QCD and collider phenomenology
part two

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Plan

Yesterday's lecture

- Colour ordering
- Helicity amplitudes
- On-shell recursions at tree level

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- Colour ordering
- Helicity amplitudes
- On-shell recursions at tree level

This lecture

- Factorization of amplitudes
- Colour ordering at one loop
- Supersymmetry
- Unitarity
- Bootstrap approach at one loop

Theory developments (I)

- Efficient techniques for computing tree amplitudes exist
  - recursion relations Berends, Giele '87

QCD amplitudes at NLO

- Straightforward in principle
  - draw all Feynman diagrams and evaluate them,
  - use standard reduction techniques for tree/loop amplitudes
- (Extremely) hard in practice
  - intermediate expressions more complicated than final results
- Known bottlenecks
  - **too many diagrams** — many diagrams are related by gauge invariance
  - **too many terms in each diagram** — nonabelian gauge boson self-interactions are complicated
  - **too many kinematic variables** — allowing the construction of arbitrarily complicated expressions

Theory developments (II)

Progess at NLO

- Lessons from tree level amplitudes $\mathcal{A}$
  - colour ordering
  - helicity amplitudes Parke, Taylor '86
- New (old?) methods at NLO
  - Analyticity
  - Unitarity (cutting rules)
  - Factorization (soft/collinear limits)
- Constructive approach at NLO (very promising)
  - a lot of recent activity Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu; *+ many others*

On-shell recursions (I)

Basic idea

- Parameter-dependent $[j,l\rangle$ shift of external massless spinors j and l

$$[j,l\rangle : \qquad \begin{aligned} \tilde{\lambda}_j &\to \tilde{\lambda}_j - z\tilde{\lambda}_l \\ \lambda_l &\to \lambda_l + z\lambda_j \end{aligned}$$

  - define $\lambda_j = u_+(k_j)$ and $\tilde{\lambda}_l = u_-(k_l)$
  - complex parameter z
- Shift in spinors corresponds to shifting momenta to complex values

$$k_j^\mu \quad \to \quad k_j^\mu(z) = k_j^\mu - \frac{z}{2}\left\langle j^- \middle| \gamma^\mu \middle| l^- \right\rangle$$

$$k_l^\mu \quad \to \quad k_l^\mu(z) = k_l^\mu + \frac{z}{2}\left\langle j^- \middle| \gamma^\mu \middle| l^- \right\rangle$$

  - momenta remain massless $k_j^2(z) = k_l^2(z) = 0$
  - momentum conservation maintained

On-shell recursions (II)

- Construction of physical amplitude $A(0)$ with $[j, l\rangle$ shift

$$A(0) = C_\infty + \sum_{r,s,h} A_L^h(z = z_{rs}) \frac{\mathrm{i}}{K_{r\dots s}^2} A_R^{-h}(z = z_{rs})$$

  - put shifted leg j in A_L (left) and shifted leg l in A_R (right) of pole in $K_{r\dots s}^2 = (k_r + k_{r+1} + \cdots + k_{s-1} + k_s)^2$
  - sum over r, s (all cyclic orderings of remaining $n - 2$ legs)
  - sum over $h = \pm 1$ (helicity states)
  - evaluate amplitudes A_L and A_R at $z = z_{rs} = \dfrac{K_{r\dots s}^2}{\langle j|K_{r\dots s}|l\rangle}$ (residue)
  - $C_\infty = 0$ if $A(z) \to 0$ as $z \to \infty$ (no ‘surface term’)

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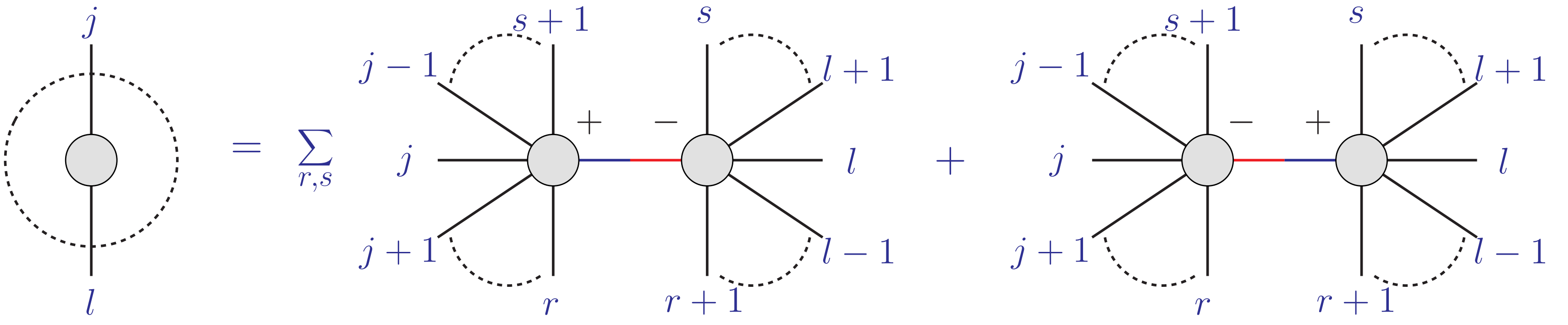

MHV rules

- Standard MHV and $\overline{\text{MHV}}$ expressions
  - three-gluon primitive amplitude
  - quark-gluon-antiquark primitive amplitude

$$A_3^{\rm tree}(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \qquad A_3^{\rm tree}(1^+,2^+,3^-) = -\frac{[12]^3}{[23][31]}\,,$$

$$A_3^{\rm tree}(1_q^-,2^-,3_{\bar q}^+) = \frac{\langle 12\rangle^2}{\langle 13\rangle}\,, \qquad A_3^{\rm tree}(1_q^-,2^+,3_{\bar q}^+) = -\frac{[23]^2}{[13]}\,,$$

