mu\, (v_f + a_f\, \gamma_5) \otimes \gamma^\mu\, (v_h + a_h\, \gamma_5)\, \bar{\Delta}^T_{ZZ} \Big]
\end{aligned}
\tag{91}
$$

where f and h are fermions with quantum numbers $Q_I, I_{3i},\; i = f, h$;

furthermore we have introduced

$$v_f = I_{3f} - 2\,Q_f\,s^2, \qquad a_f = I_{3f}, \tag{92}$$

with $e^2 = g^2\,s^2$. Always neglecting terms proportional to fermion masses it is useful to introduce an effective weak-mixing angle as follows:

Definition

$$s^2_{\rm eff} = s^2\left[1 - \frac{\Pi_{AZ\,;\,\rm ext}}{1 - s^2\,\Pi_{AA\,;\,\rm ext}}\right], \quad V_f = I_{3f} - 2\,Q_f\,s^2_{\rm eff}. \tag{93}$$

The amplitude of Eq.(91) can be cast into the following form:

$$\mathcal{M}(\bar{f}f \to \bar{h}h) = (2\pi)^4\, i\, \Big[-\gamma^\mu \otimes \gamma^\mu\, \frac{1}{1 - s^2\, \Pi_{AA\,;\,\mathrm{ext}}}\, \frac{e^2\, Q_f Q_h}{p^2} - \frac{g^2}{4\, c^2}\, \gamma^\mu\, (V_f + a_f\, \gamma_5) \otimes \gamma^\mu\, (V_h + a_h\, \gamma_5)\, \bar{\Delta}^T_{zz} \Big]. \quad (94)$$

The functions $\Pi_{AA\,;\,\mathrm{ext}}$, $\Pi_{AZ\,;\,\mathrm{ext}}$ and $\Sigma_{ZZ\,;\,\mathrm{ext}}$ start at $\mathcal{O}\left(g^2\right)$ in perturbation theory. Eq.(94) shows the nice effect of absorbing – to all orders – non-diagonal transitions into a redefinition of s^2 and forms the basis for introducing renormalization equations in the neutral sector, e.g. the one associated with the fine-structure constant α. Questions related to gauge-parameter independence of Dyson re-summation, e.g. in Eq.(93), will not be addressed here.