Fractional APT in QCD

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OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT**
- Resolution — **FAPT**: Completed set $\{\mathcal{A}_\nu; \mathfrak{A}_\nu\}_{\nu\in\mathbb{R}}$ and its properties
- Technical development of **FAPT**: higher loops, convergence, accuracy
- Applications: Phenomenological analysis of Higgs decay $H^0 \to b\bar{b}$
- Conclusions

Recent Related Publications

- A. B., Mikhailov, Stefanis – **hep-ph/0607040**
- A. B., Mikhailov, Stefanis – **PRD 72 (2005) 074014**
- A. B., Karanikas, Stefanis – **PRD 72 (2005) 074015**
- A. B., Stefanis – **NPB 721 (2005) 50**
- A. B., Passek-Kumerički, Schroers, Stefanis – **PRD 70 (2004) 033014**
- Karanikas&Stefanis – **PLB 504 (2001) 225**

Analytic Perturbation Theory

in

QCD

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- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to
  $$Z \sim a^{\nu}(L)\Big|_{\nu\,=\,\nu_0\,\equiv\,\gamma_0/(2b_0)}$$

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  m – power $\quad \Rightarrow \quad m$ – index

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Spectral representation

By **analytization** we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \mathbf{Im}\left[f(-\sigma)\right]/\pi$.

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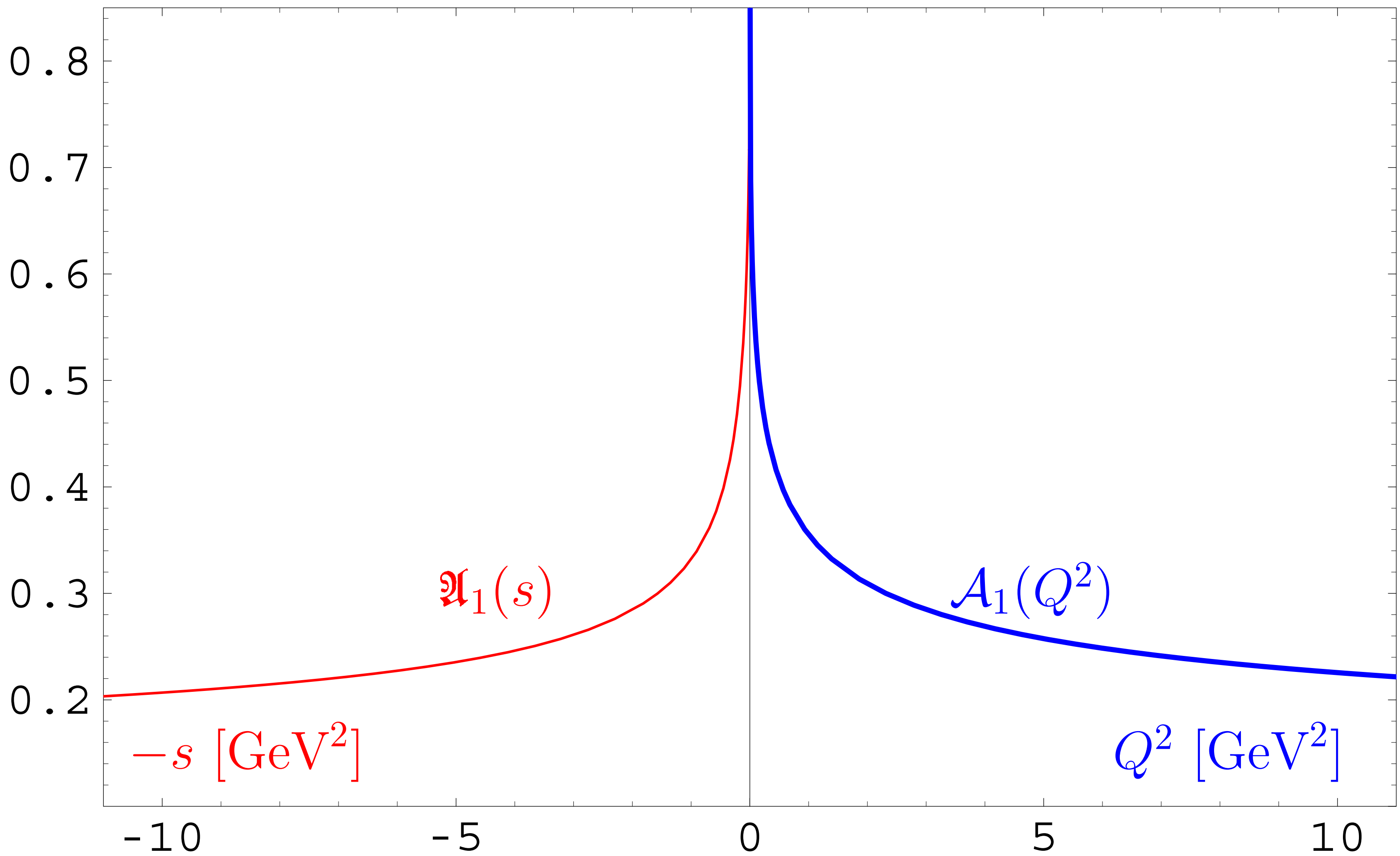
$$a_s^n(L) = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s(L)$$

APT *graphics: Distorting mirror*

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



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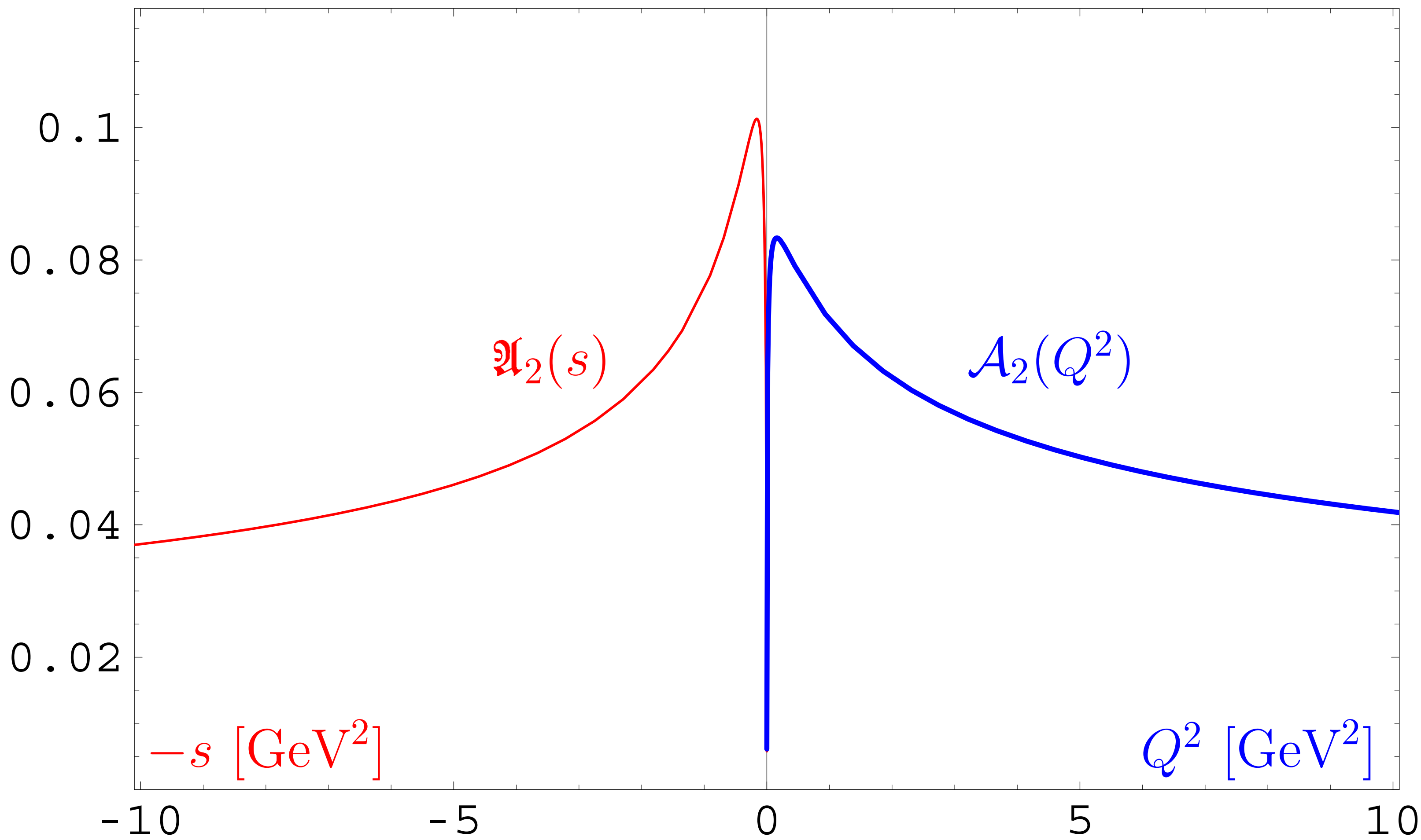
Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



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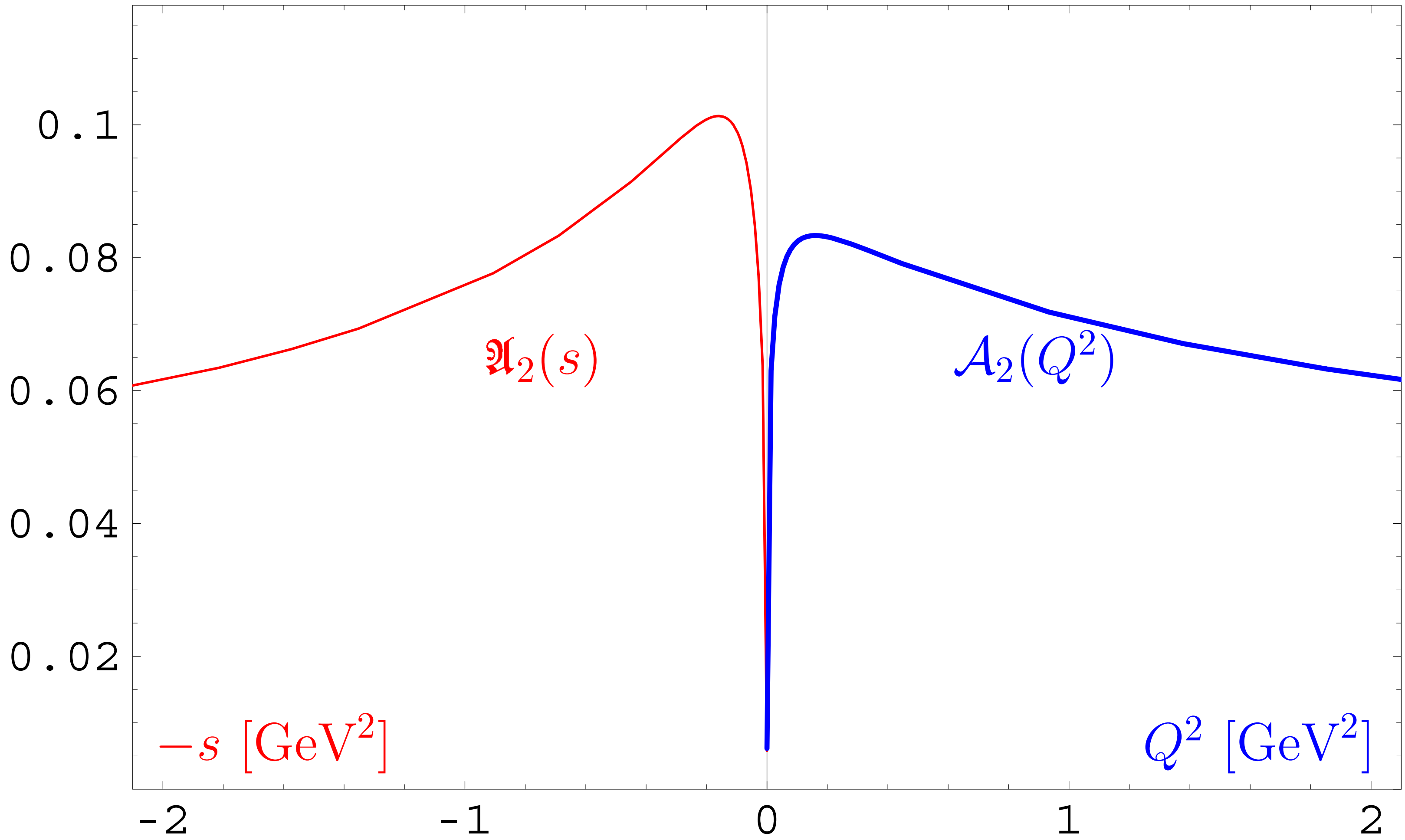

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- “Analytization” of multi-scale amplitudes beyond LO of pQCD
- Appearance of additional logs depending on scale that serves as factorization or renormalization scale
- Evolution induces logarithms to some non-integer, fractional, powers of coupling constant
- Resummation of gluonic corrections, giving rise to Sudakov factors, under “Analytization” difficult task **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