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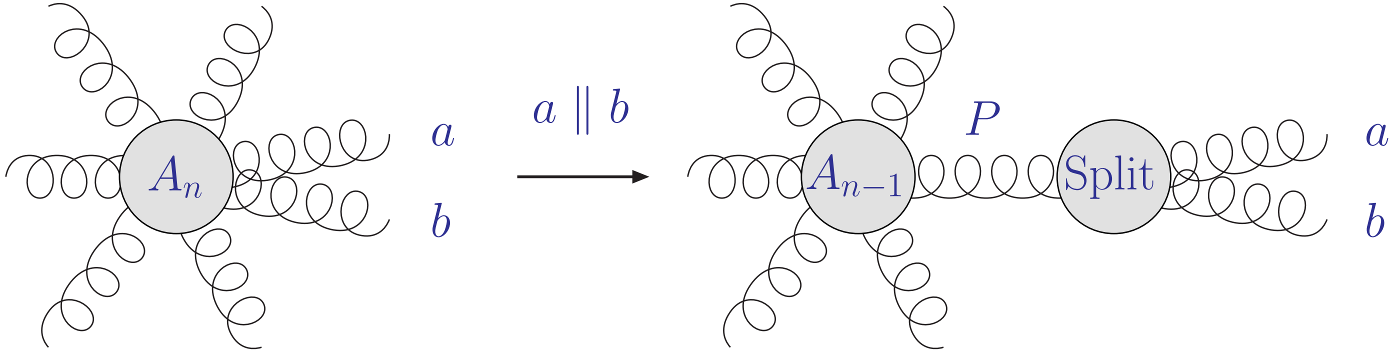
- Complex momenta k_i
  - three-point amplitudes do not vanish on-shell

Factorization (I)

- Factorization of amplitudes in soft/collinear limits
  - consistency checks on correctness of calculation
  - guiding principle in construction of amplitudes
- Pole behaviour of amplitudes (kinematic invariants vanish due to almost on-shell intermediate particle)
- Colour-ordered amplitudes only have poles in channels for sum of **cyclically adjacent** momenta $P^2_{i,j} \to 0$ with $P^{\mu}_{i,j} \equiv (k_i + k_{i+1} + \cdots + k_j)^{\mu}$



- Determination of universal splitting amplitudes (here $\text{Split}^{\text{tree}}_{\pm}(a^{\pm}, b^{\pm})$)

Factorization (II)

- Example: limit of $A_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+)$ for k_4 and k_5 parallel

$$A_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+) = i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$\stackrel{4\|5}{\longrightarrow} \frac{1}{\sqrt{z(1-z)}\langle 45\rangle} \times i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3P\rangle\langle P1\rangle}$$

$$= \text{Split}_-^{\rm tree}(4^+,5^+) \times A_4^{\rm tree}(1^-,2^-,3^+,P^+)$$

- Splitting amplitudes govern soft/collinear limits
  - Example: full set of $g \to gg$ amplitudes at tree level Mangano, Parke '90

$$\text{Split}_-^{\rm tree}(a^-,b^-) = 0 \qquad \text{Split}_-^{\rm tree}(a^+,b^+) = \frac{1}{\sqrt{z(1-z)}\langle ab\rangle}$$

$$\text{Split}_+^{\rm tree}(a^+,b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\langle ab\rangle} \qquad \text{Split}_-^{\rm tree}(a^+,b^-) = -\frac{z^2}{\sqrt{z(1-z)}[ab]}$$

  - splitting amplitudes up to two loops Campbel, Glover '97; Catani, Grazzini'98; Badger, Glover '04; Bern, Dixon, Kosower '04; +*many others*

Factorization (III)

- Collinear limits of QCD amplitudes responsible for parton evolution
- Evolution formulates dependence of cross sections for observable on **momentum transfer**
- General structure of factorized cross section
  - large momentum scale Q, factorization scale μ, hadron scale m
  - convolution $\otimes$ in suitable kinematical variables

$$Q^2\, \sigma_{\rm phys}\,(Q,m) = \hat{\sigma}_{\rm pt}\,(Q/\mu, \alpha_s(\mu))\, \otimes\, PDF\,(\mu, m)$$

- **DGLAP** Altarelli, Parisi '77

$$\mu \frac{dPDF(\mu,m)}{d\mu} \quad = \quad P\,(\alpha_s(\mu))\, \otimes\, PDF(\mu,m) + \mathcal{O}\left(\frac{1}{Q^2}\right)$$

- Square of a splitting amplitudes Split$_\pm(a^\pm, b^\pm)$ give (color-stripped) (un)-polarized splitting functions $P_{\rm ab}$ Uwer, Kosower '02

Colour ordering (I)

One loop amplitudes

- Color decomposition at one loop similar to tree level
  - up to two traces over generators t^a
  - sum over different spins J
- Example: all internal particle in loop in adjoint representation

$$\mathcal{A}_n^{\text{1-loop}}(\{k_i, \lambda_i, a_i\}) = g^n \sum_{J=0,\frac{1}{2},1} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\, A_{n;c}^{[J]}(\sigma)$$

- distinguish color structures and corresponding partial amplitudes
  - leading color $\mathrm{Gr}_{n;1}(\sigma)$ and partial amplitude $A_{n;1}^{[J]}(\sigma)$

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  - one-loop subleading color partial amplitudes given by sum over permutations of leading color ones

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- distinguish color structures and corresponding partial amplitudes
  - leading color $\text{Gr}_{n;1}(\sigma)$ and partial amplitude $A_{n;1}^{[J]}(\sigma)$
  - one-loop subleading color partial amplitudes given by sum over permutations of leading color ones
- **Upshot:** compute only leading color partial amplitudes $A_{n;1}^{[J]}$

Colour ordering (II)

- In detail:
  one-loop amplitude with internal particles in adjoint representation

$$\mathcal{A}_n^{\text{1-loop}}(\{k_i, \lambda_i, a_i\}) = g^n \sum_{J=0,\frac{1}{2},1} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathsf{Gr}_{n;c}(\sigma)\, A_{n;c}^{[J]}(\sigma)$$

  - sum over spins J; number of particles n_J of spin J
  - $\lfloor x \rfloor$ largest integer less than or equal to x (floor)
  - leading color-structure factor (N times tree level color factor)

$$\mathsf{Gr}_{n;1}(1) = N\, \mathrm{Tr}(T^{a_1} \cdots T^{a_n}),$$

  - subleading color structures

$$\mathsf{Gr}_{n;c}(1) = \mathrm{Tr}(T^{a_1} \cdots T^{a_{c-1}})\, \mathrm{Tr}(T^{a_c} \cdots T^{a_n})$$

  - S_n set of all permutations of n objects
    $S_{n;c}$ subset leaving $\mathsf{Gr}_{n;c}$ invariant

Supersymmetry (I)

