Hadronic Effects in Electroweak Precision Observables

Examples related the Muon Anomalous Magnetic Moment

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Outline of Lectures:

Part I:

① **Hadronic Effects in Electroweak Observables**

② **The role of $\alpha(M_Z)$ in precision physics**

③ **Evaluation of $\alpha(M_Z)$**

Part II:

④ **Hadronic vacuum polarization effects in $g-2$**

⑤ **Evaluation of a_μ^{had}**

⑥ **Light-by-Light scattering contribution to $g-2$**

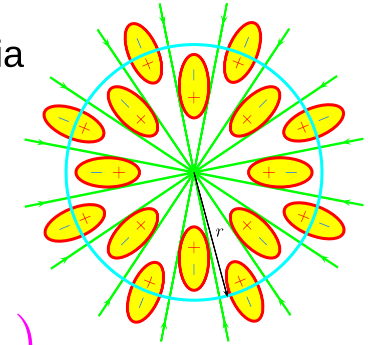
⑦ **Leading hadronic electroweak effects in $g-2$**

⑧ **Outlook**

Topics not covered is hadronic effects related to the top quark (see other lectures related to top; usually perturbative except for threshold region)

Part I: **Hadronic Contributions in gauge boson physics**

① Hadronic Effects in Electroweak Observables



Non-perturbative hadronic effects in electroweak precision observables, main effect via

effective fine-structure "constant" $\alpha(E)$

(charge screening by vacuum polarization)

Of particular interest:

$$\alpha(M_Z) \text{ and } a_\mu \equiv (g-2)_\mu/2 \Leftrightarrow \alpha(m_\mu)$$

- **electroweak effects (leptons etc.) calculable in perturbation theory**
- **strong interaction effects (hadrons/quarks etc.) perturbation theory fails**

$\Longrightarrow$ **Dispersion integrals over e^+e^-–data**

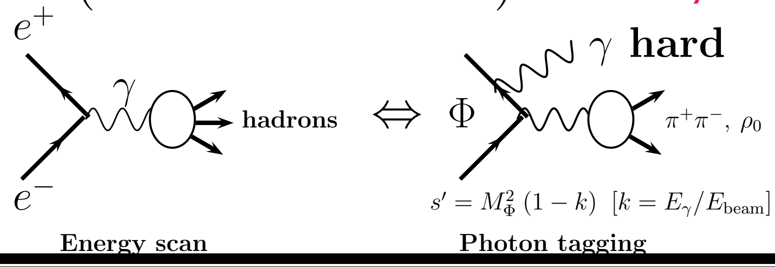
encoded in $$R_\gamma(s) \equiv \frac{\sigma(e^+e^-\to\gamma^*\to\text{hadrons})}{\sigma(e^+e^-\to\gamma^*\to\mu^+\mu^-)}$$

Errors of data $\Longrightarrow$ theoretical uncertainties !!!

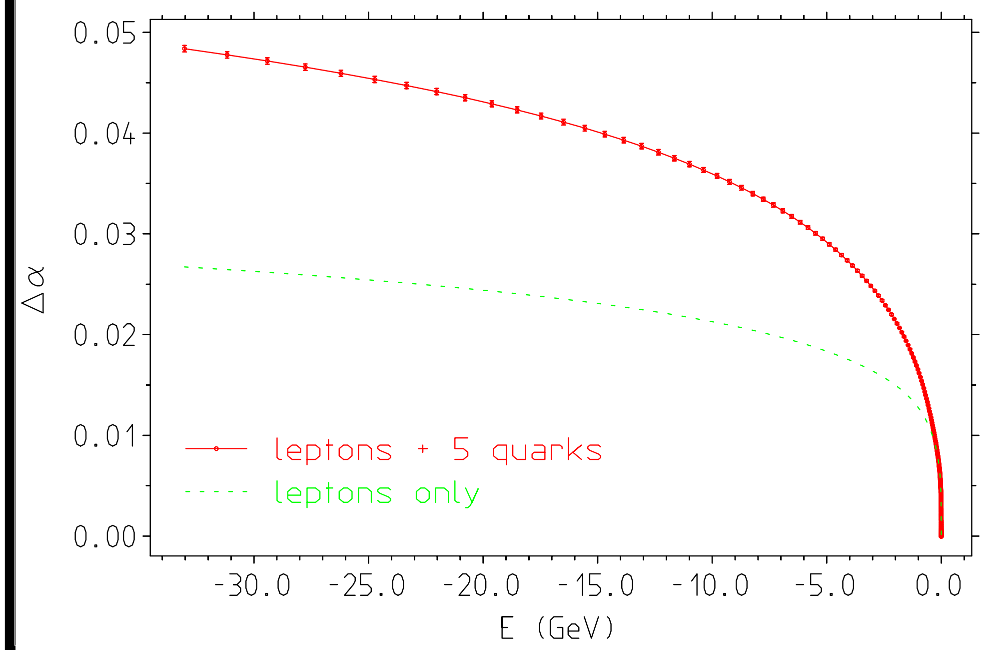
The art of getting precise results from non-precision measurements !

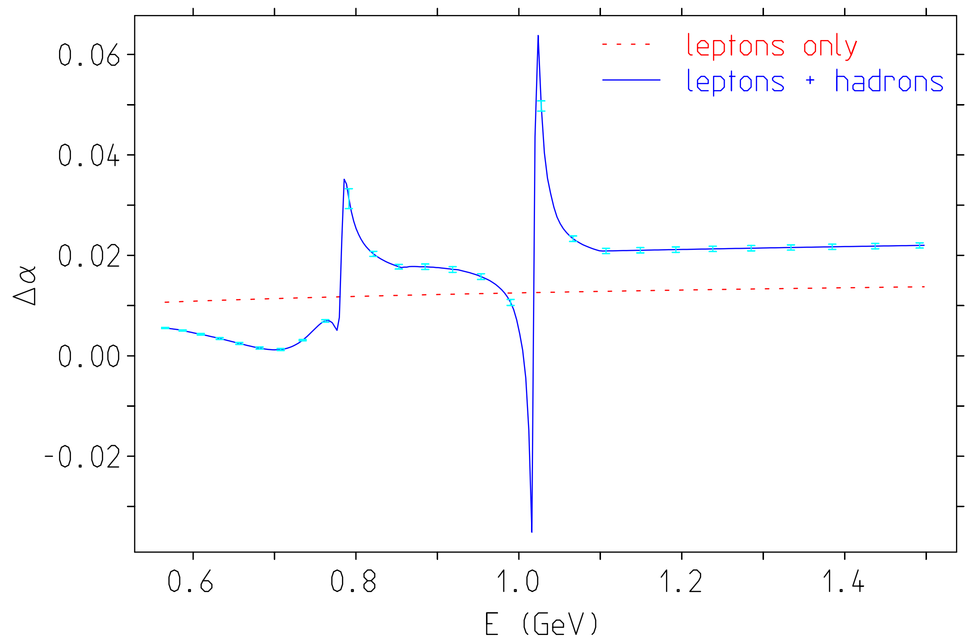
New challenge for precision experiments on $\sigma(e^+e^- \to hadrons)$ KLOE, BABAR, ...

σ_{hadronic} **via radiative return:**



- **Need to know running of $\alpha_{\rm QED}$ very precisely.**
  Large corrections, steeply increasing at low E

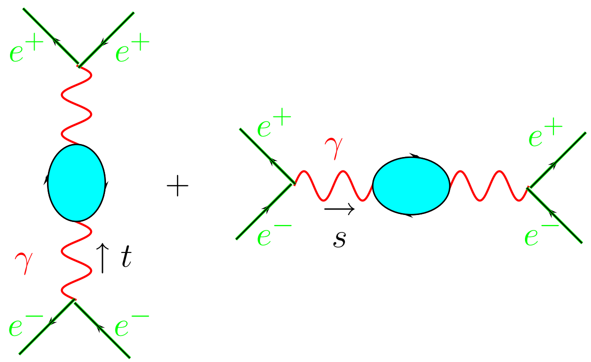




The running of α. The “negative” E axis is chosen to indicate space-like momentum transfer. The vertical bars at selected points indicate the uncertainty. In the time-like region the resonances lead to pronounced variations of the effective charge (shown in the $\rho-\omega$ and ϕ region).

Questions: why not measure $\alpha_{\rm eff}(E)$ directly, like QCD running coupling $\alpha_s(s)$?

Problem: any measurement requires normalizing process like Bhabha,



depends itself on $\alpha_{\rm eff}(t)$ and $\alpha_{\rm eff}(s)$, always measure something like

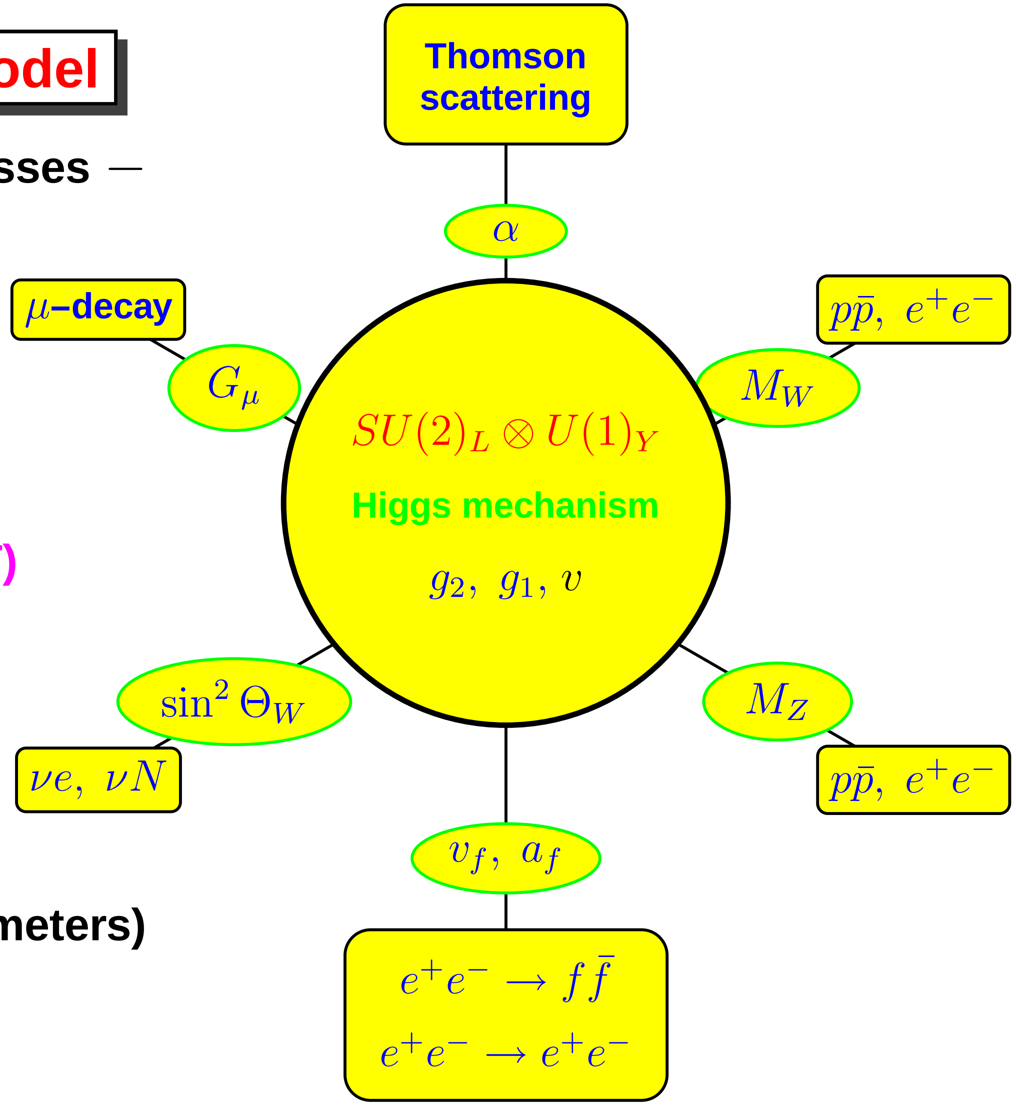
$$r(E) \propto \left(\alpha_{\rm eff}(s)/\alpha_{\rm eff}(t)\right)^2 \quad , \quad t = -\frac{1}{2}\left(s-4m_e^2\right)\left(1-\cos\theta\right)$$

where large part of the effect drops out, especially the strongly raising low energy piece, which includes substantial non-perturbative effects.

Higher energies: for all processes which are not dominated by a **single one photon exchange**, $\alpha_{\rm eff}(E)$ enters in complicated way in observables and cannot by measured in any direct way (see below).

The Parameters of the Standard Model

— in four fermion and vector boson processes —



unlike in QED and QCD in SM (SBGT)

parameter interdependence

☞

only 3 independent quantities

(besides fermion masses and mixing parameters)

α, G_μ, M_Z

$\Downarrow$

parameter relationships between very precisely measurable quantities

precision tests, possible sign of new physics

② The role of $\alpha(M_Z)$ in precision physics

Uncertainties of hadronic contributions to effective α are a problem for electroweak precision physics:

$\alpha\ ,\ G_\mu, M_Z$ most precise input parameters

50%
non-perturbative
relationship

$\Updownarrow$

precision predictions $\sin^2\Theta_f, v_f, a_f, M_W, \Gamma_Z, \Gamma_W, \cdots$

$\alpha(M_Z), G_\mu, M_Z$ best effective input parameters for VB physics (Z,W) etc.

$$\frac{\delta\alpha}{\alpha} \sim 3.6 \times 10^{-9}$$
$$\frac{\delta G_\mu}{G_\mu} \sim 8.6 \times 10^{-6}$$
$$\frac{\delta M_Z}{M_Z} \sim 2.4 \times 10^{-5}$$
$$\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} \sim 1.6 \div 6.8 \times 10^{-4} \quad (\mathrm{present: lost}\ 10^5\ \mathrm{in\ precision!})$$
$$\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} \sim 5.3 \times 10^{-5} \quad (\texttt{ILC requirement})$$

LEP/SLD: $\sin^2\Theta_{\rm eff} = (1 - g_{Vl}/g_{Al})/4 = 0.23148 \pm \boxed{0.00017}$

$$\delta\Delta\alpha(M_Z) = 0.00036 \qquad \Rightarrow \qquad \delta\sin^2\Theta_{\rm eff} = \boxed{0.00013}$$

affects Higgs mass bounds, precision tests and new physics searches!!!

For perturbative QCD contributions very crucial: precise QCD parameters $\alpha_s,\ m_c,\ m_b,\ m_t \Rightarrow$ Lattice-QCD

Indirect

Higgs boson mass "measurement"

$m_H = 88^{+53}_{-35}$ **GeV**

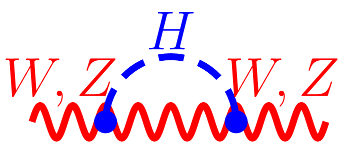
$e^+e^- \to \tau$**:** $\Rightarrow \delta m_H \sim -19$ **GeV**

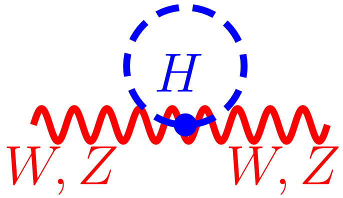
Direct lower bound:

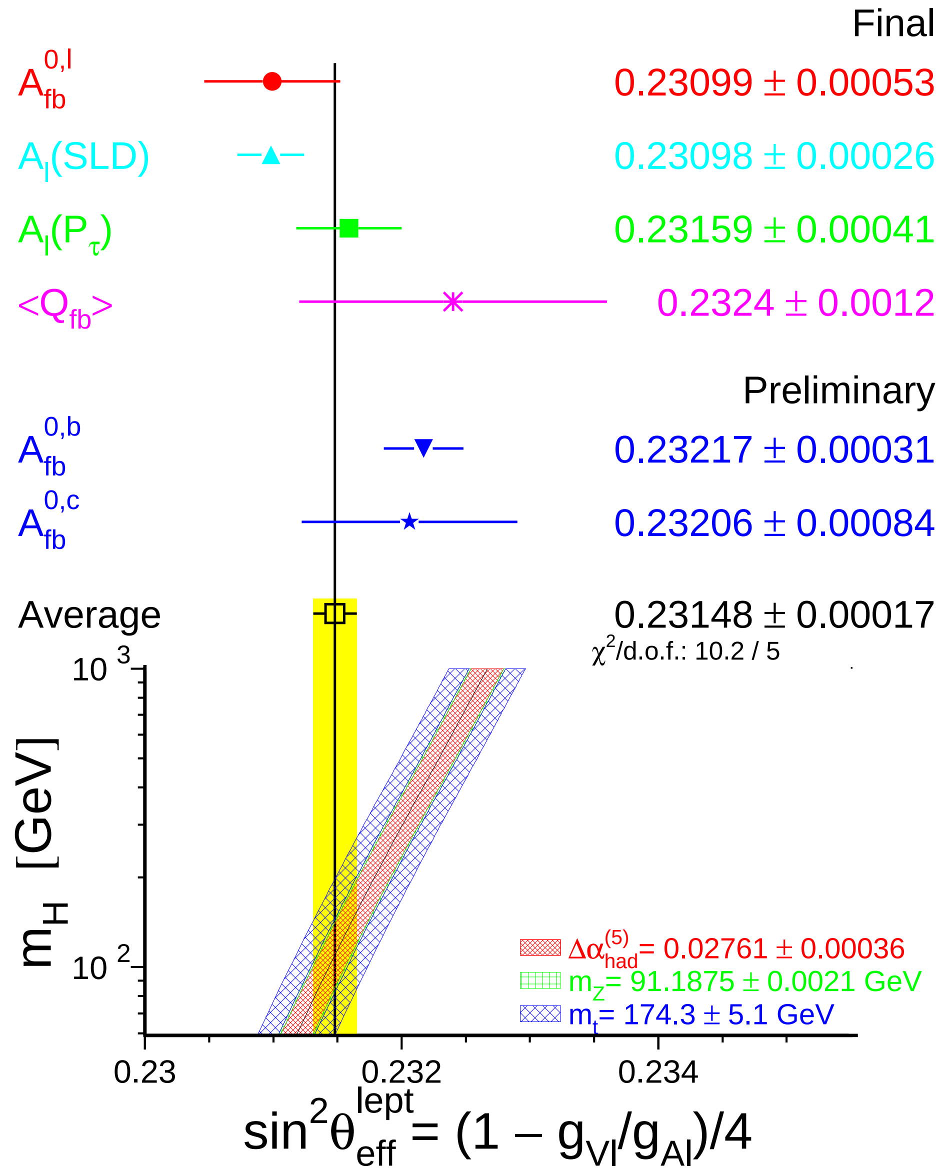
$\boxed{m_H > 114}$ **GeV at 95% CL**

Indirect upper bound:

$\boxed{m_H < 193}$ **GeV at 95% CL**







(LEP Electroweak Working Group: D. Abbaneo et al. 05)

Input parameter for ILC physics:

$$\begin{array}{cccccccccc} \frac{\delta\alpha}{\alpha} & \sim & 3.6 & \times & 10^{-9} & \qquad \frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 1.6 \div 6.8 & \times & 10^{-4} \\ \frac{\delta G_\mu}{G_\mu} & \sim & 8.6 & \times & 10^{-6} & \qquad \frac{\delta M_Z}{M_Z} & \sim & 2.4 & \times & 10^{-5} \end{array}$$

accuracy in $\delta\alpha(M_Z)$ roughly one order of magnitude worse than M_Z !

$$\sin^2\Theta_i\,\cos^2\Theta_i = \frac{\pi\alpha}{\sqrt{2}\,G_\mu\,M_Z^2}\frac{1}{1-\Delta r_i}$$

where

$$\Delta r_i = \Delta r_i(\alpha,\,G_\mu,\,M_Z,m_H,\,m_{f\neq t},\,m_t)$$

quantum corrections from gauge boson self-energies, vertex– and box–corrections.

Propagation of uncertainty: $\delta\Delta\alpha \Rightarrow \delta M_W,\, \delta\sin^2\Theta_f$:

$$\frac{\delta M_W}{M_W} \sim \frac{1}{2}\frac{\sin^2\Theta_W}{\cos^2\Theta_W-\sin^2\Theta_W}\,\delta\Delta\alpha \sim 0.23\,\delta\Delta\alpha$$

$$\frac{\delta\sin^2\Theta_f}{\sin^2\Theta_f} \sim \frac{\cos^2\Theta_f}{\cos^2\Theta_f-\sin^2\Theta_f}\,\delta\Delta\alpha \sim 1.54\,\delta\Delta\alpha$$

e.g., obscure in particular the indirect bounds on the Higgs mass obtained from electroweak precision measurements.

Precision predictions:

$$\begin{array}{lrcl}
M_W : & \sin^2\Theta_W & = & 1-\frac{M_W^2}{M_Z^2} \\
g_2 : & \sin^2\Theta_g & = & e^2/g_2^2 = \frac{\pi\alpha}{\sqrt{2}\,G_\mu\,M_W^2} \\
a_f : & \sin^2\Theta_f & = & \frac{1}{4|Q_f|}\left(1-\frac{v_f}{a_f}\right)\;,\quad f\neq\nu \\
a_f : & \rho_f & = & \frac{1}{1-\Delta\rho}\;,\quad \text{independent on}\quad \alpha
\end{array}$$

for the most important cases and the general form of Δr_i reads

$$\Delta r_i \quad = \quad \Delta\alpha - f_i(\sin^2\Theta_i)\,\Delta\rho + \Delta r_{i\,\mathrm{remainder}}$$

with a universal term $\Delta\alpha$ which affects the predictions for M_W, A_{LR}, A_{FB}^f, Γ_f, etc.

