BRASIL-JINR FORUM 2015

Unified Formulation of Classical and Quantum Behaviors in a Variational Principle

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Collaboration with
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Variational Principle in Class. Mech.

$$I(x) = \int_{t_I}^{t_F} dt \left[\frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 - V(x(t)) \right]$$

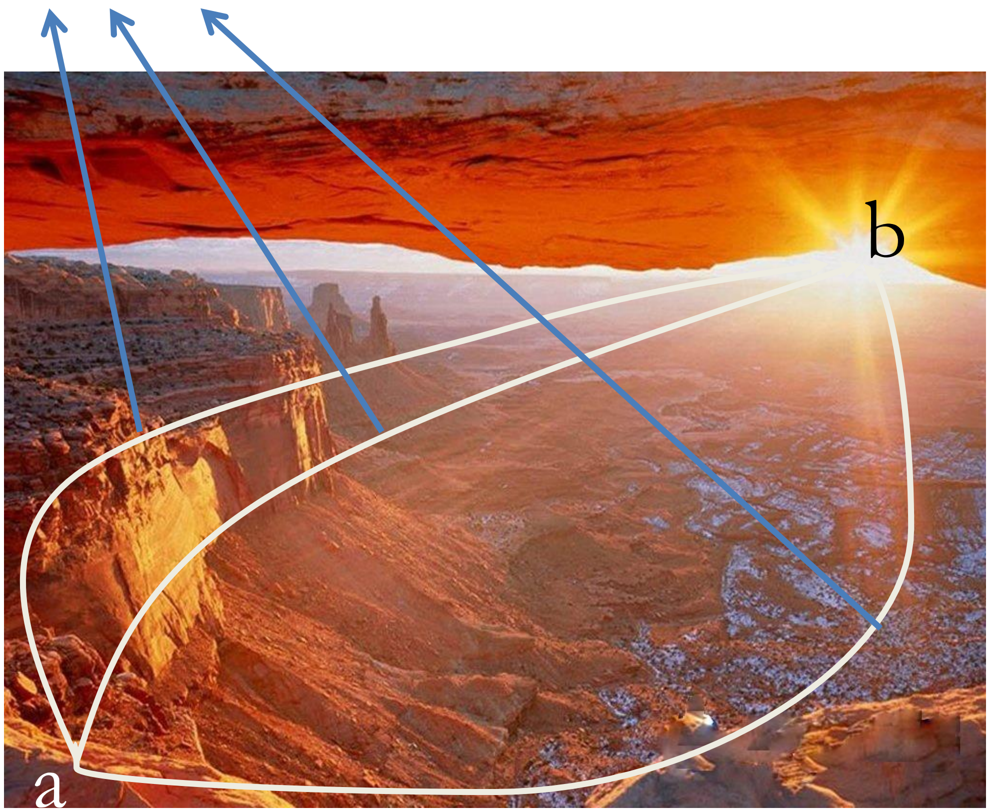
OPTIMIZATION → Newton equation

$$x(t_I) = a, \qquad x(t_F) = b$$

Path Integral Approach

$$\langle a|b\rangle = \int_a^b\int [Dx]\exp(iI)$$

All paths contribute!



Even quantum path still satisfies the law of optimization.

↓

To see this, we need to extend the formulation of the variational method.

HOW ?

A

B

Optimized path ?