- Supersymmetric decomposition of one-loop amplitudes
- $\mathcal{N} = 4$ multiplet (1 gluon, 4 Weyl fermions, 6 real scalars)
- chiral $\mathcal{N} = 1$ multiplet (1 Weyl fermion, 2 real scalars)
  - amplitude with gluon in loop A^g (and all external gluons)

$$A^g = \underbrace{\left(A^g + 4A^f + 3A^s\right)}_{\mathcal{N}=4\,\text{SUSY}} - 4\underbrace{\left(A^f + A^s\right)}_{\mathcal{N}=1\,\text{chiral SUSY}} + \underbrace{A^s}_{\mathcal{N}=0\,\text{scalar}}$$

    - complex scalar loop ($\mathcal{N} = 0$ contribution)
  - amplitude with fermion in loop A^f (and all external gluons)

$$A^f = \underbrace{\left(A^f + A^s\right)}_{\mathcal{N}=1\,\text{chiral SUSY}} - \underbrace{A_s}_{\mathcal{N}=0\,\text{scalar}}$$

- QCD amplitudes obtained from one-loop amplitude with scalar in loop (given knowledge of $\mathcal{N} = 4$ and $\mathcal{N} = 1$ supersymmetric results)

Supersymmetry (II)

- Soft/collinear limits of one-loop leading-color n-point amplitudes
  - dimensional regularization $D = 4 - 2\epsilon$
- Constraints from supersymmetric decomposition
  - tree level amplitude A^{tree} factorizes, V divergent, F finite

$$
\begin{aligned}
A_n^s &= c_\Gamma \left(V_n^s A_n^{\text{tree}} + \mathrm{i} F_n^s \right) \\
A_n^f &= -c_\Gamma \left(\left\{ V_n^f + V_n^s \right\} A_n^{\text{tree}} + \mathrm{i} \left\{ F_n^f + F_n^s \right\} \right) \\
A_n^g &= c_\Gamma \left(\left\{ V_n^g + V_n^f + V_n^s \right\} A_n^{\text{tree}} + \mathrm{i} \left\{ F_n^g + F_n^f + F_n^s \right\} \right)
\end{aligned}
$$

  - define $c_\Gamma = \dfrac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}$.
  - assume supersymmetry preserving regularization
  - otherwise (e.g. conventional dim. reg.) get additional contribution proportional to $\epsilon = (D-4)/2$

Supersymmetry (III)

- Structure of singularities for leading-color amplitudes
  - $\mathcal{N} = 4$ supersymmetry: $F_n^g = 0$

$$A_n^{\mathcal{N}=4} \quad = \quad c_\Gamma \, A_n^{\text{tree}} \, V_n^g$$

  - example: 5-point amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$

$$V_5^g \quad = \quad \sum_{j=1}^{5} \left[-\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{j,j+1}} \right)^{\epsilon} + \ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}} \right) \ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}} \right) + \frac{5}{6}\pi^2 \right]$$

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  - $\mathcal{N} = 1$ supersymmetry: $F_n^f = \# \ln(-s_{ij})$ (purely logarithmic)

$$A_n^{\mathcal{N}=1} \quad = \quad -c_\Gamma \left(A_n^{\rm tree} \, V_n^f + {\rm i} F_n^f \right)$$

  - example: 5-point amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$

$$V_5^f \quad = \quad -\frac{1}{2\epsilon} \left[\ln \left(\frac{\mu^2}{-s_{23}} \right) + \ln \left(\frac{\mu^2}{-s_{51}} \right) \right] - 2$$

Supersymmetry (IV)

- Finite terms F_n^f in $\mathcal{N} = 1$ supersymmetry proportional to logarithms
  - example: 5-point amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$

$$F_5^f \quad = \quad -\frac{1}{2} \frac{\langle 12 \rangle^2 \left(\langle 23 \rangle [34] \langle 41 \rangle + \langle 24 \rangle [45] \langle 51 \rangle \right)}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{\ln \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51} - s_{23}}$$

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- Structure of singularities for leading-color amplitudes
  - scalar in loop ($\mathcal{N}=0$)

$$A_n^{\mathcal{N}=0} \quad = \quad c_\Gamma\left(A_n^{\text{tree}}\,V_n^s+\mathrm{i}F_n^s\right)$$

  - example: 5-point amplitude $A_5(1^-,2^-,3^+,4^+,5^+)$

$$V_5^s \quad = \quad -\frac{1}{3}V_5^f+\frac{2}{9}$$

Supersymmetry (V)

- Finite terms for $\mathcal{N} = 0$ contain logarithmic and also rational terms R
  - structure $F_n^s = \# \ln(-s_{ij}) + R_n$
  - example: 5-point amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$

$$F_5^s = \widehat{R}_5 - \frac{1}{3} F_5^f$$

$$-\frac{1}{3} \frac{[34]\langle 41\rangle\langle 24\rangle[45]\left(\langle 23\rangle[34]\langle 41\rangle + \langle 24\rangle[45]\langle 51\rangle\right)}{\langle 34\rangle\langle 45\rangle} \frac{\left(\ln\left(\frac{-s_{23}}{-s_{51}}\right) - \frac{1}{2}\left(\frac{s_{23}}{s_{51}} - \frac{s_{51}}{s_{23}}\right)\right)}{(s_{51} - s_{23})^3}$$

  - rational term $\widehat{R}_5$ explicit

$$\widehat{R}_5 = -\frac{1}{3} \frac{\langle 35\rangle[35]^3}{[12][23]\langle 34\rangle\langle 45\rangle[51]} + \frac{1}{3} \frac{\langle 12\rangle[35]^2}{[23]\langle 34\rangle\langle 45\rangle[51]} + \frac{1}{6} \frac{\langle 12\rangle[34]\langle 41\rangle\langle 24\rangle[45]}{s_{23}\langle 34\rangle\langle 45\rangle s_{51}}$$

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- Increasing complexity
  - $\mathcal{N}=4$ simpler than $\mathcal{N}=1$ supersymmetric component
  - $\mathcal{N}=1$ supersymmetric simpler than scalar component

Supersymmetry (VI)

- Construction of QCD amplitude (leading-color, one loop)

$$A_n^{\rm QCD} \quad = \quad c_\Gamma \left[\left(V_n^g + 4V_n^f + V_n^s \right) A_n^{\rm tree} + {\rm i} \left(F_n^f + F_n^s \right) \right. \\ \left. - \frac{n_f}{N} \left(\left(V_n^f + V_n^s \right) A_n^{\rm tree} + {\rm i} \left(F_n^f + F_n^s \right) \right) \right]$$

- All terms with logarithmic dependence cut-constructible
  - unitarity
- Real difficulty: rational terms R_n in finite terms of $A_n^{\mathcal{N}=0}$

Supersymmetry (VI)

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$$A_n^{\mathrm{QCD}} = c_\Gamma \Big[\left(V_n^g + 4V_n^f + V_n^s \right) A_n^{\mathrm{tree}} + \mathrm{i} \left(F_n^f + F_n^s \right) - \frac{n_f}{N} \left(\left(V_n^f + V_n^s \right) A_n^{\mathrm{tree}} + \mathrm{i} \left(F_n^f + F_n^s \right) \right) \Big]$$

- All terms with logarithmic dependence cut-constructible
  - unitarity
- Real difficulty: rational terms R_n in finite terms F_n^s of $A_n^{\mathcal{N}=0}$

Unitarity (I)

- Unitarity: a fundamental concept in quantum field theory
  - unitarity S-matrix with $S^\dagger S = 1$
  - define T-matrix via $S = 1 + \mathrm{i}T$ and $T = 0$ if no interaction
  - from unitarity

$$1 = S^\dagger S \;=\; (1 - \mathrm{i}T^\dagger)(1 + \mathrm{i}T)$$

    or (equivalently)

$$\mathrm{i}(T - T^\dagger) = -T^\dagger T$$

  - evaluate in complete basis in Hilbert space $\{|a\rangle\}$, $\sum_a |a\rangle\langle a| = \mathbf{1}$

    and obtain

$$\mathrm{i}\left(\langle b|T|a\rangle - \langle b|T^\dagger|a\rangle\right) = -\sum_c \langle b|T^\dagger|c\rangle\langle c|T|a\rangle$$

    or (equivalently)

$$\mathrm{Im}\left(\langle b|T|a\rangle\right) = \frac{1}{2}\sum_c \langle b|T^\dagger|c\rangle\langle c|T|a\rangle$$

Unitarity (II)

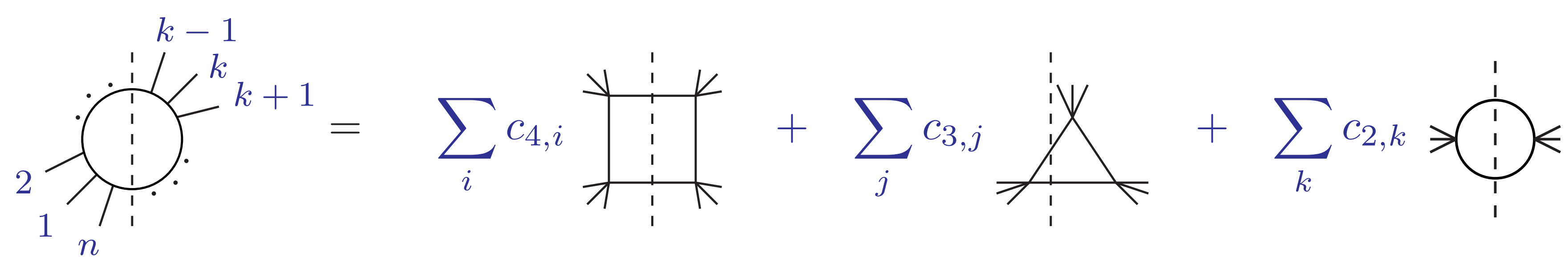
Applications in QFT

- Unitarity in quantum field theory used to formulate calculational rules
- Cutting rules for Feynman diagram calculations (optical theorem)
  Cutkosky; . . .
- In this lecture: fusing rules Bern, Dixon, Dunbar, Kosower '94; . . .
  - practical and efficient computational method for reconstructing (parts of) dimensionally regularized n-loop amplitudes from $n-1$-loop amplitudes
  - obtain cut-constructible terms

Unitarity (III)

- At one-loop exploit knowledge about integral basis
  - standard reduction techniques, e.g. Passarino, Veltman ‘79
  - express any amplitude in basis of scalar integral functions
  - boxes (I_4), triangles (I_3), and bubbles (I_2) where subfix F denotes finite part
- Example: one-loop n-gluon amplitude

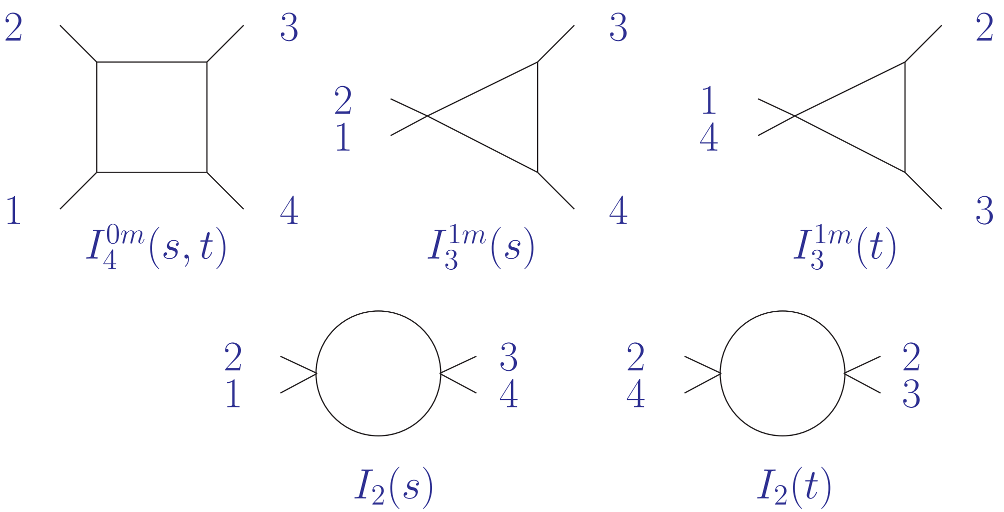
$$\mathcal{A}_n^{\mathcal{N}=0} = \sum \left(c_2 I_2 + c_3^{3m} I_3^{3m} + c_4^{1m} I_{4F}^{1m} + c_4^{2m\ e} I_{4F}^{2m\ e} + c_4^{2m\ h} I_{4F}^{2m\ h} + c_4^{3m} I_{4F}^{3m} + c_4^{4m} I_4^{4m} \right)$$



Unitarity (IV)

- Example: one-loop four point function
  - external momenta $k_1, \ldots k_4$ massless
  - Mandelstam variables $s = (k_1 + k_2)^2$ and $s = (k_2 + k_3)^2$

integral	unique function
$I_4^{0m}(s,t)$	$\ln(-s)\ln(-t)$
$I_3^{1m}(s)$	$\ln^2(-s)$
$I_3^{1m}(t)$	$\ln^2(-t)$
$I_2^{(}s)$	$\ln(-s)$
$I_2^{(}t)$	$\ln(-t)$



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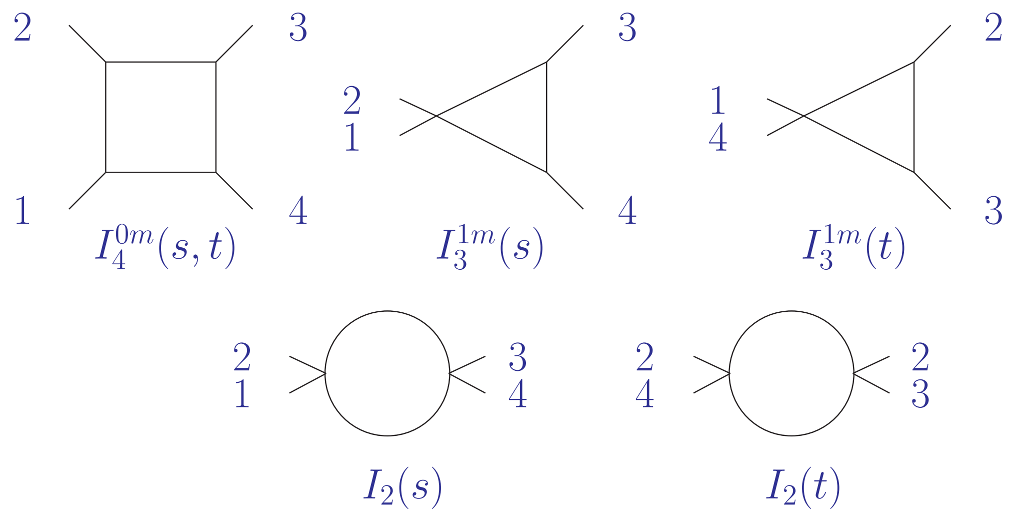
$I_4^{0m}(s,t)$	$\ln(-s)\ln(-t)$
$I_3^{1m}(s)$	$\ln^2(-s)$
$I_3^{1m}(t)$	$\ln^2(-t)$
$I_2^{(}s)$	$\ln(-s)$
$I_2^{(}t)$	$\ln(-t)$



Upshot

- At one-loop all terms with logarithms or polylogarithms can be constructed from evaluation of the cuts
- Supersymmetric amplitudes, e.g. $\mathcal{N} = 1$ or $\mathcal{N} = 1$ are completely cut-constructible Bern, Dixon, Dunbar, Kosower '94

NLO Bootstrap approach (I)

- Perform again $[j, l\rangle$ shift of external massless spinors j and l with complex parameter z
- Structure of shifted one-loop amplitude $A_n(z)$
  - $A_n(z)$ develops new features at one loop, e.g. branch cuts

$$A_n(0) = -\sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{A_n(z)}{z} - \int_{B_0}^{\infty} \frac{dz}{z} \, \text{Disc}_B A_n(z)$$

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Basic idea

- Distinguish cut-constructible and rational terms
  - define 'pure-cut'-term $C_n(z) = \frac{1}{c_\Gamma} A_n(z) \Big|_{\text{ln,Li},\pi^2}$
  - define rational term $R_n(z) = \frac{1}{c_\Gamma} A_n(z) \Big|_{\text{ln,Li},\pi^2 \to 0}$
- On-shell recursion relation for $C_n(z)$ and $R_n(z)$ (similar to tree level)

NLO Bootstrap approach (II)

- Amplitude has logarithms of kinematical invariants
- $A_n(z)$ has branch cuts
  - example: $[1,2\rangle$ shift and logarithm $\ln(-s_{23})$

$$\ln(-s_{23} - z\langle 1|3|2\rangle) = \ln\left([23](\langle 23\rangle + z\langle 13\rangle)\right)$$

  - branch cut in z starts at $z = -\dfrac{\langle 23\rangle}{\langle 13\rangle}$

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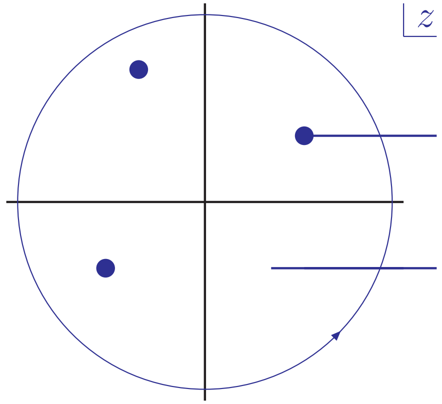
  - branch cut in z starts at $z = -\dfrac{\langle 23\rangle}{\langle 13\rangle}$
- Branch cuts in z may have end-point singularities for terms $\dfrac{\ln(-s_{ab})}{\langle ab\rangle}$
  - choice of shifts in z to avoid end-point singularities



NLO Bootstrap approach (III)

- Spurious singularities of kinematical invariants cancel between rational terms and logarithms
- Example: logarithm $\frac{\ln(r)}{(1-r)^2}$ from $C_n(z)$, $\frac{1}{1-r}$ from $R_n(z)$
- Define 'cut-completed' terms $\widehat{C}_n(z)$, put $\frac{\ln(r)+1-r}{(1-r)^2}$ in $\widehat{C}_n(z)$
  - $\widehat{C}_n(z) = C_n(z) + \widehat{CR}_n(z)$ always with appropriate rational terms
  - define new rational term $\widehat{R}_n(z) = R_n(z) - \widehat{CR}_n(z)$

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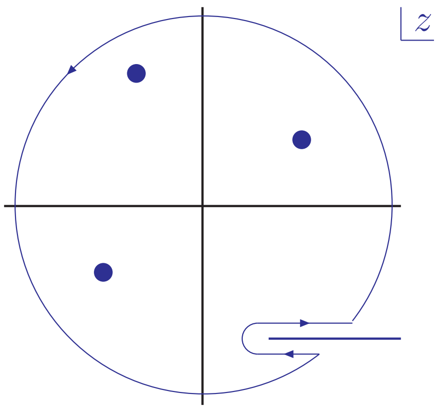
  - define new rational term $\widehat{R}_n(z) = R_n(z) - \widehat{CR}_n(z)$
- Structure of 'cut-completed' term $\widehat{C}_n(z)$
  - evaluate discontinuity across branch cut

$$\widehat{C}_n(0) = - \sum_{\text{poles }\alpha} \text{Res}_{z=z_\alpha} \frac{\widehat{C}_n(z)}{z} - \int_{B_0}^{\infty} \frac{dz}{z} \, \text{Disc}_B \widehat{C}_n(z)$$



NLO Bootstrap approach (IV)

- Structure of rational term $\widehat{R}_n(z)$
  - one-loop physical-pole recursion for the rational terms $R_n(z)$

$$R_n^D = - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{R_n(z)}{z}$$

Physical amplitude

- Structure of Physical amplitude $A_n(0)$

$$A_n(0) \quad = \quad c_\Gamma \left[\widehat{C}_n(0) + R_n^D + \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{\widehat{CR}_n(z)}{z} \right]$$

- Cut-completed physical part $\widehat{C}_n(0)$, rational terms R_n^D
- Overlap terms in sum of residues of $\widehat{CR}_n$ from double counting between cut constructible and rational terms
  - recall: spurious singularities cancel between rational terms and logarithms

NLO Bootstrap approach (V)

Remarks

- Checks
  - test amplitude (in particular rational term R_n) with known factorization properties of amplitude A_n in soft/collinear limit

Other new features at one loop

- Unreal poles may appear in shifts $\frac{[jl]}{\langle jl\rangle}$ for complex momenta
  - become pure phase for real momenta
- Double poles may appear in shifts $\frac{[jl]}{\langle jl\rangle^2}$
  - search for channel with standard factorizations
- $A(z) \to 0$ for $z \to \infty$ may not hold for some complex shifts
  - apply auxiliary recursions

Example: five gluon scattering at NLO (I)

- Construction of 5-gluon amplitude for leading-color (scalar in loop)
  $A_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$

Example: five gluon scattering at NLO (I)

- Construction of 5-gluon amplitude for leading-color (scalar in loop) $A_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$

Step 1

- Choose $[1, 2\rangle$ shift, $\tilde{\lambda}_1 \to \tilde{\lambda}_1 - z\tilde{\lambda}_2$, $\lambda_2 \to \lambda_2 + z\lambda_1$
- Collect ingredients
  - 3-point vertex (with complex momenta)

$$A_3^{\text{tree}}(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

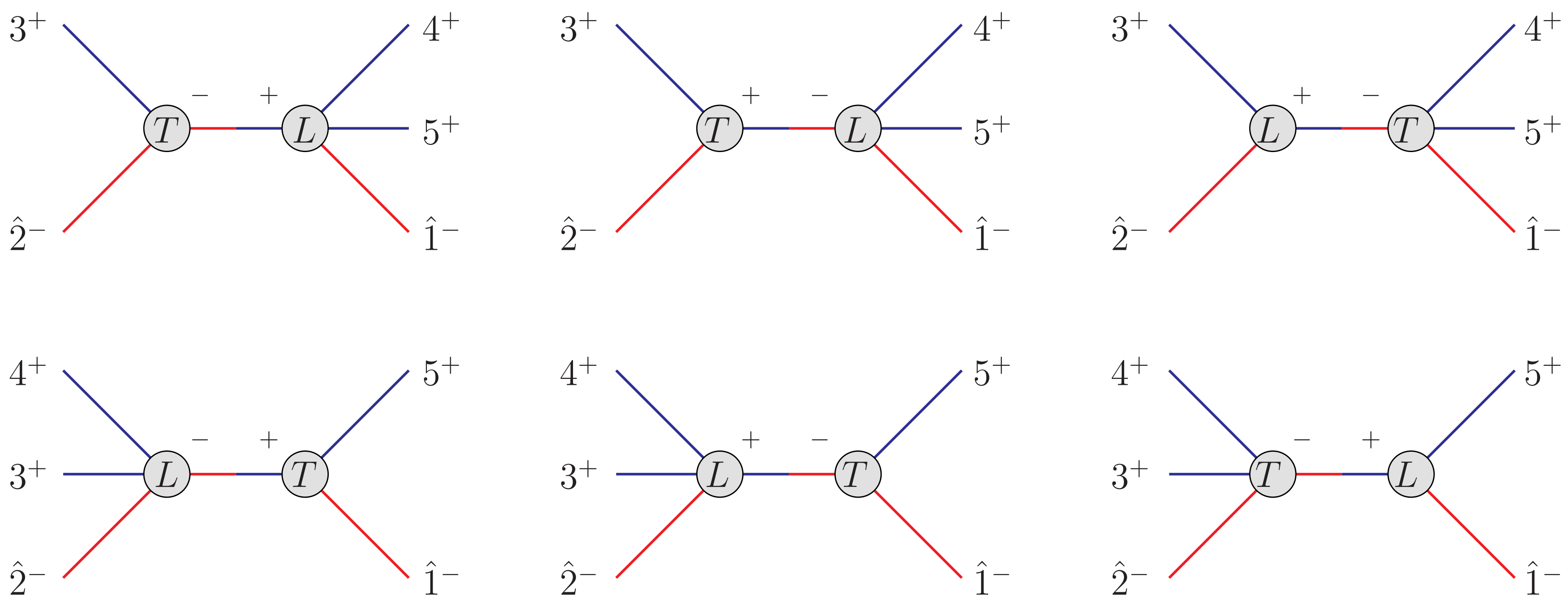
  - rational term $R_4^s(1^-, 2^+, 3^+, 4^+)$ of leading-color 4-gluon amplitude $A_4^s(1^-, 2^+, 3^+, 4^+)$ (scalar in loop)

$$\begin{aligned} R_4^s(1^-, 2^+, 3^+, 4^+) &= \frac{1}{\mathrm{i}c_\Gamma} A_4^s(1^-, 2^+, 3^+, 4^+) \\ &= \frac{1}{3}\frac{\langle 24\rangle[24]^3}{[12]\langle 23\rangle\langle 34\rangle[41]} \end{aligned}$$

Example: five gluon scattering at NLO (II)

Step 2

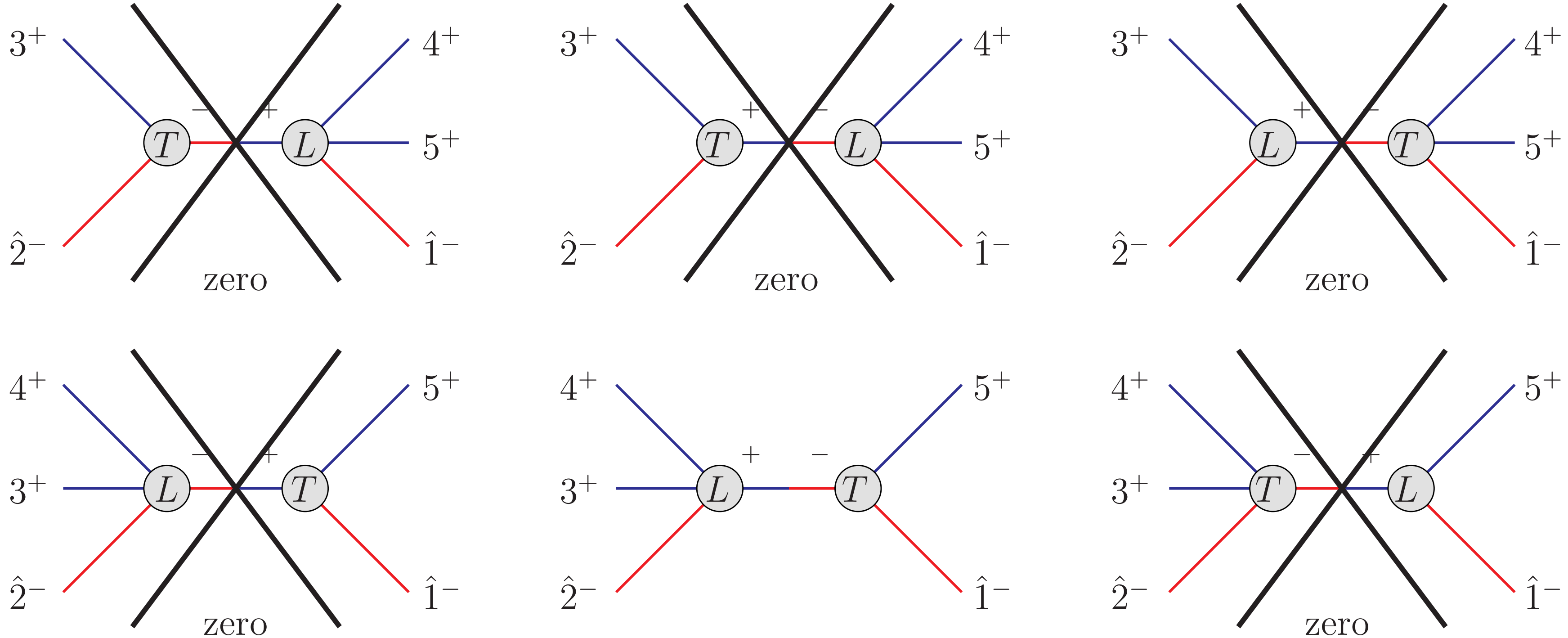
- Draw all diagrams
  - notation: L (loop) and T (tree level)



Example: five gluon scattering at NLO (III)

Step 3

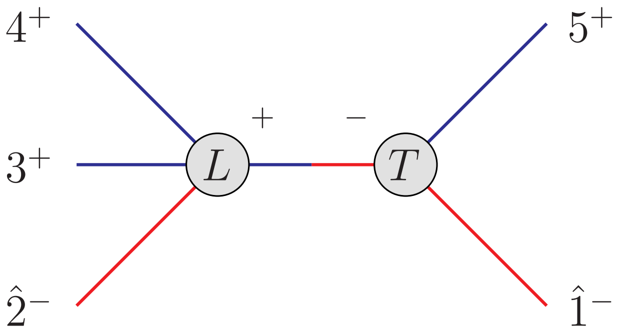
- Most diagrams vanishing for various reasons, e.g.
  - 3-gluon vertex $A_3^{\text{tree}}(2^-, 3^+, -(\hat{2}^- + 3^-))$ vanishes (first diagram)
  - 3-gluon rational term $R_3(2^-, 3^+, -(\hat{2}^+ \pm 3^\pm))$ vanishes (third diagram)



Example: five gluon scattering at NLO (IV)

Step 4

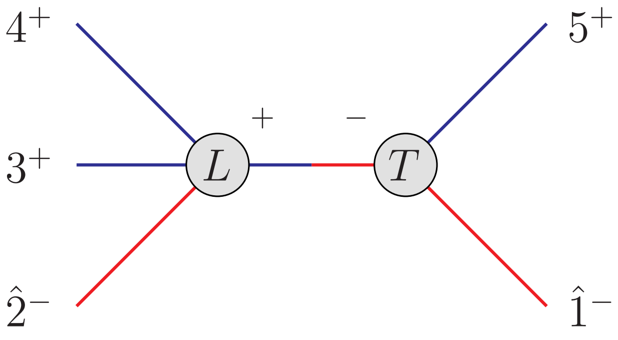
- Calculate non-vanishing diagram



Example: five gluon scattering at NLO (IV)

Step 4

- Calculate non-vanishing diagram
- Combine on-shell ingredients
  - define $P = k_1 + k_5$
  - z-dependent (hatted) momenta $\hat{1}, \hat{2}, \hat{P}$
  - pole at $z = \frac{P^2}{\langle 1|P|2\rangle} = -\frac{[15]}{[25]}$

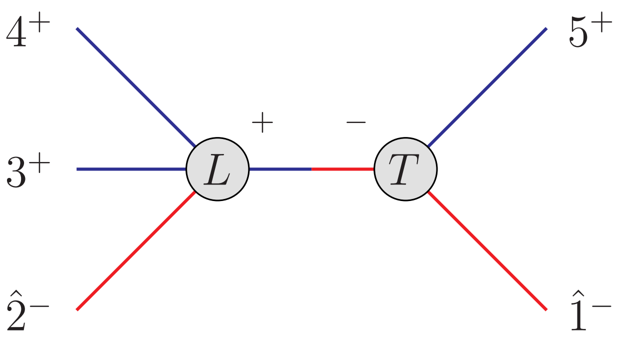


$$A_3^{\rm tree}(5^+, \hat{1}^-, -\hat{P}^-) \times \frac{\rm i}{P^2} \times R_4(\hat{2}^-, 3^+, 4^+, \hat{P}^+) =$$

$$= \quad -\frac{1}{3} \frac{\langle \hat{1}\hat{P}\rangle^3}{\langle 5\hat{1}\rangle\langle \hat{P}5\rangle} \frac{\rm i}{P^2} \frac{\langle 3\hat{P}\rangle [3\hat{P}]^3}{[\hat{2}3]\langle 34\rangle\langle 4\hat{P}\rangle [\hat{P}\hat{2}]}\Bigg|_{z=-\frac{[15]}{[25]}}$$

Example: five gluon scattering at NLO (IV)

Step 4



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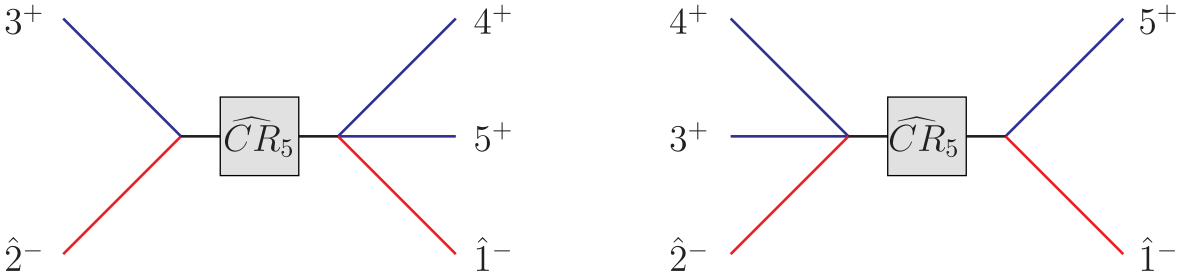
- Spinor helicity algebra, e.g. $\langle \hat{1}\hat{P}\rangle = \langle 1^-|5|2^-\rangle = \langle 15\rangle[52]$ for $[1, 2\rangle$ shift
- Work hatted momenta $\hat{1}, \hat{2}, \hat{P}$ away

Example: five gluon scattering at NLO (V)

Step 5

- Overlap contribution O_5



- Overlap contribution from rational terms in cut-completed part $\widehat{CR}_5$

$$\widehat{CR}_5 = -\frac{1}{6}\frac{s_{15}+s_{23}}{s_{23}s_{15}(s_{15}-s_{23})^2}\frac{[34]\langle 41\rangle\langle 24\rangle[45]\Big(\langle 23\rangle[34]\langle 41\rangle + \langle 24\rangle[45]\langle 51\rangle\Big)}{\langle 34\rangle\langle 45\rangle}$$

- Evaluate $\widehat{CR}_5(z)/z$ at residues
  - $z = -\frac{\langle 23\rangle}{\langle 13\rangle}$ (left diagram), $z = -\frac{[15]}{[25]}$ (right diagram)
- Result (very simple)

$$O_5^{\text{left}} = -\frac{1}{6}\frac{\langle 12\rangle^2\langle 14\rangle[34]}{\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle[23]} \qquad O_5^{\text{right}} = \frac{1}{6}\frac{\langle 14\rangle[34][35]\big(\langle 14\rangle[34]-\langle 15\rangle[35]\big)}{\langle 15\rangle\langle 34\rangle\langle 45\rangle[15][23]^2}$$

Example: five gluon scattering at NLO (VI)

Step 6

- Assemble results for QCD amplitude (leading-color, one loop) $A_5^{\rm QCD}(1^-, 2^-, 3^+, 4^+, 5^+)$
- Recall rearrangement from supersymmetry

$$A_5^{\rm QCD} = c_\Gamma \Big[\left(V_5^g + 4V_5^f + V_5^s\right) A_5^{\rm tree} + {\rm i}\left(F_5^f + F_5^s\right) - \frac{n_f}{N}\left(\left(V_5^f + V_5^s\right) A_5^{\rm tree} + {\rm i}\left(F_5^f + F_5^s\right)\right)\Big]$$

- Rational terms $\widehat{R}_n$ in F_n^s
  - $\widehat{R}_n = R_n + O_5^{\rm left} + O_5^{\rm right}$

State of the art

- Status of six-gluon amplitude
  - analytic computation of one-loop corrections Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0$ cut	$\mathcal{N}=0$ rat
$--++++$	BDDK '94	BDDK '94	BDDK '94	BDK '94
$-+-+++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$-++-++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$---+++$	BDDK '94	BBDD '04	BBDI '05 BFM '06	BBDFK '06
$--+-++$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06
$-+-+-+$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06

- Numerical evaluation Ellis, Giele, Zanderighi '06

Tools for NLO jet cross sections

- Some general purpose tools for observables with jets (and heavy quarks, weak gauge bosons, ...)
  - NLOjet++ Nagy
    (multipurpose C++ library for calculating jet cross sections)
  - MCFM Campbell, Ellis
    (vector bosons, Higgs and jets at hadron colliders)
  - MC@NLO Frixione, Nason, Webber
    (combines Monte Carlo event generator with NLO calculations)
  - PHOX family
    Aurenche, Binoth, Fontannaz, Guillet, Heinrich, Pilon, Werlen
    (processes involving photons, hadrons and jets)
- Complete list (more or less), also with tools for NLO electroweak corrections *Les Houches 2005 [hep-ph/0604120]*

Summary (part II)

Theory developments

- Analytical results for amplitudes at one loop
  - exploit supersymmetry decompostion
  - unitarity relations for boxes, triangles, etc
- Recent progress
  - on-shell recursion relations for NLO amplitudes
  - constructive approach for rational part of multi leg amplitudes
- Cutting edge
  - $2 \to 4$ processes (electroweak corrections) by ‘traditional’ methods

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Outlook

- Challenges
  - assemble new NLO helicty amplitudes in numerical codes for cross section calculations (efficiency !!)
  - amplitudes with multiple scales (particles with different masses, e.g. $M_W, M_Z, M_{\rm top}, M_{\rm SUSY}, \dots$)

Exercise (I)

- Calculate the leading order unpolarized gluon-gluon splitting function $P_{\mathrm{gg}}^{(0)}$ from the square of the respective splitting amplitudes $\text{Split}_{\pm}^{\mathrm{tree}}(a^{\pm}, b^{\pm})$ summed over all helicities. Verify

$$P_{\mathrm{gg}}^{(0)}(z) \;\propto\; \frac{2}{1-z} + \frac{2}{z} - 4 + 2z - 2z^2$$

Neglect virtual corrections, i.e. the '+'-prescription and the $\delta(1-z)$-term.

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- for discussions on the content of these lectures to

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