Equally important in:

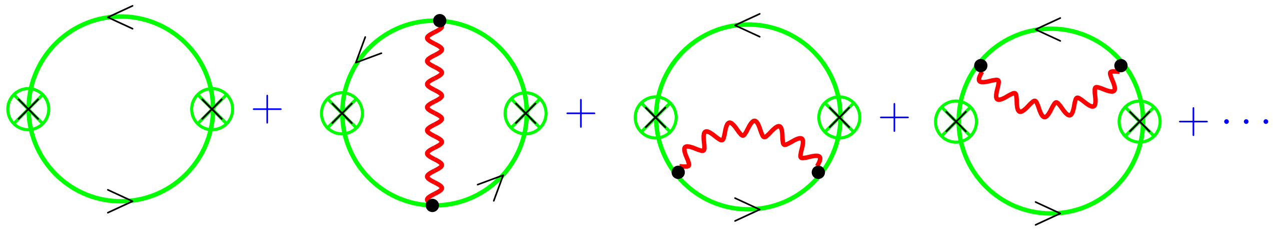
- Bhabha scattering ($\alpha(t)$)
- V_{ud} superallowed β–decay ($\alpha(m_p)$)
- any precise cross section prediction/measurement
- $\cdots$

Other quark/hadron loop contributions (in observables mentioned above)

In leptonic sector at one-loop only via gauge boson self-energies

$\Rightarrow$ QCD vacuum polarization functions $\Pi^{NC}_{V,A}(s,m_i)$ and $\Pi^{CC}_{V,A}(s,m_1,m_2)$ of the SM vector (V) and axial-vector (A) currents: pQCD applies if evaluated at high enough scale or if particle is heavy (top) and away from threshold

- **pQCD calculations of vacuum polarization amplitudes**



up to 4–loops massless — **Gorishny, Kataev, Larin 91**

up to 3–loops massive — **Chetyrkin, Kühn et al. 97**

up to 2–loops massive BF–MOM RG — **FJ, Tarasov 98**

Also needed for evaluation of Adler function: for testing non–perturbative effects (see below)

Form of self-energies: in red non-perturbative light quark terms

$$\begin{aligned}
\Pi^{\gamma\gamma} &= e^2 Q_i^2\, \Pi_V^{NC}\,(s, m_i) \\
\Pi^{Z\gamma} &= \left(\frac{eg}{4c_W}|Q_i| - \frac{e^2 s_W}{c_W} Q_i^2\right) \Pi_V^{NC}\,(s, m_i) \\
\Pi^{ZZ} &= \left(\frac{g^2}{16c_W^2}\right) \left(\left(1 - 8s_W^2|Q_i| + 16 s_W^4 Q_i^2\right) \Pi_V^{NC}\,(s, m_i) + \Pi_A^{NC}\,(s, m_i)\right) \\
\Pi^{WW} &= \left(\frac{g^2}{8}\right) \left(\Pi_V^{CC}\,(s, m_1, m_2) + \Pi_A^{CC}\,(s, m_1, m_2)\right)
\end{aligned}$$

Defining (according to QCD symmetry breaking patterns)

$$\begin{aligned}
\Delta\rho &= g^2 \left(\hat{\Pi}_{33}(0) - \hat{\Pi}_{\pm}(0)\right) / M_W^2 \\
\Delta_1 &= g^2 \left(\hat{\Pi}'_{3\gamma}(M_Z^2) - \hat{\Pi}'_{33}(M_Z^2)\right) \\
\Delta_2 &= g^2 \left(\hat{\Pi}'_{33}(M_Z^2) - \hat{\Pi}'_{\pm}(M_W^2)\right) \\
\Delta\alpha &= e^2 \left(\hat{\Pi}'_{\gamma\gamma}(0) - \hat{\Pi}'_{\gamma\gamma}(M_Z^2)\right) \\
\Delta\alpha_2 &= g^2 \left(\hat{\Pi}'_{3\gamma}(0) - \hat{\Pi}'_{3\gamma}(M_Z^2)\right)
\end{aligned}$$

in terms of vacuum matrix elements of the currents, we may write

Gauge boson self-energy contributions to various correction terms:

$$\Delta\rho = \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\frac{s_W}{c_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}$$

$$\Delta\hat{\rho} = \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_W(M_W^2)}{M_W^2} + \frac{s_W}{c_W}\frac{\Pi_{\gamma Z}(M_Z^2)+\Pi_{\gamma Z}(0)}{M_Z^2} = \Delta\rho - \frac{s_W^2}{c_W^2}\Delta_1 + \Delta_2$$

$$\Delta\bar{\rho} = \Delta\rho + \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_Z(0)}{M_Z^2} - \left(\frac{d\Pi_Z}{dq^2}\right)(M_Z^2) = \Delta\rho + \Delta_Z$$

$$\Delta_W = \frac{\Pi_W(M_W^2)}{M_W^2} - \frac{\Pi_W(0)}{M_W^2} - \left(\frac{d\Pi_W}{dq^2}\right)(M_W^2)$$

$$\Delta\kappa = \frac{c_W^2}{s_W^2}\Delta\hat{\rho} = \frac{c_W^2}{s_W^2}\Delta\rho - \Delta_1 + \frac{c_W^2}{s_W^2}\Delta_2$$

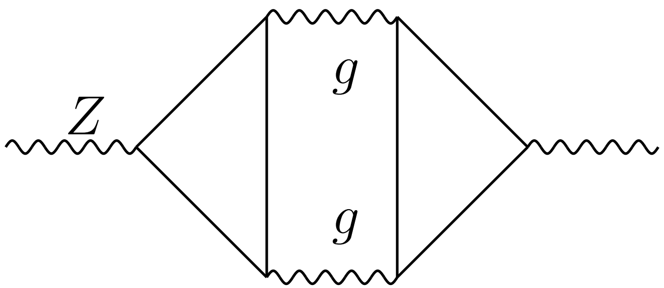
$$\Delta e = \Pi'_\gamma(0) - \Pi'_W(M_W^2) + \frac{c_W}{s_W}\Pi'_{\gamma Z}(M_Z^2) = \Delta\alpha + \Delta_1 + \Delta_2$$

$$\bar{\Delta} = \frac{c_W}{s_W}\left\{\frac{\Pi_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{\Pi_{\gamma Z}(0)}{M_Z^2} - \left(\frac{d\Pi_{\gamma Z}}{dq^2}\right)(0)\right\} = \Delta\alpha - \Delta\alpha_2$$

Using this notation we have, for example,

$$\begin{aligned} \Delta r &= \Delta e - \Delta\kappa = \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + 2\Delta_1 + \left(1 - \frac{c_W^2}{s_W^2}\right)\Delta_2 \\ \Delta\bar{r} &= \Delta e - \Delta\hat{\rho} = \Delta\alpha - \Delta\rho + \frac{1}{c_W^2}\Delta_1 \;. \end{aligned}$$

Other QCD contribution: singlet contribution to the weak axial current (Kniehl, Kühn 89)



Besides $\alpha(s)$ also the other gauge coupling $\alpha_2(s)$ cannot be calculated in pQCD, also here no direct evaluation in terms of experimental data possible. Way out: isospin and flavor separation (FJ 1968) The shift $\Delta\alpha_2$ in the $SU(2)_L$ coupling $\alpha_2 = \frac{g^2}{4\pi}$ is analogous to $\Delta\alpha$

$$\Delta\alpha_2 \;=\; \Pi'_{3\gamma}(0) - \Pi'_{3\gamma}(M_Z^2)$$

$$= \quad \frac{\alpha_2}{12\pi}\Sigma_l|Q_l|(\ln\frac{M_Z^2}{m_l^2}-\frac{5}{3}) + \Delta\alpha^{(5)}_{2,\mathrm{hadrons}}$$

where the sum extends over the light leptons and

$$\begin{aligned}\Delta\alpha^{(5)}_{2,\mathrm{hadrons}}(s) \quad &= \quad 0.0567 \pm 0.0006 \\ &\quad +0.006184 \cdot \{\ln(s_0/s) + 0.005513 \cdot (s/s_0 - 1)\}\end{aligned}$$

is the hadronic contribution of the 5 known light quarks u,d,s,c,b ($\sqrt{s_0} = 91.19$ GeV).

Appears in the relation

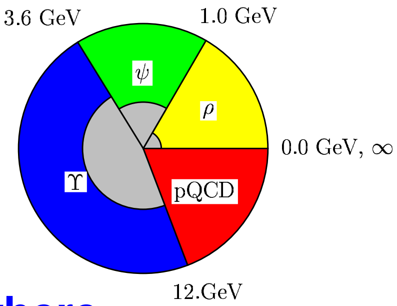
$$\sin^2\Theta_e = \left\{\frac{1-\Delta\alpha_2}{1-\Delta\alpha} + \Delta_{\nu_\mu e, vertex+box} + \Delta\kappa_{e,vertex}\right\}\sin^2\Theta_{\nu_\mu e}$$

LEP $\sin^2\theta^{\mathrm{lep}}_{\mathrm{eff}}$ versus $\sin^2\Theta_{\nu_\mu e}$ of neutrino scattering

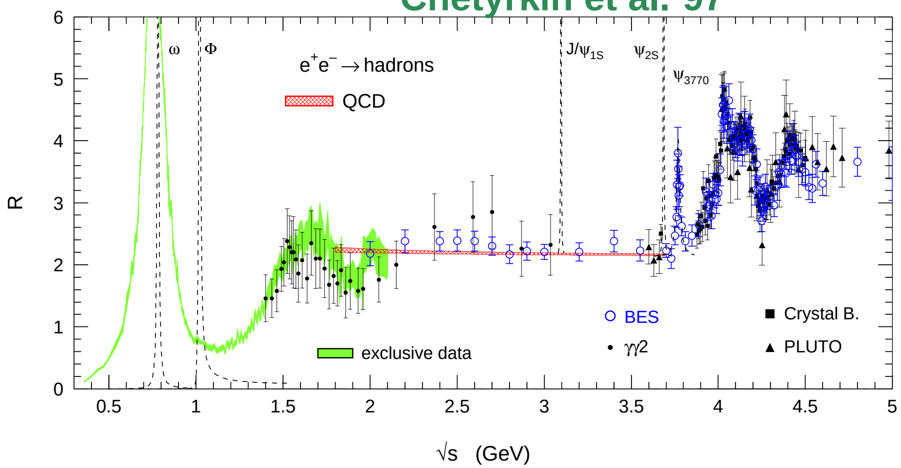
③ Evaluation of $\alpha(M_Z)$

Non-perturbative hadronic contributions $\Delta\alpha^{(5)}_{\rm had}(s)$ can be evaluated in terms of $\sigma(e^+e^- \to \mathrm{hadrons})$ data via dispersion integral:

$$\Delta\alpha^{(5)}_{\rm had}(s) = -\frac{\alpha s}{3\pi}\left(\ \mathrm{P}\!\!\int_{4m_\pi^2}^{E^2_{\rm cut}} ds' \frac{R^{\rm data}_\gamma(s')}{s'(s'-s)} + \mathrm{P}\!\!\int_{E^2_{\rm cut}}^{\infty} ds' \frac{R^{\rm pQCD}_\gamma(s')}{s'(s'-s)}\right)$$

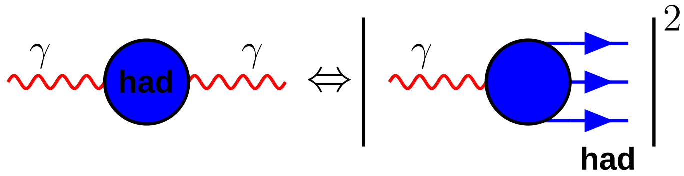


Compilation: Davier, Eidelman et al. 02
Theory = pQCD: Gorishny et al. 91, Chetyrkin et al. 97

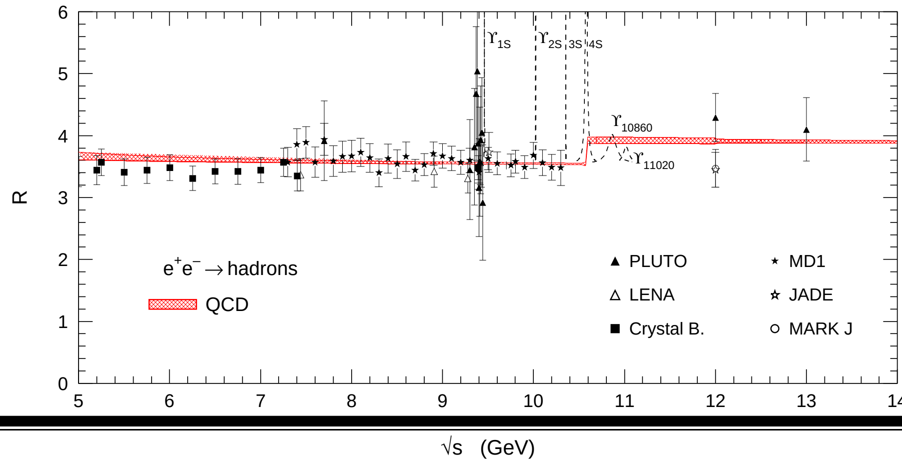


where

$$R_\gamma(s) \equiv \frac{\sigma^{(0)}(e^+e^-\to\gamma^*\to\mathrm{hadrons})}{\frac{4\pi\alpha^2}{3s}}$$



$\Pi'_\gamma(q^2)$ $\sim \sigma^{\rm had}_{\rm tot}(q^2)$



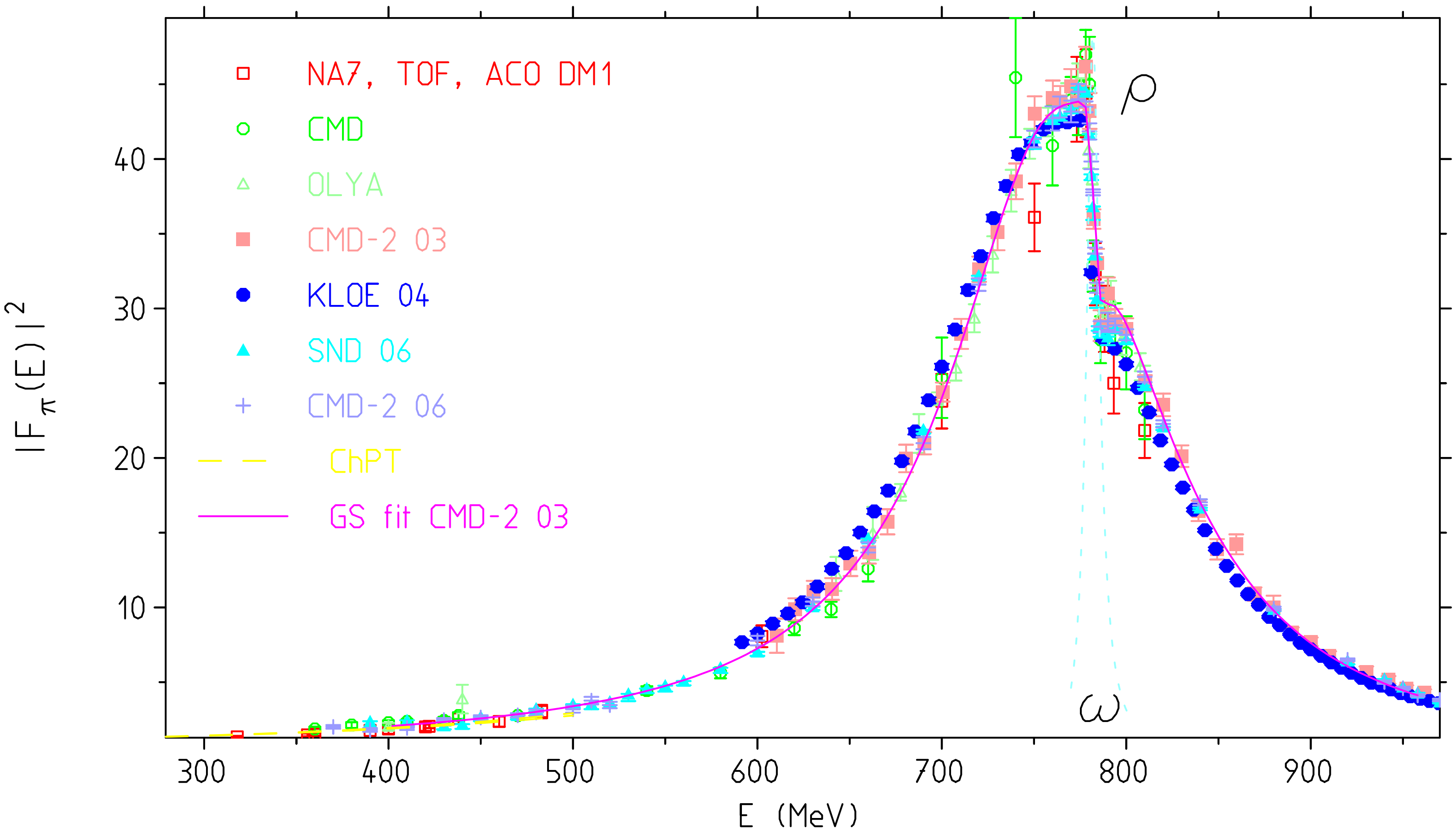
Evaluation FJ 2006 update: at $M_Z = 91.19$ GeV

- $R(s)$ data up to $\sqrt{s} = E_{cut} = 5$ GeV
  and for Υ resonances region between 9.6 and 13 GeV

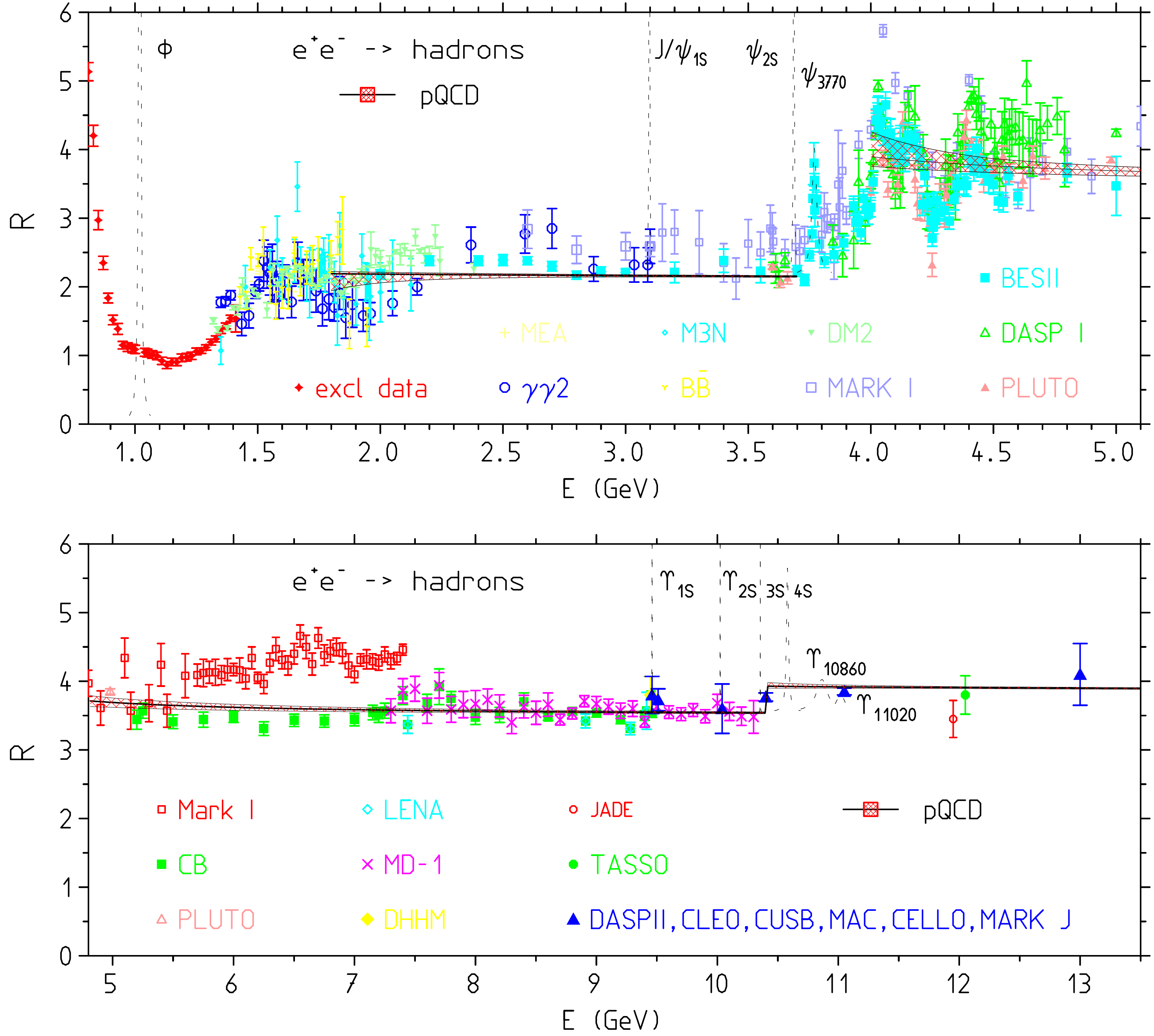
- perturbative QCD from 5.0 to 9.6 GeV
  and for the high energy tail above 13 GeV

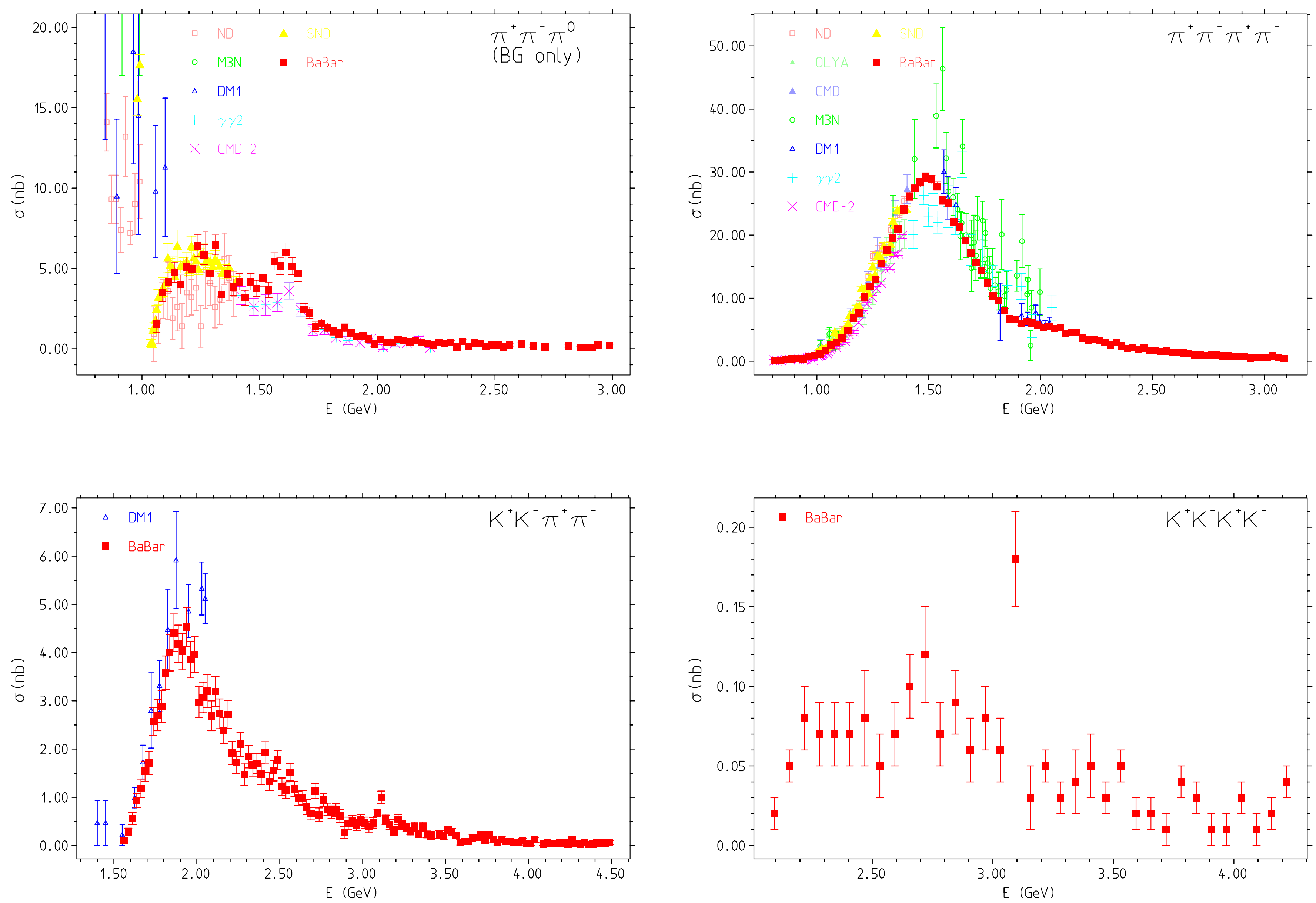
$$
\begin{array}{rclr}
\Delta\alpha^{(5)}_{\rm hadrons}(M_Z^2) & = & 0.027605 \pm 0.000275 & \\
 & & 0.027562 \pm 0.000218 & \text{Adler} \\
\alpha^{-1}(M_Z^2) & = & 128.948 \pm 0.038 & \\
 & & 128.937 \pm 0.024 & \text{Adler}
\end{array}
$$

A look at the e^+e^-–data

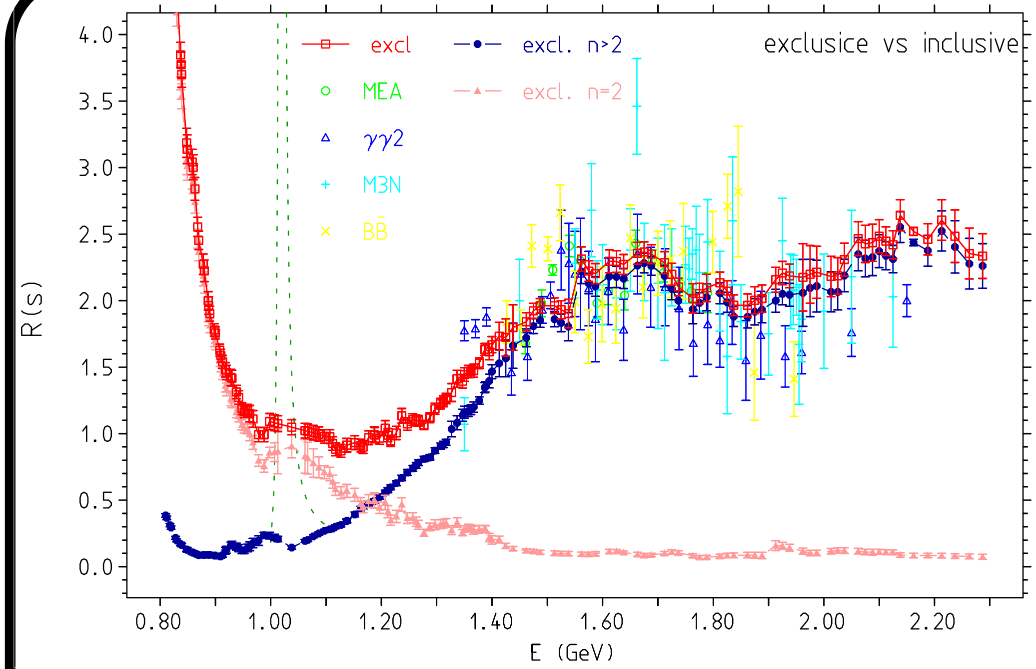


Below 1 GeV deviations between data sets much larger than errors claimed by experiments! CMD-2 vs. KLOE vs. SND somewhat confusing; will be settled by ongoing experiments


R
E (GeV)
e+e- -> hadrons
pQCD
Φ
J/ψ1S
ψ2S
ψ3770
BESII
MEA
M3N
DM2
DASP I
excl data
γγ2
BB̄
MARK I
PLUTO
e+e- -> hadrons
ϒ1S
ϒ2S 3S 4S
ϒ10860
ϒ11020
Mark I
LENA
JADE
pQCD
CB
MD-1
TASSO
PLUTO
DHHM
DASPII, CLEO, CUSB, MAC, CELLO, MARK J



BaBar radiative return measurements


excl
excl. n>2
MEA
excl. n=2
γγ2
M3N
BB
exclusice vs inclusive
R(s)
E (GeV)

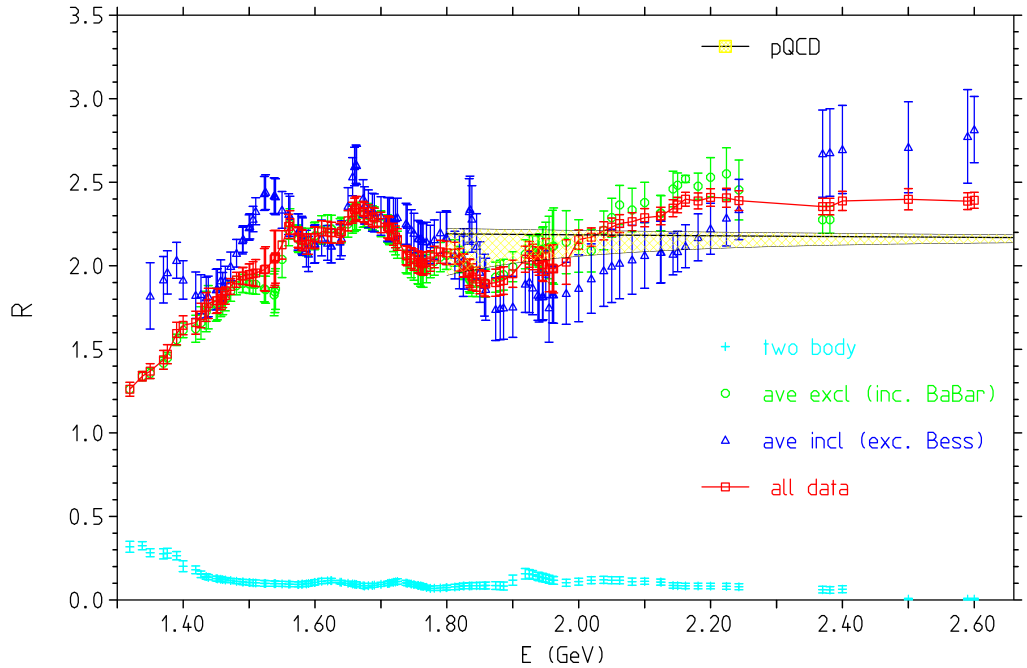
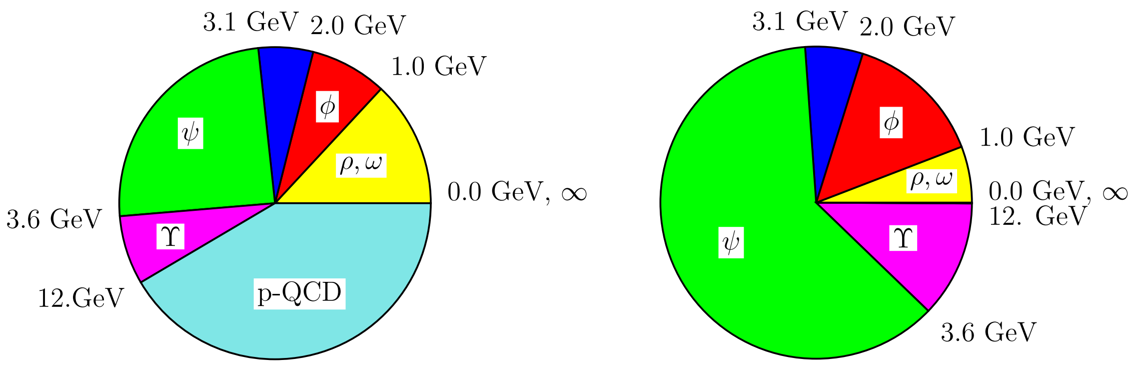

pQCD
two body
ave excl (inc. BaBar)
ave incl (exc. Bess)
all data
R
E (GeV)

Table 1: Contributions and uncertainties $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)^{\rm data} \cdot 10^4$. Direct integration method. In red the results relevant for VEPP-2000/DAFNE-II.

	$\Delta\alpha^{(5)}_{\rm had} \times 10^4$	rel. err.	abs. err.
ρ, ω ($E < 2M_K$)	36.43 [13.2](0.95)	2.6 %	14.3 %
$2M_K < E < 2$ GeV	21.80 [7.9](1.54)	7.1 %	37.6 %
2 GeV $< E < M_{J/\psi}$	15.73 [5.7](0.88)	5.6 %	12.2 %
$M_{J/\psi} < E < M_\Upsilon$	66.95 [24.2](0.85)	1.3 %	11.5 %
$M_\Upsilon < E < E_{\rm cut}$	19.69 [7.1](1.24)	6.3 %	24.1 %
$E_{\rm cut} < E$ pQCD	115.66 [41.9](0.11)	0.1 %	0.2 %
$E < E_{\rm cut}$ data	160.60 [58.1](2.52)	1.6 %	99.8 %
total	276.26 [100.0](2.52)	0.9 %	100.0 %

Table 2: Contributions and uncertainties $\Delta\alpha^{(5)}_{\rm had}(-M_0^2)^{\rm data} \cdot 10^4$ ($M_0 = 2.5$ GeV). Adler function method. In red the results relevant for VEPP-2000/DAFNE.

	$\Delta\alpha^{(5)}_{\rm had} \times 10^4$	rel. err.	abs. err.
ρ, ω ($E < 2M_K$)	33.46 [45.3](0.94)	2.8 %	37.2 %
$2M_K < E < 2$ GeV	16.45 [22.3](1.10)	6.7 %	51.3 %
2 GeV $< E < M_{J/\psi}$	7.91 [10.7](0.44)	5.6 %	8.3 %
$M_{J/\psi} < E < M_\Upsilon$	13.95 [18.9](0.27)	2.0 %	3.1 %
$M_\Upsilon < E < E_{\rm cut}$	0.96 [1.3](0.06)	6.2 %	0.2 %
$E_{\rm cut} < E$ pQCD	1.09 [1.5](0.00)	0.1 %	0.0 %
$E < E_{\rm cut}$ data	72.73 [98.5](1.54)	2.1 %	100.0 %
total	73.82 [100.0](1.54)	2.1 %	100.0 %



contributions

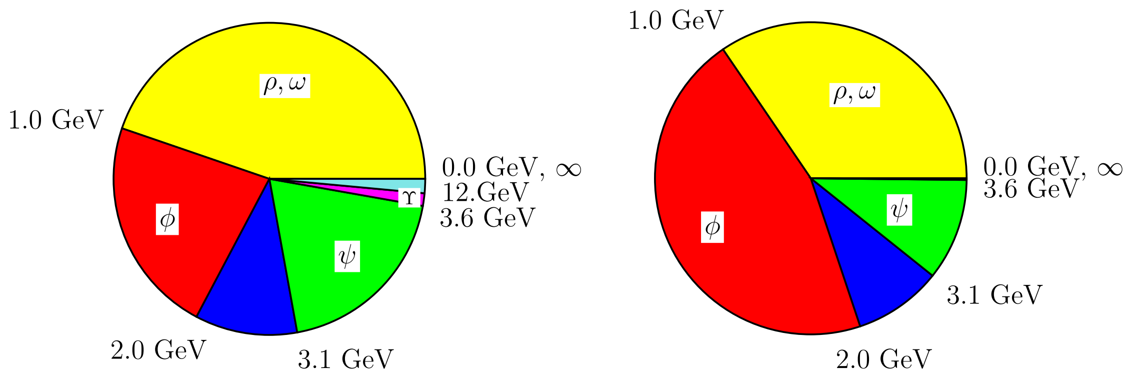
error^2

direct integration of data

$$\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2)$$

integration via Adler function

$\Delta\alpha^{(5)}_{\mathrm{had}}(-M_0^2)^{\mathrm{data}}$ **(**$M_0 = 2.5$ **GeV)**



contributions

error^2

present distribution of contributions and errors

$\Delta\alpha^{\rm had}$ via the Adler function

✗ **use old idea: Adler function: Monitor for comparing theory and data**

$$D(-s) \doteq \frac{3\pi}{\alpha}\, s\, \frac{d}{ds}\Delta\alpha_{\rm had}(s) = -\left(12\pi^2\right)\; s\, \frac{d\Pi'_\gamma(s)}{ds}$$

$$\Rightarrow \quad D(Q^2) = Q^2 \int\limits_{4m_\pi^2}^{\infty} ds\, \frac{R(s)}{(s+Q^2)^2}$$

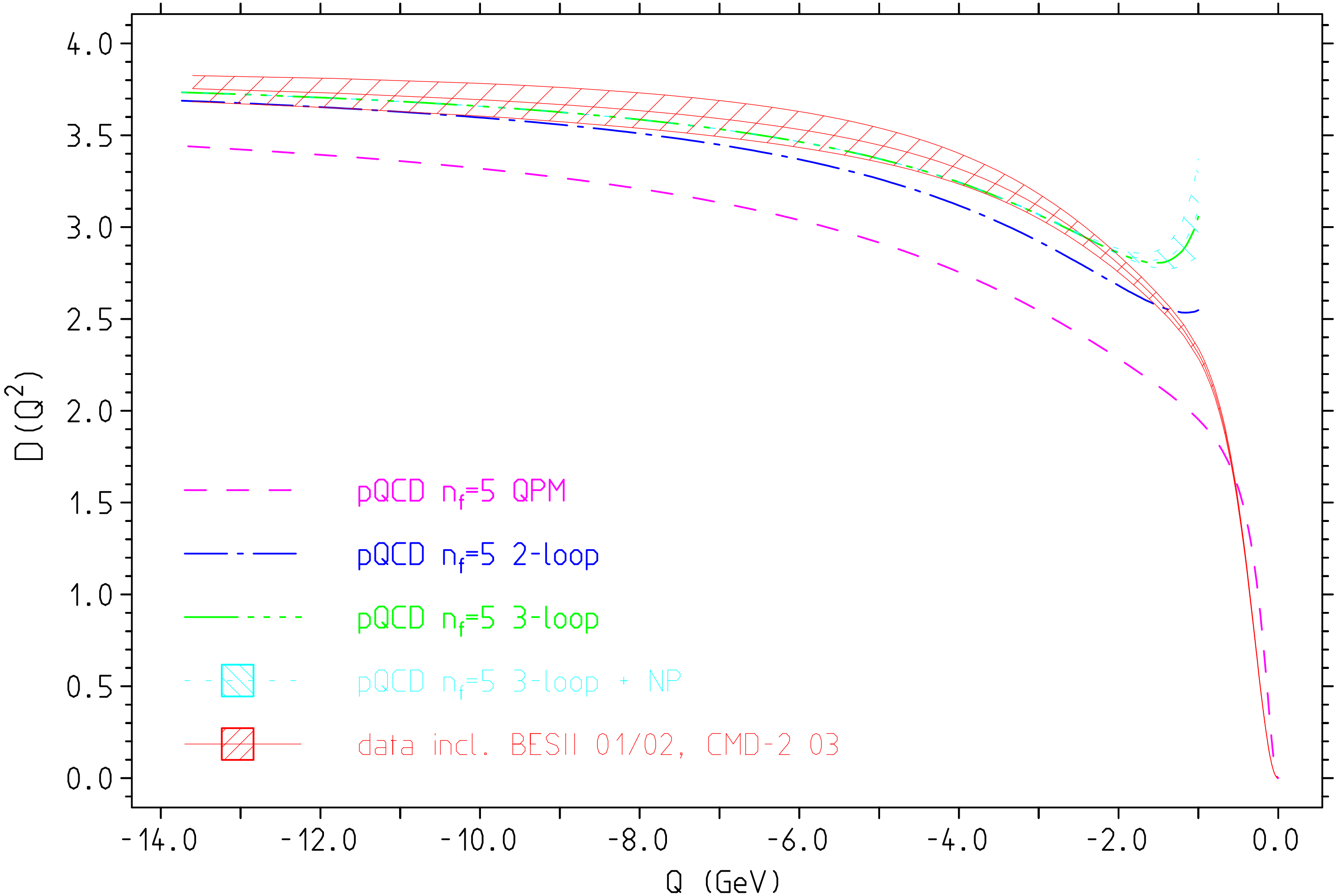
pQCD $\leftrightarrow R(s)$	**pQCD** $\leftrightarrow D(Q^2)$
very difficult to obtain in theory $R^{\rm pQCD} = R^{\rm quarks} \neq R^{\rm hadrons} = R^{\rm true}$	**smooth simple function** **in Euclidean region**

Conservative conclusion:

- **time-like approach: pQCD works well in "perturbative windows"**
  3.00 - 3.73 GeV, 5.00 - 10.52 GeV and 11.50 - ∞
  (Kühn,Steinhauser)
- **space-like approach: pQCD works well for** $\sqrt{Q^2=-q^2} > 2.5$ **GeV (see plot)**

"Experimental" Adler–function versus theory (pQCD + NP)

Error includes statistical + systematic here (in contrast to most R-plots showing statistical errors only)!



(Eidelman, FJ, Kataev, Veretin 98, FJ 05 update (BES, CMD-2)) theory based on results by Chetyrkin, Kühn et al.

$\Rightarrow$ **pQCD works well to predict $D(Q^2)$ down to $s_0 = (2.5\,\text{GeV})^2$; use this to calculate**

$$\Delta\alpha_{\rm had}(-Q^2) \sim \frac{\alpha}{3\pi}\int dQ'^2\,\frac{D(Q'^2)}{Q'^2}$$

$$\boxed{\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) \;=\; & \left[\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-s_0)\right]^{\rm pQCD} \\ & + \Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data}\end{aligned}}$$

and obtain, for $s_0 = (2.5\,\text{GeV})^2$: **(FJ 98/05)**

$$\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data} &= 0.007366 \pm 0.000159 \\ \Delta\alpha^{(5)}_{\rm had}(-M_Z^2) &= 0.027562 \pm 0.000159 \pm 0.000149\end{aligned}$$

parameter	range	pQCD uncertainty	total error
α_s	0.117 ... 0.123	0.000051	0.000155
m_c	1.550 ... 1.750	0.000087	0.000170
m_b	4.600 ... 4.800	0.000011	0.000146

Future: ILC requirement: improve by factor 10 in accuracy

- **direct integration of data: 58% from data 42% p-QCD**

  $\Delta\alpha^{(5)\,\mathrm{data}}_{\mathrm{had}} \times 10^4 = 162.72 \pm 4.13$ **(2.5%)**

  1% overall accuracy ± 1.63

  1% accuracy for each region (divided up as in table)

  added in quadrature: ± 0.85

  Data: [4.13] vs. [0.85] $\Rightarrow$ **improvement factor 4.8**

  $\Delta\alpha^{(5)\,\mathrm{pQCD}}_{\mathrm{had}} \times 10^4 = 115.57 \pm 0.12$ **(0.1%)**

  Theory: no improvement needed !

- **integration via Adler function: 26% from data 74% p-QCD**

  $\Delta\alpha^{(5)\,\mathrm{data}}_{\mathrm{had}} \times 10^4 = 073.61 \pm 1.68$ **(2.3%)**

  1% overall accuracy ± 0.74

  1% accuracy for each region (divided up as in table)

added in quadrature: ± 0.41

Data: [2.25] vs. [0.46] $\Rightarrow$ **improvement factor 4.9 (Adler vs Adler)**

[4.13] vs. [0.46] $\Rightarrow$ **improvement factor 9.0 (Standard vs Adler)**

$$\Delta\alpha_{\rm had}^{(5)\ \rm pQCD} \times 10^4 = 204.68 \pm 1.49$$

Theory: (QCD parameters) has to improve by factor 10 ! $\longrightarrow \pm 0.20$

Requirement may be realistic:

- **pin down experimental errors to 1% level in all non-perturbative regions up to 10 GeV**
- **switch to Adler function method**
- **improve on QCD parameters** α_s, m_c **and** m_b **[pQCD/lattice]**

Conclusion and Strategy

- **Recent and future high precision experiments on $a_\mu = (g-2)/2$ (BNL/KEK project may gain factor 10?) and $\sin^2\Theta_{\rm eff}$, etc. (LEP/SLD$\longrightarrow$TESLA/ILC) imposed and further impose a lot of pressure to theory and experiment to improve, in particular, in reducing the hadronic uncertainties which mainly are due to the experimental errors of $R(s)^{\rm exp}_{\rm had}$.**

- **In electroweak precision physics at non-zero energies (note $E \sim m_\mu$ in $(g-2)_\mu$) there is now way around determining $\alpha_{\rm eff}(E)$ via precision measurements of $\sigma_{\rm hadronic}$ or lattice QCD simulations via Adler function approach (which is a very challenging long term project).**

- **Needs for linear collider (like ILC): requires $\sigma_{\rm had}$ at 1% level up to the $\Upsilon \Rightarrow \delta\alpha(M_Z)/\alpha(M_Z) \sim 5 \times 10^{-5}$. New cross section measurements at VEPP-2000, DAFNE-II, radiative return measurements at DAFNE-I, BABAR and Belle, and at CLEOc and BESS-III are able to reduce the hadronic uncertainties to ???? for a_μ and to ???? for $\alpha(M_Z)$. Together with improved measurements of the top mass m_t from LHC, at present this would allow to get much better Higgs boson mass limits but** much more than that a new quality of sensitivity to new physics.

- Future precision physics requires dedicated effort on $\sigma_{\rm had}$ experimentally as well as theoretically (radiative corrections, final state radiation from hadrons etc.)

- Improving hadronic cross section measurements must be seen as a global effort in particular in the context of ILC project, which primarily makes sense as a high precision physics project. The $\sigma_{\rm hadronic}$ efforts have to be pushed at any machine able to perform such a measurement up tp 10 GeV! One has to see this activity as an integral part of the international linear collider (ILC) project and to ask for support by the international community. However, a more precise $\alpha_{\rm eff}$, besides in high precision VB physics, will be needed at many other places: $\alpha_{\rm eff}(m_\mu)$, $\alpha_{\rm eff}(m_p)$, Bhabha scattering,...

- Projects VEPP-2000 and DAFNE–II can play a major role in this respect. What is required is a scan measurement with a good energy calibration (preferable using resonance depolarization). In radiative return at higher energies and multiplicities one has to precisely reconstruct the invariant mass event by event which I think is a difficulty. Dedicated Monte Carlo simulations has to be done to study what precision in which scenario can be achieved.

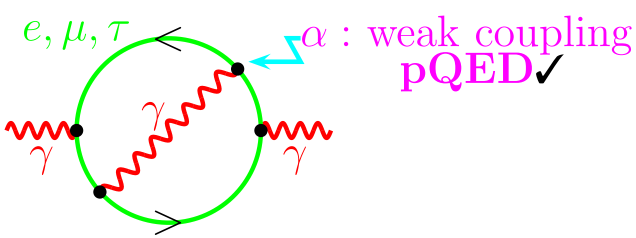
- **Don't believe people claiming very small errors and that everything has been solved already or that some other lab is already doing the same; in high precision physics any experiment becomes a real challenge and I think at least two experiments should be performed for cross check.**
- **Note: complementary approach important! direct $R(s)$ integration vs. Adler $D(Q^2)$; in particular for the latter as well as for $(g-2)_\mu$ projects like VEPP 2000 and DAFNE–II are a real need!**

Part II: **Hadronic Contributions in g-2**

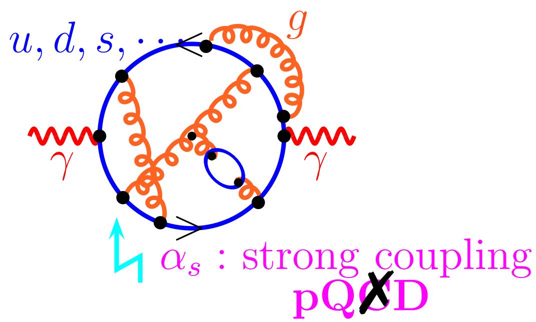
General problem in electroweak precision physics:

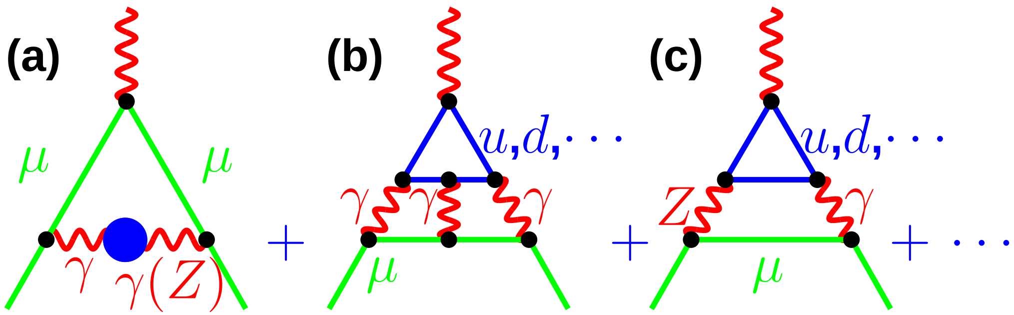
contributions from hadrons (quark loops) at low energy scales

Leptons



Quarks



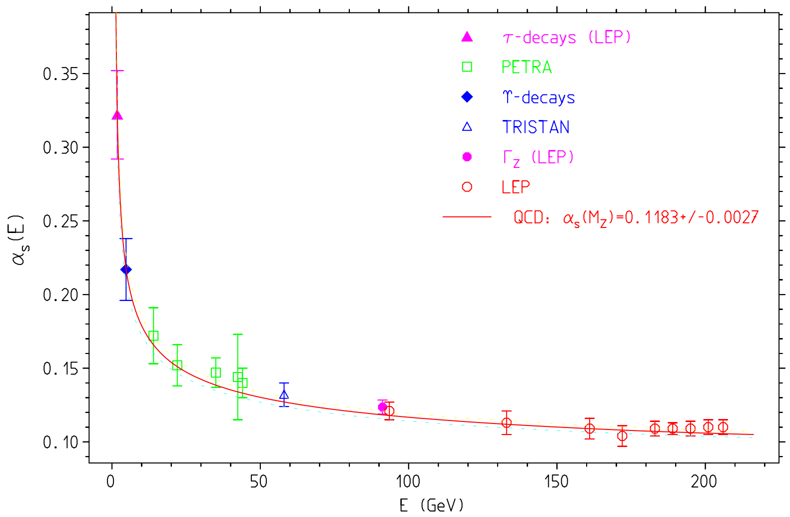


(a) **Hadronic vacuum polarization** $O(\alpha^2), O(\alpha^3)$

(b) **Hadronic light-by-light scattering** $O(\alpha^3)$

(c) **Hadronic effects in 2-loop EWRC** $O(\alpha G_F m_\mu^2)$

Light quark loops $\longrightarrow$ **Hadronic "blob"**



Compilation of α_s measurements (Bethke 04).

$\alpha_s(M_\tau) = 0.322 \pm 0.030$ **at** $M_\tau = 1.78$ **GeV**

pQCD not only fails due to strong coupling (non-convergence of expansion) but because of spontaneous chiral symmetry breaking (100% non-perturbative) responsible for the existence of pions and quark condensates, which are missing to all orders in pQCD, i.e. pQCD fails to correctly describe the low energy structure of QCD

Must use non-perturbative methods:

- **Dispersion relations, sum rules**
- **Chiral perturbation theory extended to include spin 1 vector states**
- **QCD inspired models: extended Nambu Jona-Lasinio model (ENJL), hidden local symmetry (HLS) model**
- **large N_c QCD approach (dual to infinite series of narrow vector resonances)**
- **lattice QCD in future**

④ Hadronic vacuum polarization effects in $g-2$

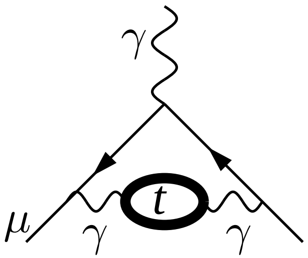
Role of the two point correlator:

- key object $< 0|j^{\mu\,\mathrm{had}}_{\mathrm{em}}(x)\; j^{\mu\,\mathrm{had}}_{\mathrm{em}}(0)|0 >$
- hadronic electromagnetic current

$$j^{\mu\,\mathrm{had}}_{\mathrm{em}} = \sum_c \left(\frac{2}{3}\bar{u}_c\gamma^\mu u_c - \frac{1}{3}\bar{d}_c\gamma^\mu d_c - \frac{1}{3}\bar{s}_c\gamma^\mu s_c + \frac{2}{3}\bar{c}_c\gamma^\mu c_c - \frac{1}{3}\bar{b}_c\gamma^\mu b_c + \frac{2}{3}\bar{t}_c\gamma^\mu t_c \right) \;,$$

- hadronic part on photon self-energy $\Pi'^{\;\mathrm{had}}_{\gamma}(s) \Leftrightarrow < 0|j^{\mu\,\mathrm{had}}_{\mathrm{em}}(x)\; j^{\mu\,\mathrm{had}}_{\mathrm{em}}(0)|0 >$
- hadronic vacuum polarization due to the 5 "light" quarks $q = u, d, s, c, b$

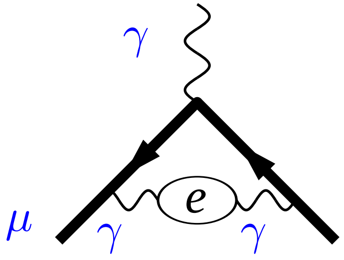
- top quark [mass $m_t \simeq 178$ GeV] pQCD applies [$\alpha_s(m_t)$ small]
- in fact t is irrelevant by decoupling theorem [heavy particles decouple in QED/QCD]
- t like τ VP loop extra factor $N_c Q_t^2 = 4/3$:



$$a_\mu^{(4)}(\mathrm{vap}, top) = \frac{4}{3}\left[\frac{1}{45}\left(\frac{m_\mu}{m_t}\right)^2 + \cdots\right]\left(\frac{\alpha}{\pi}\right)^2 \sim 6.3\cdot 10^{-16}\;.$$

Compare with leptons: different regimes for the mass dependent effects:

− **LIGHT internal masses** $\Rightarrow$ **large logarithms [of mass ratios] singular in the limit** $m_{\rm light} \to 0$



$$a_\mu^{(4)}({\rm vap},e) = \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + O\left(\frac{m_e}{m_\mu}\right)\right]\,\left(\frac{\alpha}{\pi}\right)^2$$

note large log $\ln\frac{m_\mu}{m_e} \simeq 6.285$

exact two–loop result [errors due to uncertainty in mass ratio (m_e/m_μ)**]**

$$a_\mu^{(4)}({\rm vap},e) \simeq 1.094\,258\,3111(84)\,\left(\frac{\alpha}{\pi}\right)^2 = 5.90406007(5)\times 10^{-6}\ .$$

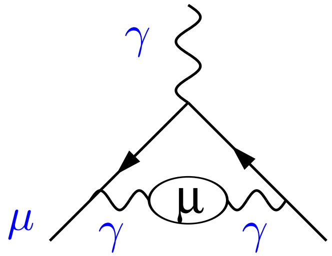
LL UV log; m_μ **serves as UV cut–off, electron mass as IR cut–off, relevant integral**

$$\int_{m_e}^{m_\mu}\frac{{\rm d}E}{E} = \ln\frac{m_\mu}{m_e}$$

may be obtained by renormalization group replace in one–loop result $\alpha \to \alpha(m_\mu)$

$$a_\mu = \frac{1}{2}\frac{\alpha}{\pi}\,(1+\frac{2}{3}\frac{\alpha}{\pi}\,\ln\frac{m_\mu}{m_e})$$

— **EQUAL internal masses yields pure number**

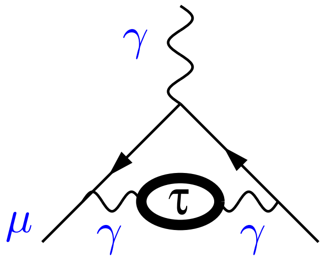


$$a_\mu^{(4)}(\mathrm{vap},\mu) = \left[\frac{119}{36} - \frac{\pi^2}{3}\right] \left(\frac{\alpha}{\pi}\right)^2$$

large cancellation between rational [$3.3055\ldots$] and transcendental π^2 term [$3.2899\ldots$], result 0.5% of individual terms:

$$a_\mu^{(4)}(\mathrm{vap},\mu) \simeq 0.015\,687\,4219 \left(\frac{\alpha}{\pi}\right)^2 = 8.464\,1332 \times 10^{-8} \ .$$

— **HEAVY internal masses decouple in the limit $m_{\mathrm{heavy}} \to \infty$, small power correction**



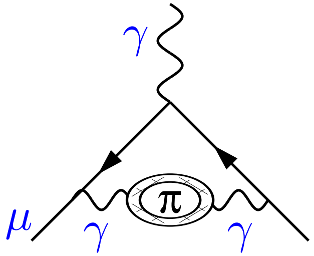
$$a_\mu^{(4)}(\mathrm{vap},\tau) = \left[\frac{1}{45}\left(\frac{m_\mu}{m_\tau}\right)^2 + O\left(\frac{m_\mu^4}{m_\tau^4}\ln\frac{m_\tau}{m_\mu}\right)\right] \left(\frac{\alpha}{\pi}\right)^2$$

Note "heavy physics" contributions, from mass scales $M \gg m_\mu$, typically are proportional to m_μ^2/M^2. This means that besides the order in α there is an extra suppression factor, e.g. $O(\alpha^2) \to Q(\alpha^2 \frac{m_\mu^2}{M^2})$ in our case. To unveil new heavy states thus requires a corresponding high precision in theory and

experiment. τ contribution tiny

$$a_\mu^{(4)}(\mathrm{vap},\tau) \simeq 0.000\,078\,064(25)\,\left(\frac{\alpha}{\pi}\right)^2 = 4.211\,935\,34(87)\times 10^{-10}\,,$$

Vacuum polarization by hadrons: CHPT equivalently scalar QED



$$a_\mu^{(4)}(\mathrm{vap},\pi^\pm) \simeq 1.4154\cdot 10^{-8}$$

charged spin 0 pions $\pi^\pm$ assumed to couple to photons via minimal coupling [good for soft photons only], point-like pion approximation; relevant parameters $N_{ci}Q_i^2$ and mass m_π

underestimating the effect dramatically

(pion of mass m_μ: $2.2913[5.8420\,\mathrm{muon}\,]\cdot 10^{-8}$)

compare quark parton model treat quarks like leptons, strong interactions switched off $\Rightarrow$ sum u and d quark $4.4925[5.9041\,\mathrm{electron}\,]\cdot 10^{-6}$ note large difference between the $\pi^\pm$ result and the (u,d) doublet result

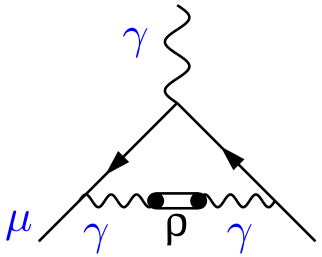
illustrates the dilemma with naive perturbative approaches

huge contribution on the quark level using current quark masses $m_u \sim 3$**MeV**, $m_d \sim 8$**MeV** which appear in the QCD Lagrangian resorting to effective "constituent quark masses" [concept not very well–defined] e.g.

$m_u \sim m_d \sim 300$ MeV (about 1/3 of the proton mass) one gets $2.2511 \cdot 10^{-8}$

$m_u \sim m_d \sim m_\pi$ would yield $8.8817 \cdot 10^{-8}$ (another non-sense result)

All these fail: missing the pronounced ρ^0 spin 1 resonance $e^+e^- \to \rho^0 \to \pi^+\pi^-$



$$a_\mu^{(4)}(\mathrm{vap}, \rho^0) \simeq 4.2099 \cdot 10^{-8}$$

integrating a non-relativistic Breit-Wigner $m_\rho \sim 775$ MeV, $\Gamma_\rho \sim 145$ MeV resonance in the range (280,810) MeV $\Rightarrow$ remarkably good result

directly integrating the experimental data in the same range around the ρ (cross section enhanced about a factor 50): $a_\mu^{(4)\;exp}(\mathrm{vap}, [280, 810]) \simeq 4.2666 \cdot 10^{-8}$

Lesson:

- pQCD fails; QPM result arbitrary (quark masses)
- ChPT (only knows pions) fails; reason only converge for $p \lesssim$ 400 MeV
- dominating is spin 1 resonance ρ^0 at $\simeq 775$ MeV (VDM); cries for large N_c QCD
- lattice QCD may solve the problem once one can simulate at physical quark masses [future]
- resort on sum rule type semi-phenomenological approach Dispersion Relations (DR) and experimental data.

Basic analyticity and unitarity yield first principle relation for evaluating photon vacuum polarization reliably via DRs using experimental data $e^+e^- \to$ hadrons:

$$a_\mu^{(4)}(\mathrm{vap},\mathrm{had}) = 694.70(1.97)(6.16)[6.46] \cdot 10^{-10}$$

where the errors are statistical, systematic and total.

Table 3: The various types of contributions to a_μ in units 10^{-6}, ordered according to their size (L.O. lowest order, H.O. higher order, LBL. light–by–light)

L.O. universal	$1161.409\,73$	(0)
e**–loops**	$6.194\,57$	(0)
H.O. universal	$-1.757\,55$	(0)
L.O. hadronic	$0.069\,46$	(81)
L.O. weak	$0.001\,95$	(0)
H.O. hadronic	$-0.001\,00$	(6)
LBL. hadronic	$0.000\,80$	(40)
τ**–loops**	$0.000\,43$	(0)
H.O. weak	$-0.000\,41$	(2)
e**+**τ**–loops**	$0.000\,01$	(0)
theory	$1165.917\,99$	(91)
experiment	$1165.920\,8$	(6)

Dispersion relations and VP insertions in g-2

Starting point:

- ***Optical Theorem*** **(unitarity) for the photon propagator**

$$\mathrm{Im}\Pi'_\gamma(s) = \frac{s}{4\pi\alpha}\,\sigma_{\rm tot}(e^+e^- \to \text{anything})$$

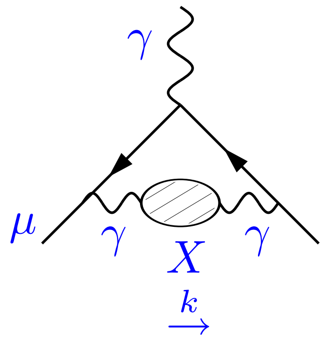
- ***Analyticity*** **(causality), may be expressed in form of a so–called (subtracted) dispersion relation**

$$\Pi'_\gamma(k^2) - \Pi'_\gamma(0) = \frac{k^2}{\pi}\int\limits_0^\infty \mathrm{d}s\,\frac{\mathrm{Im}\Pi'_\gamma(s)}{s\,(s-k^2-\mathrm{i}\varepsilon)} \quad .$$

– **based on general principles**

– **holds beyond perturbation theory**

Use of DRs in g-2 calculations, prototype example: diagram of the type



"blob" = full photon propagator $g^{\mu\nu}$ term of the full photon propagator, carrying loop momentum k, reads

$$\frac{-\mathrm{i}g^{\mu\nu}}{k^2\,(1+\Pi'_\gamma(k^2))} \simeq \frac{-\mathrm{i}g^{\mu\nu}}{k^2}\,\left(1-\Pi'_\gamma(k^2)+\left(\Pi'_\gamma(k^2)\right)^2-\cdots\right)$$

and the renormalized photon self–energy may be written as

$$-\frac{\Pi'_{\gamma\;\mathrm{ren}}(k^2)}{k^2} = \int\limits_0^\infty \frac{\mathrm{d}s}{s}\,\frac{1}{\pi}\,\mathrm{Im}\,\Pi'_\gamma(s)\,\frac{1}{k^2-s}\quad .$$

– k dependence under the convolution integral shows up in free propagator only

– free photon propagator in next higher order is replace by

$$-\mathrm{i}g_{\mu\nu}/k^2 \to -\mathrm{i}g_{\mu\nu}/(k^2-s)$$

= exchange of a photon of mass square s.

– afterwards convoluted with imaginary part of the photon vacuum polarization

In a first step we have to calculate the contributions from the massive photon which may be calculated exactly as in the massless case. This is possible up to the 3–loop QED contribution to a_μ. As discussed above $F_{\mathrm{M}}(0)$ most simply may be calculated using the projection method directly at $q^2=0$. The result is

$$K_\mu^{(2)}(s) \equiv a_\mu^{(2)\;\mathrm{heavy}\;\gamma} = \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,\frac{x^2\,(1-x)}{x^2+(s/m_\mu^2)(1-x)}$$

which is the second order contribution to a_μ from an exchange of a photon with square mass s. Note that for $s=0$ we get the known Schwinger result.

The contribution from the "blob" to $g-2$ then reads

$$a_\mu^{(X)} = \frac{1}{\pi}\int\limits_0^\infty \frac{\mathrm{d}s}{s}\,\text{Im}\,\Pi_\gamma^{'(X)}(s)\,K_\mu^{(2)}(s)\ \ .$$

If we exchange integrations and evaluating the DR we arrive at

$$\begin{aligned} a_\mu^{(X)} &= \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\int\limits_0^\infty \frac{\mathrm{d}s}{s}\,\frac{1}{\pi}\,\text{Im}\,\Pi_\gamma^{'(X)}(s)\,\frac{x^2}{x^2+(s/m_\mu^2)(1-x)} \\ &= \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\left[-\Pi_\gamma^{'(X)}(s_x)\right] \end{aligned}$$

where

$$s_x = -\frac{x^2}{1-x}\,m_\mu^2\ \ .$$

The last simple representation in terms of $\Pi_\gamma^{'(X)}(s_x)$ follows using

$$\frac{x^2}{x^2+(s/m_\mu^2)(1-x)} = -s_x\,\frac{1}{s-s_x}\ \ .$$

A one-fold integral representation of the VP function reads

$$\Pi'^{\ell}_{\gamma\,\mathrm{ren}}\left(\frac{-x^2}{1-x}m_\mu^2\right) = -\frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}z\, 2z\,(1-z)\,\ln\left(1+\frac{x^2}{1-x}\frac{m_\mu^2}{m_\ell^2}\,z\,(1-z)\right)\,,$$

which yields a two–fold integral representation of the VP contribution by lepton loops at two–loop order.

This kind of dispersion integral representation can be generalized to higher order and sequential VP insertions corresponding to the powers of $\Pi'(k^2)$. Denoting $\rho(s) =$ Im $\Pi'_{\gamma\,\mathrm{ren}}(s)/\pi$ we may write

$$\begin{aligned} a_\mu^{(X)} &= \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\left(\int\limits_0^\infty \frac{\mathrm{d}s}{s}\,\rho(s)\,\frac{-s_x}{s-s_x}\right)^n \\ &= \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\left(-\Pi'_{\gamma\,\mathrm{ren}}(s_x)\right)^n \end{aligned}$$

We are thus able to write formally the result for the one–loop muon vertex when we replace the free photon propagator by the full transverse propagator as

$$a_\mu^{(X)} = \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\left(\frac{1}{1+\Pi'_{\gamma\,\mathrm{ren}}(s_x)}\right)$$

$$= \frac{1}{\pi} \int_0^1 \mathrm{d}x \, (1-x) \, \alpha(s_x) \; ,$$

which is equivalent to the contribution of a free photon interacting with dressed charge (effective fine structure constant) Landau pole problem!.

Expand the 1PI photon self–energy into order by order contributions

$$\Pi'_{\gamma\,\mathrm{ren}}(k^2) = \Pi'^{(2)}_{\gamma\,\mathrm{ren}}(k^2) + \Pi'^{(4)}_{\gamma\,\mathrm{ren}}(k^2) + \cdots$$

and also write $\rho = \rho^{(2)} + \rho^{(4)} + \cdots$ for the spectral densities.

Other form of single VP insertion

$$\Pi'^{\,\ell}_{\gamma\,\mathrm{ren}}\left(q^2/m^2\right) = -\frac{\alpha}{\pi}\frac{q^2}{m^2} \int_0^1 \mathrm{d}t \, \frac{\rho_2(t)}{\frac{q^2}{m^2} - 4/(1-t^2)} ,$$

with

$$\rho_2(t) = \frac{t^2\,(1-t^2/3)}{1-t^2} \; ,$$

and we may write

$$a_\mu^{(X)} \quad = \quad \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \mathrm{d}x \, (1-x) \int_0^1 \mathrm{d}t \frac{\rho_2(t)}{W_t(x)}$$

where

$$W_t(x) = 1 + \frac{4m^2}{(1-t^2)\,m_\mu^2}\,\frac{1-x}{x^2}\quad .$$

If n equal loops are inserted one gets

$$a_\mu^{(X)} \quad = \quad \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\left(\frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}t\,\frac{\rho(t)}{W_t(x)}\right)^n$$

in terms of the spectral function $\rho(t)$=imaginary part of 1PI photon self–energy

- 1–loop $\rho_2(t)$
- 2–loop $\rho_4(t)$ (Källén Sabry 55)

$$\begin{aligned}\rho_4(t) \quad = \quad & \frac{2t}{3\,(1-t^2)}\Bigg\{\frac{(3-t^2)\,(1+t^2)}{2}\left(\mathrm{Li}_2(1)+\ln\frac{1+t}{2}\;\ln\frac{1+t}{1-t}\right.\\ & +2\,\left(\mathrm{Li}_2(\frac{1-t}{1+t})+\mathrm{Li}_2(\frac{1+t}{2})-\mathrm{Li}_2(\frac{1-t}{2})\right)-4\,\mathrm{Li}_2(t)+\mathrm{Li}_2(t^2)\Bigg)\\ & +\left(\frac{11}{16}(3-t^2)\,(1+t^2)+\frac{1}{4}t^4-\frac{3}{2}t\,(3-t^2)\right)\,\ln\frac{1+t}{1-t}\\ & +t\,(3-t^2)\,\left(3\ln\frac{1+t}{2}-2\ln t\right)+\frac{3}{8}t\,(5-3t^2)\Bigg\}\quad .\end{aligned}$$

- **3–loop (Hoang et al 95)**
- **4–loop (Broadhurst, Kataev, Tarasov 93) approximate form**

Basic for analytical, numerical and a large class of higher order calculations. Works for VP insertions in any higher order QED calculation by replacing photons by heavy ones.

Formalism is key tool to evaluate non-perturbative *VP type hadronic contributions* applying the optical theorem

$$\sigma_{\mathrm{had}}(s) = \frac{4\pi^2\alpha}{s}\frac{1}{\pi}\mathrm{Im}\,\Pi_\gamma^{\prime\,\mathrm{had}}(s)$$

together with experimental data. The basic integral representation is

$$a_\mu^{\mathrm{had}} = \frac{\alpha}{\pi}\int\limits_0^\infty \frac{\mathrm{d}s}{s}\int\limits_0^1 \mathrm{d}x\,\frac{x^2\,(1-x)}{x^2+(1-x)\,s/m_\mu^2}\,\frac{\alpha}{3\pi}\,R(s)\quad .$$

Integration in x yields standard representation [time-like]

$$a_\mu^{\mathrm{had}} = \frac{\alpha}{3\pi}\int\limits_0^\infty \frac{\mathrm{d}s}{s}\,K_\mu^{(2)}(s)\,R(s)$$

Interchange of the order of integrations yields

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} \int\limits_0^1 {\rm d}x\,(1-x)\,\Delta\alpha^{\rm had}\left(-Q^2(x)\right)$$

where $Q^2(x) \equiv \frac{x^2}{1-x} m_\mu^2$ is the space–like square momentum–transfer or

$$x = \frac{Q^2}{2m_\mu^2}\left(\sqrt{1+\frac{4m_\mu^2}{Q^2}}-1\right) \quad .$$

Alternatively, by writing $(1-x) = -\frac{1}{2}\frac{d}{{\rm d}x}(1-x)^2$ and performing a partial integration one finds

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} m_\mu^2 \int\limits_0^1 {\rm d}x\, x\,(2-x)\,\left(D(Q^2(x))/Q^2(x)\right)$$

where $D(Q^2)$ is the *Adler-function* defined as a derivative of the shift of the fine structure constant

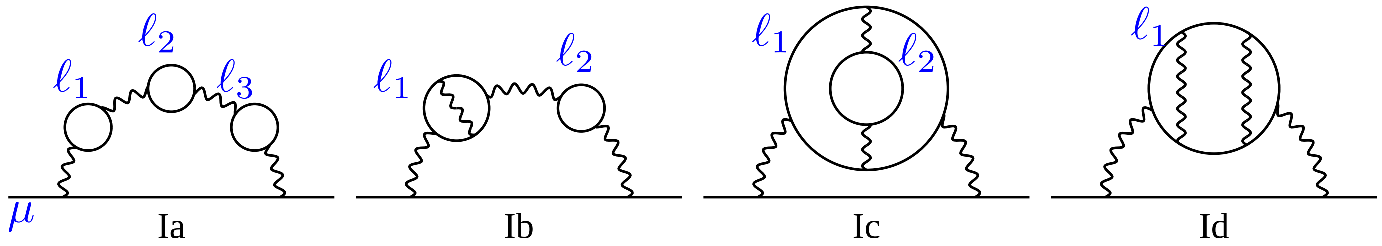
$$D(-s) = -(12\pi^2)\,s\,\frac{{\rm d}\Pi'_\gamma(s)}{{\rm d}s} = \frac{3\pi}{\alpha} s \frac{d}{{\rm d}s}\Delta\alpha_{\rm had}(s) \quad .$$

The Adler-function is represented by [space-like]

$$D(Q^2) = Q^2\left(\int_{4m_\pi^2}^\infty \frac{R(s)}{(s+Q^2)^2}{\rm d}s\right)$$

Application to four loop QED contribution [Kinoshita]:

Group I: **49 diagrams obtained from the 1–loop muon vertex by inserting 1–, 2– and 3–loop lepton VP sub-diagrams, i.e., the internal photon line is replaced by the full propagator at 3–loops. The group is subdivided into four gauge invariant subclasses I(a), I(b), I(c) and I(d).**



Typical diagrams of subgroups Ia (7 diagrams), Ib (18 diagrams), Ic (9 diagrams) and Id (15 diagrams). The lepton lines represent fermions propagating in an external magnetic field. ℓ_i denote VP insertions

Results for this group have been obtained by numerical and analytic methods

Subgroup Ia has a the integral representation

$$A^{(8)}_{2\,\mathrm{Ia}} = \int_0^1 \mathrm{d}x\,(1-x)\left(\int_0^1 \mathrm{d}t\,\frac{\rho_2(t)}{1+[4/(1-t^2)](1-x)/x^2}\right)^3$$

where $\rho_2(t)$ has been given above. Carrying out the t integral one obtains

$$
\begin{aligned}
A_{2\,\mathrm{Ia}}^{(8)} &= \int\limits_0^1 \mathrm{d}x\,(1-x)\left[-\frac{8}{9}+\frac{a^2}{3}+\left(\frac{a}{2}-\frac{a^3}{6}\right)\,\ln\frac{a+1}{a-1}\right]^3 \\
&\quad \frac{1}{3^3}\int\limits_0^1 \mathrm{d}x\,(1-x)\left[-\frac{5}{3}-\frac{4}{x}+\frac{4}{x^2}+\left(1-\frac{6}{x^2}+\frac{4}{x^3}\right)\,\ln(1-x)\right]^3
\end{aligned}
$$

with $a=2/(1-x)$. In this case also the last integration may be carried out analytically. Similarly, subgroup Ib has the representation

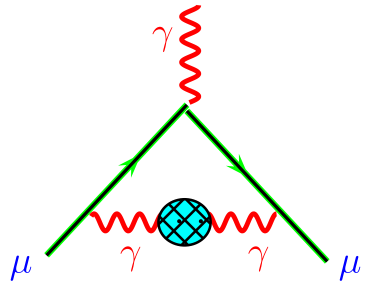
$$
\begin{aligned}
A_{2\,\mathrm{Ib}}^{(8)} &= 2\int\limits_0^1 \mathrm{d}x\,(1-x)\left(\int\limits_0^1 \mathrm{d}t_1\,\frac{\rho_2(t_1)}{1+[4/(1-t_1^2)](1-x)/x^2}\right) \\
&\qquad \times\left(\int\limits_0^1 \mathrm{d}t_2\,\frac{\rho_4(t_2)}{1+[4/(1-t_2^2)](1-x)/x^2}\right)
\end{aligned}
$$

with ρ_2 and ρ_4 given above.

⑤ Evaluation of $a_\mu^{\rm had}$

Leading non-perturbative hadronic contributions $a_\mu^{\rm had}$ can be obtained in terms of $R_\gamma(s) \equiv \sigma^{(0)}(e^+e^- \to \gamma^* \to \mathrm{hadrons})/\frac{4\pi\alpha^2}{3s}$ data via dispersion integral:

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left(\int\limits_{4m_\pi^2}^{E^2_{\rm cut}} ds\, \frac{R_\gamma^{\rm data}(s)\;\hat{K}(s)}{s^2} + \int\limits_{E^2_{\rm cut}}^{\infty} ds\, \frac{R_\gamma^{\rm pQCD}(s)\;\hat{K}(s)}{s^2} \right)$$

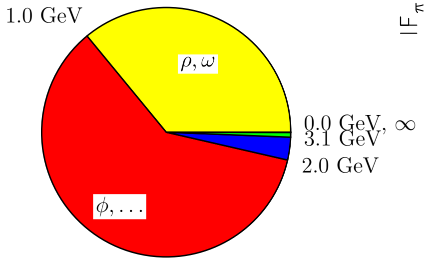


Data:
CMD-2, SND, KLOE

- **Experimental error implies theoretical uncertainty!**
- **Low energy contributions enhanced: $\sim 36\%$ of error on $a_\mu^{\rm had}$ comes from region $4m_\pi^2 < m_{\pi\pi}^2 < M_\Phi^2$, 60% from 1 to 2 GeV**

$$a_\mu^{\rm had(1)} = (694.7 \pm 6.5)\; 10^{-10}$$

e^+e^-–data based



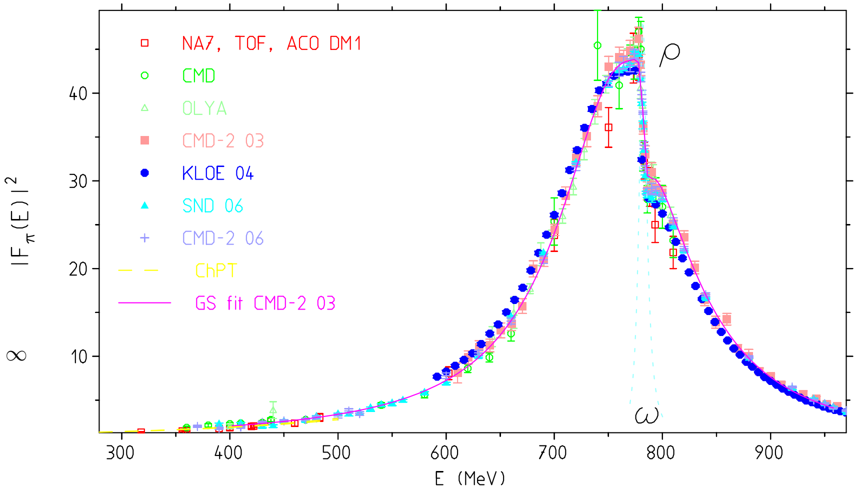
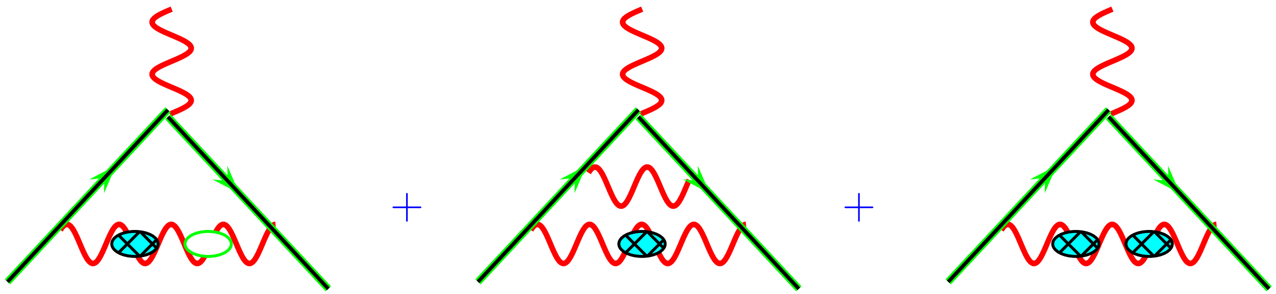


Table 4: Contributions and uncertainties $a_\mu^{\rm had}[\%](\rm error) \times 10^{10}$. In red the results relevant for VEPP-2000/DAFNE.

	$a_\mu^{\rm had}[\%](\rm error) \times 10^{10}$	rel. err.	abs. err.
ρ, ω ($E < 2M_K$)	540.84 [77.9](4.29)	0.8 %	36.0 %
$2M_K < E < 2$ GeV	102.33 [14.7](5.57)	5.4 %	60.4 %
2 GeV $< E < M_{J/\psi}$	22.13 [3.2](1.23)	5.6 %	3.0 %
$M_{J/\psi} < E < M_\Upsilon$	26.41 [3.8](0.57)	2.2 %	0.6 %
$M_\Upsilon < E < E_{\rm cut}$	1.40 [0.2](0.09)	6.2 %	0.0 %
$E_{\rm cut} < E$ pQCD	1.53 [0.2](0.00)	0.1 %	0.0 %
$E < E_{\rm cut}$ data	693.11 [99.8](7.16)	1.0 %	100.0 %
total	694.64 [100.0](7.16)	1.0 %	100.0 %

Higher order hadronic VP contributions

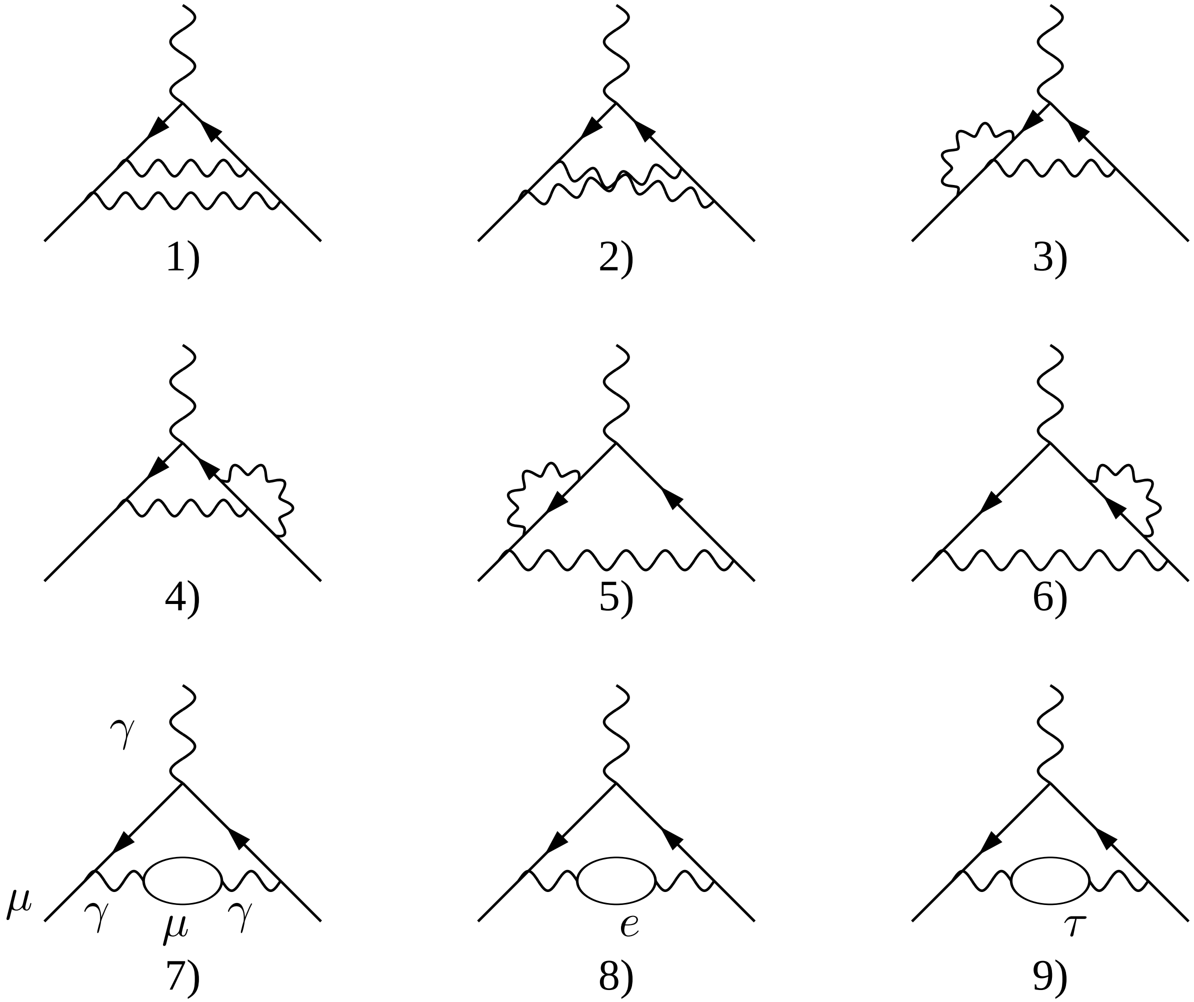


(Kinoshita, Nizic, Okamoto 1985)

$$a_\mu^{\mathrm{had}(2)} = -(100 \pm 6) \times 10^{-11}$$ **(Krause 96)**

$$a_\mu^{\mathrm{had}(2)} = -(\ 98 \pm 1) \times 10^{-11}$$ **(Hagiwara et al. 03)**

Two-Loop QED Contribution

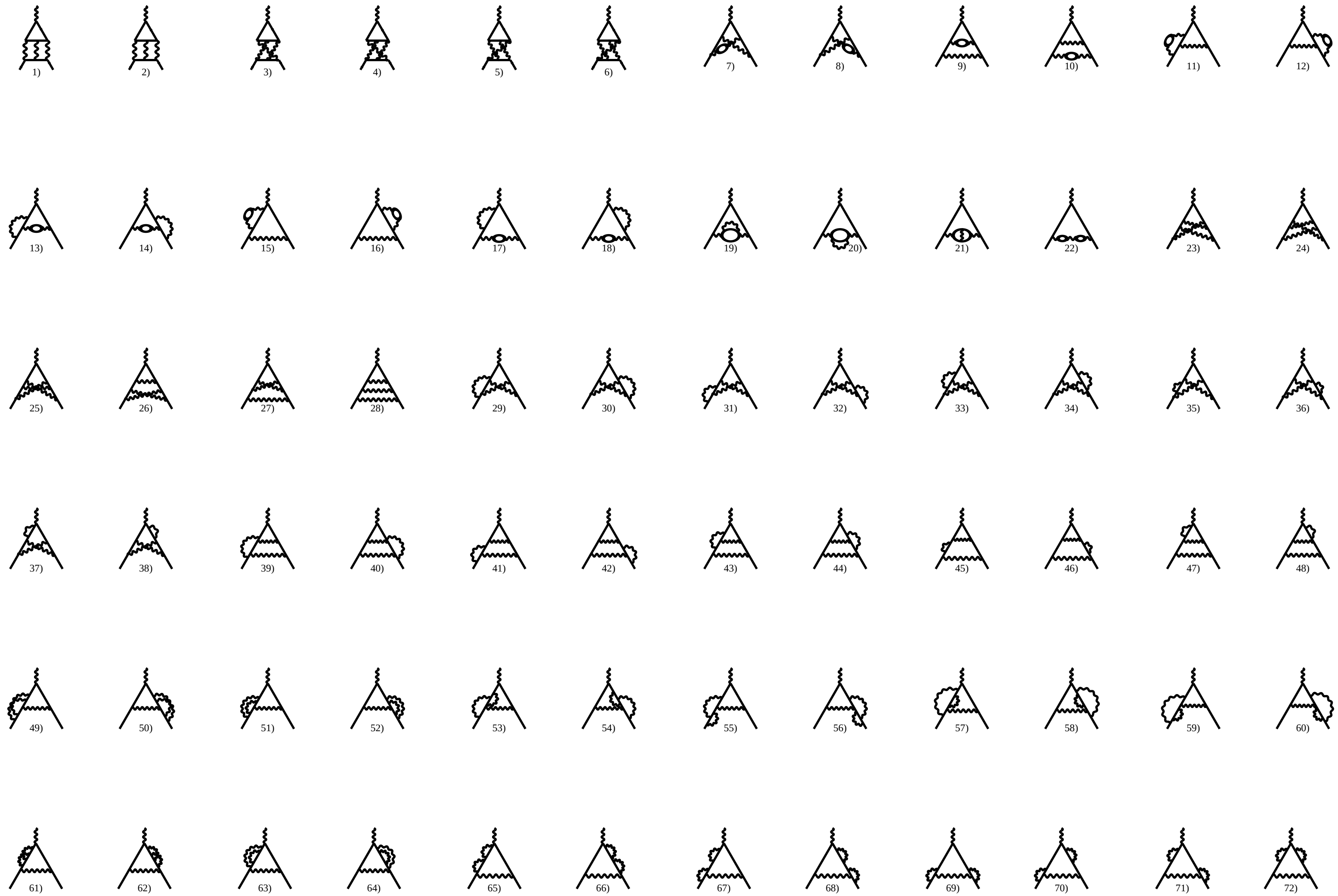


$$a_{\ell\ \mathrm{universal}}^{(4)}[1-7] = \left[\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Peterman 57, Sommerfield 57

$$a_\ell^{(6)} = \left[\frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) + \frac{100}{3}\left\{\mathrm{Li}_4(\frac{1}{2}) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} - \frac{239}{2160}\pi^4 + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5)\right]\left(\frac{\alpha}{\pi}\right)^3$$

Laporta & Remiddi 96



Class (a) yields a contribution of the type

$$a_\mu^{(6)\,\mathrm{had}_{[(a)]}} = \left(\frac{\alpha}{\pi}\right)^3 \frac{2}{3} \int\limits_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \, R(s) \, K^{[(a)]}\left(s/m_\mu^2\right)$$

with $K^{[(a)]}(s/m_\mu^2)$ a QED function which was obtained analytically by Barbieri, Remiddi 74. The kernel function is the contribution to a_μ of the 14 two–loop diagrams obtained from diagrams 1) to 7) by replacing one of the two photons by a "heavy photon" of mass $\sqrt{s}$. The convolution with $R(s)$ then provides the hadronic VP insertion.

$$\begin{aligned}
K^{[(a)]}(b) \quad = \quad & -\frac{139}{144} + \frac{115}{72}\,b + \left(\frac{19}{12} - \frac{7}{36}\,b + \frac{23}{144}\,b^2 + \frac{1}{b-4}\right) \ln b \\
& + \left(-\frac{4}{3} + \frac{127}{36}\,b - \frac{115}{72}\,b^2 + \frac{23}{144}\,b^3\right) \frac{\ln y}{\sqrt{b(b-4)}} \\
& + \left(\frac{9}{4} + \frac{5}{24}\,b - \frac{1}{2}\,b^2 - \frac{2}{b}\right) \zeta(2) + \frac{5}{96}\,b^2 \ln^2 b \\
& + \left(-\frac{1}{2}\,b + \frac{17}{24}\,b^2 - \frac{7}{48}\,b^3\right) \frac{\ln y}{\sqrt{b(b-4)}} \ln b \\
& + \left(\frac{19}{24} + \frac{53}{48}\,b - \frac{29}{96}\,b^2 - \frac{1}{3b} + \frac{2}{b-4}\right) \ln^2 y \\
& + \left(-2\,b + \frac{17}{6}\,b^2 - \frac{7}{12}\,b^3\right) \frac{D_p(b)}{\sqrt{b(b-4)}}
\end{aligned}$$

$$+\left(\frac{13}{3}-\frac{7}{6}\,b+\frac{1}{4}\,b^2-\frac{1}{6}\,b^3-\frac{4}{b-4}\right)\,\frac{D_m(b)}{\sqrt{b(b-4)}}$$
$$+\left(\frac{1}{2}-\frac{7}{6}\,b+\frac{1}{2}\,b^2\right)\,T(b)$$

where

$$y=\frac{\sqrt{b}-\sqrt{b-4}}{\sqrt{b}+\sqrt{b-4}}$$

and

$$\begin{aligned}
D_p(b) &= \mathrm{Li}_2(y)+\ln y\;\ln(1-y)-\frac{1}{4}\,\ln^2 y-\zeta(2)\;,\\
D_m(b) &= \mathrm{Li}_2(-y)+\frac{1}{4}\,\ln^2 y+\frac{1}{2}\,\zeta(2)\;,\\
T(b) &= -6\,\mathrm{Li}_3(y)-3\,\mathrm{Li}_3(-y)+\ln^2 y\;\ln(1-y)\\
&\quad+\frac{1}{2}\left(\ln^2 y+6\,\zeta(2)\right)\,\ln(1+y)+2\ln y\,(\mathrm{Li}_2(-y)+2\mathrm{Li}_2(y))\quad.
\end{aligned}$$

Class (b) consists of 2 diagrams only, obtained from diagram 22) of 3–loop QED, and one may write this contribution in the form

$$a_\mu^{(6)\,\mathrm{had}_{[(b)]}} = \left(\frac{\alpha}{\pi}\right)^3 \frac{2}{3} \int\limits_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} R(s)\, K^{[(b)]}(s/m_\mu^2)$$

$$K^{[(b)]}(s/m_\mu^2) = \int\limits_0^1 \mathrm{d}x\, \frac{x^2\,(1-x)}{x^2+(1-x)\,s/m_\mu^2} \left[-\hat{\Pi}_\gamma^{'\,e}\left(-\frac{x^2}{1-x}\frac{m_\mu^2}{m_e^2}\right)\right]$$

where we have set $\Pi^{'} = \frac{\alpha}{\pi}\hat{\Pi}^{'}$. ($z = -\frac{x^2}{1-x}\frac{m_\mu^2}{m_e^2}$),

$$\hat{\Pi}_\gamma^{'\,e}(z) = -\,2\int\limits_0^1 \mathrm{d}y\, y\,(1-y)\,\ln\left(1-z\,y\,(1-y)\right)$$

$$= \frac{8}{9} - \frac{\beta^2}{3} + \left(\frac{1}{2}-\frac{\beta^2}{6}\right)\beta\ln\frac{\beta-1}{\beta+1}$$

with $\beta = \sqrt{1+4\frac{1-x}{x^2}\frac{m_e^2}{m_\mu^2}}$.

Here the kernel function is the contribution to a_μ of the 2 two–loop diagrams obtained from diagrams 8) by replacing one of the two photons by a “heavy photon” of mass $\sqrt{s}$.

In diagram b) $\frac{m_f^2}{m^2} = (m_e/m_\mu)^2$ is very small and one may expand β in terms of this small parameter. The x–integration afterwards may be performed analytically. Up to terms of order $\mathcal{O}(\frac{m_f^2}{m^2})$ the result reads (Krause

96)

$$\begin{aligned} K^{[(b)]}(s) &= -\left(\frac{5}{9}+\frac{1}{3}\ln\frac{m_f^2}{m^2}\right)\times\left\{\frac{1}{2}-(x_1+x_2)\right. \\ &\left.+\frac{1}{x_1-x_2}\left[x_1^2(x_1-1)\ln\left(\frac{-x_1}{1-x_1}\right)-x_2^2(x_2-1)\ln\left(\frac{-x_2}{1-x_2}\right)\right]\right\} \\ &-\frac{5}{12}+\frac{1}{3}(x_1+x_2)+\frac{1}{3(x_1-x_2)}\left\{x_1^2(1-x_1)\left[\mathrm{Li}_2\left(\frac{1}{x_1}\right)-\frac{1}{2}\ln^2\left(\frac{-x_1}{1-x_1}\right)\right]\right. \\ &\left.-x_2^2(1-x_2)\left[\mathrm{Li}_2\left(\frac{1}{x_2}\right)-\frac{1}{2}\ln^2\left(\frac{-x_2}{1-x_2}\right)\right]\right\}\;, \end{aligned}$$

with $x_{1,2}=\frac{1}{2}(b\pm\sqrt{b^2-4b})$ and $b=\frac{s}{m^2}$.

Class (c) includes the double hadronic VP insertion, which is given by

$$a_\mu^{(6)\,\mathrm{had}_{[(c)]}} = \left(\frac{\alpha}{\pi}\right)^3\frac{1}{9}\int\limits_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}s'}{s'}R(s)\,R(s')\,K^{[(c)]}(s,s')$$

where

$$K^{[(c)]}(s,s')=\int\limits_0^1\mathrm{d}x\,\frac{x^4\,(1-x)}{[x^2+(1-x)\,s/m_\mu^2][x^2+(1-x)\,s'/m_\mu^2]}\;.$$

This integral may be performed analytically. Setting $b = \frac{s}{m^2}$ and $c = \frac{s'}{m^2}$ one obtains for $b \neq c$

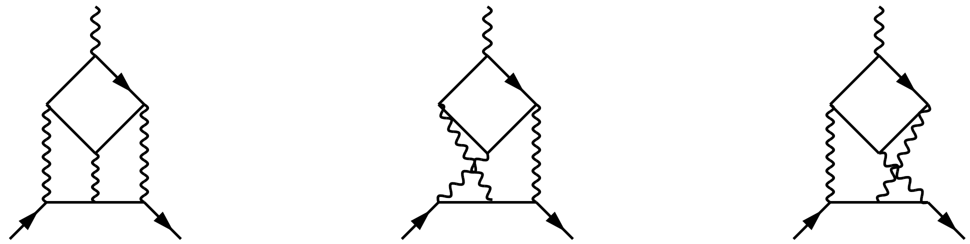
$$
\begin{aligned}
K^{[(c)]}(s,s') \quad = \quad & \frac{1}{2} - b - c - \frac{(2-b)\;b^2\;\ln(b)}{2\;(b-c)} - \frac{b^2\;\left(2 - 4\,b + b^2\right)\;\ln(\frac{b+\sqrt{-(4-b)\;b}}{b-\sqrt{-(4-b)\;b}})}{2\;(b-c)\;\sqrt{-\;(4-b)\;b}} \\
& - \frac{(-2+c)\;c^2\;\ln(c)}{2\;(b-c)} + \frac{c^2\;\left(2 - 4\,c + c^2\right)\;\ln(\frac{c+\sqrt{-(4-c)\;c}}{c-\sqrt{-(4-c)\;c}})}{2\;(b-c)\;\sqrt{-\;(4-c)\;c}}\;,
\end{aligned}
$$

and for $b = c$

$$
\begin{aligned}
K^{[(c)]}(s,s') \quad = \quad & \frac{1}{2} - 2\,c + \frac{c}{2}\,(-2 + c - 4\;\ln(c) + 3\,c\;\ln(c)) + \frac{c\;\left(-2 + 4\,c - c^2\right)}{2(-4+c)} \\
& + \frac{c\;\left(12 - 42\,c + 22\,c^2 - 3\,c^3\right)\;\ln(\frac{c+\sqrt{(-4+c)\;c}}{c-\sqrt{(-4+c)\;c}})}{2\,(-4+c)\;\sqrt{(-4+c)\;c}}\;.
\end{aligned}
$$

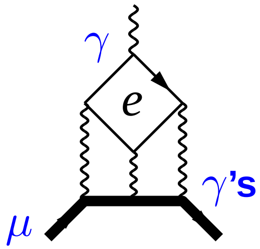
⑥ Light-by-Light scattering contribution to $g-2$

6 diagrams related by permutation of photon lines attached to muon:



Again, different regimes:

— LIGHT internal masses also in this case give rise to potentially large logarithms of mass ratios which get singular in the limit $m_{\mathrm{light}} \to 0$

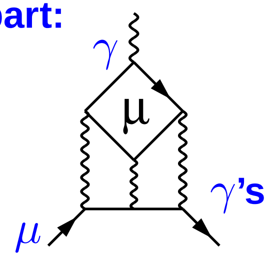


$$a_\mu^{(6)}(\mathrm{lbl},e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$

Again a light loop which yields a unexpectedly large contribution

$$a_\mu^{(6)}(\mathrm{lbl},e) \simeq 20.947\,924\,89(16)\,\left(\frac{\alpha}{\pi}\right)^3 = 2.625\,351\,02(2)\times 10^{-7}\ .$$

— **EQUAL internal masses** case which yields a pure number which is usually included in the $a_\ell^{(6)}$ universal part:



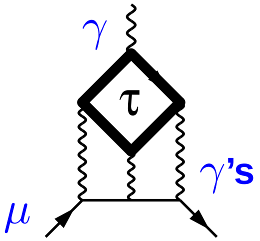
$$a_\mu^{(6)}(\mathrm{lbl},\mu) = \left[\frac{5}{6}\,\zeta(5) - \frac{5}{18}\,\pi^2\,\zeta(3) - \frac{41}{540}\pi^4 - \frac{2}{3}\pi^2\ln^2 2 + \frac{2}{3}\ln^4 2 + 16a_4 - \frac{4}{3}\,\zeta(3) - 24\pi^2\ln 2 + \frac{931}{54}\pi^2 + \frac{5}{9}\right]\left(\frac{\alpha}{\pi}\right)^3$$

where a_4 is a known constant. The single scale QED contribution is much smaller

$$a_\mu^{(6)}(\mathrm{lbl},\mu) \simeq 0.371005293\,\left(\frac{\alpha}{\pi}\right)^3 = 4.64971651\times 10^{-9}$$

but is still a substantial contributions at the required level of accuracy.

— **HEAVY internal masses** again decouple in the limit $m_{\mathrm{heavy}}\to\infty$ and thus only yield small power correction



$$a_\mu^{(6)}(\mathrm{lbl},\tau) = \left[\left[\frac{3}{2}\,\zeta(3) - \frac{19}{16}\right]\left(\frac{m_\mu}{m_\tau}\right)^2 + O\left(\frac{m_\mu^4}{m_\tau^4}\ln^2\frac{m_\tau}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$

As expected this heavy contribution is power suppressed yielding

$$a_\mu^{(6)}(\mathrm{lbl},\tau) \simeq 0.002\,142\,90(69)\,\left(\frac{\alpha}{\pi}\right)^3 = 2.685\,65(86)\times 10^{-11}\ .$$

Turn to hadronic part:

Which energy region contribute most to LBL graphs?

Check for muon loop and cutting off high energy contributions by a cut-off Λ:

Λ [GeV]	0.5	0.7	1.0	2.0
$a_\mu \times 10^{10}$	24	26	38	45

$\Rightarrow$ sensitive to effects from high scales! Only when the cut-off exceeds about 2 GeV the correct result $a_\mu^{(6)}(\mathrm{lbl},\mu) \simeq 46.50 \cdot 10^{-10}$ is well approximated. Naively one would expect the muon mass $\sim$ 105 MeV to set the scale $\Rightarrow$ sensitive to intermediate energies!

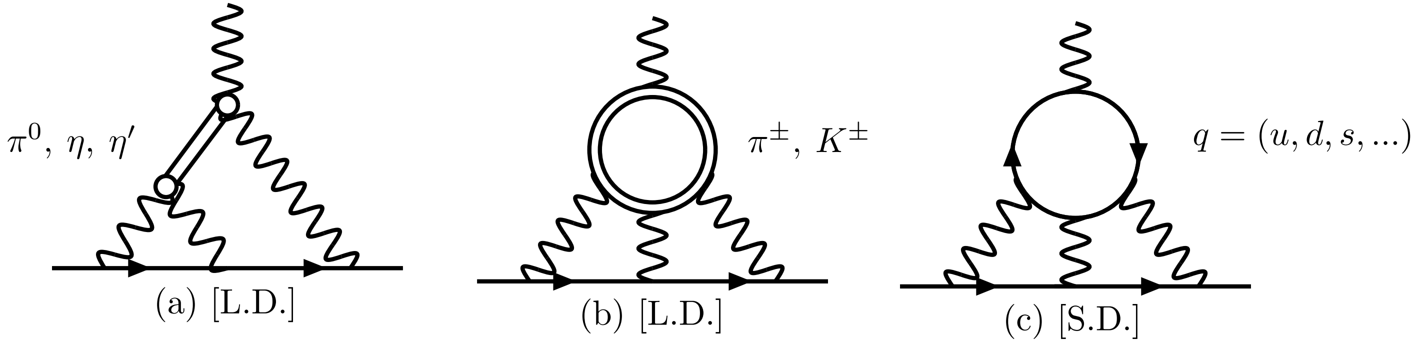
A constituent quark loop (adopting constituent quark masses)

CQM estimates of $a_\mu^{(6)}(\mathrm{lbl},q) \cdot 10^{11}$

0.3 GeV lepton	$[ud]$	s	c	$[uds]$	$[udsc]$	method
79.0	49.7	1.1	2.1	50.8	52.9	KNO84 numerical
94.2	59.3	1.3	2.2	60.6	62.8	expansion 1st term
81.9	51.6	1.2	2.2	52.7	55.0	expansion 2 terms

For the light quarks the numerical results are certainly more trustable while for the heavier quarks, like the c, the asymptotic expansion becomes more reliable. Note with current quark masses one would get

$a_\mu^{(6)}(\mathrm{lbl}, u+d) = 8229.34 \cdot 10^{-11}$ **and** $a_\mu^{(6)}(\mathrm{lbl}, s) = 17.22 \cdot 10^{-11}$ **by adapting color, charge and mass in leptonic results. Meaning of these result more than questionable. Missing proper low energy structure of QCD. Need low energy effective theory mentioned before** $\Rightarrow$ **amount to calculate the following diagrams**



Hadronic light–by–light scattering diagrams in a low energy effective model description. Diagrams (a) and (b) represent the long distance [L.D.] contributions at momenta $p \leq \Lambda$**, diagram (c) involving a quark loop yield the leading short distance [S.D.] part at momenta** $p \geq \Lambda$ **with** $\Lambda \sim 1$ to 2 **GeV an UV cut-off**

Orders with respect to $1/N_c$ **and chiral expansion of typical leading contributions**

Diagram	$1/N_c$ expansion	p expansion	type
Fig (a)	N_c	p^6	π^0, η, η' exchange
Fig (a)	N_c	p^8	a_1, ρ, ω exchanges
Fig (b)	1	p^4	meson loops ($\pi^\pm,\ K^\pm$)
Fig (c)	N_c	p^8	quark loops

Originally VMD model (Kinoshita et al 85):

- ρ mesons play an important role in the game (see hadronic VP) $\Rightarrow$ looks natural to apply a vector meson dominance (VDM) model
- Naive VDM

$$\frac{\mathrm{i}}{q^2} \to \frac{\mathrm{i}}{q^2}\,\frac{m_\rho^2}{q^2-m_\rho^2} = \frac{\mathrm{i}}{q^2} - \frac{\mathrm{i}}{q^2-m_\rho^2} \quad .$$

- provides a damping at high energies, ρ mass as an effective cut-off (physical version of a Pauli-Villars cut-off) $\Rightarrow$ photons $\longrightarrow$ dressed photons!
- naive VDM violates electromagnetic WT-identity

Correct implementation in accord with low energy symmetries of QCD:

- vector meson extended CHPT (EχPT) model (Ecker et al 89)
- hidden local symmetry (HLS) model (Bando et al 85)
- extended NJL (ENJL) model (Dhar et al 85, Ebert, Reinhardt 86, Bijnens 96)

to large extend equivalent

Problem: matching L.D. with S.D. $\Rightarrow$ results depend on matching cut off Λ $\Rightarrow$ model dependence (non-renormalizable low energy effective theory vs. renormalizable QCD)

Novel approach: refer to quark–hadron duality of large-N_c QCD, hadrons spectrum known, infinite series of narrow spin 1 resonances ('t Hooft 79) $\Rightarrow$ no matching problem (resonance representation has to

match quark level representation) (De Rafael 94, Knecht, Nyffeler 02)

$\Rightarrow$ ***lowest meson dominance (LMD) plus one vector state (V)*** **approximation to large-N_c QCD allows for correct matching**

$\Rightarrow$ **"LMD+V" parametrization of $\pi^0\gamma\gamma$ form–factor:**

$$F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{4\pi^2F_\pi^2}{N_c}\,\frac{q_1^2q_2^2(q_1^2+q_2^2)-h_2q_1^2q_2^2+h_5(q_1^2+q_2^2)+(N_cM_1^4M_2^4/4\pi^2F_\pi^2)}{(q_1^2+M_1^2)(q_1^2+M_2^2)(q_2^2+M_1^2)(q_2^2+M_2^2)},$$

with parameters fixed by high energy and phenomenological constraints

Based on such models, major efforts in estimating a_μ^{LbL} were made by

- (Hayakawa, Kinoshita, Sanda) **(HKS 1995)**, (Hayakawa, Kinoshita) **(HK 1998)** [HLS model]
- (Bijnens, Pallante and Prades) **(BPP 1995)** [ENJL model]
- (Knecht, Nyffeler) **(KN 2001)** [LMD+V model] **discovered a sign mistake in the π^0,η,η' exchange contribution, which changed the central value by $+167\cdot10^{-11}$! at that time.**
- (Melnikov,Vainshtein) **(MV 2004)** [LMD+V model] **found additional inconsistencies in previous calculations, this time in the short distance constraints (QCD/OPE) used in matching the high energy behavior of the effective models used for the π^0,η,η' exchange contribution, shifts central value by $+53\cdot10^{-11}$!.**

a) Leading is the π^0, η, η' contribution

Light-by-Light: π^0, η, η'

Model for $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \cdot 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \cdot 10^{11}$
Point form factor $\mathcal{F}_{\pi^0\gamma\gamma}$	330	$-$
ENJL[BPP]	59(11)	85(13)
HLS [HKS,HK]	57(4)	83(6)
LMD+V[KN] ($h_2 = 0$)	58(10)	83(12)
LMD+V[MV](1) ($h_2 = -10$ GeV^2)	63(10)	83(12)
LMD+V[MV](2) ($h_2 = -10$ GeV^2)+new S.D.	77(5)	114(10)

b) Sub-leading are pion and kaon loops

c) S.D. quark loop high energy complement of HLS and ENJL models

Light-by-Light: $\pi^{\pm}, K^{\pm}$ & quark loops

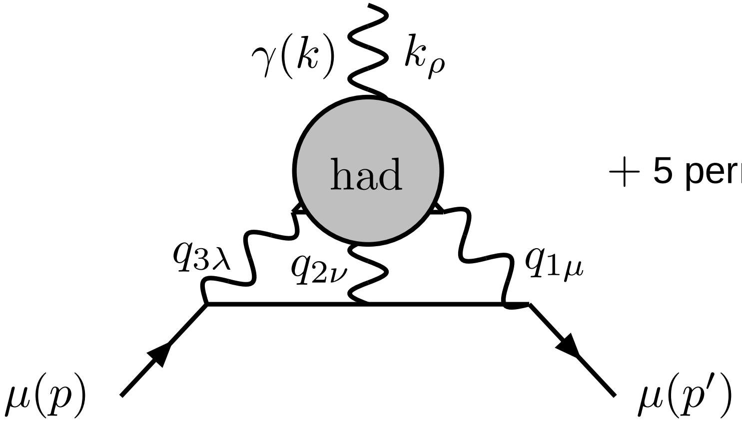
Model $\pi^+\pi^-\gamma^*(\gamma^*)$	$a_\mu(\pi^{\pm}) \cdot 10^{11}$	$a_\mu(\pi^{\pm}, K^{\pm}) \cdot 10^{11}$	$a_\mu(\mathrm{quarks}) \cdot 10^{11}$
Point	-45	—	62(3)
VDM	-16	—	—
ENJL[BPP]	-18(13)	-19(13)	21(3)
HLS [HKS,HK]	-4(8)	-4.5(8.1)	9.7(11.1)
guesstimate [MV]	0(10)	0(10)	-

In the large-N_c resonance saturation approach (LMD) the S.D. behavior is incorporated as a boundary condition and no separate quark loops contributions has to be accounted for.

Summary of most recent results

$10^{11}\,a_\mu$	**BPP**	**HKS**	**KN**	**MV**
π^0,η,η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10
π, K **loops**	-19 ± 13	-4.5 ± 8.1		0 ± 10
axial vector	2.5 ± 1.0	1.7 ± 0.0		$22 \pm \ 5$
scalar	-6.8 ± 2.0	-	-	-
quark loops	$21 \pm \ 3$	9.7 ± 11.1	-	-
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25

Hadrons in $< 0|T\{A^\mu(x_1)A^\nu(x_2)A^\rho(x_3)A^\sigma(x_4)\}|0 >$



$+$ 5 permutations of the q_i

Figure 1: Setup for the calculation of the hadronic contribution of the light-by-light scattering to the muon electromagnetic vertex

Key object full rank-four hadronic vacuum polarization tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3) = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_3\,\mathrm{e}^{\mathrm{i}\,(q_1x_1+q_2x_2+q_3x_3)} \times\langle\,0\,|\,T\{j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3)j_\rho(0)\}\,|\,0\,\rangle\ .$$

Ward-Takahashi identities

$$\{q_1^\mu;q_2^\nu;q_3^\lambda;k^\rho\}\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3) = 0\,,$$

with $k=(q_1+q_2+q_3)$. This implies that

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2)=-k^\sigma(\partial/\partial k^\rho)\,\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2)\ \ .$$

Contribution of $\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3)$ to the electromagnetic vertex

$$\begin{aligned}\langle\mu^-(p')|(\mathrm{i}e)j_\rho(0)|\mu^-(p)\rangle \quad &= \quad (-\mathrm{i}e)\,\bar{u}(p')\,\Pi_\rho(p',p)\,u(p)\\
&= \int\frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{\mathrm{d}^4q_2}{(2\pi)^4}\frac{(-\mathrm{i})^3}{q_1^2\,q_2^2\,(q_1+q_2-k)^2}\frac{\mathrm{i}}{(p'-q_1)^2-m^2}\frac{\mathrm{i}}{(p-q_1-q_2)^2-m^2}\\
&\times(-\mathrm{i}e)^3\,\bar{u}(p')\,\gamma^\mu(\not p'-\not q_1+m)\,\gamma^\nu\,(\not p-\not q_1-\not q_2+m)\,\gamma^\lambda\,u(p)\\
&\times(\mathrm{i}e)^4\,\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2)\,,\end{aligned}$$

with $k_\mu=(p'-p)_\mu$. For the contribution to the form factors

$$\bar{u}(p')\,\Pi_\rho(p',p)\,u(p)\ =\ \bar{u}(p')\left[\gamma_\rho F_{\mathrm{E}}(k^2)\ +\ \mathrm{i}\ \frac{\sigma_{\rho\tau}k^\tau}{2m_\mu}F_{\mathrm{M}}(k^2)\right]u(p)\,,$$

WTIs imply $\Pi_\rho(p',p)=k^\sigma\Pi_{\rho\sigma}(p',p)$ with

$$\begin{aligned}\bar{u}(p')\,\Pi_{\rho\sigma}(p',p)\,u(p) \quad &= \quad -\mathrm{i}e^6\times\\
&\int\frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{\mathrm{d}^4q_2}{(2\pi)^4}\frac{1}{q_1^2\,q_2^2\,(q_1+q_2-k)^2}\frac{1}{(p'-q_1)^2-m^2}\frac{1}{(p-q_1-q_2)^2-m^2}\\
&\times\ \bar{u}(p')\,\gamma^\mu\,(\not p'-\not q_1+m)\,\gamma^\nu\,(\not p-\not q_1-\not q_2+m)\,\gamma^\lambda\,u(p)\end{aligned}$$

$$\times \ \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) \, .$$

WT–identity $\Rightarrow k^\rho k^\sigma \bar{u}(p')\Pi_{\rho\sigma}(p', p)\, u(p) \ = 0 \Rightarrow \delta^{\rm lbl} F_{\rm E}(0) = 0$ and $V_\rho(p) = \Pi_\rho(p', p)|_{k=0} = 0$ and $T_{\rho\sigma}(p) = \Pi_{\rho\sigma}(p', p)|_{k=0}$. Hadronic light-by-light contribution to the muon anomalous magnetic moment is equal to

$$F_{\rm M}(0) \ = \ \frac{1}{48m} \,\mathrm{Tr}\, \{(\not p + m)[\gamma^\rho, \gamma^\sigma](\not p + m)\Pi_{\rho\sigma}(p, p)\} \ .$$

- 8-dim integral
- 3 may be performed analytically
- remain 5 dim integration, 2 moduli, 3 angles
- many scales involved ($m_\mu, m_\pi, M_\rho, \cdots$), expansion techniques fail

Hadronic tensor $\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$:

- non-perturbative physics
- general covariant decomposition involves 138 Lorentz structures of which
- 32 can contribute to $g - 2$
- fortunately, dominated by the pseudoscalar exchanges $\pi^0, \eta, \eta', \ldots$ described by the effective Wess-Zumino Lagrangian

- generally, pQCD useful to evaluate the short distance (S.D.) tail
- the dominant long distance (L.D.) part must be evaluated using some low energy effective model which includes the pseudoscalar Goldstone bosons as well as the vector mesons play key role

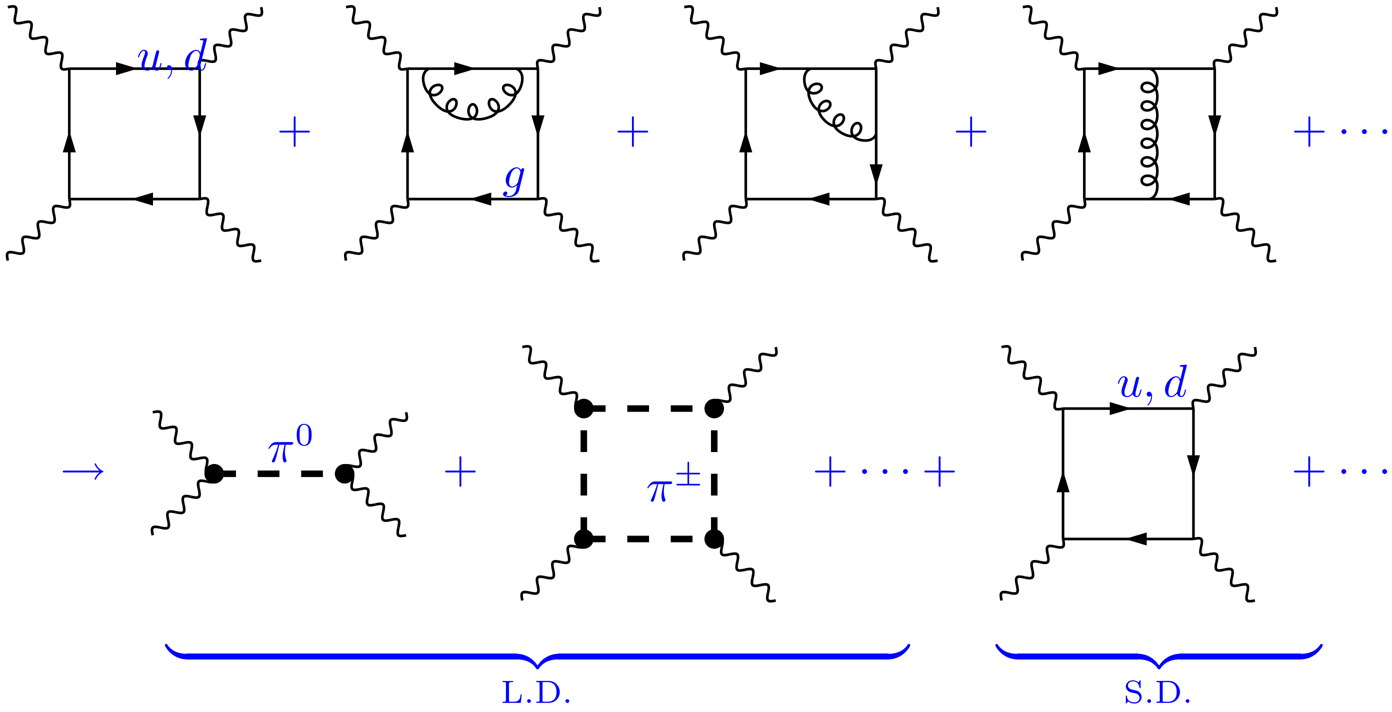


Figure 2: Hadronic light–by–light scattering is dominated by π^0 exchange in the odd parity channel, pion loops etc. at long distances (L.D.) and quark loops incl. hard gluonic corrections at short distances (S.D.)

The most interesting and at the same time leading phenomenon is the non-perturbative π^0 exchange, which is a consequence of the low energy structure of QCD, in particular the completely non-perturbative spontaneous breakdown of the chiral symmetry, which implies the existence of the pseudoscalar Goldstone bosons in the chiral limit. Thus, in spite of the fact that in pQCD, our hadronic tensor $\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k-q_1-q_2)$ only involves parity conserving vector interactions (γ^μ-type), in full QCD, due to the spontaneous breakdown of the chiral symmetry, as well as by the existence of the ABJ anomaly related via PCAC to the pseudoscalar states, the parity violating axial vector interactions ($\gamma^\mu\gamma_5$-type) are ruling the game, as illustrated in

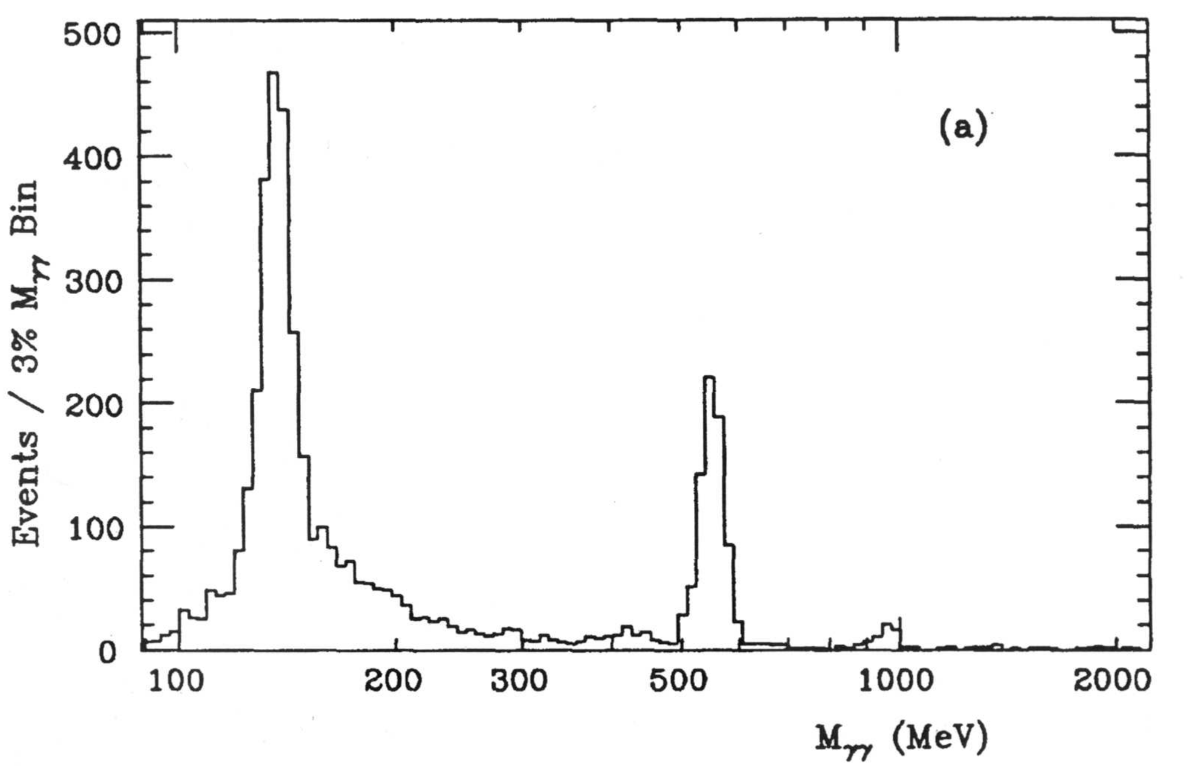


Figure 3: The spectrum of invariant $\gamma\gamma$ masses obtained with the Crystal Ball detector. The three rather pronounced spikes seen are the $\gamma\gamma \to$ pseudoscalar (PS) $\to \gamma\gamma$ excitations: PS=π^0, η, η'

A much more problematic set of hadronic corrections are related to hadronic light-by-light scattering contributions which fortunately set in only at order $O(\alpha^3)$, however, we know from the leptonic counterpart that these contribution could be dramatically enhanced.

Pion-pole contribution **dominating hadronic contributions = neutral pion exchange**

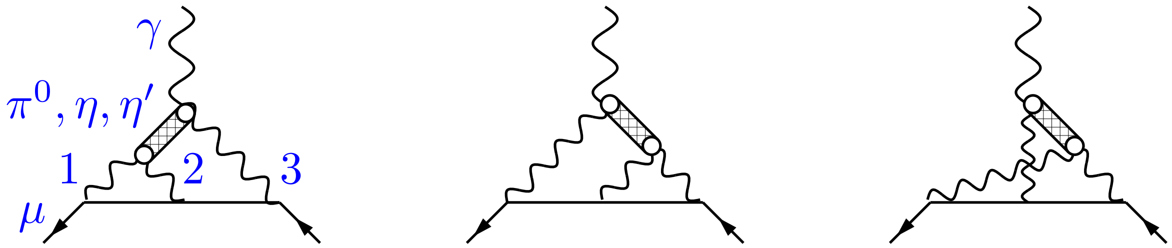


Figure 4: Leading hadronic light–by–light scattering diagrams

The key object here is the $\pi^0\gamma\gamma$ form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ which is defined by the matrix element

$$\mathrm{i}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(p)\rangle = \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q^2, (p-q)^2) \ .$$

Properties:

- **bose symmetric** $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(s, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(s, q_2^2, q_1^2)$
- **need it off-shell in integrals; if** $(s, q_1^2, q_2^2) \neq (m_\pi^2, 0, q^2)$ **in fact not known**
- **pion pole approximation in mind (pion pole dominance)** $s = m_\pi^2$
- **for generality use vertex function (not matrix element)**

$$\mathrm{i}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{j_\mu(x)\, j_\nu(0)\, \tilde{\varphi}_{\pi^0}(p)\}|0\rangle = \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(p^2, q^2, (p-q)^2) \times \frac{\mathrm{i}}{p^2 - m_\pi^2} \ ,$$

with $\tilde{\varphi}(p) = \int \mathrm{d}^4y\, \mathrm{e}^{\mathrm{i}px} \varphi(y)$ the Fourier transformed π^0–field. $\Rightarrow$ representation in terms of form factors

$$\begin{aligned} \mathrm{i}\ \Pi^{(\pi^0)}_{\mu\nu\lambda\rho}(q_1,q_2,q_3) &= \\ & \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_3}^2,q_1^2,q_2^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_3}^2,q_3^2,k^2)}{{q'_3}^2-m_\pi^2}\ \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\ \varepsilon_{\lambda\rho\sigma\tau}\, q_3^\sigma {q'_3}^\tau \\ &+ \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_1}^2,q_2^2,q_3^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_1}^2,q_1^2,k^2)}{{q'_1}^2-m_\pi^2}\ \varepsilon_{\mu\rho\alpha\beta}\, q_1^\alpha {q'_1}^\beta\ \varepsilon_{\nu\lambda\sigma\tau}\, q_2^\sigma q_3^\tau \\ &+ \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_2}^2,q_1^2,q_3^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}({q'_2}^2,q_2^2,k^2)}{{q'_2}^2-m_\pi^2}\ \varepsilon_{\mu\lambda\alpha\beta}\, q_1^\alpha q_3^\beta\ \varepsilon_{\nu\rho\sigma\tau}\, q_2^\sigma {q'_2}^\tau \end{aligned}$$

with $q'_i = q_i + k$. To compute $a_\mu^{\mathrm{LbL};\pi^0} \equiv F_{\mathrm{M}}(0)|_{\text{pion pole}}$, we need $\mathrm{i}\ \frac{\partial}{\partial k^\rho}\Pi^{(\pi^0)}_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2)$ at $k=0$ where $p_3=-(p_1+p_2)$. Computing the Dirac traces yields

$$\begin{aligned} a_\mu^{\mathrm{LbL};\pi^0} &= -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2(q_1+q_2)^2[(p+q_1)^2-m^2][(p-q_2)^2-m^2]} \\ & \times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2,q_1^2,q_3^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2,q_2^2,0)}{q_2^2-m_\pi^2}\ T_1(q_1,q_2;p) \right. \\ & \left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2,q_1^2,q_2^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_3^2,q_3^2,0)}{q_3^2-m_\pi^2}\ T_2(q_1,q_2;p) \right], \end{aligned}$$

where $T_1(q_1,q_2;p)$ and $T_2(q_1,q_2;p)$ are scalar kinematics factors; two terms unified by bose symmetry.

The pion-photon-photon transition form factor

- **Form factor function $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(s,s_1,s_2)$ is largely unknown**
- **Fortunately some experimental data is available:**
- **The constant $\boxed{\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2,0,0)}$ is well determined by the $\pi^0 \to \gamma\gamma$ decay rate**

The invariant matrix element reads

$$\mathcal{M}\left[\pi^0(q) \to \gamma(p_1,\lambda_1)\,\gamma(p_2,\lambda_2)\right] = e^2\,\varepsilon^{\mu *}(p_1,\lambda_1)\,\varepsilon^{\nu *}(p_2,\lambda_2)\,\varepsilon_{\mu\nu\alpha\beta}\,p_1^\alpha p_2^\beta\,\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q^2,p_1^2,p_2^2)$$

The on-shell transition amplitude in the chiral limit follows from the Wess-Zumino-Witten interaction term

$$\mathcal{L}^{(4)} = -\frac{\alpha N_c}{12\pi F_0}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}A^\rho\partial^\sigma\pi^0 + \cdots$$

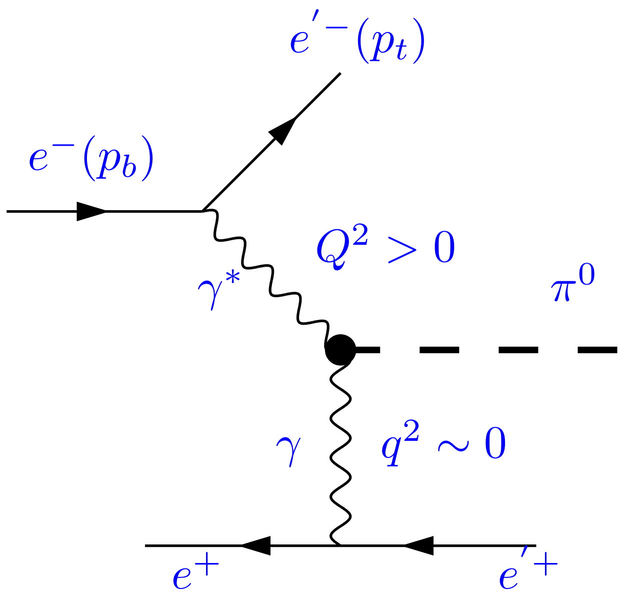
which reproduces the Adler-Bell-Jackiw anomaly and provides an important constraint in estimating the leading hadronic LBL contribution. The calculation yields ($f_\pi \sim F_0$)

$$M_{\pi^0\gamma\gamma} = e^2\,\mathcal{F}_{\pi^0\gamma\gamma}(0,0,0) = \frac{e^2 N_c}{12\pi^2 f_\pi} = \frac{\alpha}{\pi f_\pi} \approx 0.025\ \text{GeV}^{-1}\ ,$$

and with $f_\pi \sim 93$ MeV and quark color number $N_c = 3$ rather accurately predicts the experimental result

$$|M^{\exp}_{\pi^0\gamma\gamma}| = \sqrt{64\pi\Gamma_{\pi^0\gamma\gamma}/m_\pi^3} = 0.025 \pm 0.001\ \text{GeV}^{-1}\ .$$

Information on $\boxed{\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2,-Q^2,0)}$ comes from experiments $e^+e^- \to e^+e^-\pi^0$ where the electron (positron) gets tagged to high Q^2 values.



Measurement of the π^0 form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2,-Q^2,0)$ at high space–like Q^2

Data for $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2,-Q^2,0)$ is available from CELLO and CLEO. Brodsky–Lepage interpolating formula

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2,-Q^2,0) \simeq \frac{1}{4\pi^2 f_\pi}\,\frac{1}{1+(Q^2/8\pi^2 f_\pi^2)}$$

gives an acceptable fit to the data.

Assuming the pole approximation this FF has been used by all authors (HKS,BPP,KN) in the past, but has

been criticized recently (MV).

In fact in $g-2$ we are at zero momentum such that only the FF $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q^2,-Q^2,0) \neq \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2,-Q^2,0)$ is consistent with kinematics. Unfortunately, this off–shell form factor is not known and in fact not measurable.

An alternative way to look at the problem is to use the anomalous PCAC relation and to relate $\pi^0\gamma\gamma$ to directly the ABJ anomaly. The VVA three point function

$$\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \mathrm{i}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}\,(q_1\cdot x_1+q_2\cdot x_2)}\,\langle\, 0\,|\,\mathsf{T}\{V_\mu(x_1)V_\nu(x_2)A_\rho(0)\}\,|\,0\,\rangle$$

of the flavor and color diagonal fermion currents

$$V_\mu \;=\; \overline{\psi}\gamma_\mu\,\psi \quad , \quad A_\mu \;=\; \overline{\psi}\gamma_\mu\gamma_5\,\psi$$

where $\psi(x)$ is a quark field. The vector currents are strictly conserved $\partial_\mu V^\mu(x)=0$, while the axial vector current satisfies a PCAC relation plus the anomaly (indexed by $_0$ are bare parameters),

$$\partial_\mu A^\mu(x) = 2\,\mathrm{i}\,m_0\bar{\psi}\gamma_5\psi(x) + \frac{\alpha_0}{4\pi}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}(x) \;\; .$$

We will be mainly interested in the properties of strongly interacting quark flavor currents in perturbative QCD. To leading order the correlator of interest is associated with the one–loop triangle diagram plus its crossed ($q_1,\mu \leftrightarrow q_2,\nu$) partner. The covariant decomposition of $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$ into invariant functions to

four terms

$$\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \frac{1}{8\pi^2} \left\{ w_L \left(q_1^2,q_2^2,q_3^2\right) \, (q_1+q_2)_\rho \, \varepsilon_{\mu\nu\alpha\beta} \, q_1^\alpha q_2^\beta \; + \; \mathrm{transversal} \right\}$$

The longitudinal part is entirely fixed by the anomaly,

$$w_L \left(q_1^2,q_2^2,q_3^2\right) \; = \; -\frac{2N_c}{q_3^2}$$

which is exact to all orders of perturbation theory, the famous Adler-Bardeen non-renormalization theorem. In order to obtain the coupling to pseudoscalars we have to take the derivative as required by the PCAC relation we obtain

$$(q_1+q_2)^\rho \mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \frac{1}{8\pi^2} \, \varepsilon_{\mu\nu\alpha\beta} \, q_1^\alpha q_2^\beta w_L \left(q_1^2,q_2^2,q_3^2\right) \, q_3^2 = -\frac{N_c}{4\pi^2} \, \varepsilon_{\mu\nu\alpha\beta} \, q_1^\alpha q_2^\beta \;\; .$$

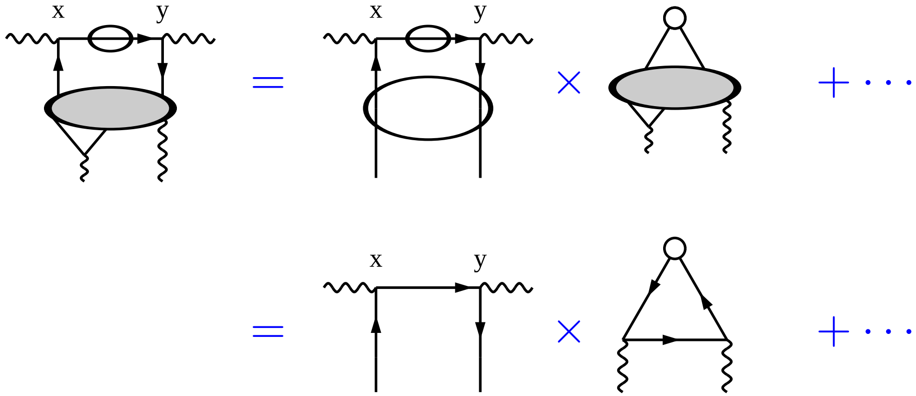
This holds to all orders and for arbitrary momenta.

High energy behavior of $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ is best investigated by OPE of

$$\langle \, 0 \, | \, T\{ j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) \} \, | \, \gamma(k) \, \rangle$$

taking into account that the external photon is in a physical state. Looking at first diagram:

- q_1 **and** q_2 **as independent loop integration momenta**
- **most important region** $q_1^2 \sim q_2^2 \gg q_3^2$, $\Leftrightarrow$ **short distance expansion of** $T\{j_\mu(x_1) j_\nu(x_2)\}$ **for** $x_1 \to x_2$.



Usual structure of OPE

"perturbative *hard* short distance coefficient function" $\times$ "*soft* long distance matrix element"

Surprisingly, the leading possible term in is just given by the ABJ anomaly diagram known exact to all orders and given by the lowest order (one–loop) result. This requires of course that the leading operator in the short distance expansion must involve the divergence axial current. Indeed,

$$\mathrm{i}\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2\, \mathrm{e}^{\mathrm{i}\,(q_1x_1+q_2x_2)}\, T\{j_\mu(x_1)j_\nu(x_2)X\} \;=\; \int \mathrm{d}^4z\, \mathrm{e}^{\mathrm{i}\,(q_1+q_2)\,z}\, \frac{2\mathrm{i}}{\hat{q}^2}\, \varepsilon_{\mu\nu\alpha\beta}\hat{q}^\alpha T\{j_5^\beta(z)X\} + \cdots$$

with $j_5^\mu = \bar{q}\hat{Q}^2\gamma^\mu\gamma_5 q$ the relevant axial current and $\hat{q} = (q_1-q_2)/2 \approx q_1 \approx -q_2$; the momentum flowing through the axial vertex is q_1+q_2 and in the limit $k \to 0$ of our interest $q_1+q_2 \to -q_3$ which is assumed to be much smaller than $\hat{q}$ ($q_1^2 - q_2^2 \sim -2q_3\hat{q} \sim 0$).

After the large q_1, q_2 behavior has been factored out the remaining soft matrix element to be calculated is

$$T_{\lambda\beta} = \mathrm{i} \int \mathrm{d}^4 z \, \mathrm{e}^{\mathrm{i}\, q_3 z} < 0|T\{j_{5\beta}(z) j_\lambda(0)\}|\gamma(k) > ,$$

which is precisely the well known VVA triangle correlator

$$\begin{aligned} T^{(a)}_{\lambda\beta} &= -\frac{\mathrm{i}\, e N_c}{4\pi^2} \times \\ & w_L(q_3^2)\, q_{3\beta} q_3^\sigma \tilde{f}_{\sigma\lambda} + w_T(q_3^2) \left(-q_3^2 \tilde{f}_{\lambda\beta} + q_{3\lambda} q_3^\sigma \tilde{f}_{\sigma\beta} - q_{3\beta} q_3^\sigma \tilde{f}_{\sigma\lambda} \right) \end{aligned}$$

Both amplitudes the longitudinal w_L as well as the transversal w_T are calculable from the triangle fermion one–loop diagram. In the chiral limit they are given by

$$w_L^{(a)}(q^2) = 2 w_T^{(a)}(q^2) = -2/q^2 \quad .$$

as discovered by (Vainshtein 03). So everything is perturbative with no radiative corrections at all!

At this stage of the consideration it looks like a real mystery what all this has to do with π^0 exchange, as everything looks perfectly controlled by perturbation theory. The clue is that as a low energy object we may evaluate this matrix element at the same time perfectly in terms of hadronic spectral functions by saturating it by a sum over intermediate states. For the positive frequency part we have

$$< 0|j_5^\beta(z) j_\lambda(0)|\gamma(k) > = \int \frac{\mathrm{d}^4 p_n}{(2\pi)^3} \int\!\!\!\!\!\!\sum n < 0|j_5^\beta(z)|n >< n|j_\lambda(0)|\gamma(k) >$$

where the lowest state contributing is the π^0, thus

$$< 0|j_5^\beta(z)j_\lambda(0)|\gamma(k) > \;=\; \int \frac{\mathrm{d}^3p}{(2\pi)^3\, 2\omega(\vec{p})} < 0|j_5^\beta(z)|\pi^0(p) >< \pi^0(p)|j_\lambda(0)|\gamma(k) > \\ +\text{subleading terms} \;\; .$$

Here we have

$$\begin{aligned} < 0|j_5^\beta(z)|\pi^0(p) > &= \mathrm{e}^{\mathrm{i}\,pz}\, 2\mathrm{i} f_\pi p^\beta \\ < \pi^0(p)|j_\lambda(0)|\gamma(k) > &= -4e g_{\pi^0\gamma\gamma} p^\alpha \tilde{f}_{\alpha\lambda} \end{aligned}$$

with $\tilde{f}_{\alpha\lambda} = k_\alpha \varepsilon_\lambda - k_\lambda \varepsilon_\alpha$

$$g_{\pi^0\gamma\gamma} = \frac{N_c \mathbf{Tr}\,[\lambda_3 \hat{Q}^2]}{16\pi^2 f_\pi}$$

As a result we find

$$< 0|j_5^\beta(z)j_\lambda(0)|\gamma(k) >= \int \frac{\mathrm{d}^4p}{(2\pi)^3}\, \Theta(p^0)\delta(p^2-m_\pi^2)\, \mathrm{e}^{\mathrm{i}\,pz}\, 2\mathrm{i} f_\pi p^\beta \frac{N_c \mathbf{Tr}\,[\lambda_3 \hat{Q}^2]}{16\pi^2 f_\pi} p^\alpha \tilde{f}_{\alpha\lambda}$$

and finally for the time ordered correlation

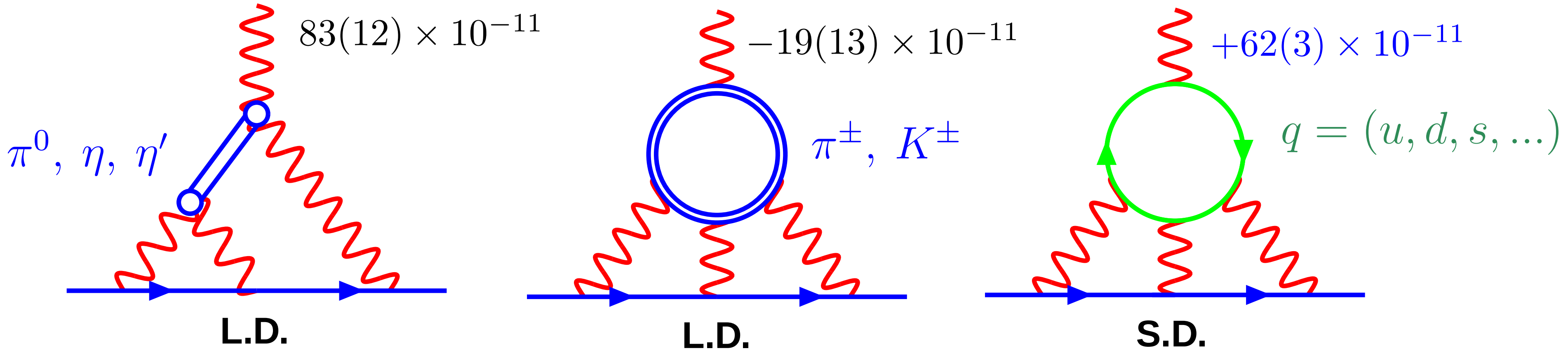
$$< 0|T\{j_5^\beta(z)j_\lambda(0)\}|\gamma(k) >= \int \frac{\mathrm{d}^4p}{(2\pi)^3}\, \frac{1}{\pi}\, \frac{\mathrm{i}}{p^2-m_\pi^2+\mathrm{i}\varepsilon}\, \mathrm{e}^{\mathrm{i}\,pz}\, 2\mathrm{i} f_\pi p^\beta \frac{N_c \mathbf{Tr}\,[\lambda_3 \hat{Q}^2]}{16\pi^2 f_\pi} p^\alpha \tilde{f}_{\alpha\lambda}$$

Melnikov, Vainshtein: vertex with external photon must be non-dressed! i.e. no VDM damping $\Rightarrow$ result increases by $\boxed{50\%}$!

Light-by-light-scattering contribution: summary

Melnikov-Vainshtein approach vs EJLN/HGS approach:

- **Hadronic light–by–light scattering** $a_\mu^{\rm lbl} = (\,80 \pm 40)\ \times 10^{-11}$ **(Knecht & Nyffeler 02)**
  $a_\mu^{\rm lbl} = (136 \pm 25)\ \times 10^{-11}$ **(Melnikov & Vainshtein 03)** **shift by** $\boxed{+56 \times 10^{-11}}$



Low energy effective theory: e.g. ENJL

MV and KN utilize the same model LMD+V form factor:

$$F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{4\pi^2 F_\pi^2}{N_c}\ \frac{q_1^2 q_2^2 (q_1^2 + q_2^2) - h_2 q_1^2 q_2^2 + h_5 (q_1^2 + q_2^2) + (N_c M_1^4 M_2^4 / 4\pi^2 F_\pi^2)}{(q_1^2 + M_1^2)(q_1^2 + M_2^2)(q_2^2 + M_1^2)(q_2^2 + M_2^2)},$$

where $M_1 = 769\ \mathrm{MeV}$, $M_2 = 1465\ \mathrm{MeV}$, $h_5 = 6.93\ \mathrm{GeV}^4$.

with two modifications:

- **form factor: undressed soft photon (non-renormalization of ABJ)**
- $h_2 = 0 \pm 20$ **GeV^2 (KN) vs.** $h_2 = -10$ **GeV^2 (MV) fixed by twist 4 in OPE** $(1/q^4)$

	π^0, η, η'[π^0]	a_1[f_1, f_1^*]	$\pi^\pm$	pQCD/QPM	tot
HK	83(06)	1.7 [a_1	-4.5(8.5)	10(11)	90(15)
BPP	85(13)	-4(3) [$a_1 + f_0$]	-19(5)	21(3)	83(32)
KN	83(12)			80(40)	
MV	114.5[76.5]	22[7]	0	0	136(25)

Is this the final answer? How to improve? A limitation to more precise $g-2$ tests?

Looking for new ideas to get ride of modeldependence

In principle lattice QCD could provide an answer [far future]

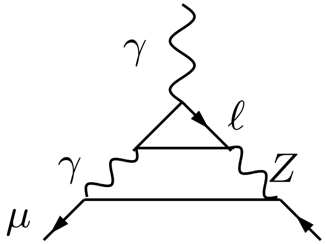
⑦ Leading hadronic electroweak effects in $g-2$ and the VVA anomaly

Electroweak two–loop calculations: leptons yield large terms proportional to

$$\sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

enhanced by a large logarithm (Kukhto, Kuraev, Schiller, Silagadze, 1992)

The most important diagrams VVA triangle diagrams ($VVV=0, VVA\neq 0$)



$$a_\mu^{(4)\,\mathrm{EW}}(\ell) \simeq -\frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\left[3\ln\frac{M_Z^2}{m_\ell^2}+C_\ell\right]$$

Anomaly cancellation by lepton quark duality: $\sum_f N_{cf} Q_f^2 T_{3f} = 0$!

Need consider complete family: $\Rightarrow$ hadronic effects on quarks triangle graphs?

Quarks, parton model: **(Peris et al. 1995, Czarnecki, Krause, Marciano, 1995)**

First family in QPM adopting constituent quark masses would yield

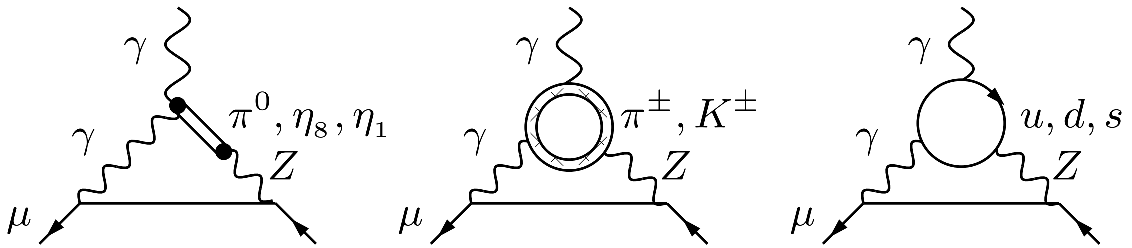
$$a_\mu^{(4)\,\mathrm{EW}}([e,u,d])_{\mathrm{QPM}} \simeq -\frac{\sqrt{2}G_\mu\, m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\,\left[\ln\frac{m_u^8}{m_\mu^6 m_d^2}+\frac{17}{2}\right]$$

need to know ill-defined constituent quark masses.

Note: large logs $\sim \ln M_Z$ **have dropped!**

Quarks $\longrightarrow$ **low energy effective field theory (LEFT):**

(Peris, Perrottet, de Rafael, 1995, Knecht, Peris, Perrottet, de Rafael, 2002)



The two leading χPT diagrams (L.D.) and the QPM diagram (S.D.)

in unitary gauge and in the chiral limit yields

$$a_\mu^{(4)\,\mathrm{EW}}([u,d,s];p<M_\Lambda)_{\mathrm{CHPT}} = \frac{\sqrt{2}G_\mu\,m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\,\left[\frac{4}{3}\ln\frac{M_\Lambda^2}{m_\mu^2}+\frac{2}{3}+O\left(\frac{m_\mu^2}{M_\Lambda^2}\ln\frac{M_\Lambda^2}{m_\mu^2}\right)\right]$$

$$a_\mu^{(4)\,\mathrm{EW}}([u,d,s];p>M_\Lambda)_{\mathrm{QPM}} = \frac{\sqrt{2}G_\mu\,m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\,\left[2\ln\frac{M_Z^2}{M_\Lambda^2}\right]$$

Note that the last diagram of in fact takes into account the leading term of w_T which is protected by Vainshtein's relation (below).

Including the finite contributions from e, μ and c :

$$a_\mu^{(4)\,\mathrm{EW}}([e,\mu,c])_{QPM} = \frac{\sqrt{2}G_\mu\,m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\,\left[-6\ln\frac{M_Z^2}{m_\mu^2}+4\ln\frac{M_Z^2}{m_c^2}-\frac{37}{3}+\frac{8}{9}\pi^2\right]$$

the complete answer for the 1st plus 2nd family reads (Peris et al. 1995)

$$a_\mu^{(4)\,\mathrm{EW}}\left(\left[\begin{array}{c} e,u,d \\ \mu,c,s \end{array}\right]\right)_{\mathrm{CHPT}} = \frac{\sqrt{2}G_\mu\,m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\,\left[-\frac{14}{3}\ln\frac{M_\Lambda^2}{m_\mu^2}+4\ln\frac{M_\Lambda^2}{m_c^2}-\frac{35}{3}+\frac{8}{9}\pi^2\right]$$

$$\simeq -7.09(13)\cdot 10^{-11}\ .$$

+ refinements (Marseille group, Czarnecki, Marciano, Vainshtein, 2003)

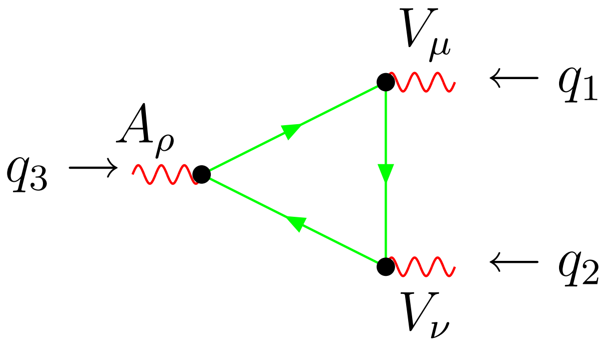
VVA correlation function

$$\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = i \int d^4x_1 d^4x_2 \, e^{i(q_1\cdot x_1+q_2\cdot x_2)} \, \langle\, 0 \,|\, \mathrm{T}\{V_\mu(x_1)V_\nu(x_2)A_\rho(0)\} \,|\, 0 \,\rangle$$

of the flavor and color diagonal fermion currents

$$V_\mu \; = \; \overline{\psi}\gamma_\mu \, \psi \quad , \quad A_\mu \; = \; \overline{\psi}\gamma_\mu\gamma_5 \, \psi$$

where ψ is a quark field. To leading order the correlator of interest is associated with the one–loop triangle diagram



plus its crossed ($q_1,\mu \leftrightarrow q_2,\nu$) partner ($q_3=-(q_1+q_2)$).

Ward identities: vector current conservation imposed

$$q_1^\mu \mathcal{W}_{\mu\nu\rho}(q_1,q_2) = q_2^\nu \mathcal{W}_{\mu\nu\rho}(q_1,q_2) = 0$$

ABJ axial current anomaly: (Adler 1969, Bell, Jackiw 1969)

$$q_3^\rho \mathcal{W}_{\mu\nu\rho}(q_1,q_2) = -\frac{1}{8\pi^2}\, w_L\, \left(q_1^2,q_2^2,q_3^2\right)\, q_3^3\, \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta = -\frac{1}{2\pi^2}\varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta \sum_f N_{cf} T_f Q_f^2$$

For the static low energy quantity $a_\mu = \frac{1}{2}(g-2)_\mu = F_{\rm M}(0)$, given by the Pauli form factor at zero momentum transfer, the VVA correlator is required in the limit

$$\begin{aligned} \mathcal{W}_{\mu\nu\rho}(q_1=k+q,q_2=-k) \;\; &= \\ -\frac{1}{8\pi^2}\left\{ w_L\left(q^2,0,q^2\right)\, q_\rho\, \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha k^\beta + \; w_T\left(q^2,0,q^2\right)\, t_{T\,\mu\nu\rho}\right\} &+ O(k^2)\,, \end{aligned}$$

with

$$t_{T\,\mu\nu\rho} = \left\{ q^2 \varepsilon_{\mu\nu\rho\sigma} k^\sigma + q_\mu \varepsilon_{\rho\nu\alpha\beta} q^\alpha k^\beta - q_\rho \varepsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta \right\}\;.$$

Indeed, in this kinematic region, the leading strong interaction effects may be parametrized by two VVA amplitudes, a longitudinal $w_L(Q^2)$ and a transversal $w_T(Q^2)$ one as functions of $Q^2=-q^2$, which contribute as

$$\Delta a_\mu^{(4)\,{\rm EW}}([f])_{\rm VVA} \simeq \frac{\sqrt{2} G_\mu\, m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\int_{m_\mu^2}^{\Lambda^2} dQ^2\, \left(w_L(Q^2) + \frac{M_Z^2}{M_Z^2+Q^2}\, w_T(Q^2)\right)\;,$$

where Λ is a cutoff to be taken to ∞ at the end. (Vainshtein and Knecht et al.) have shown that in the chiral limit the relation

$$w_T(Q^2)_{\rm pQCD}\Big|_{m=0} = \frac{1}{2}\, w_L(Q^2)\Big|_{m=0}\ ,$$

which was known to hold at one–loop (Rosenberg 1963), is valid actually to all orders of perturbative QCD. Vainshtein's theorem follows from the symmetry ($\rho, q \leftrightarrow \mu, q+k$) ($k \to 0$). Formally, discarding regularization problems, the asymptotic symmetry derives from the fact that γ_5 may be moved from the A_ρ vertex to the V_μ vertex by anti-commuting it an even number of times. Thus for the quarks the non-renormalization theorem valid beyond pQCD for the anomalous amplitude w_L (normalized to $Z^*\gamma^*\gamma^*$)

$$w_L(Q^2)\Big|_{m=0} = w_L^{1-\rm loop}(Q^2)\Big|_{m=0} = \sum_q \frac{4N_cT_qQ_q^2}{Q^2}$$

carries over to the perturbative part of the transversal amplitude. Thus in the chiral limit the perturbative QPM result for w_T is exact. This may be somewhat puzzling, since in low energy effective QCD, which encodes the non-perturbative strong interaction effects, this kind of term seems to be absent. The term is recovered however by taking into account all relevant terms in the operator product expansion.

General covariant decomposition of $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$ respecting Ward identities

$$
\begin{aligned}
-8\pi^2\,\mathcal{W}_{\mu\nu\rho}(q_1,q_2) \;&=\; w_L\left(q_1^2,q_2^2,q_3^2\right) q_{3\rho}\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta + w_T^{(+)}\left(q_1^2,q_2^2,q_3^2\right) t^{(+)}_{\mu\nu\rho}(q_1,q_2) \\
&+ w_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right) t^{(-)}_{\mu\nu\rho}(q_1,q_2) + \widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right) \widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2)\,,
\end{aligned}
$$

with $-q_3 = q_1 + q_2$ and transverse tensors given by

$$
\begin{aligned}
t^{(+)}_{\mu\nu\rho}(q_1,q_2) &= q_{1\nu}\,\varepsilon_{\mu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \;-\; q_{2\mu}\,\varepsilon_{\nu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \;-\; q_1q_2\,\varepsilon_{\mu\nu\rho\alpha}\,(q_1-q_2)^\alpha + \frac{2\,q_1q_2}{q_3^2}\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta q_{3\rho} \\
t^{(-)}_{\mu\nu\rho}(q_1,q_2) &= \left[(q_1-q_2)_\rho + \frac{q_1^2-q_2^2}{q_3^2}\,q_{3\rho}\right]\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta \\
\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2) &= q_{1\nu}\,\varepsilon_{\mu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \;+\; q_{2\mu}\,\varepsilon_{\nu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \;+\; q_1q_2\,\varepsilon_{\mu\nu\rho\alpha}\,q_3^\alpha\,.
\end{aligned}
$$

The longitudinal part is entirely fixed by the anomaly,

$$
\boxed{w_L\left(q_1^2,q_2^2,q_3^2\right) \;=\; -\frac{2N_c}{q_3^2}}
$$

exact to all orders of perturbation theory **Adler-Bardeen non-renormalization theorem 1969**

The Vainshtein relation is obtained in the limit upon identifying (with $Q^2=-q^2$)

$$
\begin{aligned}
w_L(Q^2) \;&\equiv\; w_L(q^2,0,q^2)\,, \\
w_T(Q^2) \;&\equiv\; w_T^{(+)}(q^2,0,q^2) + \widetilde{w}_T^{(-)}(q^2,0,q^2)\,.
\end{aligned}
$$

Bose symmetry ($q_1, \mu \leftrightarrow q_2, \nu$) entails

$$\begin{aligned} w_T^{(+)}\left(q_2^2, q_1^2, q_3^2\right) &= +w_T^{(+)}\left(q_1^2, q_2^2, q_3^2\right), \\ w_T^{(-)}\left(q_2^2, q_1^2, q_3^2\right) &= -w_T^{(-)}\left(q_1^2, q_2^2, q_3^2\right), \quad \widetilde{w}_T^{(-)}\left(q_2^2, q_1^2, q_3^2\right) = -\widetilde{w}_T^{(-)}\left(q_1^2, q_2^2, q_3^2\right) . \end{aligned}$$

Knecht et al. 2004 derive chiral symmetry relations between amplitudes

$$\left\{\left[w_T^{(+)} + w_T^{(-)}\right]\left(q_1^2, q_2^2, q_3^2\right) - \left[w_T^{(+)} + w_T^{(-)}\right]\left(q_3^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} = 0$$

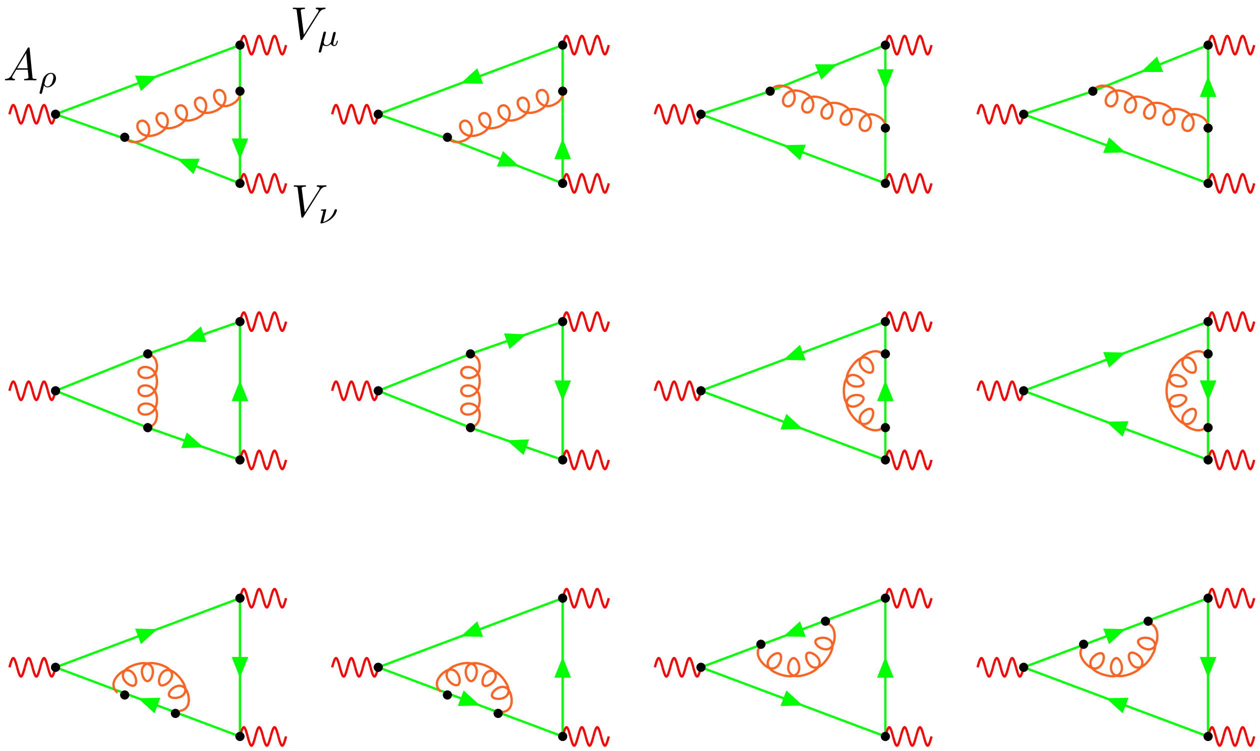
$$\left\{\left[\widetilde{w}_T^{(-)} + w_T^{(-)}\right]\left(q_1^2, q_2^2, q_3^2\right) + \left[\widetilde{w}_T^{(-)} + w_T^{(-)}\right]\left(q_3^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} = 0$$

and

$$\left\{\left[w_T^{(+)} + \widetilde{w}_T^{(-)}\right]\left(q_1^2, q_2^2, q_3^2\right) + \left[w_T^{(+)} + \widetilde{w}_T^{(-)}\right]\left(q_3^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} - w_L\left(q_3^2, q_2^2, q_1^2\right)$$

$$= -\left\{\frac{2\left(q_2^2 + q_1 \cdot q_2\right)}{q_1^2}\, w_T^{(+)}\left(q_3^2, q_2^2, q_1^2\right) - 2\,\frac{q_1 \cdot q_2}{q_1^2}\, w_T^{(-)}\left(q_3^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}},$$

hold for all values of the momentum transfers q_1^2, q_2^2 and q_3^2 ((Knecht et al. 2004)). Implies Vainshtein's relation for $q_1 = k + q, q_2 = -k$ in the limit $k \to 0$.



Two-loop QCD correction diagrams contributing to the $\langle VVA \rangle$ correlator

Method:

- **use dimensional regularization**
- **use linear covariant gauge with free gauge parameter ξ**
- **write down all fermion loops starting with the axial-vector vertex, and then perform Feynman integrals and Dirac algebra without assuming any property of γ_5 at all. In this way all diagrams will be expressed in terms of traces of 10 combinations of γ matrices:**

$$
\begin{array}{lll}
4iA_1 = \gamma_\rho\gamma_5\gamma_\mu\gamma_\nu\hat{q}_2, & 4iA_2 = \gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\gamma_\nu, & 4iA_3 = q_2^\mu\gamma_\rho\gamma_5\hat{q}_1\gamma_\nu\hat{q}_2, \\
4iA_4 = q_1^\mu\gamma_\rho\gamma_5\hat{q}_1\gamma_\nu\hat{q}_2, & 4iA_5 = -q_2^\nu\gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\hat{q}_2, & 4iA_6 = -q_1^\nu\gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\hat{q}_2, \\
4iA_7 = \gamma_\rho\gamma_5\hat{q}_1, & 4iA_8 = \gamma_\rho\gamma_5\hat{q}_2, & 4iA_9 = \gamma_\rho\gamma_5\gamma_\mu, \\
4iA_{10} = \gamma_\rho\gamma_5\gamma_\nu & &
\end{array}
$$

with $\hat{q} \equiv q_\mu\gamma^\mu$.

The prescription is sufficient to enable us to arrive at expressions in front of $A_1, \ldots A_{10}$

which have finite limits as $d \to 4$. After this the usual formulae

$$\mathrm{Tr}[\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu] = 4i\varepsilon_{\alpha\beta\mu\nu}\ , \quad \mathrm{Tr}[\gamma_5\gamma_\alpha\gamma_\beta] = 0$$

valid in $d = 4$ dimensions were used. In our convention $\varepsilon_{0123} = +1$ and $(1-\gamma_5)/2$ projects to left–handed fermion fields.

- **all scalar integrals were reduced to 6 master integrals by using the Gröbner basis technique (Tarasov 1998, 2004)**
- **expressions for the individual diagrams are sums over 21 terms, which are combinations of the 6 master integrals**

$$\begin{aligned}
I_2^{(d)}(q_j^2) &= \int \frac{d^dk_1}{[i\pi^{d/2}]}\frac{1}{k_1^2(k_1-q_j)^2}, \\
I_3^{(d)}(q_1^2,q_2^2,q_3^2) &= \int \frac{d^dk_1}{[i\pi^{d/2}]}\frac{1}{k_1^2(k_1-q_1)^2(k_1-q_2)^2} \\
J_3^{(d)}(q_j^2) &= \int\int \frac{d^dk_1d^dk_2}{[i\pi^{d/2}]^2}\frac{1}{k_1^2(k_1-k_2)^2(k_2-q_j)^2},
\end{aligned}$$

$$
\begin{aligned}
R_1(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1 d^dk_2}{[i\pi^{d/2}]^2} \frac{1}{k_1^2(k_1-k_2)^2(k_2-q_1)^2(k_2+q_2)^2} \\
R_2(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1 d^dk_2}{[i\pi^{d/2}]^2} \frac{1}{k_1^4(k_1-k_2)^2(k_2-q_1)^2(k_2+q_2)^2}, \\
P_5(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1 d^dk_2}{[i\pi^{d/2}]^2} \frac{1}{k_1^2k_2^2(k_1-k_2)^2(k_1-q_1)^2(k_2+q_2)^2}.
\end{aligned}
$$

and multiplied by ratios of polynomials in momenta and d:

$$
D_j = \sum_{k=1}^{21} M_k \frac{P_k(q_r^2,d)}{Q_k(q_s^2,d)} \quad .
$$

The momentum dependence of the denominators turns out to be rather simple:

$$
Q_k(q_s^2,d) = Q(d)(q_1^2)^{a_k}(q_2^2)^{b_k}(q_3^2)^{c_k}\Delta^{e_k}
$$

where a_k,b_k,c_k,e_k, are some numbers, $Q(d)$ is a polynomial in d and

$$
\Delta = q_1^4+q_2^4+q_3^4-2q_1^2q_2^2-2q_1^2q_3^2-2q_2^2q_3^2 \quad .
$$

- **perform expansion in $\varepsilon = (d-4)/2$ (Usyukina, Davydychev 2004)**
- **perform $\overline{\mathrm{MS}}$ renormalization**
- **sum of all diagrams is gauge parameter independent**
- **in the Feynman gauge at $q_3 = 0$ and for arbitrary d, the results of our calculation are in agreement with the ones presented by (Jones, Leveille 1982) diagram by diagram**
- **bring tensor to standard form by applying Schouten identities**

Results **Form-factors: 1 + 2 loops in $\overline{\mathrm{MS}}$ scheme**

$$w_T^{(\pm)}(q_1^2,q_2^2,q_3^2) = N_f\,N_c\,w_{1,T}^{(\pm)}(q_1^2,q_2^2,q_3^2) + \frac{\alpha_s}{4\pi}\,N_f\,N_c\,C_2(R)\,w_{2,T}^{(\pm)}(q_1^2,q_2^2,q_3^2)$$

$$w_L(q_1^2,q_2^2,q_3^2) = N_f\,N_c\,w_{1,L}(q_1^2,q_2^2,q_3^2) + \frac{\alpha_s}{4\pi}\,N_f\,N_c\,C_2(R)\,w_{2,L}(q_1^2,q_2^2,q_3^2)$$

α_s **QCD coupling** N_f **is the number of flavors**

N_c **the number of colors** $C_2(R) = 4/3$ for QCD

In addition to $\overline{\mathrm{MS}}$ renormalization factor a finite renormalization is required (Trueman 79)

$$(J_\rho^5)_r = Z_5 Z_{\overline{\mathrm{MS}}}\,(J_\rho^5)_0.$$

Our calculation:

$$Z_5 = 1 - 4\,C_2(R)\,\frac{\alpha_s}{4\pi}\quad .$$

in agreement with (Jones & Leveille 1982/83, Larin 1993)

General massless 2–loop result surprisingly simple: $x = \frac{q_1^2}{q_3^2}$ $y = \frac{q_2^2}{q_3^2}$

$$
\begin{aligned}
w_{2,L}(q_1^2,q_2^2,q_3^2) &= -8/q_3^2, \\
w_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2) &= \Big(8(x-y)\Delta \ + 8(x-y)(6xy+\Delta)\Phi^{(1)}(x,y) \\
&\quad - 4[18xy+6x^2-6x+(1+x+y)\Delta)]L_x \\
&\quad +4[18xy+6y^2-6y+(1+x+y)\Delta)]L_y\Big)\,/(q_3^2\Delta^2)
\end{aligned}
$$

$$
\begin{aligned}
w_{2,T}^{(+)}(q_1^2,q_2^2,q_3^2) &= \Big(+8\Delta \ + 8[6xy+(x+y)\Delta]\Phi^{(1)}(x,y) \\
&\quad - 4[6x+\Delta](x-y-1)L_x \\
&\quad +4[6y+\Delta](x-y+1)L_y)\,/(q_3^2\Delta^2) \\
\widetilde{w}_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2) &= -w_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2),
\end{aligned}
$$

with logarithms $L_x = \ln x, \ \ L_y = \ln y$ **and dilogarithmic function**

$$
\begin{aligned}
\Phi^{(1)}(x,y) = \frac{1}{\lambda}\Big\{\, 2\left(\mathbf{Li}_2\left(-\rho x\right) + \mathbf{Li}_2\left(-\rho y\right)\right) \\
+\ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\Big\},
\end{aligned}
$$

where

$$\lambda(x,y) \equiv \sqrt{\Delta} \quad , \quad \rho(x,y) \equiv 2\,(1-x-y+\lambda)^{-1}, \;\; \Delta = (1-x-y)^2 - 4xy \; .$$

The comparison with the results of the one-loop calculation reveals that

$$\boxed{\mathcal{W}_{\mu\nu\rho}(q_1,q_2)\,|_{two-loop} \;=\; 4\,C_2(R)\,\frac{\alpha_s}{4\pi}\,\mathcal{W}_{\mu\nu\rho}(q_1,q_2)\,|_{one-loop}}$$

Multiplying the sum of one- and two-loop terms by the finite factor Z_5 we arrive at the non-renormalization relation for the full correlator up to two–loops:

$$\boxed{\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \mathcal{W}_{\mu\nu\rho}(q_1,q_2)\,|_{one-loop}}$$

??? is this true to all orders ???

topological reasons?

we think this result is not an accident and should extend the higher orders

Conclusion on VVA

- **the result is rather unexpected; in contrast to anomalous amplitude w_L which is trivial in a sense, the momentum dependence of the two independent transversal amplitudes $w_T^{(\pm)}$ is highly non-trivial; nevertheless the properly renormalized two–loop corrections vanish.**

- **not as expected there seem to exist only 2 (instead of 3) transversal amplitudes for generic momenta. Note for $q_1^2 = q_2^2$ Bose symmetry immediately reduces the number of amplitudes.**

- **most crucial question for hadron physics: if perturbative correction are absent to all ?? orders, how come hadrons into play as inferred by the Wess-Zumino effective action?**

$$\mathcal{L}_{\mathrm{WZ}} = \frac{\alpha}{\pi} \frac{N_c}{24 f_\pi} \left(\pi^0 + \cdots\right) \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \;,$$

which reproduces the ABJ anomaly via the PCAC relation; crucial for understanding

$$\pi^0 \to \gamma\gamma$$

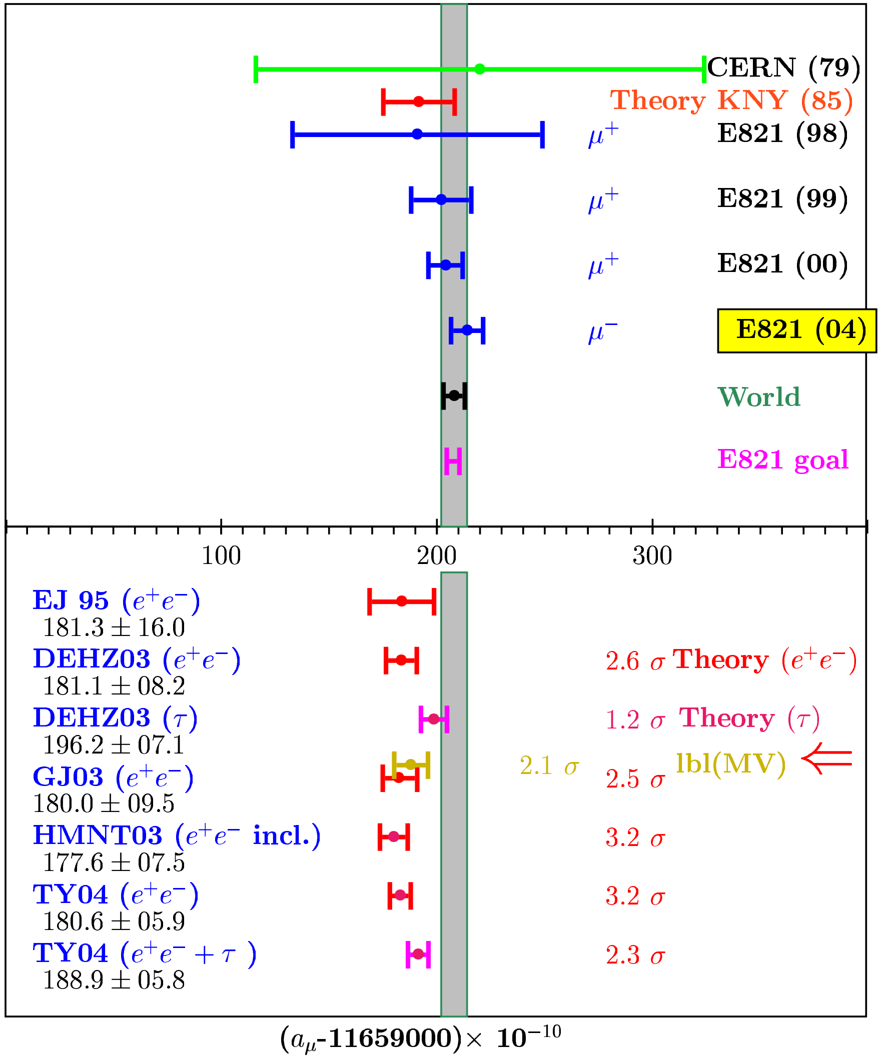
- **is all non-perturbative physics in this context related to quark condensates ? the u, d, s quark condensate $\langle \bar{\psi}\psi \rangle \neq 0$ break the degeneracy $w_T(Q^2) = \frac{1}{2} w_L(Q^2)$**

found in perturbation theory: (Knecht et al. 2002)

$$w_T(Q^2)_{\rm NP} \simeq \frac{16}{9}\pi^2 \, \frac{1}{M_\rho^2} \, \frac{\alpha_s}{\pi} \frac{\langle \bar{\psi}\psi \rangle^2}{Q^6} \;\; \mathrm{at} \;\; Q^2 \;\; \mathrm{large}$$

- **such questions should be investigated by lattice calculations**

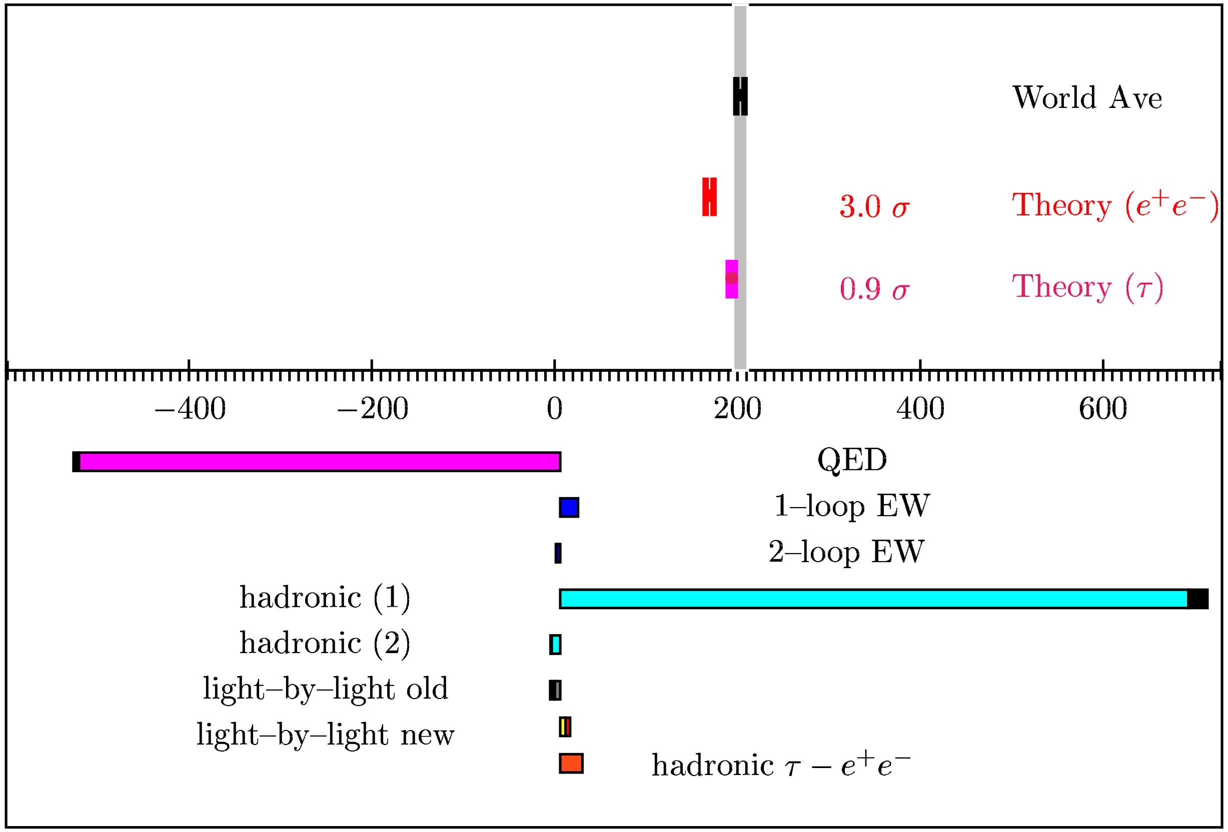
⑧ Outlook on $g-2$



Given theory results only differ by $a_\mu^{\rm had(1)}$!

a_μ: type and size of contributions

$$a_\mu = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{had(1)}} + a_\mu^{\mathrm{had(2)}} + a_\mu^{\mathrm{weak(1)}} + a_\mu^{\mathrm{weak(2)}} + a_\mu^{\mathrm{lbl}} \; (+a_\mu^{\text{new physics}})$$



All kind of physics meets !

Main problems and interesting challange: the hadronic contributions

- **VP effects require low energy experiments determining $e^+e^- \longrightarrow$ hadrons much more precisely (progress is possible mainly by VEPP-2000/DAFNE-II)**
- **LBL remains a real challenge for theory!**
- **Plans for new $g-2$ experiment!**

Present:	**BNL E821**	**0.5 ppm**
Future:	**BNL E969**	**0.2 ppm**
	J-PARC	**0.1 ppm**

History of sensitivity to various contributions