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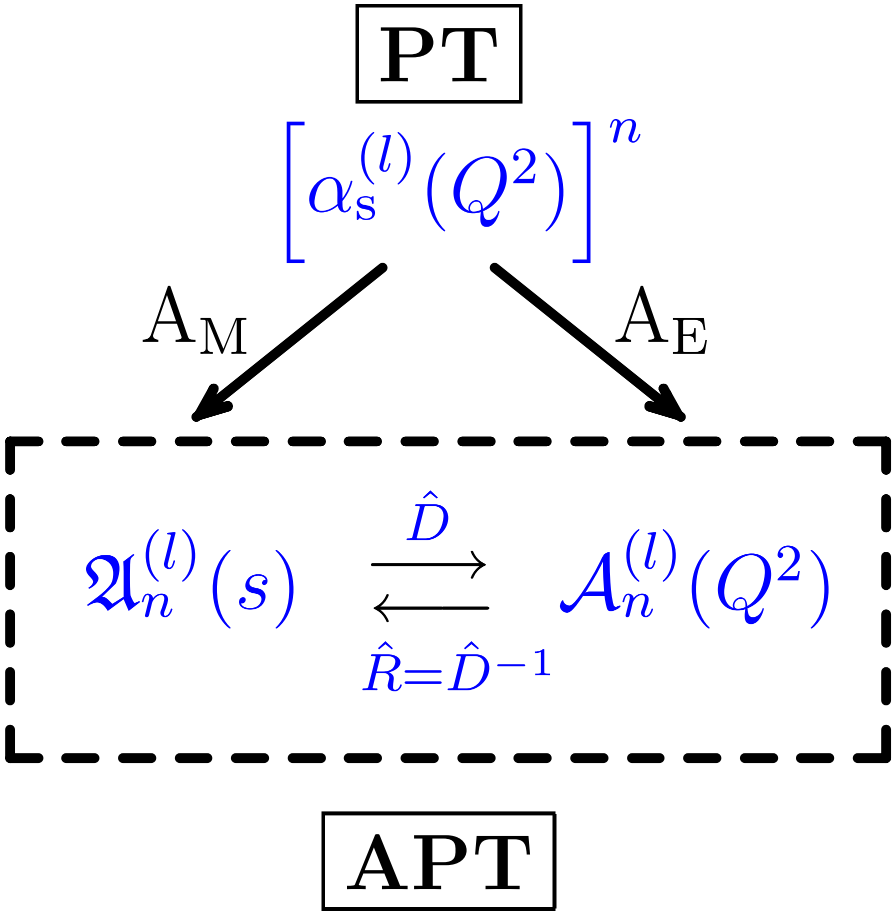
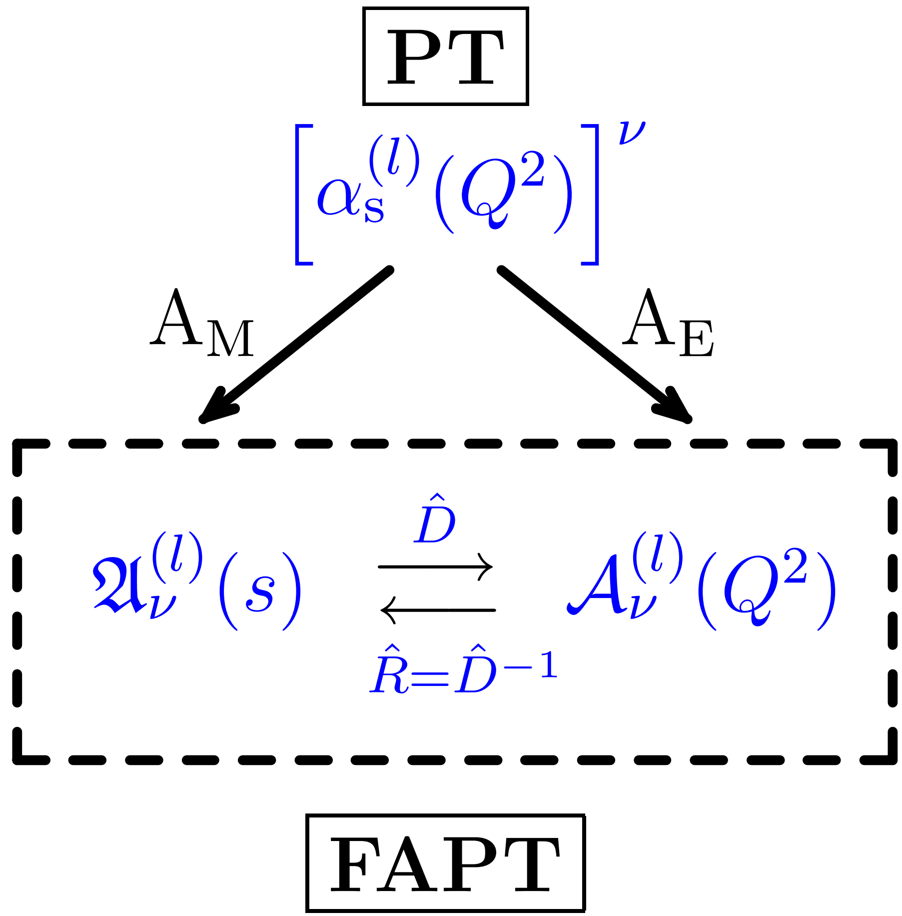
New functions: $(a_s)^\nu\,,\ (a_s)^\nu \ln(a_s),\ (a_s)^\nu L^m,\ e^{-a_s},\ \ldots$

Conceptual scheme of APT



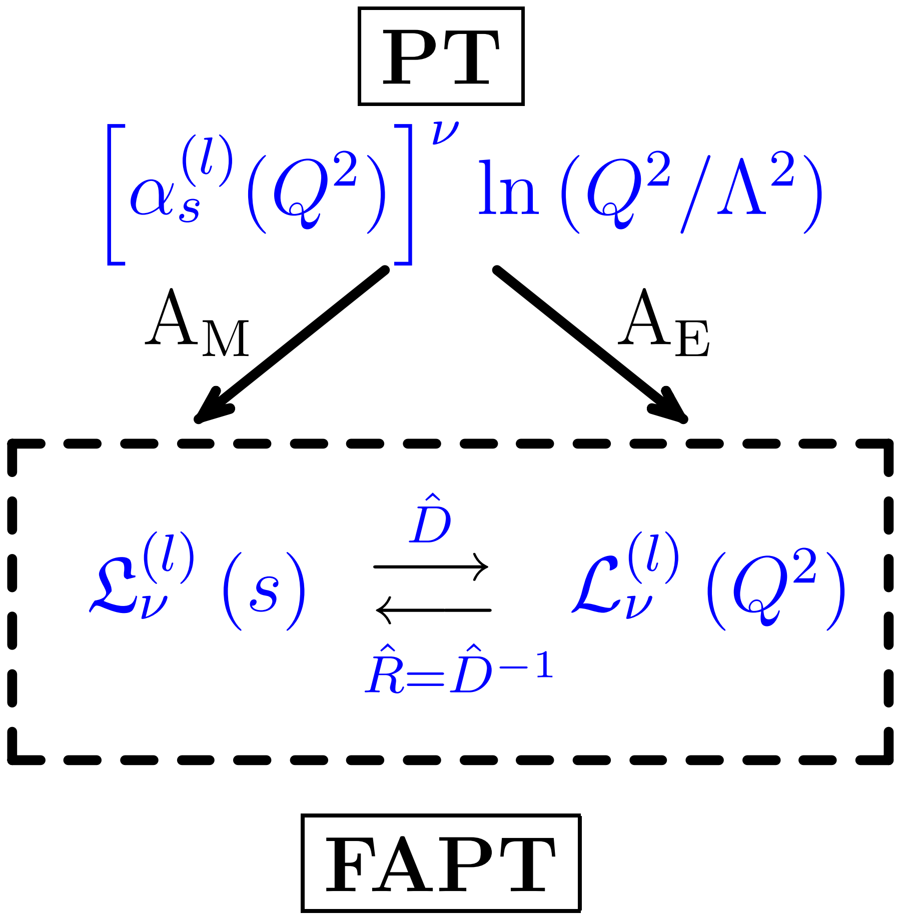
The index n in APT is restricted to integer values only.

Conceptual scheme of FAPT



In FAPT index ν can assume any real values. ✍

Conceptual scheme of FAPT



This enables "analytization" of expressions like shown in figure.

Fractional

APT

FAPT: *Construction of* $\mathcal{A}_\nu(L)$

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Only need to know $\tilde{\mathcal{A}}_1(t)$!

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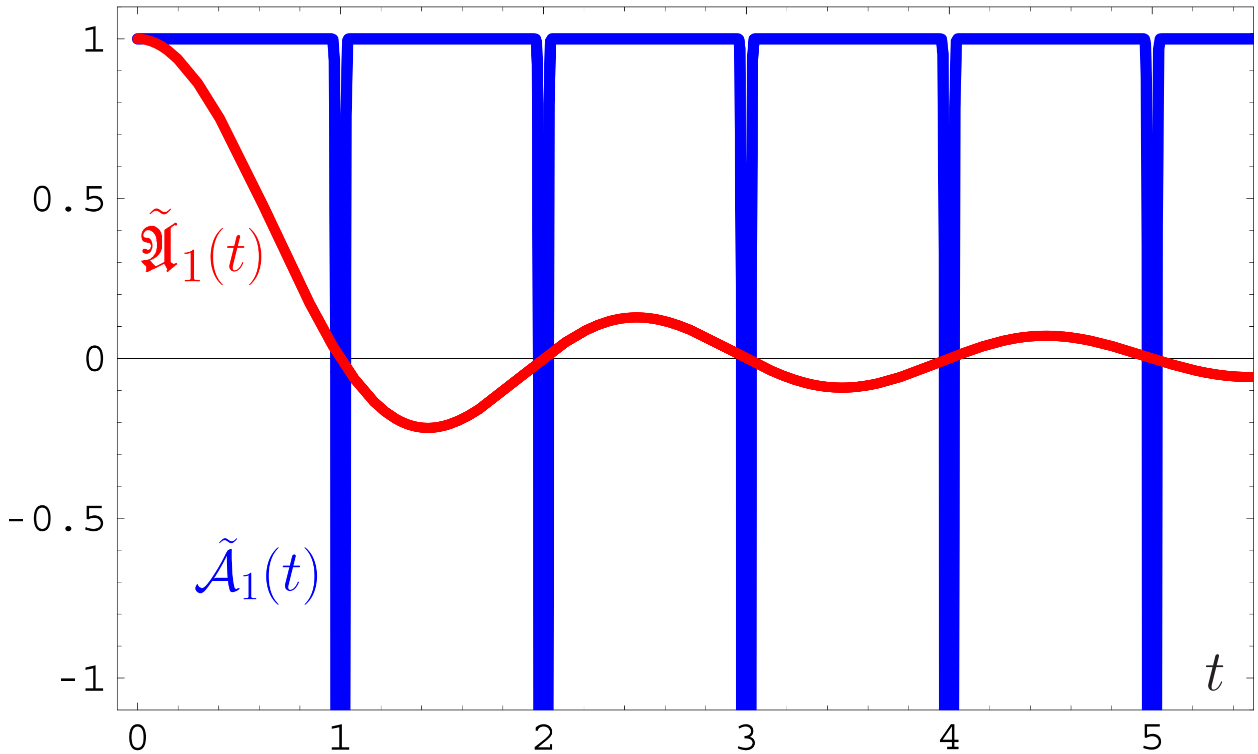
Graphics:



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- Here $F(z, \nu)$ is reduced **Lerch** transcendental function. It is analytic function in ν. Interesting: $\mathcal{A}_\nu(L)$ is entire function in ν.

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  - $\mathcal{A}_m(\pm\infty) = 0$ for $m \geq 2\,,\; m \in \mathbb{N}$; ☞
  - $\mathcal{D}^k\,\mathcal{A}_\nu \equiv \dfrac{d^k}{d\nu^k}\,\mathcal{A}_\nu$

FAPT: *Properties of* $\mathcal{A}_\nu(L)$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

- Here $F(z, \nu)$ is reduced **Lerch** transcendental function.
- Properties:
  - $\mathcal{A}_0(L) = 1$;
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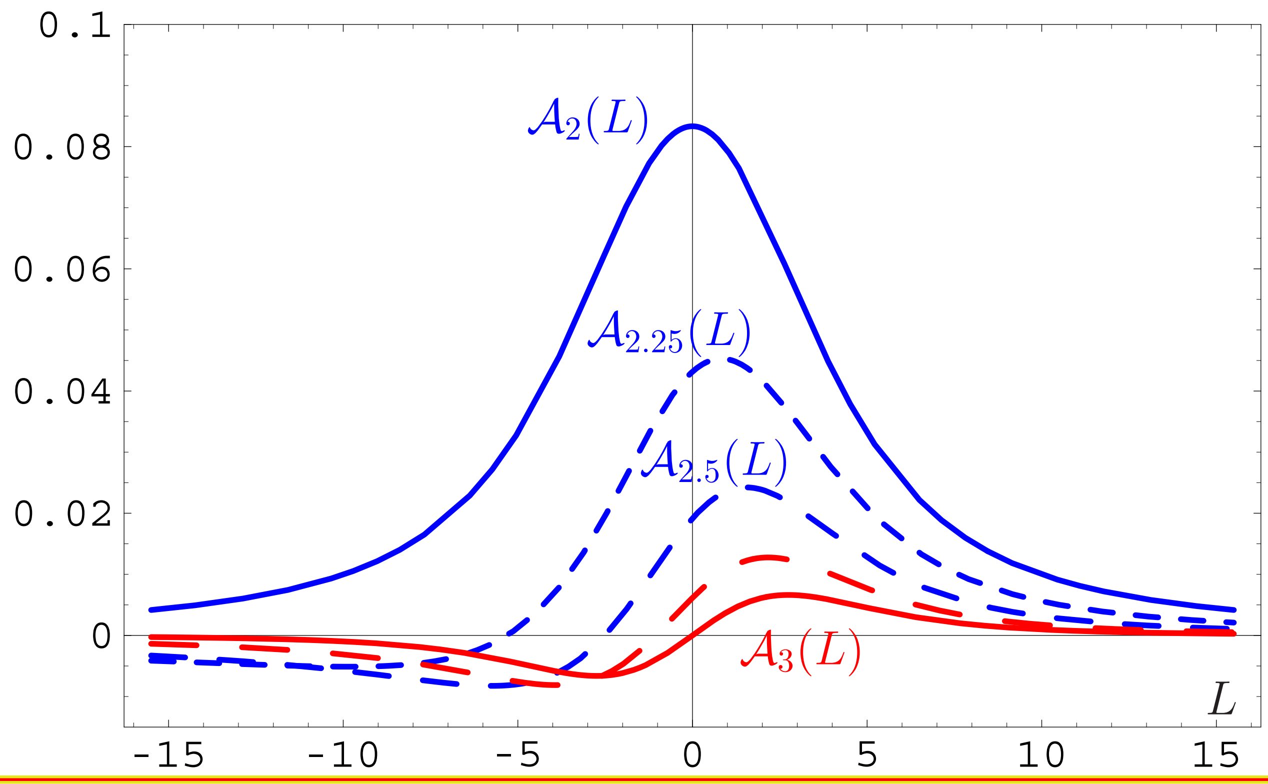
  - $\mathcal{A}_\nu(L) = -\frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty} \zeta(1-\nu-r) \frac{(-L)^r}{r!}$ for $|L| < 2\pi$

FAPT: *Graphics of* $\mathcal{A}_\nu(L)$ *vs.* L

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

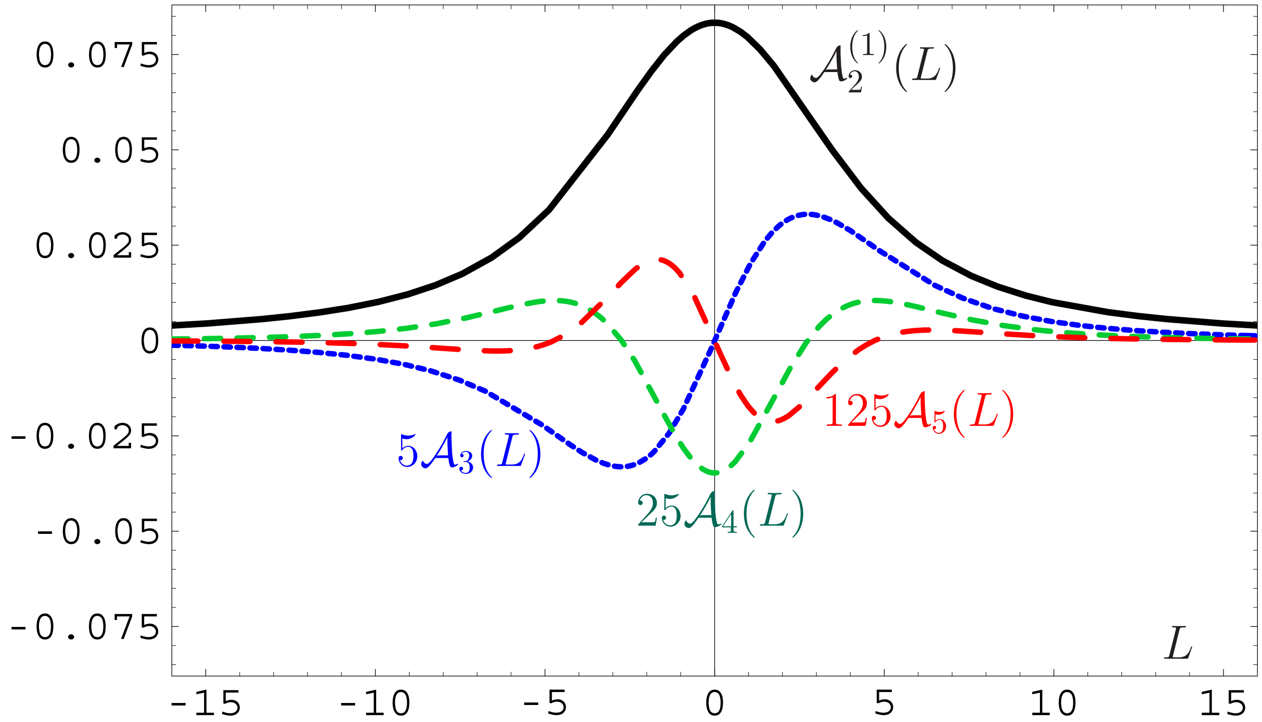
First, graphics for fractional $\nu \in [2, 3]$: ☞



FAPT: *Graphics of $\mathcal{A}_\nu(L)$ vs. L*

$$\mathcal{A}_1(0) = \frac{1}{2}\,, \quad \mathcal{A}_2(0) = \frac{1}{12}\,, \quad \mathcal{A}_4(0) = -\frac{1}{720}\,, \quad \mathcal{A}_3(0) = \mathcal{A}_5(0) = 0$$

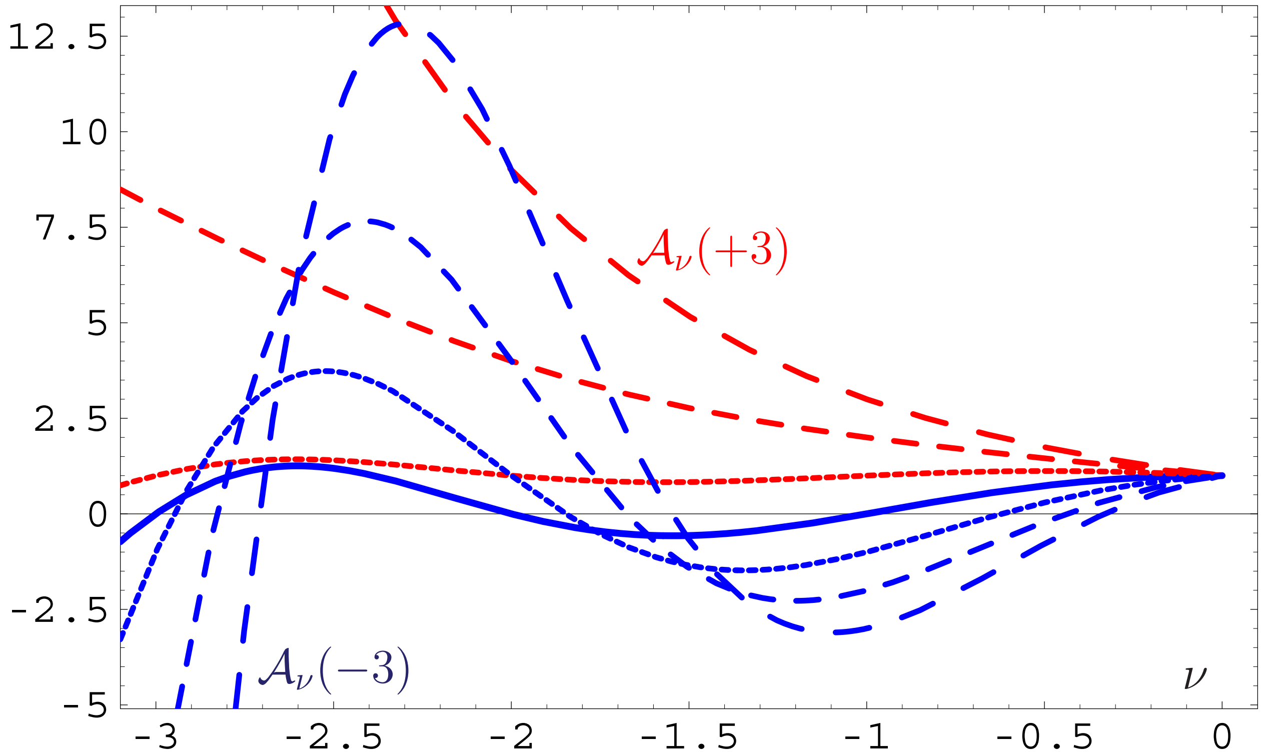
Next, graphics for $\nu = 2, 3, 4, 5$: ☞



FAPT: Graphics of $\mathcal{A}_\nu(L)$ vs. ν

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Now, graphics for $\nu \leq 0$ (note $\mathcal{A}_{-m}(L) = L^m$): ☞



FAPT: *Graphics of* $\mathcal{A}_\nu(L)$ *vs.* ν

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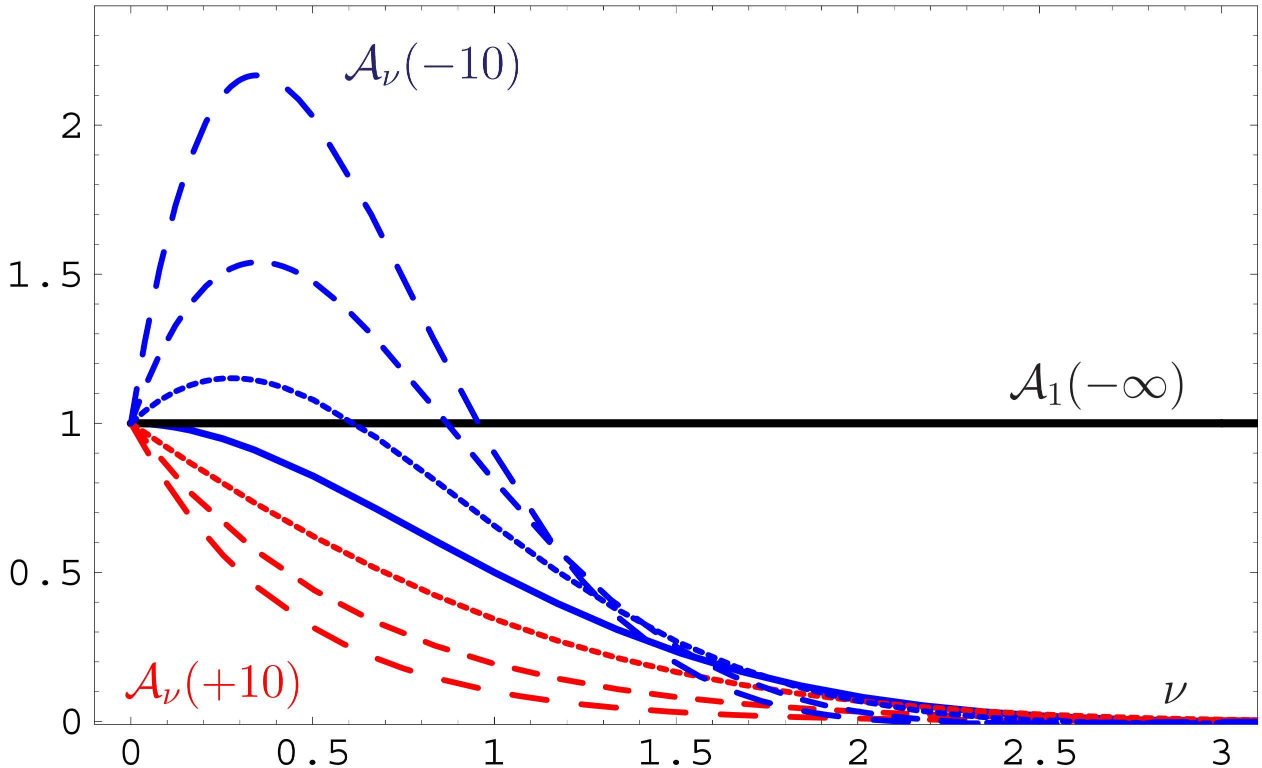
Last, graphics for $\nu \geq 0$: ☞



MFAPT: *Properties of* $\mathfrak{A}_\nu(L)$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu(L) = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

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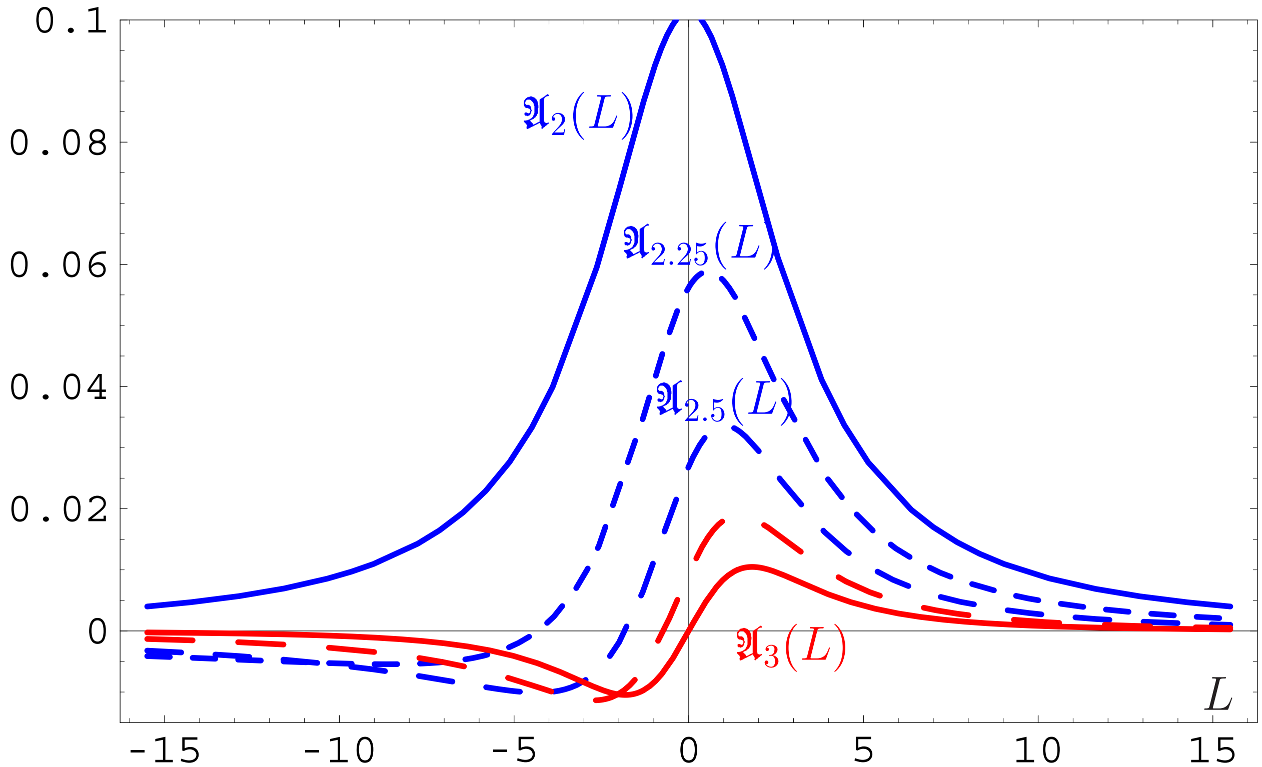
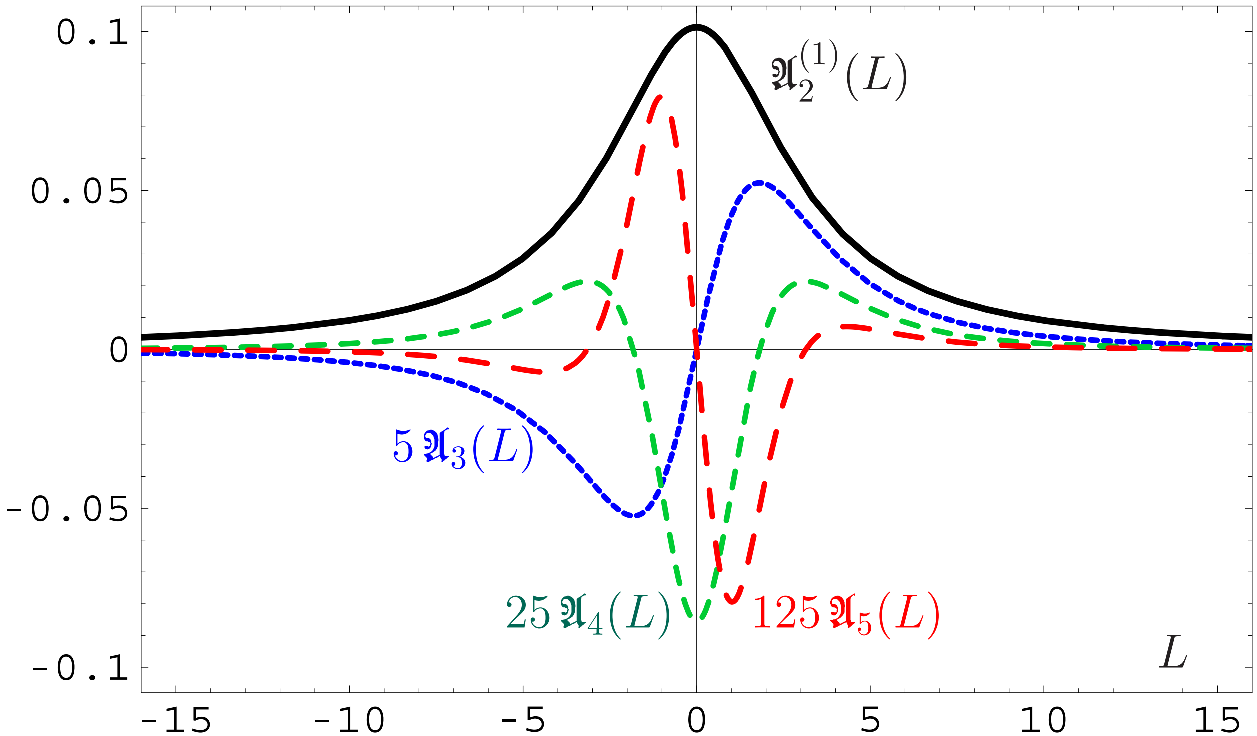


MFAPT: *Graphics of* $\mathfrak{A}_\nu(L)$ *vs.* L

$$\mathfrak{A}_1(0) = \frac{1}{2}\,, \quad \mathfrak{A}_2(0) = \frac{1}{\pi^2}\,, \quad \mathfrak{A}_4(0) = -\frac{1}{3\pi^4}\,, \quad \mathfrak{A}_3(0) = \mathfrak{A}_5(0) = 0$$

Next, graphics for $\nu = 2, 3, 4, 5$: ☞



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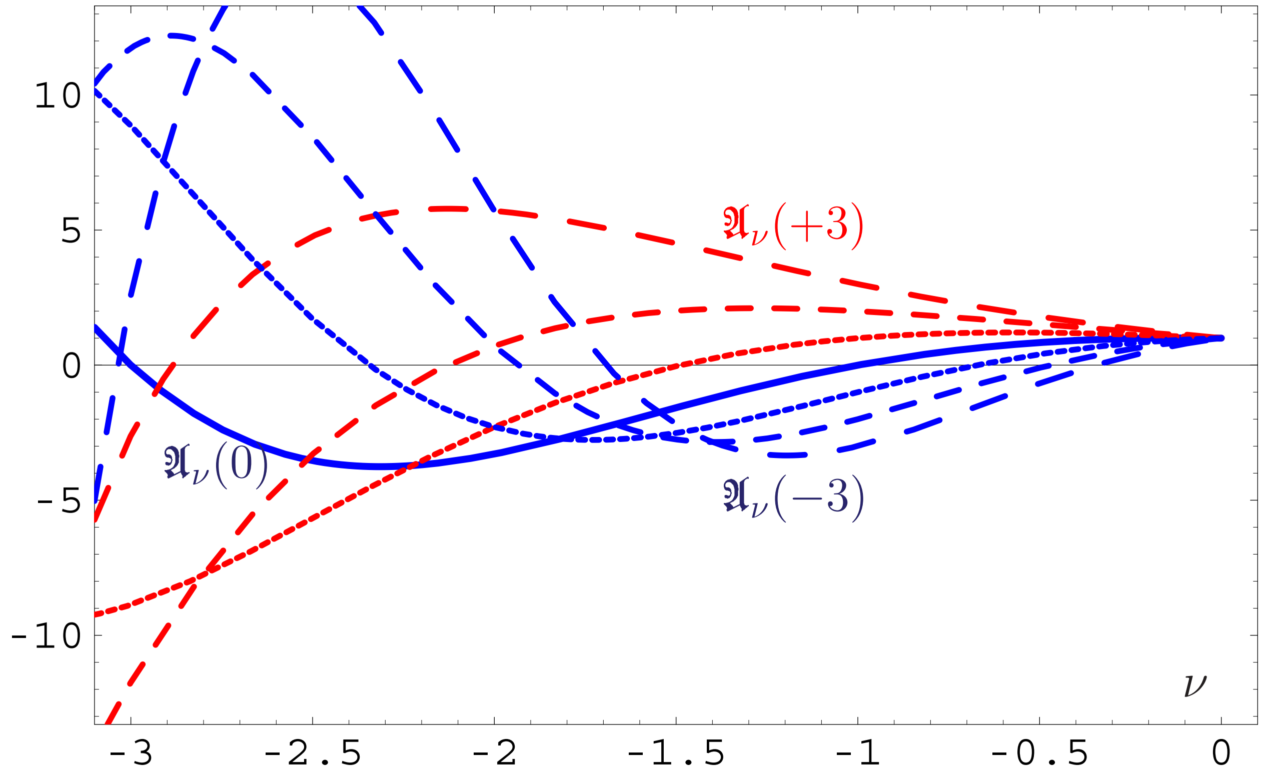
$$\mathfrak{A}_\nu(L) = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Now, graphics for $\nu \leq 0$: ☞



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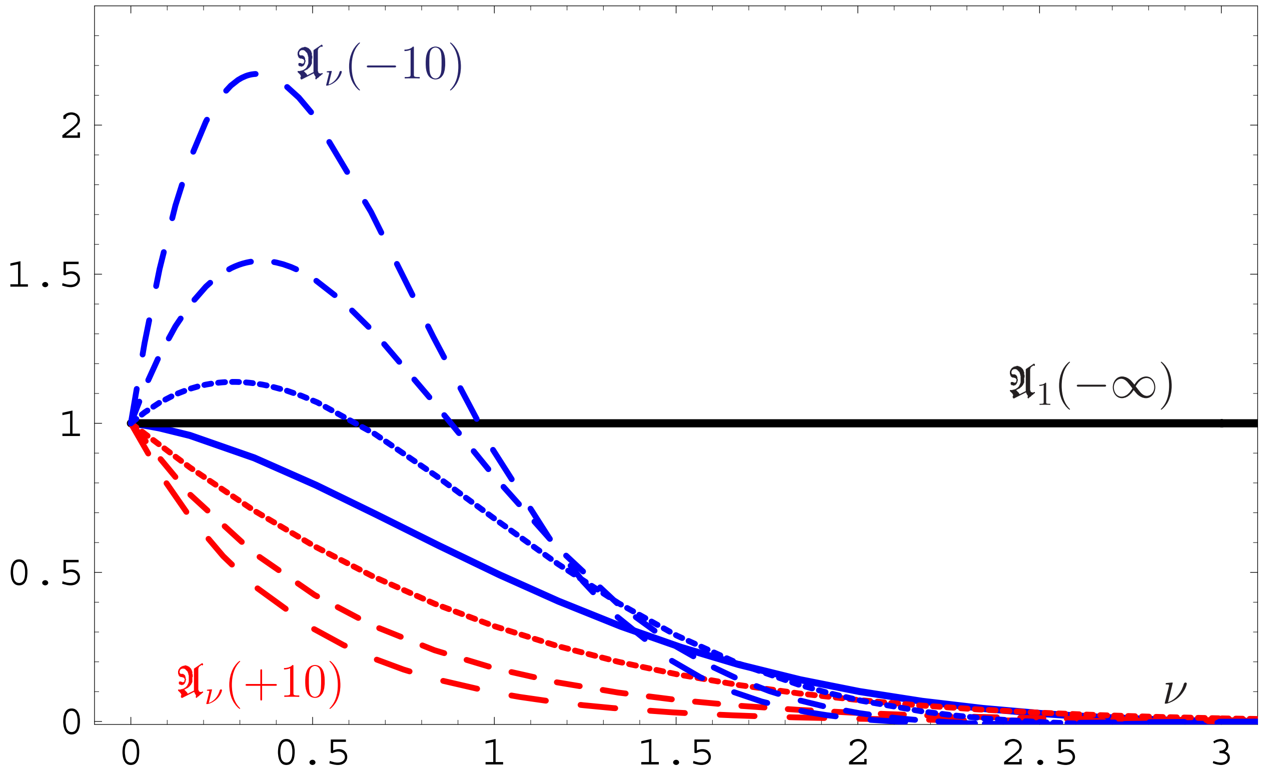
Last, graphics for $\nu \geq 0$: ☞



*Comparison of PT, **APT**, and **FAPT***

Theory	**PT**	**APT**	**FAPT**
Space	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_{\nu}\right\}_{\nu\in\mathbb{R}}$

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Series expansion	$\sum_{m} f_m\, a^m(L)$	$\sum_{m} f_m\, \mathcal{A}_m(L)$	$\sum_{m} f_m\, \mathcal{A}_m(L)$

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Inverse powers	$(a(L))^{-m}$	—	$\mathcal{A}_{-m}(L) = L^m$

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Series expansion	$\sum_{m} f_m\, \boldsymbol{a}^m(\boldsymbol{L})$	$\sum_{m} f_m\, \mathcal{A}_m(L)$	$\sum_{m} f_m\, \mathcal{A}_m(L)$
Inverse powers	$(\boldsymbol{a}(\boldsymbol{L}))^{-m}$	—	$\mathcal{A}_{-m}(L) = L^m$
Multiplication	$\boldsymbol{a}^{\mu}\boldsymbol{a}^{\nu} = \boldsymbol{a}^{\mu+\nu}$	—	—

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Inverse powers	$(a(L))^{-m}$	—	$\mathcal{A}_{-m}(L) = L^m$
Multiplication	$a^\mu a^\nu = a^{\mu+\nu}$	—	—
Index derivative	$a^\nu \ln^k a$	—	$\mathcal{D}^k \mathcal{A}_\nu$

Development of FAPT:

Higher Loops and Logs

Development of FAPT: Two-loop coupling

Two-loop equation for normalized coupling $a = b_0\,\alpha/(4\pi)$ reads

$$\frac{da_{(2)}}{dL} = -a^2_{(2)}(L)\left[1 + c_1\,a_{(2)}(L)\right] \quad \text{with } c_1 \equiv \frac{b_1}{b_0^2}$$

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RG solution of this equation assumes form:

$$\frac{1}{a_{(2)}(L)} + c_1 \ln\left[\frac{a_{(2)}(L)}{1 + c_1 a_{(2)}(L)}\right] = L = \frac{1}{a_{(1)}(L)}$$

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Expansion of $a_{(2)}(L)$ in terms of $a_{(1)}(L) = 1/L$ with inclusion of terms $\mathcal{O}(a_{(1)}^3)$:

$$a_{(2)} = a_{(1)} + c_1\, a_{(1)}^2 \,\ln a_{(1)} + c_1^2\, a_{(1)}^3 \,\left(\ln^2 a_{(1)} + \ln a_{(1)} - 1\right) + \ldots$$

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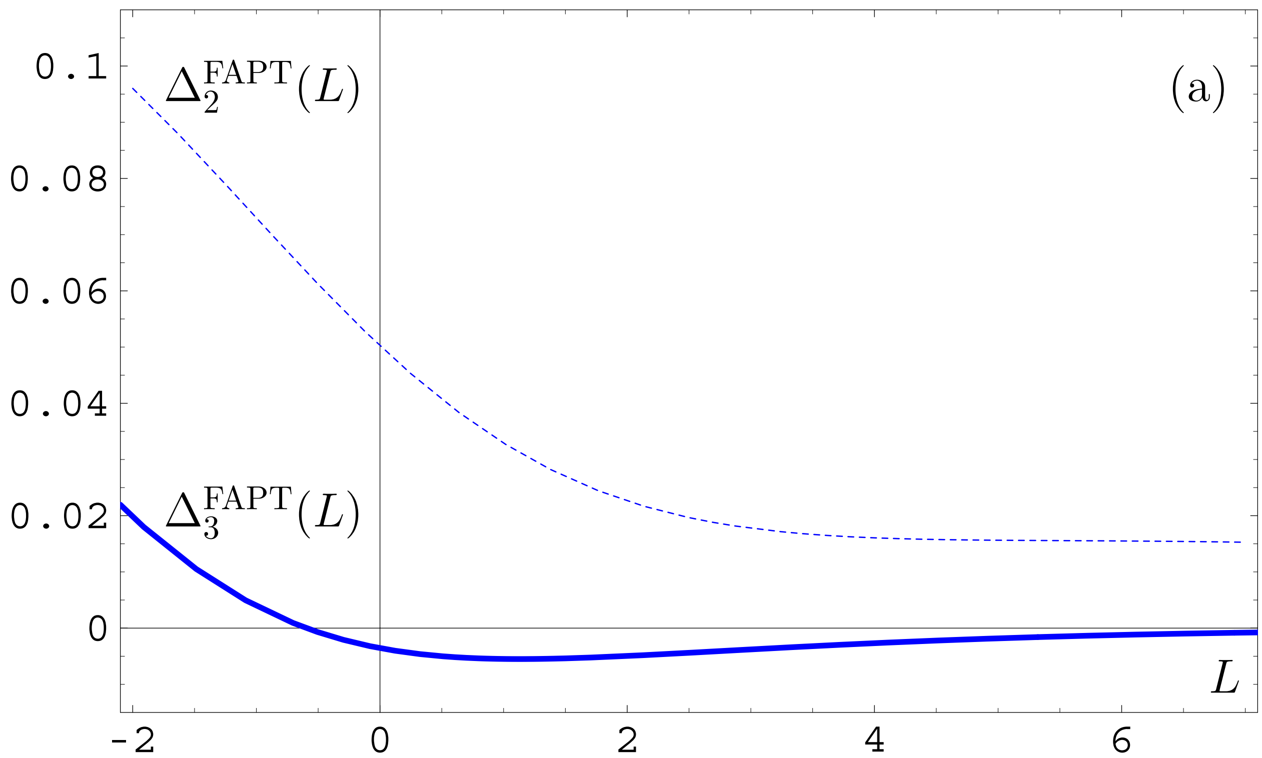
Analytic version of this expansion:

$$\mathcal{A}_1^{(2)}(L) = \mathcal{A}_1^{(1)} + c_1\, \mathcal{D}\, \mathcal{A}_{\nu=2}^{(1)} + c_1^2\, \left(\mathcal{D}^2 + \mathcal{D}^1 - 1\right)\, \mathcal{A}_{\nu=3}^{(1)} + \ldots$$

Development of FAPT: Two-loop coupling

Nice convergence of this expansion:

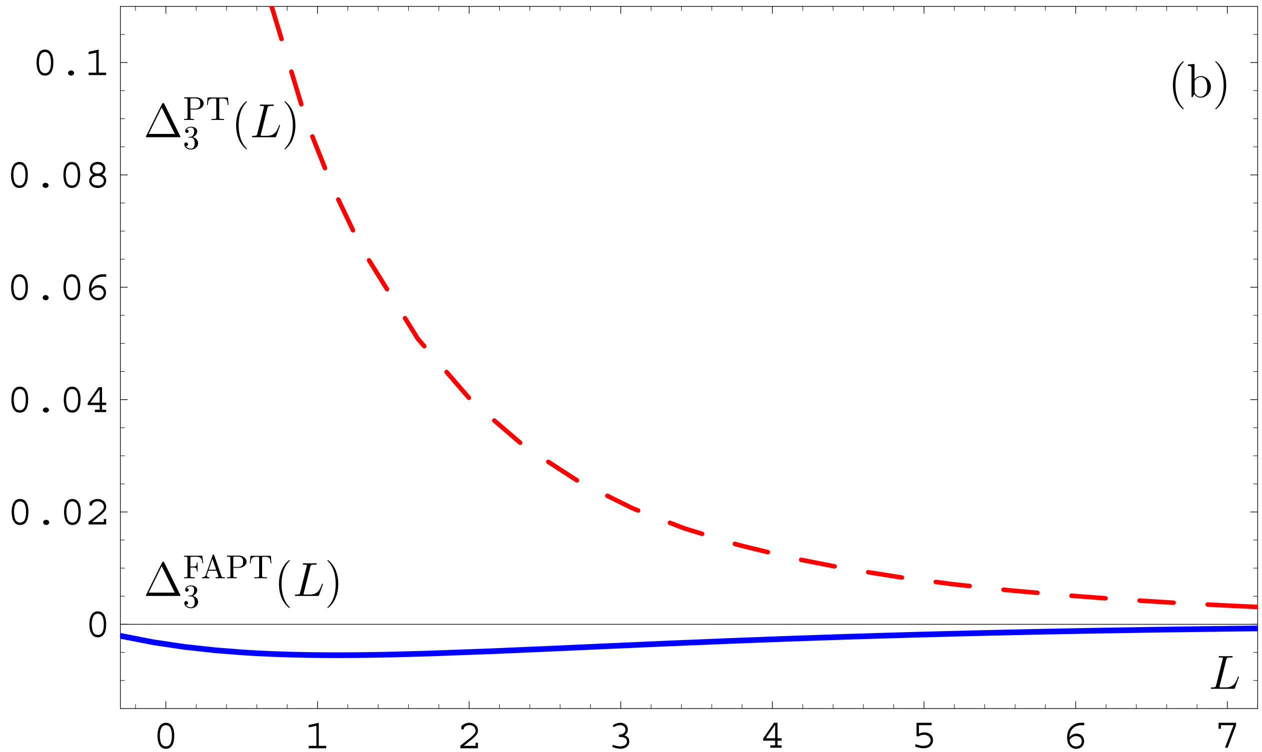
$$\Delta_2^{\text{FAPT}}(L) = 1 - \frac{\mathcal{A}_1^{(1)}(L) + c_1\,\mathcal{D}\mathcal{A}_{\nu=2}^{(1)}(L)}{\mathcal{A}_1^{(2)}(L)}$$



Development of FAPT: Two-loop coupling

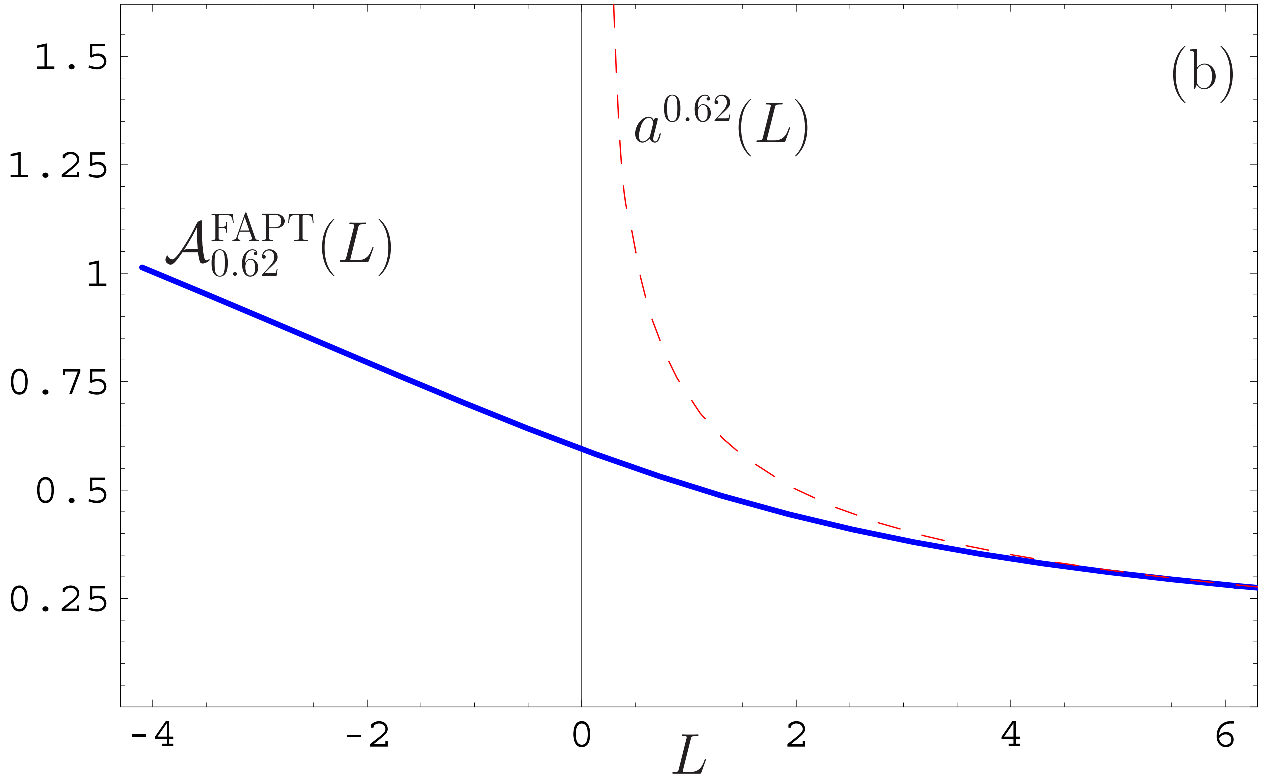
Nice convergence of this expansion:

$$\Delta_3^{\text{FAPT}}(L) = \Delta_2^{\text{FAPT}}(L) - \frac{c_1^2\left(\mathcal{D}^2 + \mathcal{D}^1 - 1\right)\mathcal{A}_{\nu=3}^{(1)}(L)}{\mathcal{A}_1^{(2)}(L)}$$



FAPT: Two-loop coupling $\mathcal{A}_{\nu}^{(2)}(L)$

$$\mathcal{A}_{\nu}^{(2)}(L) = \mathcal{A}_{\nu}^{(1)} + c_1 \nu \, \mathcal{D} \, \mathcal{A}_{\nu+1}^{(1)} + c_1^2 \, \nu \left[\frac{(\nu+1)}{2} \mathcal{D}^2 + \mathcal{D} - 1 \right] \mathcal{A}_{\nu+2}^{(1)} + \ldots$$



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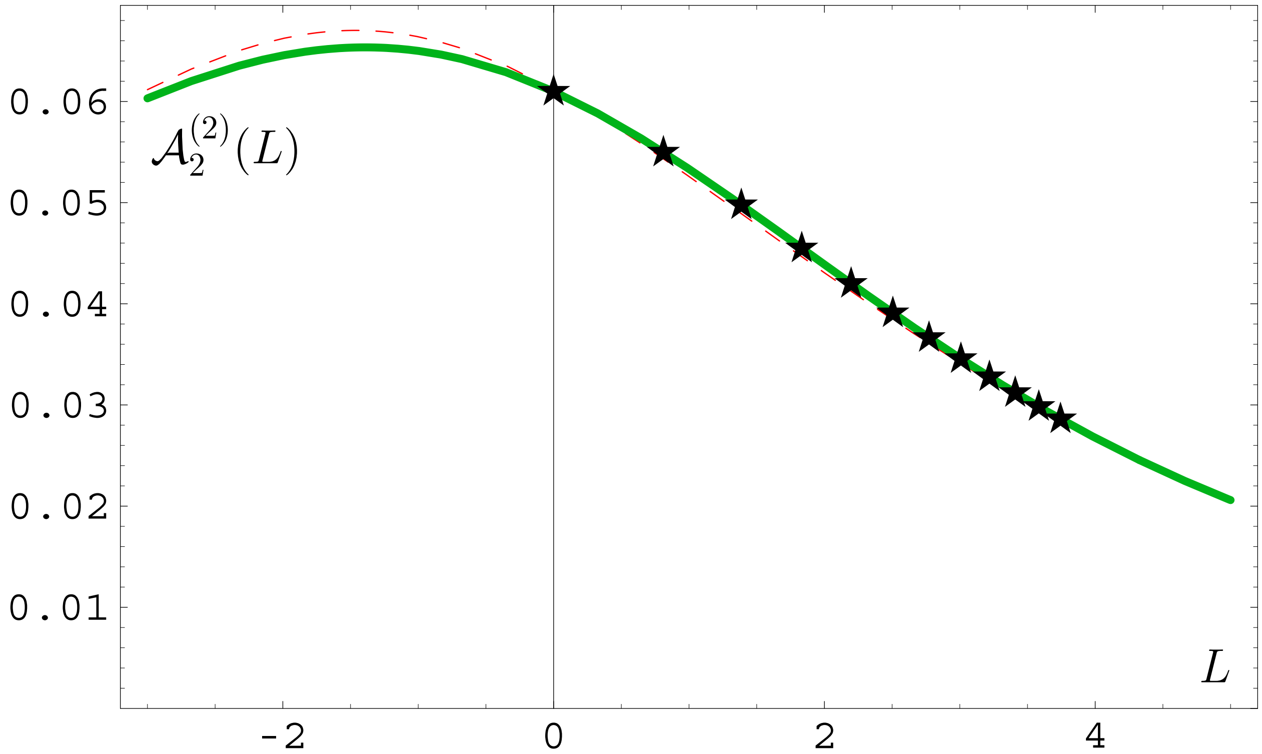
$$\mathcal{A}_2^{(2)}(L) = \mathcal{A}_2^{(1)} + 2\,c_1\,\mathcal{D}\,\mathcal{A}_{\nu=3}^{(1)} + c_1^2\,\left[3\,\mathcal{D}^2 + 2\,\mathcal{D} - 2\right]\mathcal{A}_{\nu=4}^{(1)} + \ldots$$



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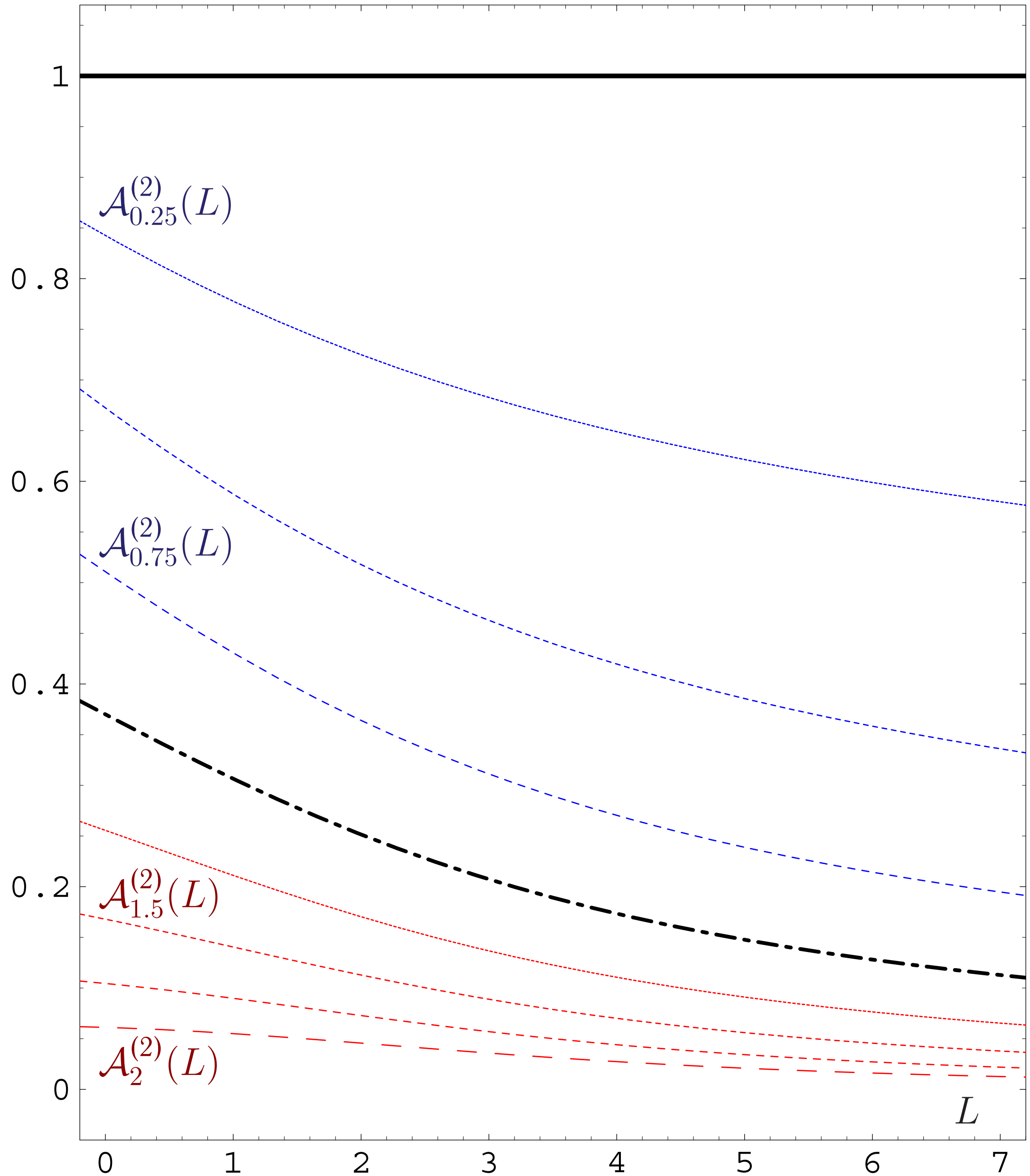

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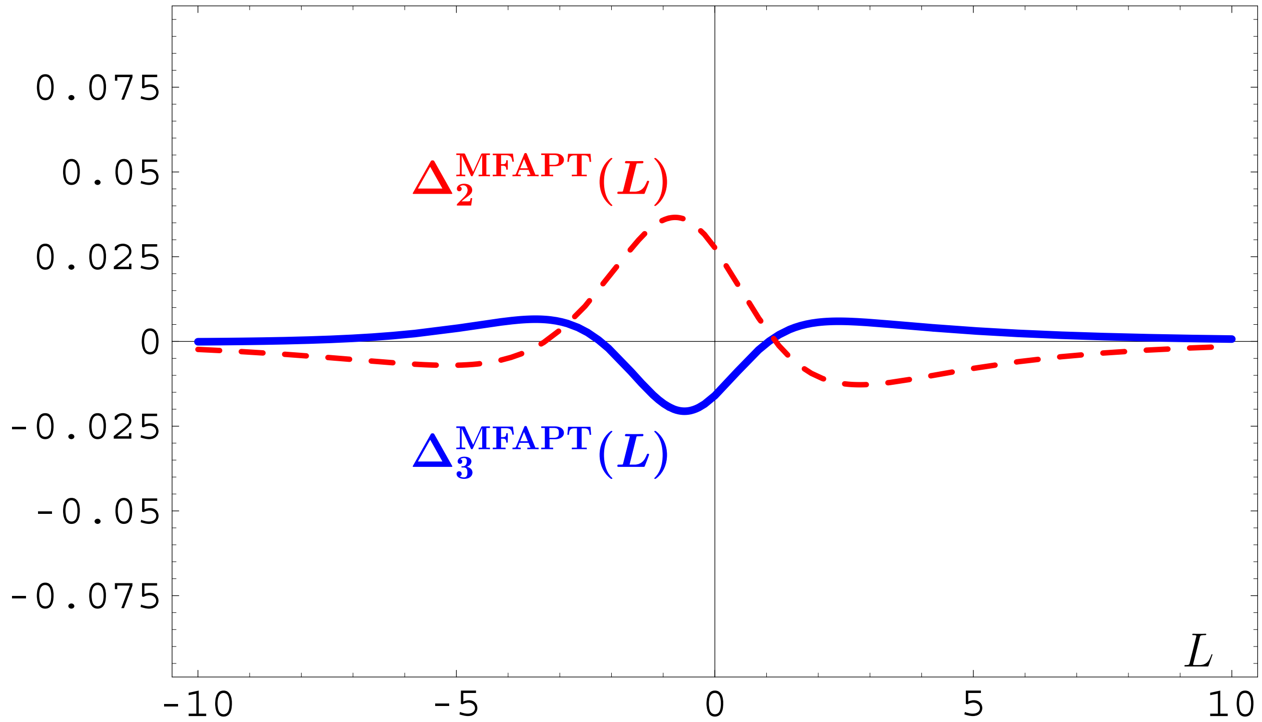

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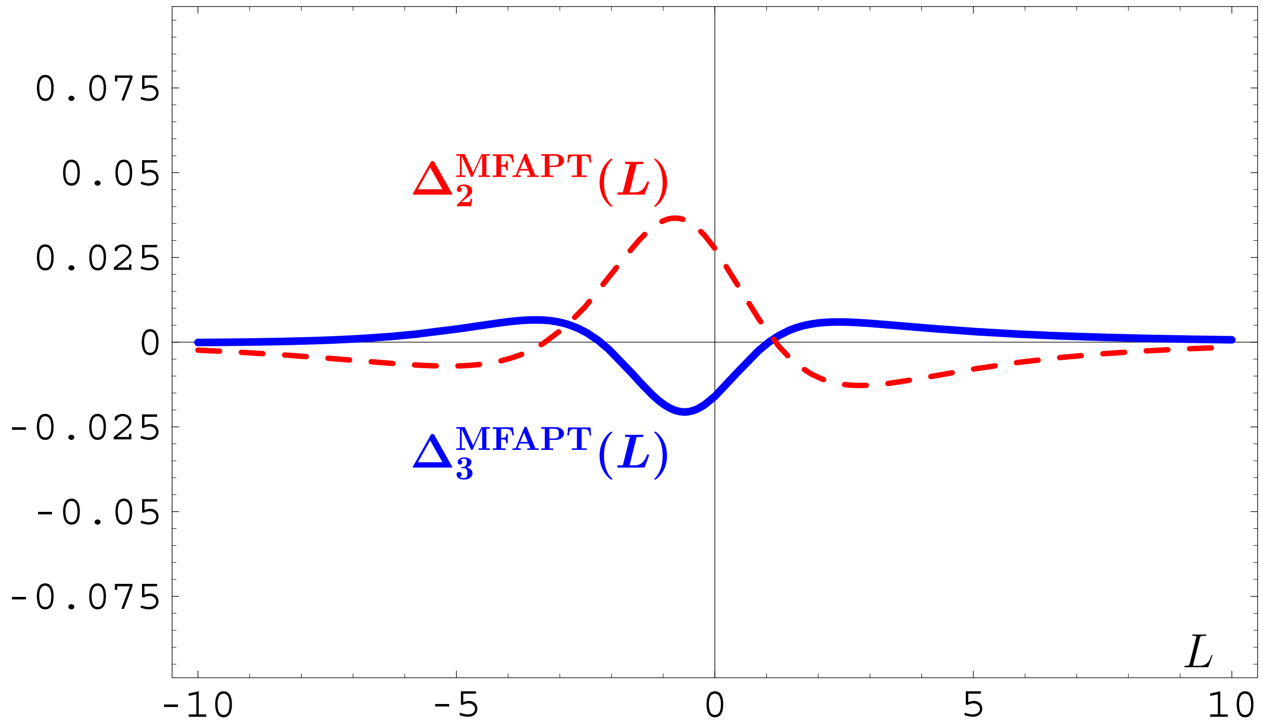
$$\Delta_3^{\rm MFAPT}(L) = \Delta_2^{\rm MFAPT}(L) - \frac{c_1^2\ [\ldots]}{\mathfrak{A}_1^{(2)}(L)}$$



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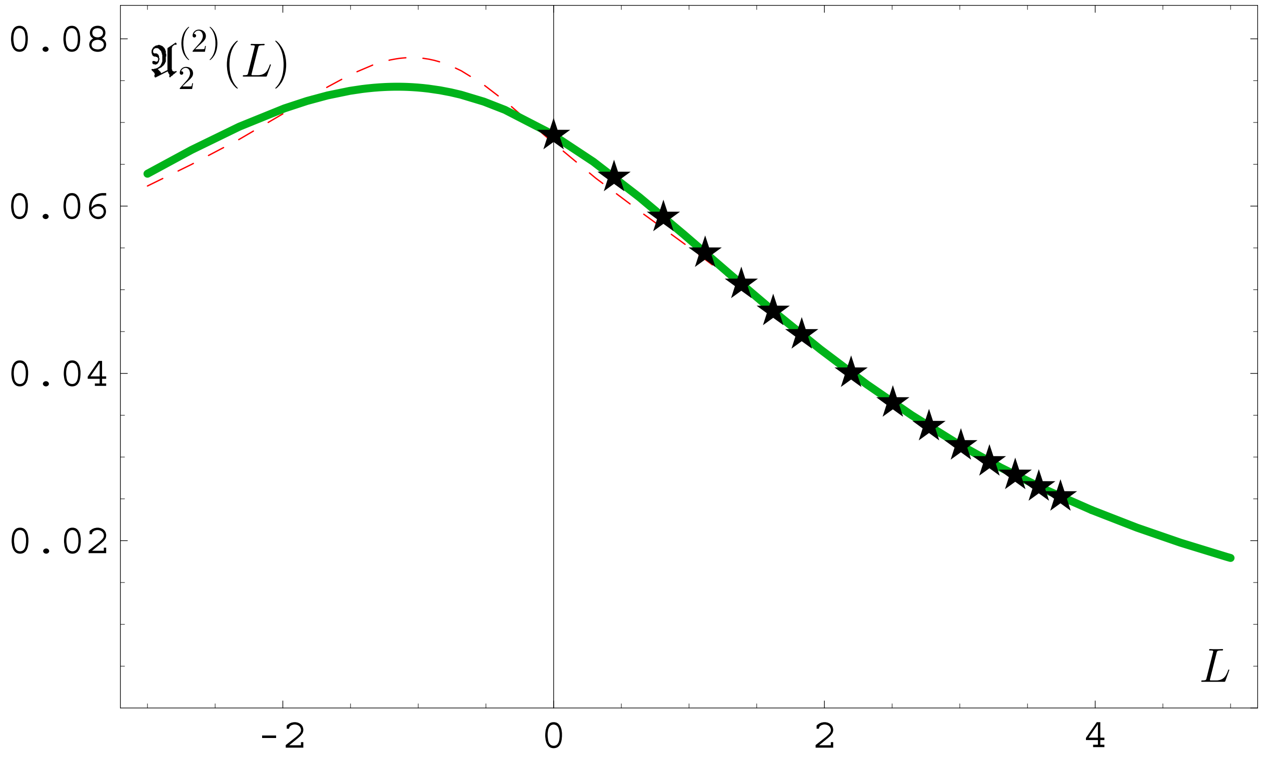
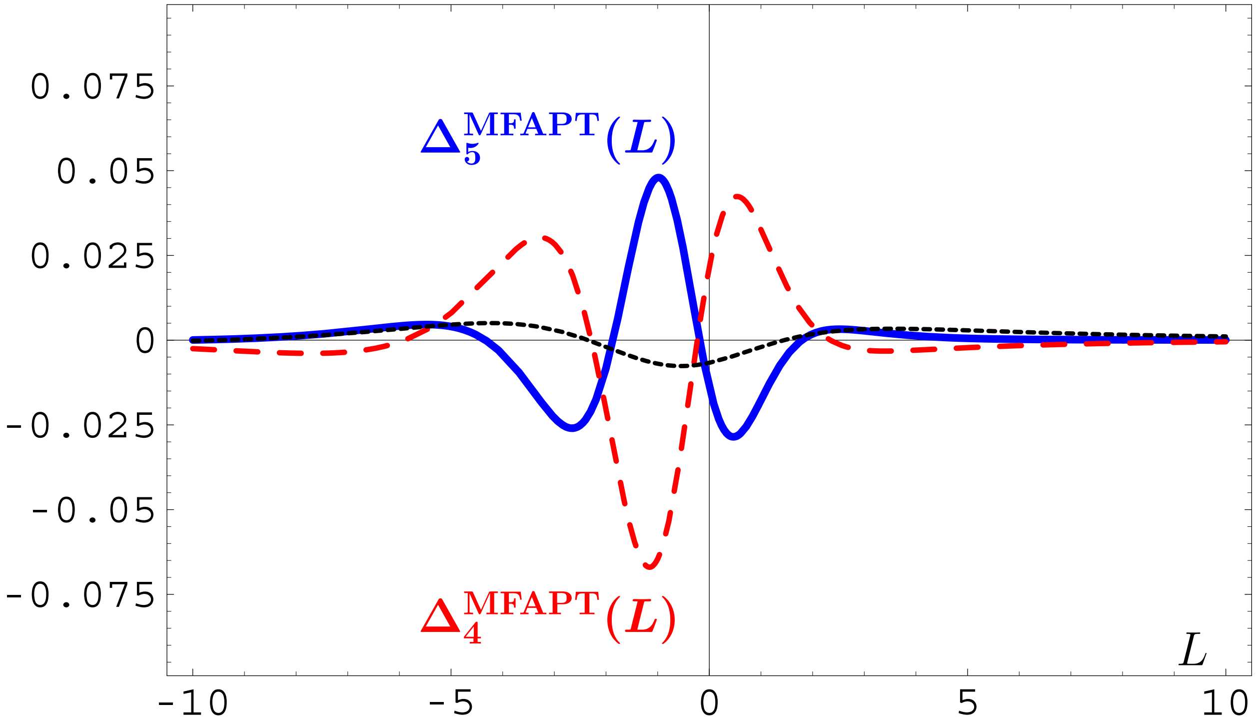


MFAPT: Two-loop coupling $\mathfrak{A}_{\nu}^{(2)}(L)$

$$\Delta_5^{\text{MFAPT}}(L) = \Delta_4^{\text{MFAPT}}(L) - \frac{c_1^4\,[\ldots]}{\mathfrak{A}_2^{(2)}(L)}$$



Application:

Pion FF in FAPT

Factorizable part of pion FF at NLO

Scaled hard-scattering amplitude truncated at NLO and evaluated at renormalization scale $\mu_R^2 = \lambda_{\rm R} Q^2$ reads

$$Q^2 T_{\rm H}^{\rm NLO}\left(x, y, Q^2; \mu_F^2, \lambda_{\rm R} Q^2\right) = \alpha_s\left(\lambda_{\rm R} Q^2\right)\, t_{\rm H}^{(0)}(x, y) + \frac{\alpha_s^2\left(\lambda_{\rm R} Q^2\right)}{4\pi}\, C_{\rm F}\, t_{\rm H,2}^{(1,\rm F)}\left(x, y; \frac{\mu_F^2}{Q^2}\right) + \frac{\alpha_s^2\left(\lambda_{\rm R} Q^2\right)}{4\pi}\left\{b_0\, t_{\rm H}^{(1,\beta)}(x, y; \lambda_{\rm R}) + t_{\rm H}^{(\rm FG)}(x, y)\right\}$$

with shorthand notation

$$t_{\rm H,2}^{(1,\rm F)}\left(x, y; \frac{\mu_F^2}{Q^2}\right) = t_{\rm H}^{(0)}(x, y)\left[2\Big(3 + \ln(\bar{x}\,\bar{y})\Big) \ln\frac{Q^2}{\mu_F^2}\right]$$

Pion Distribution Amplitude

Leading twist 2 pion DA at normalization scale $\mu_0^2 \approx 1$ GeV^2 given by

$$\varphi_\pi(x, \mu_0^2) \;=\; 6\,x\,(1-x)\Big[1 + a_2(\mu_0^2)\,C_2^{3/2}(2x-1) + a_4(\mu_0^2)\,C_4^{3/2}(2x-1) + \ldots\Big]$$

All nonperturbative information encapsulated in Gegenbauer coefficients $a_n(\mu_0^2)$.

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All nonperturbative information encapsulated in Gegenbauer coefficients $a_n(\mu_0^2)$.
To obtain factorized part of pion FF $\Rightarrow$ convolute pion DA with hard-scattering amplitude:

$$F_\pi^{\text{Fact}}(Q^2) = \varphi_\pi(x,\mu_0^2) \underset{x}{\otimes} T_{\text{H}}^{\text{NLO}}\left(x,y,Q^2;\mu_F^2,\lambda_{\text{R}}Q^2\right) \underset{y}{\otimes} \varphi_\pi(y,\mu_0^2)$$

Analyticity of Pion FF at NLO

Naive "analytization" **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

$$\left[Q^2 T_{\rm H}\left(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2\right)\right]_{\text{Nai-An}} =$$

$$\mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y) + \frac{\left(\mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\right)^2}{4\pi}\, t_{\rm H}^{(1)}\left(x,y;\lambda_{\rm R},\frac{\mu_{\rm F}^2}{Q^2}\right)$$

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Maximal "analytization" **[Bakulev, Passek, Schroers, Stefanis – PRD 70 (2004) 033014]**

$$\left[Q^2 T_{\rm H}\left(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2\right)\right]_{\text{Max-An}} =$$

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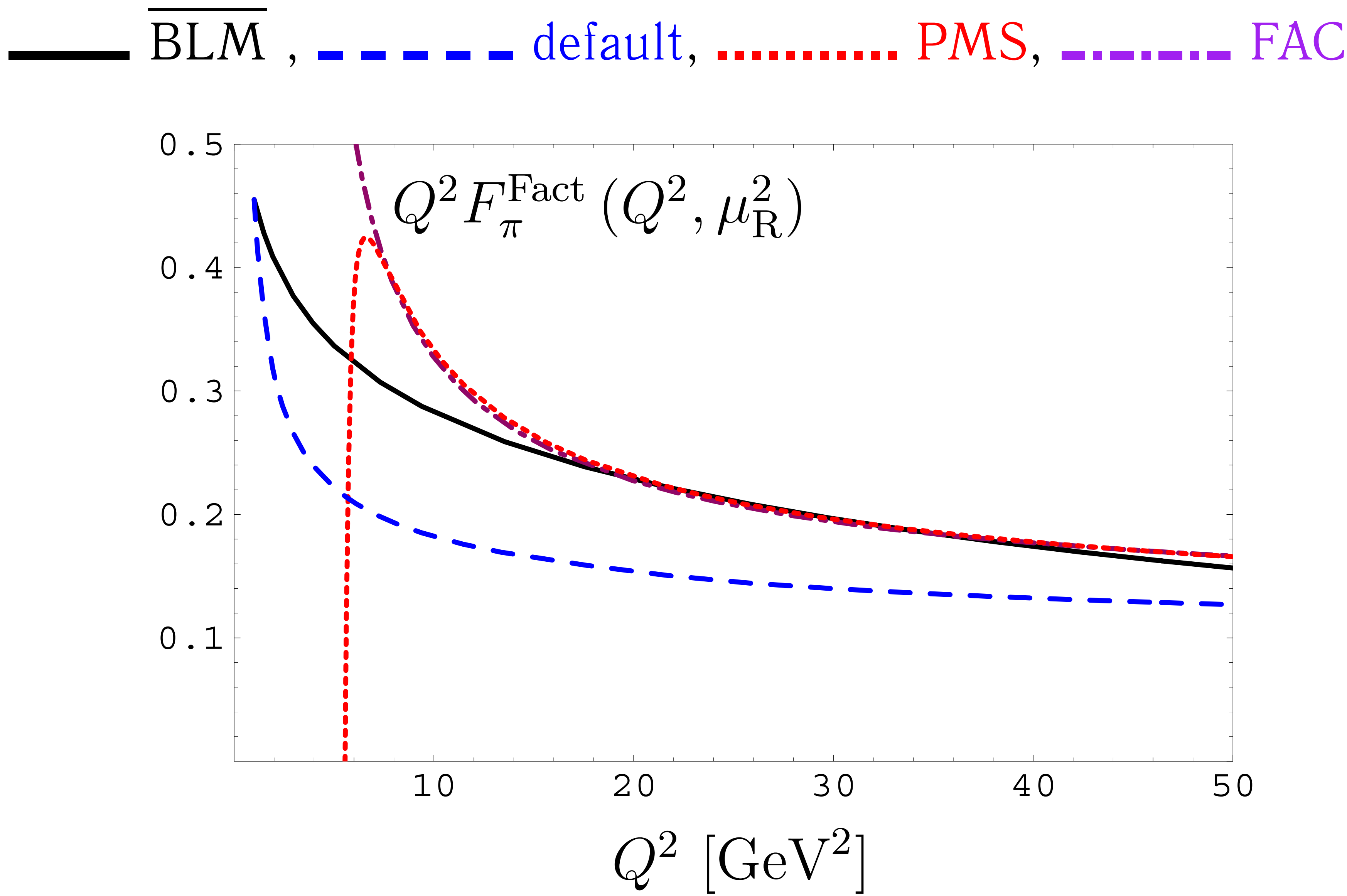
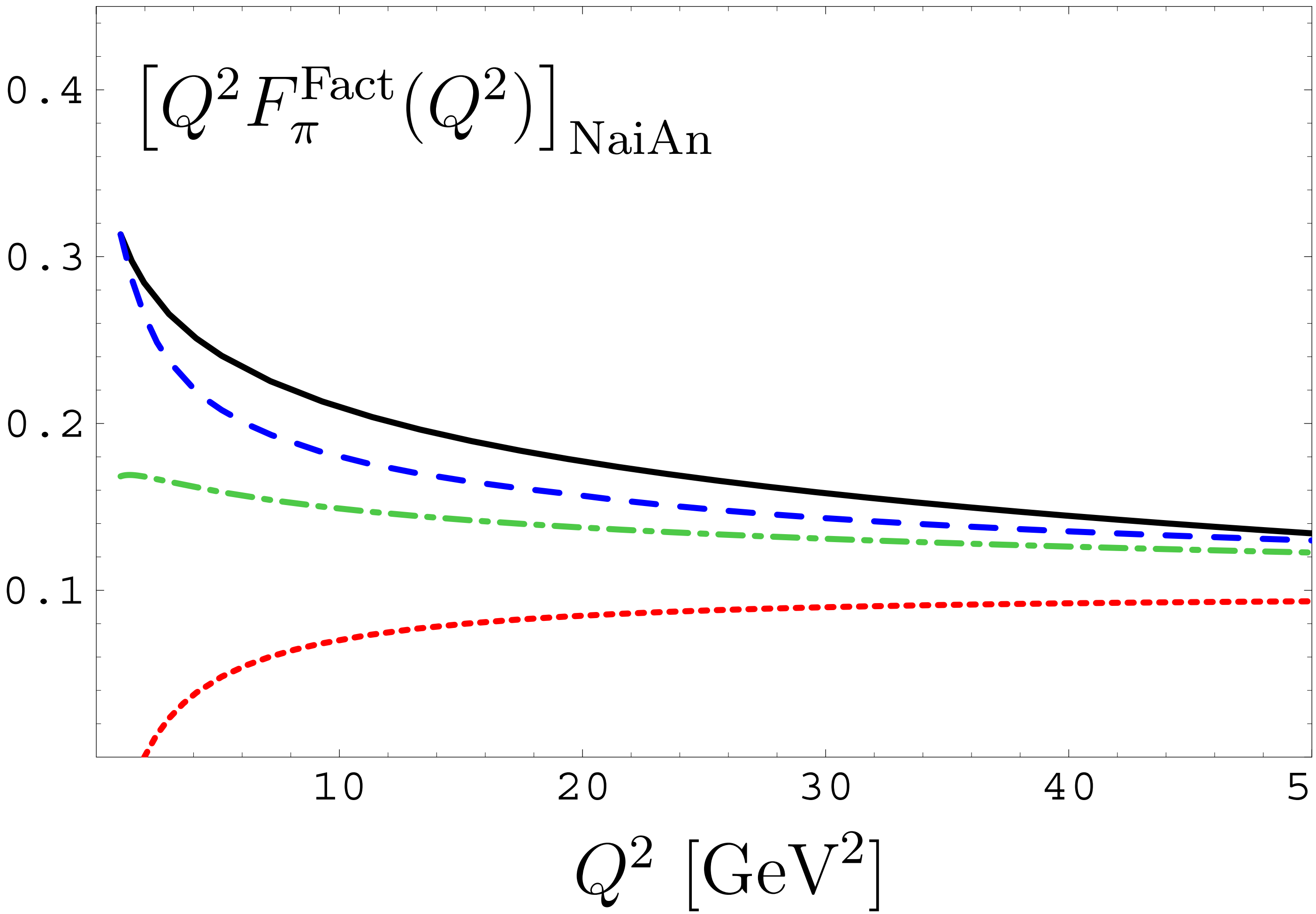
Factorized Pion FF in Standard $\overline{MS}$ scheme



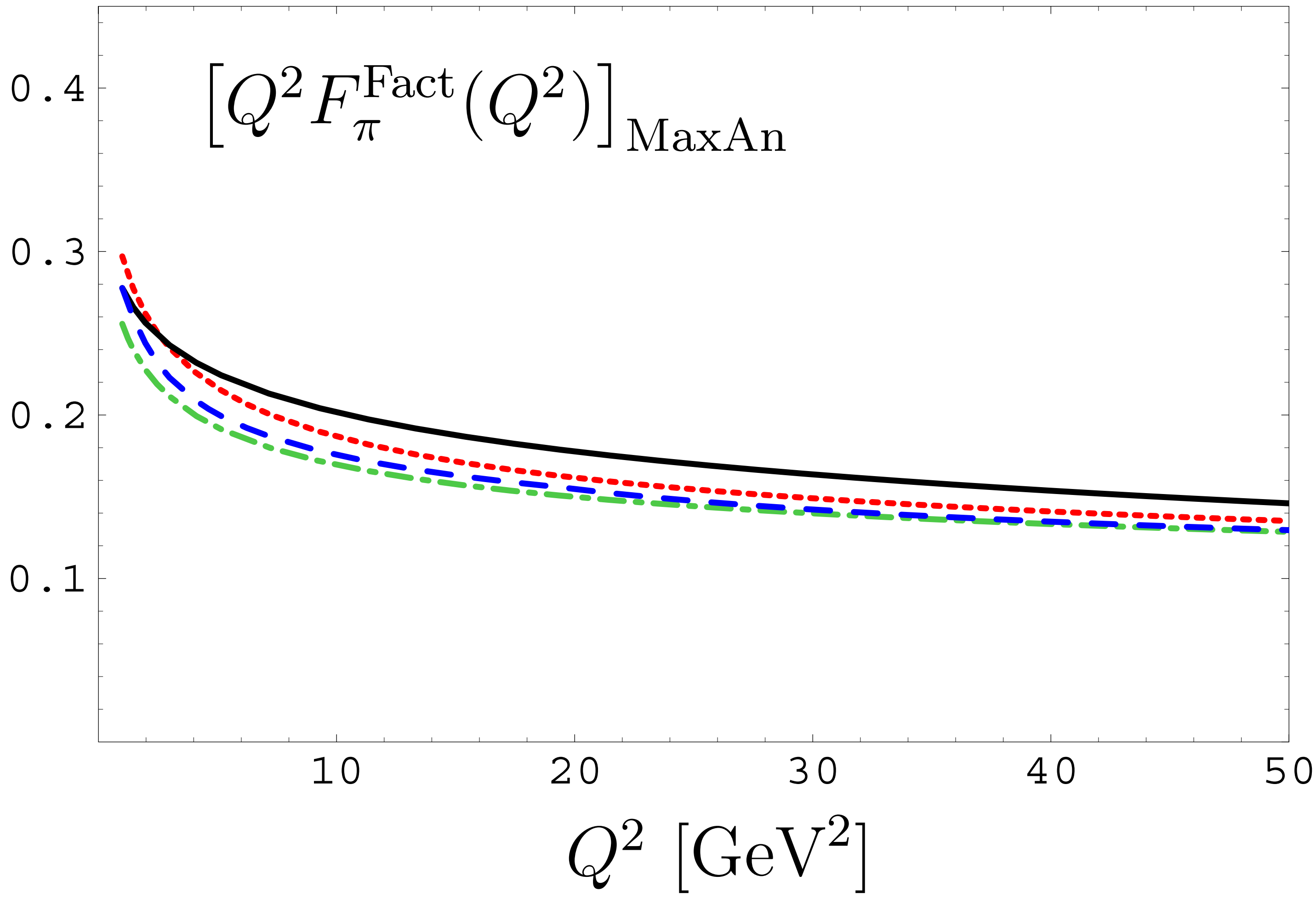
Factorized Pion FF in Naive Analyticization

$\overline{\text{BLM}}$, default, BLM, α_{V}



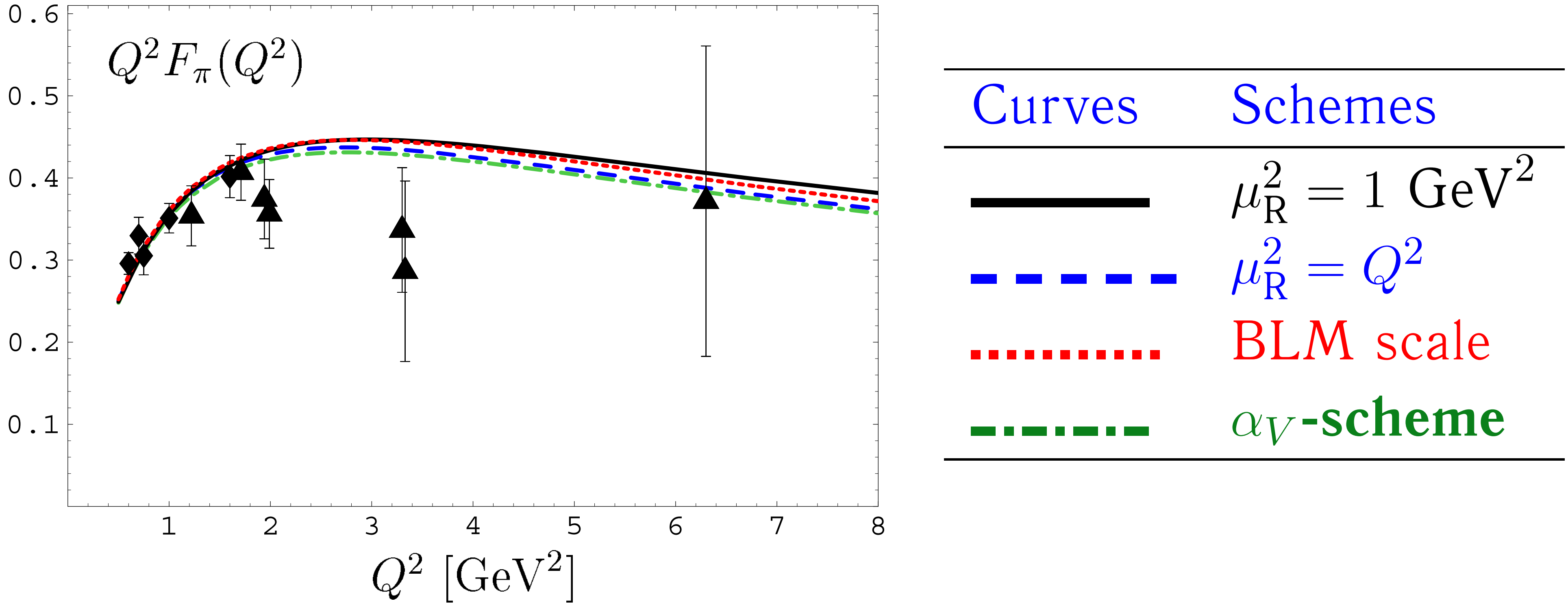
Factorized Pion FF in Max. Analyticization

$\overline{\text{BLM}}$ (solid), default (dashed), BLM (dotted), α_V (dash-dotted)



Pion form factor in analytic NLO pQCD

[AB-Passek-Schroers-Stefanis, PRD 70 (2004) 033014]



Practical independence on scheme/scale setting!

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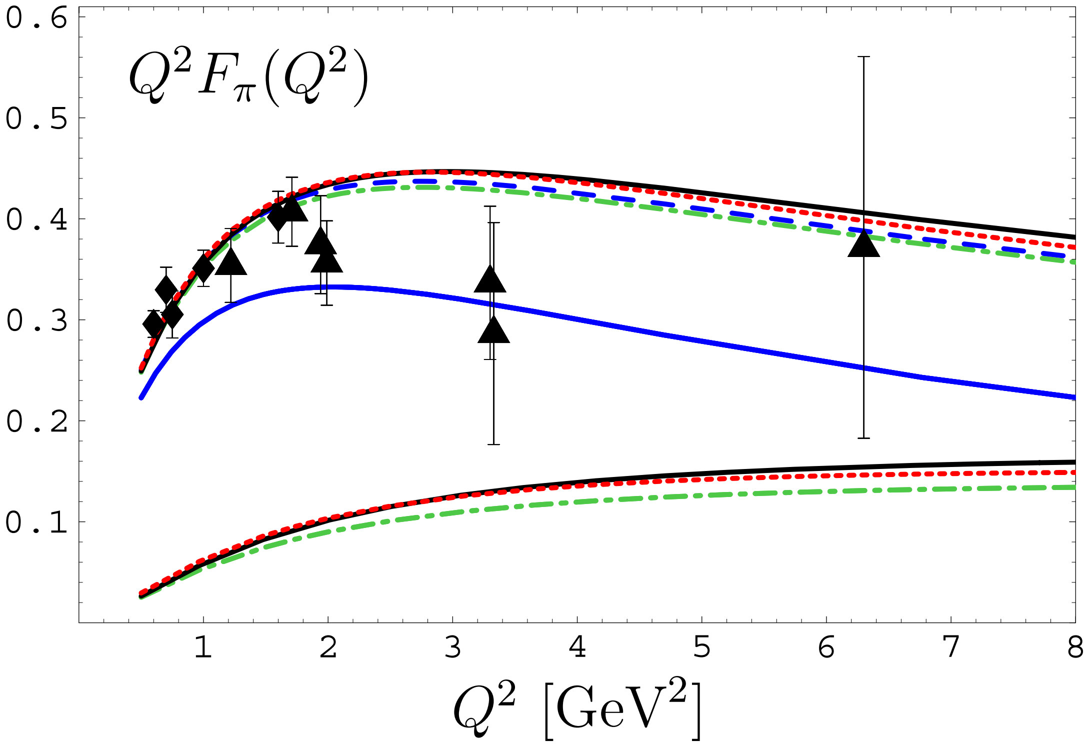

Curves	Schemes
black solid	$\mu_R^2 = 1$ GeV^2
blue dashed	$\mu_R^2 = Q^2$
red dotted	BLM scale
green dash-dotted	α_V**-scheme**
blue solid	**soft part**

Practical independence on scheme/scale setting!

Application:

Higgs decay in MFAPT

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\rm S}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\rm S}(x) J_{\rm S}(0)\,]\, |0\rangle$$

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in terms of discontinuity of its imaginary part

$$R_{\rm S}(s) = \mathbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma({\rm H} \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_{\rm H} m_b^2(M_{\rm H}) R_{\rm S}(s = M_{\rm H}^2)\,.$$

Standard PT analysis of $R_{\rm S}$

Direct multi-loop calculations are usually performed in the Euclidean region for the corresponding Adler function $D_{\rm S}$, where QCD perturbation theory works:

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The functions D and R can be related to each other via a dispersion relation without any reference to perturbation theory. This generates relations between r_n and d_n

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$$= 1 + 0.2075 + 0.0391 + 0.0020 - 0.00148\,.$$

Here $a_s = \alpha_s(M_{\rm H}^2)/\pi = 0.0366$ correspondsg to Higgs boson mass $M_{\rm H} = 120$ GeV.

MFAPT analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[a_s(Q^2)\right]^{\nu_0} \left[1 + \frac{c_1 b_0}{4\pi} a_s(Q^2)\right]^{\nu_1} .$$

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Following the procedure illustrated in ☞ we obtain

$$\widetilde{R}_{\rm S}^{(l)\rm MFAPT} = \left[3\hat{m}^2\right] \left(\frac{4}{b_0}\right)^{\nu_0} \left[\mathfrak{A}_{\nu_0}^{(l)} + \sum_{m>0} \tilde{d}_m^{(l)} \left(\frac{4}{b_0}\right)^m \mathfrak{A}_{m+\nu_0}^{(l)}\right]$$

MFAPT analysis of R_S in two loops

PT-series convergence using $M_H = 120$ GeV.

Scheme	$\widetilde{R}_S\left(M_H^2\right)$	$O(1)$	$O(a_s)$	$O(a_s^2)$	$O(a_s^3)$	$O(a_s^4)$
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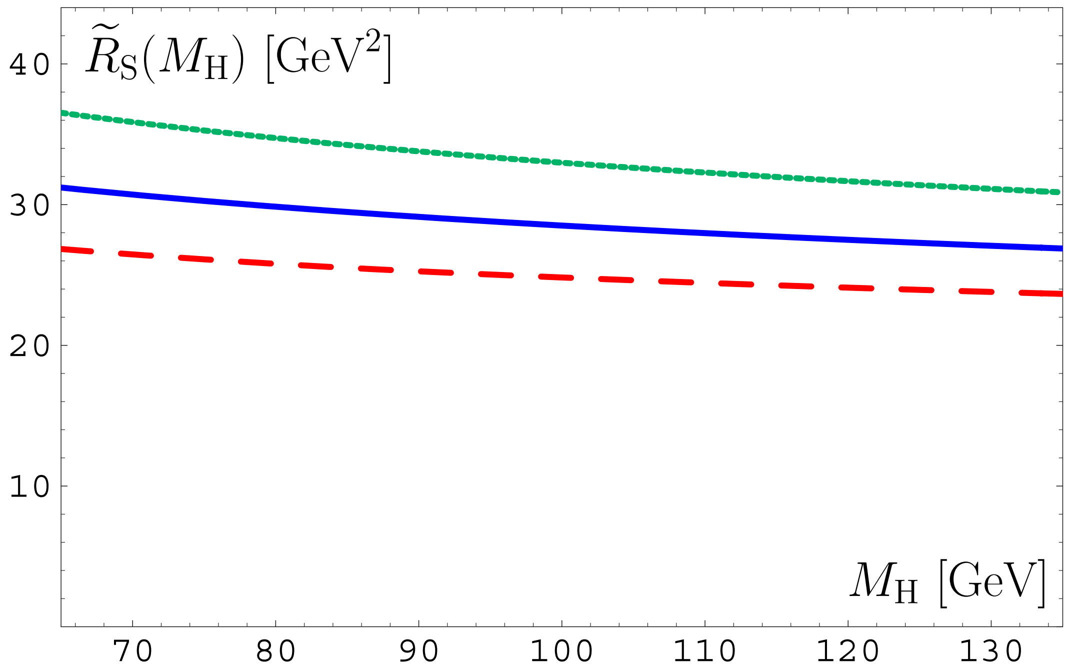
Quality of convergence for all schemes $\approx$ the same.

But: in MFAPT convergence could be traced down to $s = 1$ GeV^2.

Graphics for R_S in two loops

Illustration of $\widetilde{R}_{\rm S}(M_{\rm H}^2)$ calculation in different schemes:
2-loop QCD PT (dashed red line),
1-loop MFAPT (dotted green line), and
2-loop MFAPT (solid blue line).



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- Advantages entailed: **Minimal sensitivity** to both renormalization and factorization scale setting (pion's electromagnetic form factor);
- Advantage of applying **MFAPT** (decay $H^0 \longrightarrow b\bar{b}$) is that the coupling parameters $\mathfrak{A}_\nu$ include the resummed contribution of all π^2-terms due to analytic continuation from the Euclidean to the Minkowski space.