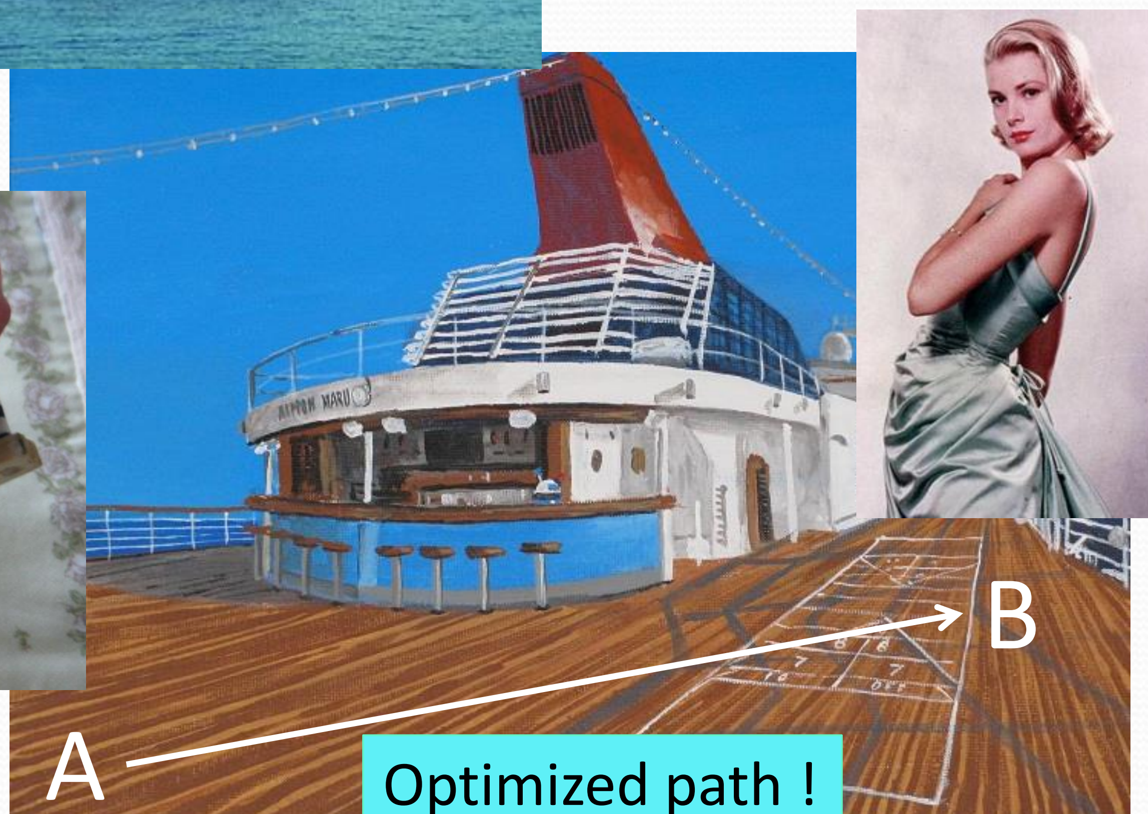

Queen Victoria
A
B
Optimized path ?

A

B

Optimized path ?


Queen Victoria

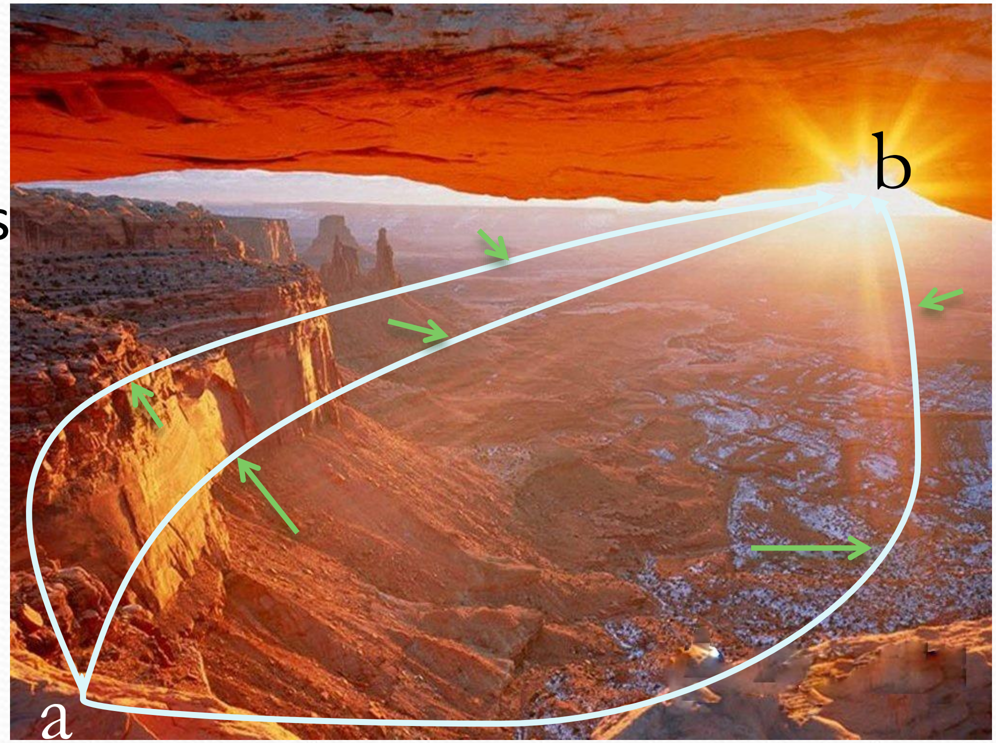

A
B
Optimized path !


KUBANSKAYA

We cannot follow the optimized path!!

Variational Principle in Class. Mech.

Effect of variables which we cannot control.



$$I(x) = \int_{t_I}^{t_F} dt \left[\frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 - V(x(t)) \right]$$

Newton equation

- Yasue, J. Funct. Anal. 41 327 (1981)
- Nelson, Quantum Fluctuations (Princeton, NJ: Princeton University Press, 1985)
- Guerra and Morato, Phys. Rev. D27 1774 (1983)
- Pavon, J. Math. Phys. 36 6774 (1995)
- Nagasawa, Stochastic Process in Quantum Physics (Bassel:Birkhaeuser, 2000)
- Cresson and Darses , J. Math. Phys. 48 072703 (2007)
- Holm, arXiv:1410.8311 [math-ph]

⋮

Stochastic Variational method

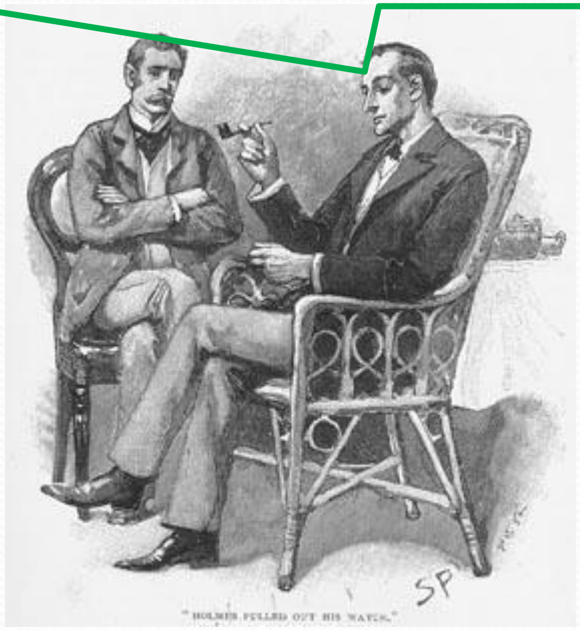
is one approach to calculate optimization including such a fluctuation.

Formulation of SVM

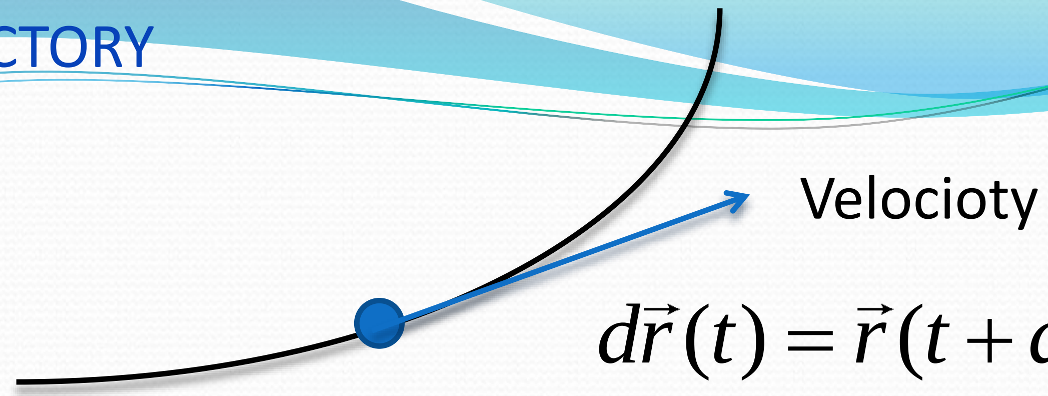
KEY POINT

Definition of velocity!

5 important steps



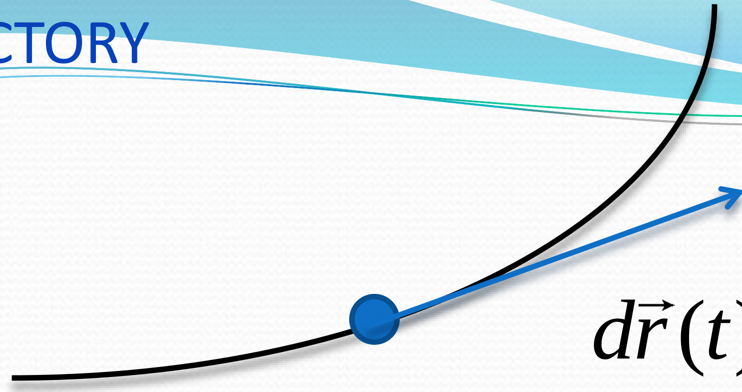
CLASSICAL TRAJECTORY



$$d\vec{r}(t) = \vec{r}(t+dt) - \vec{r}(t)$$

$$= \vec{u}dt$$

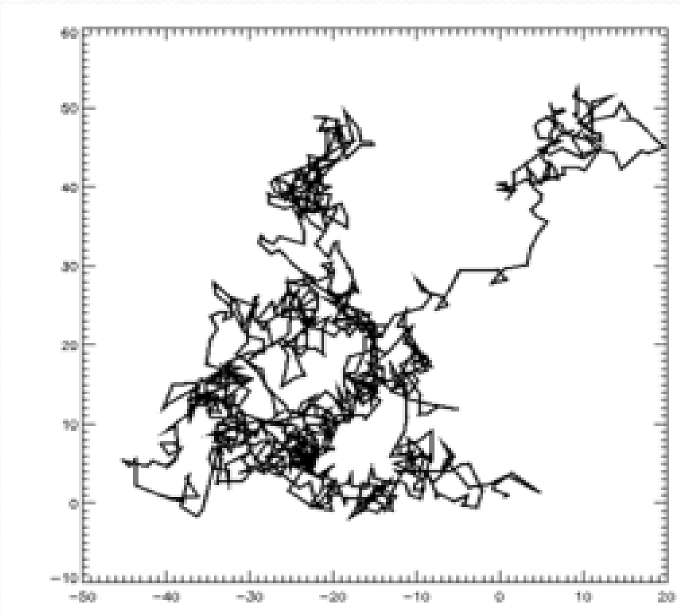
CLASSICAL TRAJECTORY



$$d\vec{r}(t) = \vec{r}(t+dt) - \vec{r}(t)$$

$$= \vec{u}dt$$

STOCHASTIC TRAJECTORY

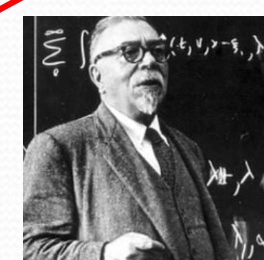


noise intensity

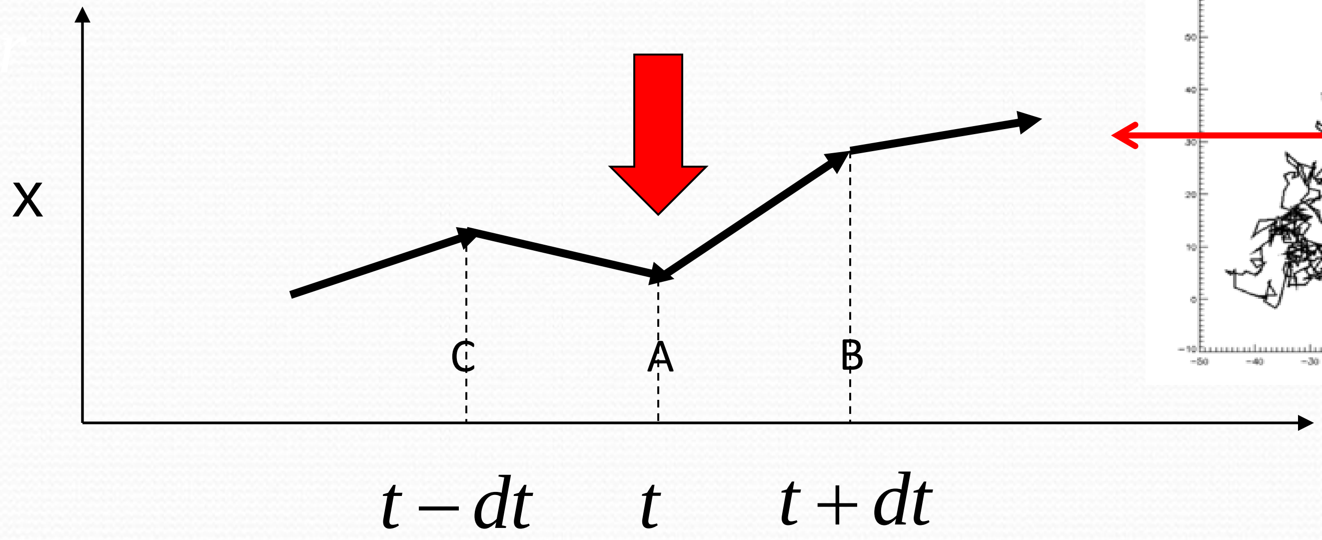
$$d\vec{r}(t) = \vec{u}(\vec{r}(t),t)dt + \sqrt{2\nu}\cdot d\vec{W}(t)$$

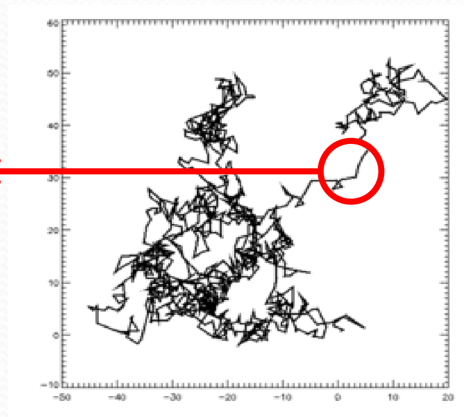
$$dt > 0$$

Gaussian white noise
(Wiener process)

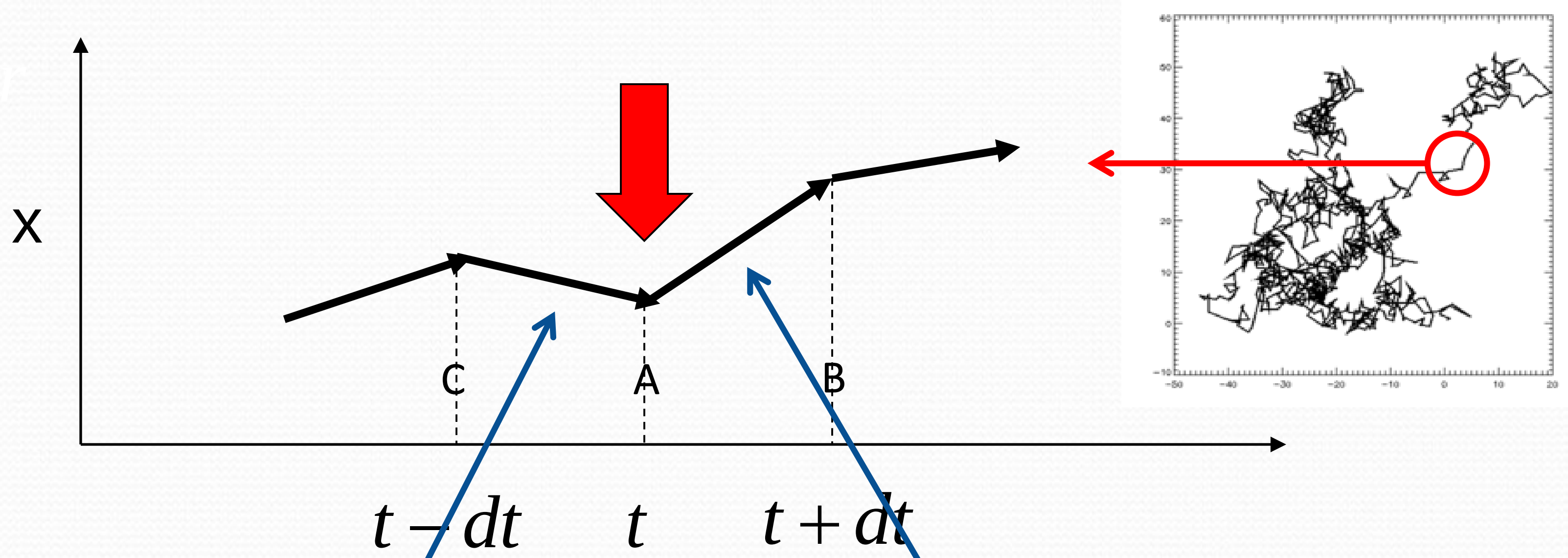


How define the velocity at A?





How define the velocity at A?



$$\lim_{dt\to 0+}\frac{\vec{r}(t+dt)-\vec{r}(t)}{dt}\Rightarrow u(\vec{r}(t),t)$$

$$\lim_{dt\to 0-}\frac{\vec{r}(t+dt)-\vec{r}(t)}{dt}\Rightarrow \tilde{u}(\vec{r}(t),t)$$

Bernstein Process

Forward stochastic differential equation $(dt > 0)$

$$d\vec{r}(t) = \vec{u}(\vec{r}(t),t)dt + \sqrt{2\nu}\cdot d\vec{W}(t)$$

Backward stochastic differential equation $(dt < 0)$

$$d\vec{r} = \tilde{\vec{u}}(\vec{r}(t),t)dt + \sqrt{2\nu}\cdot d\tilde{\vec{W}}(t)$$

We employ the variational procedure to determine these unknown functions.

Consistency Condition

$$\rho(\vec{x},t) = E\left[\delta(\vec{x}-\vec{r}(t))\right]$$

The Fokker-Plank equation (forward)

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho$$

The Fokker-Plank equation (backward)

$$\partial_t \rho = -\nabla\left(\vec{\tilde{u}} + \nu\nabla\right)\rho$$

These two should be equivalent

$$\vec{u} = \vec{\tilde{u}} + 2\nu\nabla\ln\rho$$

Caticha, JPA44, 225303(2011)
Kodama&Koide, arXiv:1412.6472

Time Drivative Operations

Because of the two different definitions of velocities, we can introduce the two different time derivatives.

(Nelson)

3

Mean forward derivative $D\vec{r} = \vec{u}$

Mean backward derivative $\tilde{D}\vec{r} = \vec{\tilde{u}}$

Partial Integration Formula

CLASSICAL

$$\int_a^b dt \frac{dX}{dt} \cdot Y = \left[X(b)Y(b) - X(a)Y(a) \right] - \int_a^b dt X \cdot \frac{dY}{dt}$$

STOCHASTIC

4

$$\int_a^b dt E\left[(DX) \cdot Y \right]$$

$$= E\left[X(b)Y(b) - X(a)Y(a) \right] - \int_a^b dt E\left[X \cdot (\tilde{D}Y) \right]$$

See, Koide et al., arXiv:1412.5865.

Ito Formula (Ito's lemma)

This is a kind of Tayler expansion for stochastic variables.

Taylor

$$d\vec{r} = \vec{u}dt$$

$$df(\vec{r}(t),t) = dt\left[\partial_t + u\cdot\nabla\right] f(\vec{r}(t),t) + O(dt^2)$$

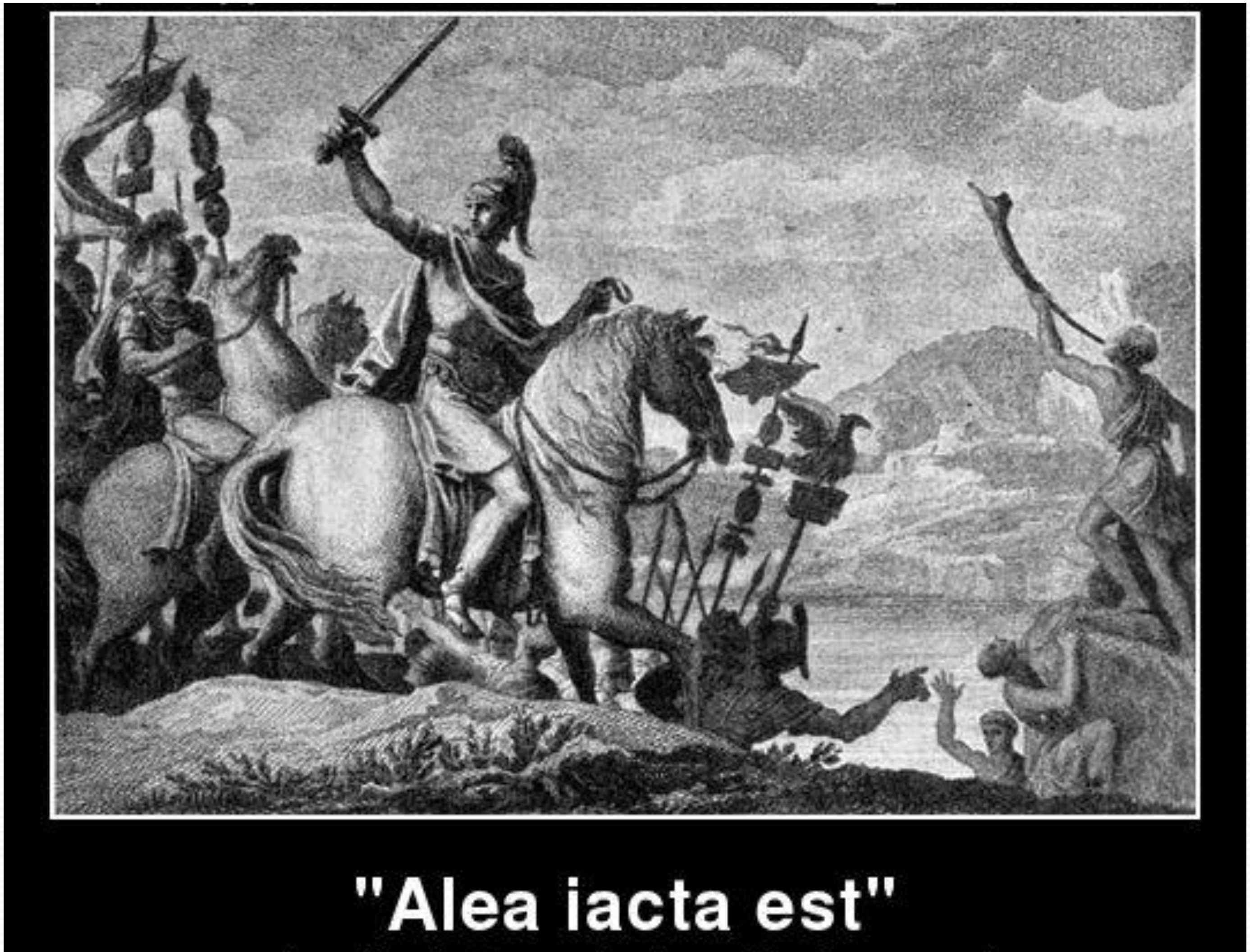
5

Ito

$$d\vec{r} = \vec{u}dt + \sqrt{2\nu}d\vec{W}$$

$$df(\vec{r}(t),t) = dt\left[\partial_t + u\cdot\nabla + \nu\nabla^2\right] f(r(t),t) + \sqrt{2\nu}\nabla f\cdot dW$$
$$+O(dt^2)$$

Let’s apply!!



Stochastic Representation of Action

Classical action
$$I_{cla} = \int_a^b dt \left(\frac{m}{2} \left(\frac{d\vec{r}(t)}{dt} \right)^2 - V(\vec{r}(t)) \right)$$

For example

$$\left(\frac{d\vec{r}}{dt} \right)^2 \Rightarrow \begin{cases} 1) \quad D\vec{r} \cdot D\vec{r} \\ 2) \quad \tilde{D}\vec{r} \cdot \tilde{D}\vec{r} \\ 3) \quad \dfrac{D\vec{r} \cdot D\vec{r} + \tilde{D}\vec{r} \cdot \tilde{D}\vec{r}}{2} \end{cases}$$

We consider 3)

Stochastic action
$$I_{sto} = \int_a^b dt E \left[\frac{m}{2} \frac{(D\vec{r})^2 + (\tilde{D}\vec{r})^2}{2} - V(\vec{r}) \right]$$

Stochastic Variation for Kinetic Term

$$\vec{r} \to \vec{r} + \delta\vec{r}$$

$$\delta\int_a^b dt\frac{m}{2}E\left[(D\vec{r})\cdot(D\vec{r})\right] = m\int_a^b dtE\left[(D\vec{r})\cdot(D\delta\vec{r})\right]$$

$$= m\int_a^b dtE\left[\vec{u}\cdot(D\delta\vec{r})\right]$$

$$= -m\int_a^b dtE\left[\tilde{D}\vec{u}\cdot\delta\vec{r}\right]$$

Ito formula $\tilde{D}\vec{u} = \left(\partial_t + \vec{\tilde{u}}\cdot\nabla - \nu\Delta\right)\vec{u}$

Variation of Action

$$\delta I = 0 \Rightarrow \left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

Variation of Action

$$\delta I = 0 \Rightarrow \quad \left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

when $\nu = 0$ ⟹

Variation of Action

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$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

$$\left(\partial_t + \vec{u}_m \cdot \nabla\right) = \frac{d}{dt}$$

when $\nu = 0$ ