QCD and collider phenomenology
part one

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Introduction

The challenge

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- Successful experimental search relies heavily on the ability to make precision predictions for hard scattering cross-sections, e.g.
  - **Higgs** production
  - new physics phenomena (**BSM**)
  - backgrounds
  - evolution of parton distributions in proton

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- Successful experimental search relies heavily on the ability to make precision predictions for hard scattering cross-sections, e.g.
  - **Higgs** production
  - new physics phenomena (**BSM**)
  - backgrounds
  - evolution of parton distributions in proton
- LHC will be a QCD machine (LEP was an electroweak machine)
  - provide accurate predictions (including QCD radiative corrections)
  - perturbative QCD is essential and established part of toolkit (we no longer “test” QCD)

QCD jets

Lessons from Tevatron

- Top search was the outstanding issue at the start of run I at Tevatron
- Initiated many developments in LO multi-parton generation for $pp \to W^\pm +$ jets (e.g. numerical recursion, algebraic generation of tree level amplitudes)
  - unexpected challenge: importance of matching issues between matrix elements and shower Monte Carlo's
  - initiated development of "numerical" partonic NLO jet Monte Carlo's

Expectations for LHC

- In the coming years
  - all new challenges for NLO are encapsulated by Higgs searches at ATLAS/CMS
  - large number of high multiplicity processes

LHC “priority” wishlist

process ($V \in \{\gamma, W^{\pm}, Z\}$)	background to
$pp \to VV + 1$ jet	$t\bar{t}H$, new physics
$pp \to H + 2$ jets	H production by vector boson fusion (VBF)
$pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
$pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
$pp \to VVb\bar{b}$	VBF $\to VV$, $t\bar{t}H$, new physics
$pp \to VV + 2$ jets	VBF $\to VV$
$pp \to V + 3$ jets	various new physics signatures
$pp \to VVV$	SUSY trilepton

Les Houches 2005 [hep-ph/0604120]

Original experimenter's wishlist

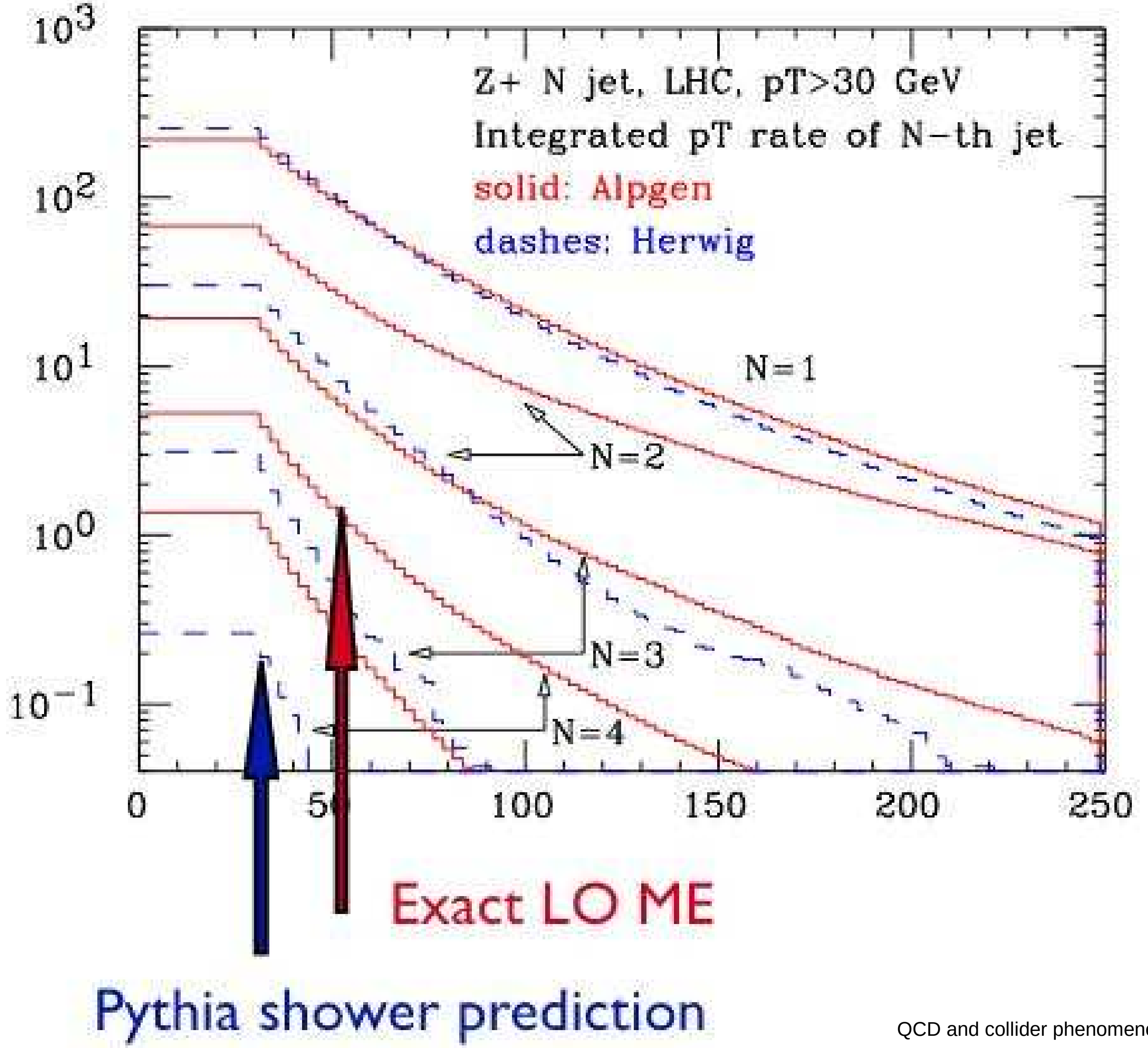
Tevatron Run II Monte Carlo workshop April 2001

Run II Monte Carlo Workshop, April 2001

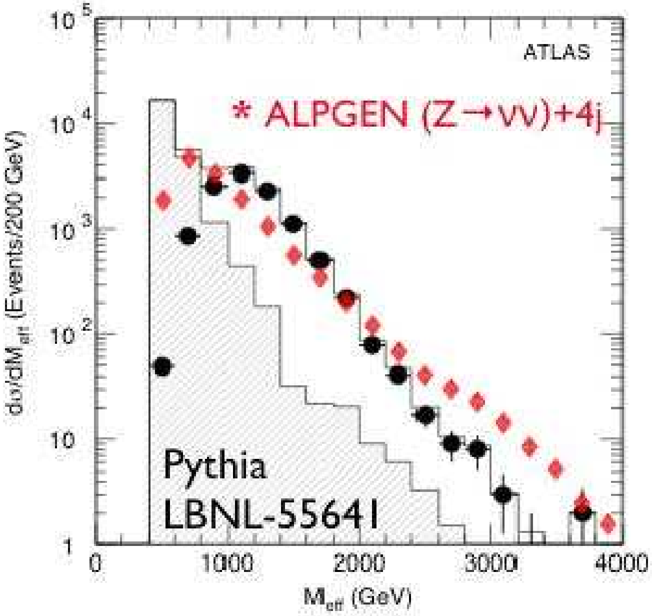
Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

How reliable are background estimates?

Gianotti, Mangano ‘05



SUSY searches (I)



- SM background in channel $pp \to Z(\to \nu\bar{\nu}) + 4$jets from Alpgen
  Gianotti, Mangano '05
  - Njet ≥ 4
  - $E_{T(1,2)} > 100$GeV
  - $E_{T(3,4)} > 50$GeV
  - MET $= M_{\text{eff}} + \max(100, M_{\text{eff}}/4)$
  - $M_{\text{eff}} = \text{MET} + \sum_i E_{Ti}$

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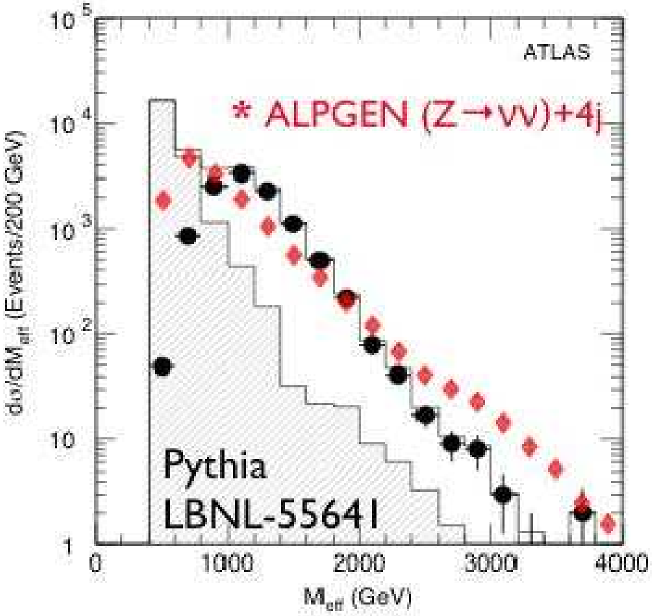

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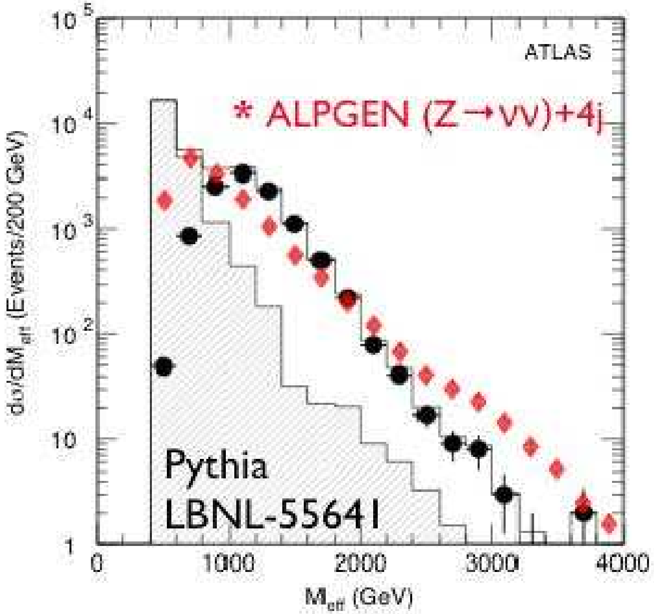
- Early ATLAS TDR studies with Pythia are overly optimistic
- Background largely underestimated in high-end tail of missing E_T (normalization of $Z \to \nu\bar{\nu} +$ jets from experiment $Z \to e^+e^- +$ jets)
  Gianotti, Mangano '05
- Shape of BSM signal indistinguishable from background shape at LO
- Significance of potential disagreement between data and Alpgen ?

SUSY searches (II)



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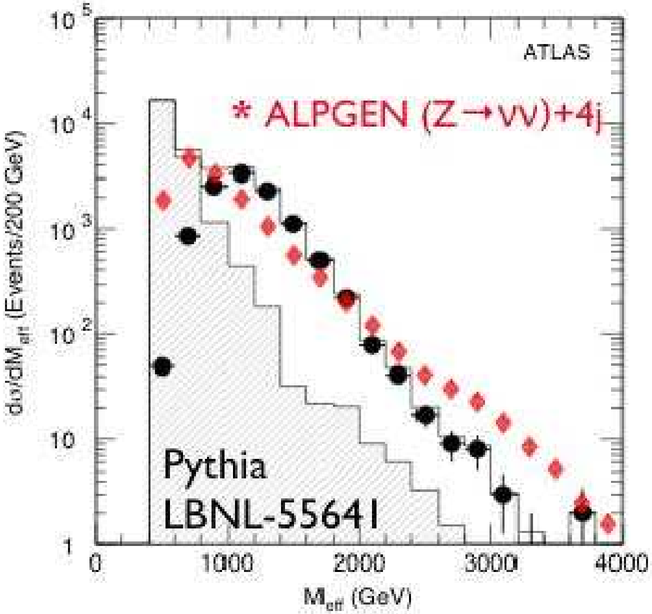
  - based on leading order matrix elements
  - models hard jets much better than e.g. Pythia

SUSY searches (II)



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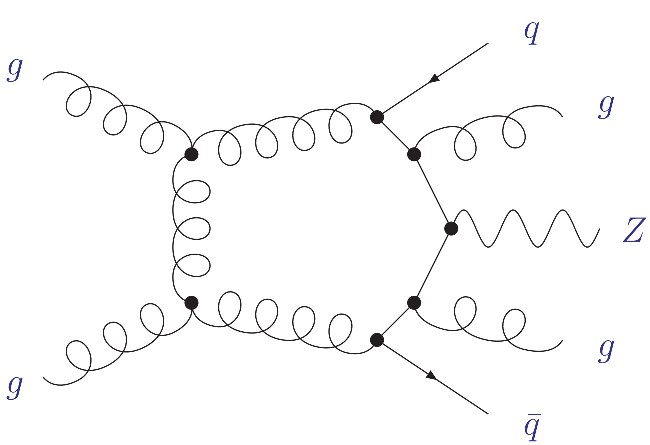

- Need for $pp \to Z + 4$ jets at NLO

Perturbative QCD corrections essential

- NLO important for rates (background); large K-factors, new parton channels may dominate beyond tree level
  - e.g. $W + 4\,$jets is $\mathcal{O}(\alpha_s^4)$ and $\Delta(\alpha_s^{\rm LO}) \simeq 10\%$ gives $\Delta(\sigma^{\rm LO}) \simeq 40\%$

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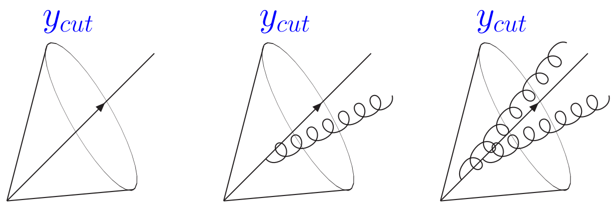
- NNLO (for di-jets) important for scale uncertainty, PDF determination, modelling of jets, ...



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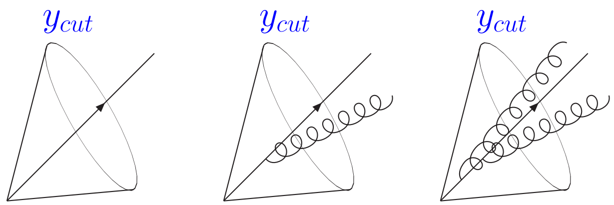


- Hadronic di-jets: large statistics even with high-p_t cuts
  - experimental calibration (HCAL uniformity, establish missing E_t)
  - gluon jets constrain gluon PDF at medium/large x
  - searches for quark sub-structure (di-jet angular correlations)

QCD theory (I)

Theory requirements

- General solution applying to any process; scalable to large numbers n of external partons
- Numerical stability; solution to be automated

Status

- Efficient techniques for computing tree amplitudes $\mathcal{A}$ exist
  - recursion relations Berends, Giele '87
- Complete NLO calculations available (codes)

  Many people; see e.g. *Les Houches 2005 [hep-ph/0604120]*
  - many $2 \to 3$ processes
  - $2 \to 4$ processes
    - electroweak corrections to $e^+e^- \to 4$ fermions

      Denner, Dittmaier, Roth, Wieders '05
    - electroweak corrections to $e^+e^- \to \nu\bar{\nu}HH$

      Boudjema, Fujimoto, Ishikawa, Kaneko, Kato, Kurihara, Shimizu, Yasui '05

QCD theory (II)

Progress

- Recent developments in amplitudes for multi-particle production
  - on-shell recursions (analyticity properties), unitarity, . . .
  - constructive approach at NLO (a lot of recent activity) Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu *+ many others*

Problems

- Multiple scales (particles with different masses, e.g. $M_W, M_Z, M_{\rm top}, M_{\rm SUSY}, \dots$)
- Numerical phase space integration (efficiency!!)
  - tedious at NLO
  - very difficult at NNLO

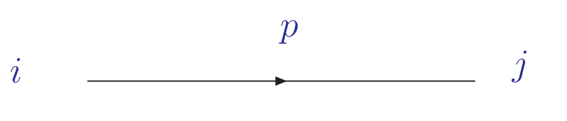
Plan

Rest of this lecture

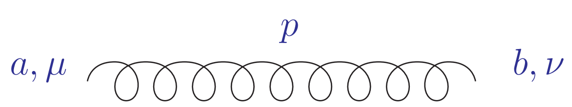
- Review of Feynman rules
- Colour ordering
- Spinor conventions
- Helicity amplitudes
- On-shell recursions at tree level

Feynman rules (I)

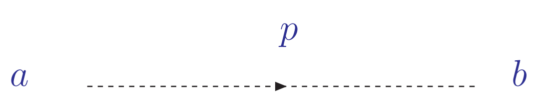
- Propagators
  - fermions, gluons, ghosts
  - covariant gauge



$$\delta^{ij}\,\frac{\mathrm{i}}{\not{p}-m}$$



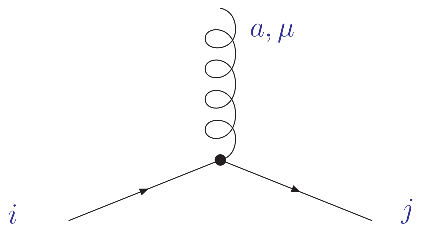
$$\delta^{ab}\,\mathrm{i}\left(\frac{-g^{\mu\nu}}{p^2}+(1-\lambda)\frac{p^\mu p^\nu}{(p^2)^2}\right)$$



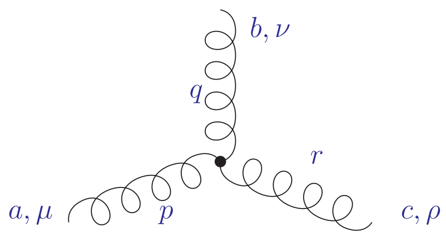
$$\delta^{ab}\,\frac{\mathrm{i}}{p^2}$$

Feynman rules (II)

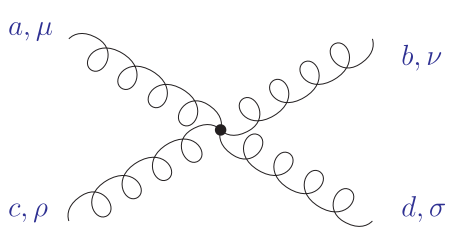
- Vertices



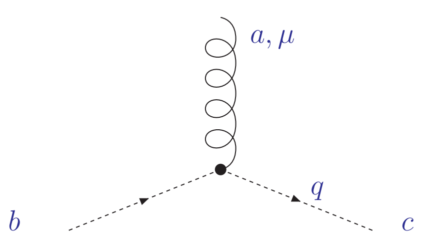
$$-\mathrm{i}\, g\, (t^a)_{ji}\, \gamma^\mu$$



$$-g\, f^{abc}\, \left((p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\mu\rho}\right)$$



$$-\mathrm{i}\, g^2\, f^{xac} f^{xbd}\, (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$-\mathrm{i}\, g^2\, f^{xad} f^{xbc}\, (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})$$
$$-\mathrm{i}\, g^2\, f^{xab} f^{xcd}\, (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$



$$g\, f^{abc}\, q^\mu$$

Multiparticle production

- Number of Feynman diagrams for n-parton amplitudes grows quickly with n
  - example n-gluon amplitudes

n	diagrams
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

- Feynman diagram evaluation very inefficient for many legs
  - too many diagrams, terms per diagram, kinematic variables

Quantum numbers

QCD amplitudes for n partons

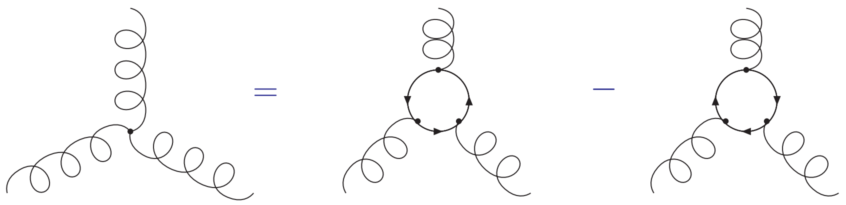
- Complete amplitude $\mathcal{A}$
  - dependence on momenta k_i, helicities λ_i and colour a_i
- Keep track of quantum phases
  - computing transition amplitude rather than cross-section
- Use helicity/colour quantum-numbers of amplitude $\mathcal{A}$
  - decompose $\mathcal{A}$ into simpler, gauge-invariant pieces (so called partial amplitudes A)
- Exploit effective supersymmetry of QCD at tree level
  - manage spins of particles propagating around the loop
- Transition to numerical evaluation at very end of calculation
  - combine the virtual and real corrections
  - square amplitudes, sum over helicities and colours and obtain unpolarized cross-sections

Colour ordering (I)

- SU(N)-generators t^a from fundamental representation
  $\mathrm{Tr}\left(t^a t^b\right) \;=\; \delta^{ab}$

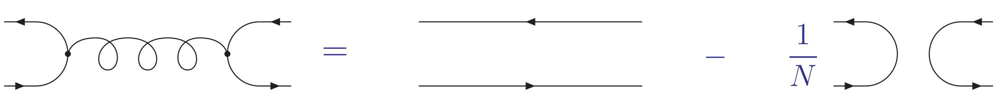
- SU(N)-generators f^{abc} of adjoint representation

$$f^{abc} \;=\; -\frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{Tr}\left(\left[t^a, t^b\right] t^c\right)$$



- Fierz identity

$$\left(t^a\right)_{i_1}^{\;j_1} \left(t^b\right)_{i_2}^{\;j_1} \;=\; \delta_{i_1}^{\;j_2}\,\delta_{i_2}^{\;j_1} - \frac{1}{N}\,\delta_{i_1}^{\;j_1}\,\delta_{i_2}^{\;j_2}$$



Partial amplitudes

- Color decomposition of n-parton amplitudes $\mathcal{A}_n$
  - colour ordered partial amplitudes A with kinematic information

Colour ordering (II)

- Tree level amplitude $\mathcal{A}^{\rm tree}$ with n external gluons
  - sum over all non-cyclic permutations S_n/Z_n of external gluons

$$\mathcal{A}_n^{\rm tree}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} {\rm Tr}(t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}) A(\sigma(1^{\lambda_1}), \dots, \sigma(n^{\lambda_n}))$$

- Tree level amplitude $\mathcal{A}^{\rm tree}$ with $q\bar{q}$ and $n-2$ external gluons

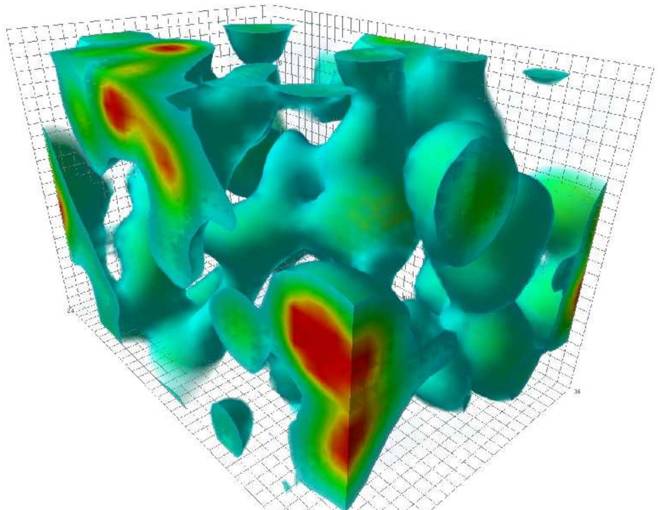
$$\mathcal{A}_n^{\rm tree}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_{n-2}} \left(t^{a_{\sigma(3)}} \dots t^{a_{\sigma(n)}}\right)_{i_2}^{j_1} A(1_{\bar{q}}^{\lambda_1}, 2_q^{\lambda_2}, \sigma(3^{\lambda_3}), \dots, \sigma(n^{\lambda_n}))$$

Instantons

- Gauge field $A_\mu(x)$ defines mapping of $S^3 \to$ SU(N)
- Non-trivial coupling of space-time and colour coordinates
- Classical (Euclidean) solution for self-dual field strength
  - $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ with $\tilde{F}_{\mu\nu} = 1/2\,\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ (local minima of the action)

$$(A_\mu)_i^j\,(x) \quad = \quad -\frac{\mathrm{i}}{g}\,\rho^2\,\frac{\left(U\,(\sigma_\mu\,(\sigma^\nu x_\nu) - x_\mu)U^\dagger\right)_i^j}{x^2(x^2+\rho^2)}$$

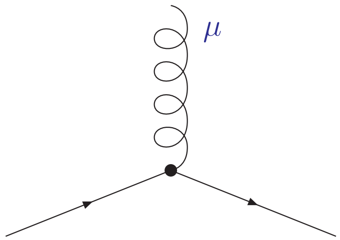
- Physical intepretation: tunneling transition between distinct vacua
- Instanton solutions depend on collective coordinate
  - position x
  - size ρ
  - colour orientation U



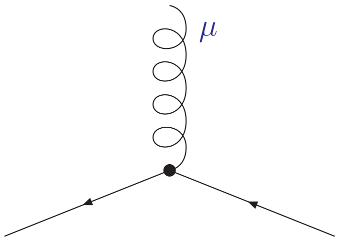
impression of QCD action density by D. Leinweber

Colour ordered rules (III)

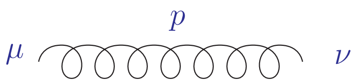
- Feynman rules for colour ordered partial amplitudes
  - vertices, propagators (omitting ghosts)



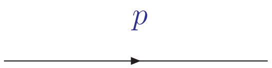
$$-\frac{\mathrm{i}}{\sqrt{2}}\,\gamma^\mu$$



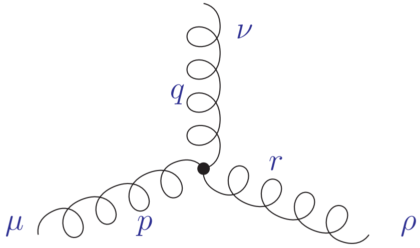
$$+\frac{\mathrm{i}}{\sqrt{2}}\,\gamma^\mu$$



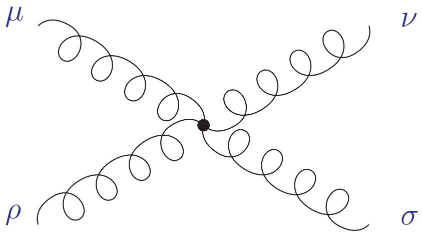
$$-\mathrm{i}\,\frac{g^{\mu\nu}}{p^2}$$



$$\frac{\mathrm{i}}{\not{p}}$$



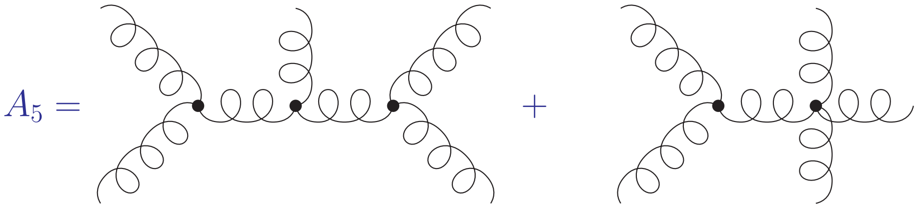
$$\frac{\mathrm{i}}{\sqrt{2}}\left((p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\mu\rho}\right)$$



$$\mathrm{i} g^{\mu\sigma} g^{\nu\rho} - \frac{\mathrm{i}}{2}\left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}\right)$$

Colour ordered rules (IV)

- Calculate only diagrams with cyclic colour ordering
  - example 5-gluon amplitude A_5 (10 diagrams instead of 25)



- In general big reduction in number of Feynman diagrams
  - example n-gluon amplitudes

n	diagrams	colour ordered diagrams
4	4	3
5	25	10
6	220	36
7	2485	133
8	34300	501
9	559405	1991
10	10525900	7335

Spinor conventions (I)

- Massless Dirac spinors $\psi(k)$ with momentum k
  - chiral representation of Dirac γ matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

  - chiral projections $\psi_\pm(k) = \frac{1}{2}(1 \pm \gamma_5)\psi(k)$
- Massless Weyl spinors $u_\pm(k)$ with $\pm$-chirality
  - construction of chiral compontents of Dirac spinor (4-dim) $\psi(k)$ from two Weyl spinors (2-dim) $u_\pm(k)$
- (Weyl) spinor inner-products

$$\begin{aligned} \langle jl \rangle &= \langle j^- | l^+ \rangle = \overline{u_-}(k_j) u_+(k_l) \\ [jl] &= \langle j^+ | l^- \rangle = \overline{u_+}(k_j) u_-(k_l) = \text{sign}(k_j^0 k_l^0) \, \langle lj \rangle^* \end{aligned}$$

- Scalar product of Lorentz-vectors

$$\langle ij \rangle [ji] = 2 k_i \cdot k_j = s_{ij}$$

Spinor conventions (II)

- Gordon identity

$$\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle \;=\; 2k_i^{\mu}$$

- Antisymmetry

$$\langle ji\rangle = -\langle ij\rangle, \qquad [ji] = -[ij], \qquad \langle ii\rangle = [ii] = 0$$

- Fierz rearrangement

$$\langle i^{+}|\gamma^{\mu}|j^{+}\rangle\langle k^{+}|\gamma_{\mu}|l^{+}\rangle \;=\; 2\,[ik]\langle lj\rangle$$

- Charge conjugation of current

$$\langle i^{+}|\gamma^{\mu}|j^{+}\rangle \;=\; \langle j^{-}|\gamma^{\mu}|i^{-}\rangle$$

- Schouten identity

$$\langle ij\rangle\langle kl\rangle \;=\; \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle$$

- Momentum conservation in n-point amplitude $\sum_{i=1}^{n} k_i^{\mu} = 0$

$$\sum_{\substack{i=1 \\ i\neq j,k}}^{n} [ji]\langle ik\rangle \;=\; 0$$

Spinor conventions (III)

- Polarization vector for massless gauge boson (helicity states ± 1)
  - spinor representation for boson with momentum k

$$\epsilon_\mu^\pm(k,q) \;=\; \pm \frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle q^\mp|k^\pm\rangle}$$

  - massless auxiliary vector q (reference momentum) reflects on-shell gauge degrees of freedom

Properties

- $\epsilon^\pm$ transverse to k for any q, that is $\epsilon^\pm(k,q)\cdot k \;=\; 0.$
- Complex conjugation reverses helicity: $(\epsilon_\mu^+)^* \;=\; \epsilon_\mu^-$
- Normalization

$$\epsilon^+\cdot(\epsilon^+)^* \;=\; \epsilon^+\cdot\epsilon^- \;=\; -\frac{1}{2}\frac{\langle q^-|\gamma_\mu|k^-\rangle\langle q^+|\gamma^\mu|k^+\rangle}{\langle qk\rangle[qk]} \;=\; -1$$

$$\epsilon^+\cdot(\epsilon^-)^* \;=\; \epsilon^+\cdot\epsilon^+ \;=\; \frac{1}{2}\frac{\langle q^-|\gamma_\mu|k^-\rangle\langle q^-|\gamma^\mu|k^-\rangle}{\langle qk\rangle^2} \;=\; 0$$

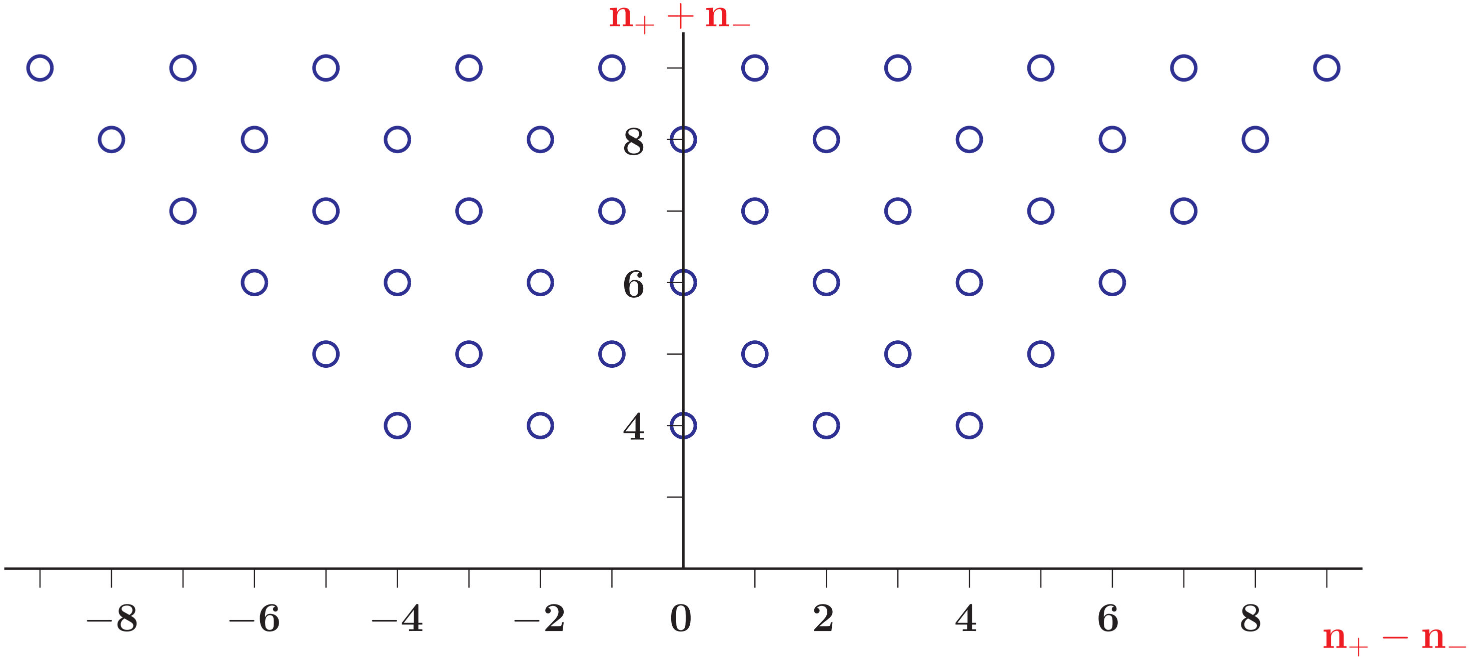
Spinor conventions (IV)

- Choice of reference momentum q can simplify calculation
  - useful identities

$$
\begin{aligned}
\epsilon^{\pm}(k_i, q) \cdot q &= 0 \\
\epsilon^{+}(k_i, q) \cdot \epsilon^{+}(k_j, q) &= \epsilon^{-}(k_i, q) \cdot \epsilon^{-}(k_j, q) \;=\; 0 \\
\epsilon^{+}(k_i, k_j) \cdot \epsilon^{-}(k_j, q) &= \epsilon^{+}(k_i, q) \cdot \epsilon^{-}(k_j, k_i) \;=\; 0 \\
\not{\epsilon}^{+}(k_i, k_j) | j^{+} \rangle &= \not{\epsilon}^{-}(k_i, k_j) | j^{-} \rangle \;=\; 0 \\
\langle j^{+} | \not{\epsilon}^{-}(k_i, k_j) &= \langle j^{-} | \not{\epsilon}^{+}(k_i, k_j) \;=\; 0
\end{aligned}
$$

Helicity amplitudes (I)

- n-gluon helicity amplitudes
  - diference between n_+ positive and n_- negative helicities



  - each row describes scattering process with n_+ positive and n_- negative helicities
  - each circle represents one allowed helicity configuration

Helicity amplitudes (II)

- Example 5-gluon amplitude A_5
  - result of computing the 25 diagrams for the five-gluon process

$$A_5^{\rm tree}(1^\pm, 2^+, \dots, 5^+) = 0$$

$$A_5^{\rm tree}(1^-, 2^-, 3^+, \dots, 5^+) = {\rm i}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle}$$

n-point amplitudes

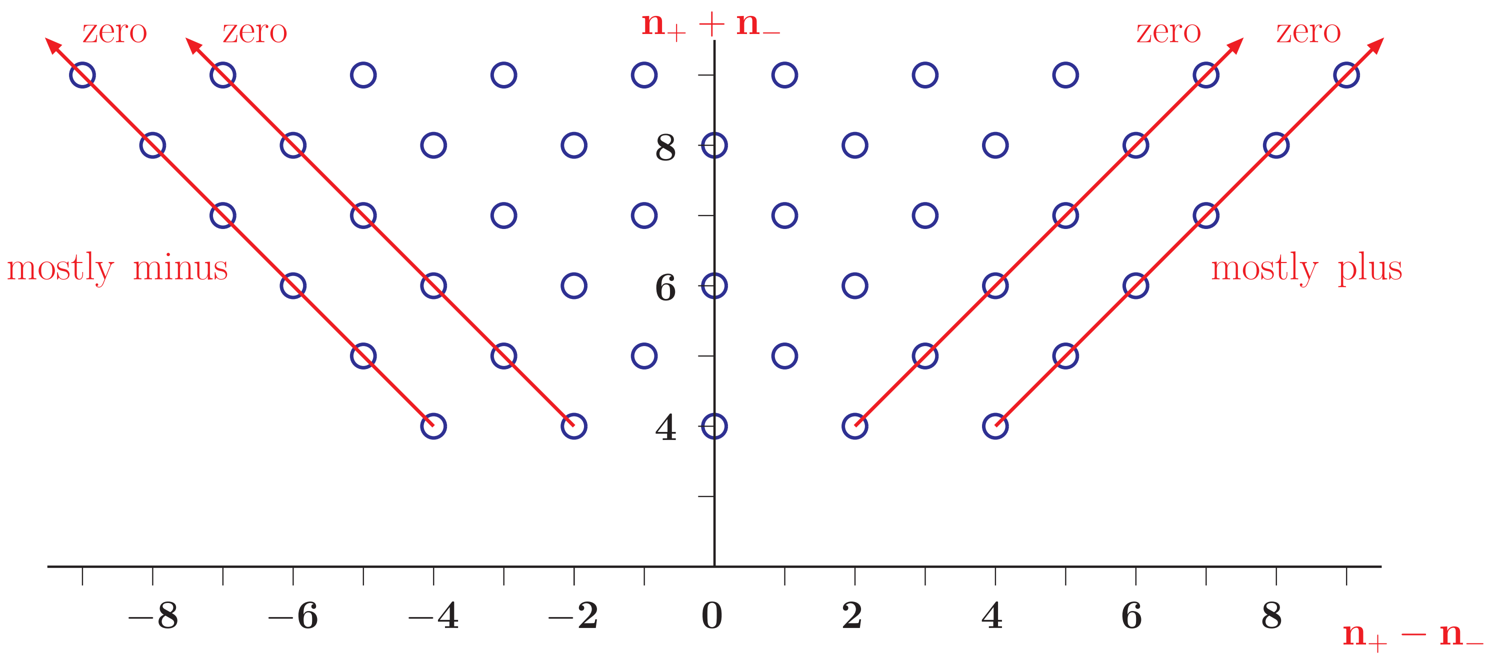
- Generally, for n-gluon amplitude A_n
  - $A_n^{\rm tree}(1^\pm, 2^+, \dots, n^+) = 0$
  - maximal helicity violating (MHV) amplitudes

    Parke, Taylor ‘86 Berends, Giele ‘87

$$A_n^{\rm tree}(1^-, 2^-, 3^+, \dots, n^+) = {\rm i}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\dots\,\langle n1\rangle}$$

Helicity amplitudes (III)

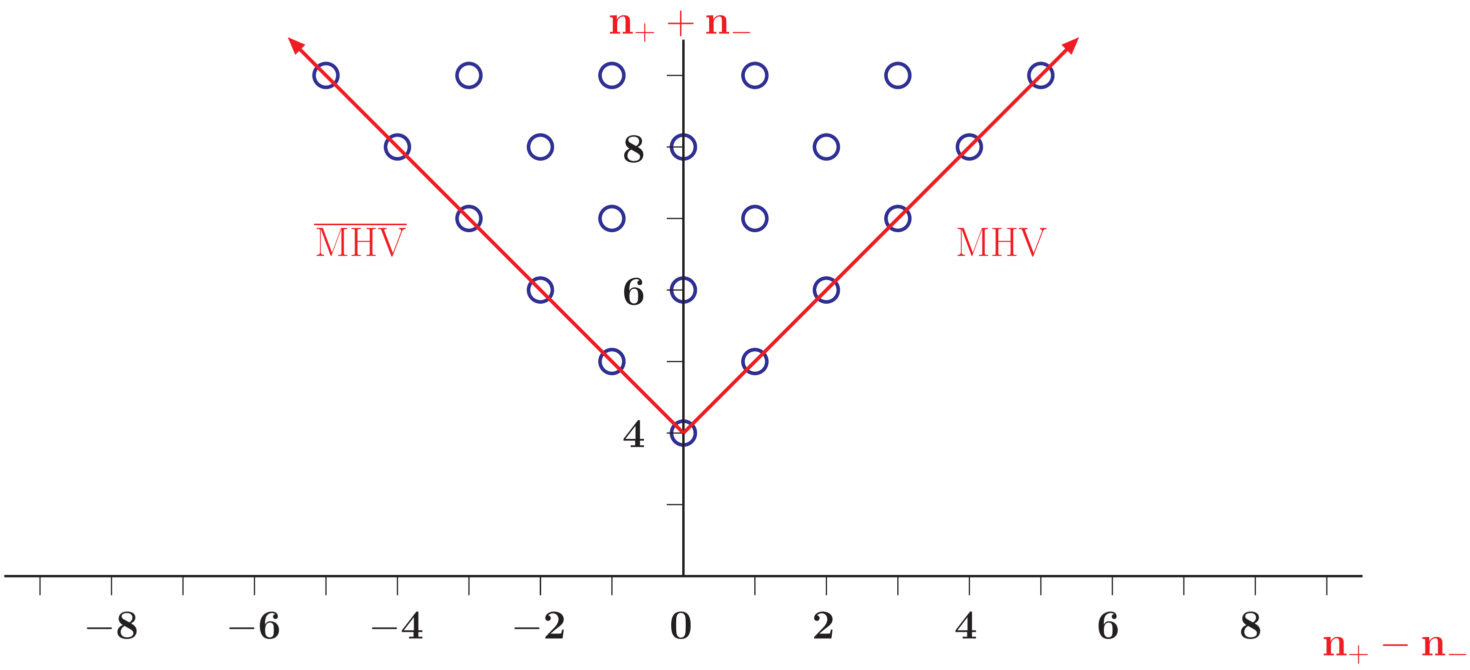
- n-gluon helicity amplitudes



- effective supersymmetry at tree level $A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) = 0$

Helicity amplitudes (IV)

- n-gluon helicity amplitudes



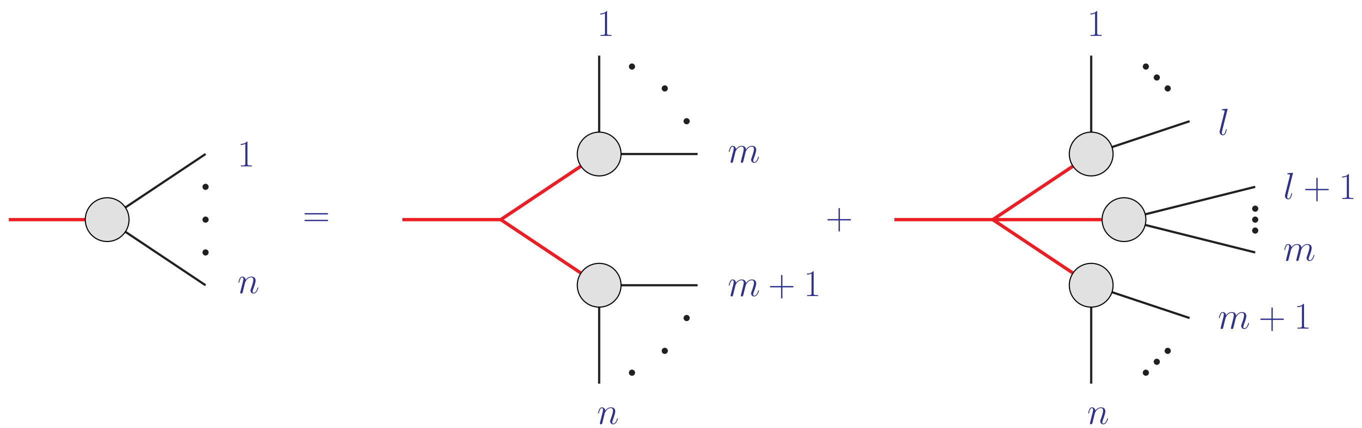
- maximal helicity violating amplitudes Parke, Taylor '86

$$A_n^{\text{tree}}(1^-, 2^-, 3^+, \dots, n^+) = \mathrm{i}\frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

Tree level amplitudes

Recursion relations

- Build up full amplitude from simpler amplitudes with fewer particles
  - recursion relation from off-shell currents (momentum conservation) Berends, Giele '87



  - **Red** gluons are off-shell, **black** gluons are on-shell

- Particularly suitable for numerical evaluation, e.g.
  - Alpgen Caravaglios, Mangano, Moretti, Piccini, Pittau, Polosa
  - HELAC/PHEGAS Draggiotis, Kleiss, Papadopoloulos

On-shell recursions (I)

- On-shell recursions in n-point process
  - (helicity) amplitudes written as sum over “factorizations” into on-shell amplitude Britto, Cachazo, Feng, Witten
- Proof exploits elementary complex analysis and general factorization properties of scattering amplitude
- Generality of proof permits extension to loop level

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Basic idea

- Parameter-dependent $[j, l\rangle$ shift of external massless spinors j and l

$$
[j, l\rangle : \qquad \begin{aligned} \tilde{\lambda}_j &\to \tilde{\lambda}_j - z\tilde{\lambda}_l \\ \lambda_l &\to \lambda_l + z\lambda_j \end{aligned}
$$

  - define $\lambda_j = u_+(k_j)$ and $\tilde{\lambda}_l = u_-(k_l)$
  - complex parameter z

On-shell recursions (II)

- Shift in spinors corresponds to shifting momenta to complex values

$$
\begin{aligned}
k_j^\mu \quad &\rightarrow \quad k_j^\mu(z) = k_j^\mu - \frac{z}{2}\left\langle j^- \middle| \gamma^\mu \middle| l^- \right\rangle \\
k_l^\mu \quad &\rightarrow \quad k_l^\mu(z) = k_l^\mu + \frac{z}{2}\left\langle j^- \middle| \gamma^\mu \middle| l^- \right\rangle
\end{aligned}
$$

  - momenta remain massless $k_j^2(z) = k_l^2(z) = 0$
  - momentum conservation maintained
- Similarly for cases with massive particles Badger, Glover, Khoze '05

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Amplitude at tree level

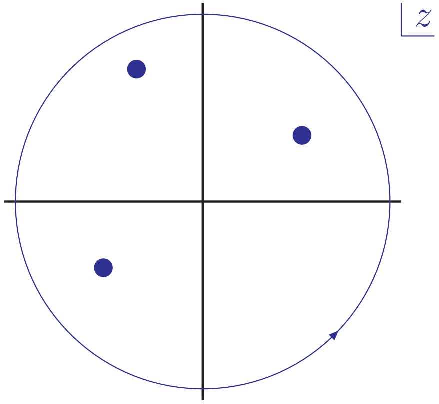
- On-shell amplitude with shifted momenta k_j and k_l becomes parameter-dependent

$$A(z) = A(k_1, \ldots, k_j(z), k_{j+1}, \ldots, k_l(z), k_{l+1}, \ldots, k_n)$$

- Physical amplitude recovered by taking $z = 0$

On-shell recursions (III)

- $A(z)$ is analytic function containing only simple poles
  - exploit Cauchy's theorem and construct $A(z)$ from its residues
- Assume $A(z) \to 0$ as $z \to \infty$
  - no 'surface term' in contour integral around circle at infinity
  - contour integral vanishes $$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{dz}{z} A(z) = 0$$



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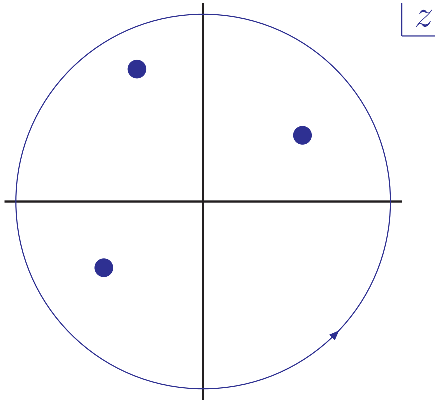

Physical amplitude

- Evaluate integral as sum of residues and solve for the amplitude $A(0)$

$$A(0) = - \sum_{\text{poles } \alpha} \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

- Requirement of vanishing $A(z)$ as $z \to \infty$ satisfied at tree level by wide classes of shifts

On-shell recursions (IV)

- Construction of physical amplitude $A(0)$ with $[j, l\rangle$ shift

$$A(0) = C_\infty + \sum_{r,s,h} A_L^h(z = z_{rs}) \frac{\mathrm{i}}{K_{r\ldots s}^2} A_R^{-h}(z = z_{rs})$$

  - put shifted leg j in A_L (left) and shifted leg l in A_R (right) of pole in $K_{r\ldots s}^2 = (k_r + k_{r+1} + \cdots + k_{s-1} + k_s)^2$
  - sum over r, s (all cyclic orderings of remaining $n - 2$ legs)
  - sum over $h = \pm 1$ (helicity states)
  - evaluate amplitudes A_L and A_R at $z = z_{rs} = \dfrac{K_{r\ldots s}^2}{\langle j|K_{r\ldots s}|l\rangle}$ (residue)
  - $C_\infty = 0$ if $A(z) \to 0$ as $z \to \infty$ (no 'surface term')

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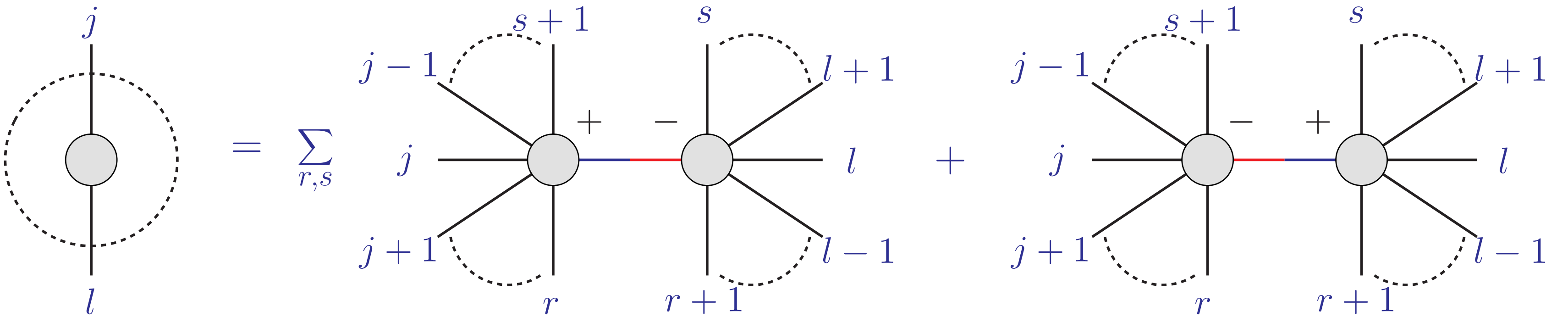

Example: six gluon scattering (I)

- Construction of six-gluon amplitude $A_6^{\text{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

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Step 1

- Choose $[3, 4\rangle$ shift, $\tilde{\lambda}_3 \to \tilde{\lambda}_3 - z\tilde{\lambda}_4$, $\lambda_4 \to \lambda_4 + z\lambda_3$

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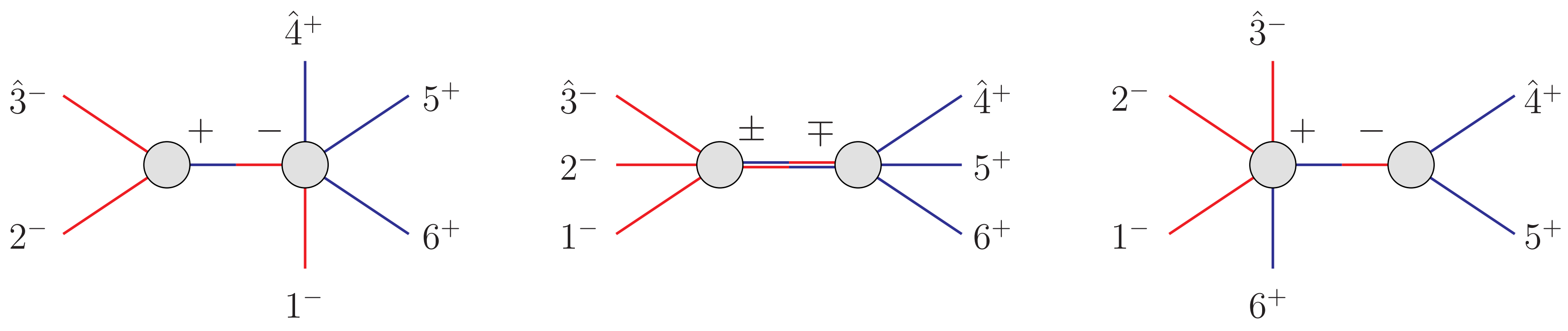
Step 1

- Choose $[3,4\rangle$ shift, $\tilde{\lambda}_3 \to \tilde{\lambda}_3 - z\tilde{\lambda}_4$, $\lambda_4 \to \lambda_4 + z\lambda_3$

Step 2

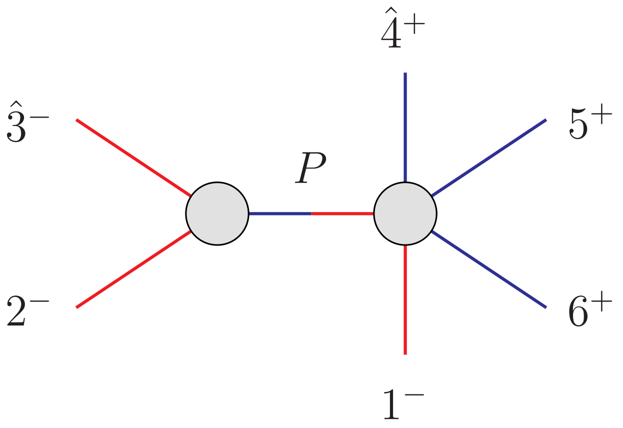
- Draw all diagrams
  - second pair of diagrams (middle ones) vanishes



Example: six gluon scattering (II)

Step 3

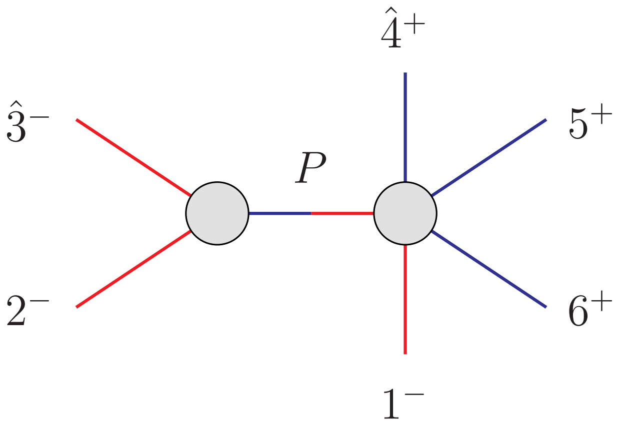
- Diagram 1



Example: six gluon scattering (II)

Step 3

- Diagram 1
- Combine on-shell amplitudes
  - define $P = k_2 + k_3$
  - z-dependent (hatted) momenta $\hat{3}, \hat{4}, \hat{P}$
  - pole at $z = \dfrac{P^2}{\langle 3|P|4\rangle}$

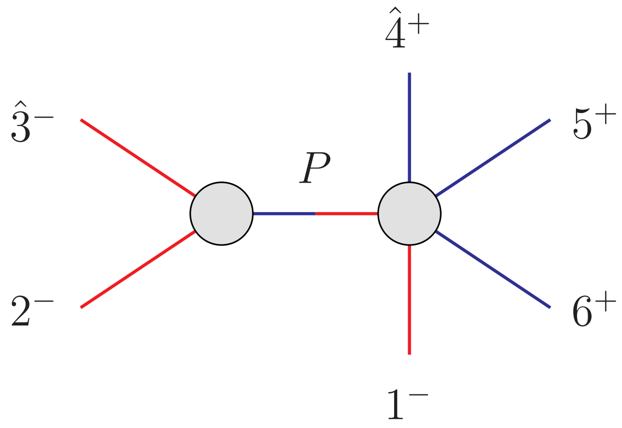


$$A_3^{\rm tree}(2^-,\hat{3}^-,\hat{P}^+) \times \frac{\rm i}{P^2} \times A_5^{\rm tree}(1^-,-\hat{P}^-,\hat{4}^+,5^+,6^+) =$$

$$\frac{\langle 2\hat{3}\rangle^3}{\langle \hat{3}\hat{P}\rangle\langle \hat{P}2\rangle}\,\frac{\rm i}{P^2}\,\frac{\langle 1\hat{P}\rangle^3}{\langle \hat{P}\hat{4}\rangle\langle \hat{4}5\rangle\langle 56\rangle\langle 61\rangle}\Bigg|_{z=\frac{P^2}{\langle 3|P|4\rangle}}$$

Example: six gluon scattering (II)

Step 3



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$$A_3^{\text{tree}}(2^-, \hat{3}^-, \hat{P}^+) \times \frac{\text{i}}{P^2} \times A_5^{\text{tree}}(1^-, -\hat{P}^-, \hat{4}^+, 5^+, 6^+) =$$

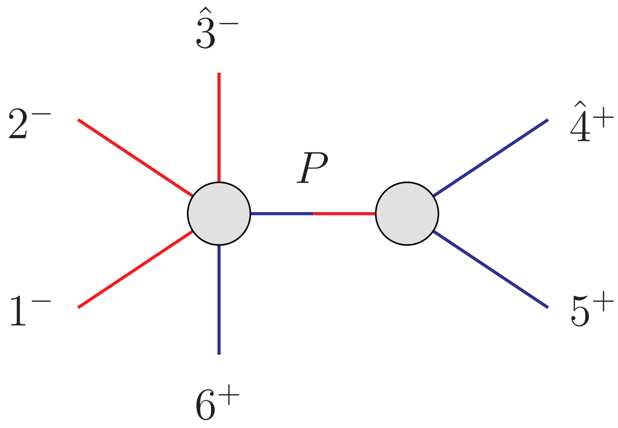
$$\frac{\langle 2\hat{3}\rangle^3}{\langle \hat{3}\hat{P}\rangle\langle \hat{P}2\rangle} \frac{\text{i}}{P^2} \frac{\langle 1\hat{P}\rangle^3}{\langle \hat{P}\hat{4}\rangle\langle \hat{4}5\rangle\langle 56\rangle\langle 61\rangle}\Bigg|_{z=\frac{P^2}{\langle 3|P|4\rangle}}$$

- Spinor helicity algebra, e.g. $\langle 2\hat{3}\rangle = \langle 23\rangle$ for $[3,4\rangle$ shift
- Work hatted momenta $\hat{3}, \hat{4}, \hat{P}$ away

Example: six gluon scattering (III)

Step 4

- Diagram 2



- Calculate from diagram 1
  - complex conjugation ($\pm \leftrightarrow \mp$, $\langle .. \rangle \leftrightarrow [..]$, etc.)
  - relabeling of momenta $(1, 2, 3, 4, 5, 6) \to (6, 5, 4, 3, 2, 1)$

Example: six gluon scattering (IV)

Step 5

- Combine everything, obtain extremely compact result
  - define $s_{234} = (k_2 + k_3 + k_4)^2$, etc

$$A_6^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+) =$$
$$\mathrm{i}\,\frac{1}{\langle 5|\not{3}+\not{4}|2\rangle}\left(\frac{\langle 1|\not{2}+\not{3}|4\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}} + \frac{\langle 3|\not{4}+\not{5}|6\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{345}}\right)$$

MHV rules

- Standard MHV and $\overline{\text{MHV}}$ expressions
  - three-gluon primitive amplitude
  - quark-gluon-antiquark primitive amplitude

$$A_3^{\rm tree}(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \qquad A_3^{\rm tree}(1^+,2^+,3^-) = -\frac{[12]^3}{[23][31]}\,,$$

$$A_3^{\rm tree}(1_q^-,2^-,3_{\bar q}^+) = \frac{\langle 12\rangle^2}{\langle 13\rangle}\,, \qquad A_3^{\rm tree}(1_q^-,2^+,3_{\bar q}^+) = -\frac{[23]^2}{[13]}\,,$$

$$A_3^{\rm tree}(1_q^+,2^-,3_{\bar q}^-) = \frac{\langle 23\rangle^2}{\langle 13\rangle}\,, \qquad A_3^{\rm tree}(1_q^+,2^+,3_{\bar q}^-) = -\frac{[12]^2}{[13]}\,.$$

- Complex momenta k_i
  - three-point amplitudes do not vanish on-shell

Summary (part I)

Standard Model

- Successful experimental program at LHC relies crucially on detailed understanding of Standard Model processes
  - e.g. $pp \to$ boson $+\, n$ jets with $n \leq 4$
- Perturbative predictions required for processes with $n \leq 7$ legs
  - amplitudes up to NLO for jets in final states (and with massive particles, e.g. W, Z or t)

Theory developments

- Analytical results for amplitudes
  - concepts of colour ordering and helicity amplitudes
- Recent progress
  - constructive approach for multi leg amplitudes
  - on-shell recursions exploit analyticity properties

Literature

- Reviews:
  - *Multiparton amplitudes in gauge theories*
    M.L. Mangano, S.J. Parke Phys.Rept.200 (1991) 301-367
- Lectures:
  - *Calculating scattering amplitudes efficiently*
    L.J. Dixon [hep-ph/9601359] (TASI '95)
  - *Lectures on twistor strings and perturbative Yang-Mills theory*
    F. Cachazo, P. Svrcek [hep-th/0504194] (S.I.S.S.A. Trieste '05)
- Original papers (my favourites):
  - *Direct proof of tree-level recursion relation in Yang-Mills theory*
    R. Britto, F. Cachazo, B. Feng, E. Witten [hep-th/0501052]
  - *Recursion relations for gauge theory amplitudes with massive vector bosons and fermions*
    S. Badger, N. Glover, V. Khoze [hep-th/0507161]
  - *On-shell recurrence relations for one-loop QCD amplitudes*
    Z. Bern, L. Dixon, D. Kosower [hep-th/0501240]

Exercise (I)

- Calculate the (nonzero) helicity amplitude $A_4^{\mathrm{tree}}(1^-, 2^-, 3^+, 4^+)$ using colour-ordered Feynman rules
  - Hint: Choose the reference momenta $q_1 = q_2 = k_4$, $q_3 = q_4 = k_1$, so that only contraction $\epsilon_2^- \cdot \epsilon_3^+$ is nonzero

Solution (I)

- Only one (with gluon exchange in s_{12} channel) graph contributes $A_4^{\text{tree}}(1^-,2^-,3^+,4^+)$

$$
\begin{aligned}
&= \left(\frac{i}{\sqrt{2}}\right)^2 \left(\frac{-i}{s_{12}}\right) \\
&\quad \times \left[\epsilon_1^- \cdot \epsilon_2^- (k_1 - k_2)^\mu + (\epsilon_2^-)^\mu \epsilon_1^- \cdot (2k_2 + k_1) + (\epsilon_1^-)^\mu \epsilon_2^- \cdot (-2k_1 - k_2)\right] \\
&\quad \times \left[\epsilon_3^+ \cdot \epsilon_4^+ (k_3 - k_4)_\mu + (\epsilon_4^+)_\mu \epsilon_3^+ \cdot (2k_4 + k_3) + (\epsilon_3^+)_\mu \epsilon_4^+ \cdot (-2k_3 - k_4)\right] \\
&= -\frac{2i}{s_{12}} (\epsilon_2^- \cdot \epsilon_3^+)(\epsilon_1^- \cdot k_2)(\epsilon_4^+ \cdot k_3) \\
&= -\frac{2i}{s_{12}} \left(-\frac{2}{2}\frac{[43]\langle 12\rangle}{[42]\langle 13\rangle}\right)\left(-\frac{[42]\langle 21\rangle}{\sqrt{2}[41]}\right)\left(+\frac{\langle 13\rangle[34]}{\sqrt{2}\langle 14\rangle}\right) \\
&= -i\,\frac{\langle 12\rangle[34]^2}{[12]\langle 14\rangle[14]} \;=\; i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}
$$

  - last line: antisymmetry, momentum conservation and $s_{34} = s_{12}$

Exercise (II)

- Calculate the helicity amplitude $A_6^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ using MHV rules
  - Hint: Choose the $[3, 4\rangle$ shift, $\tilde{\lambda}_3 \to \tilde{\lambda}_3 - z\tilde{\lambda}_4$, $\lambda_4 \to \lambda_4 + z\lambda_3$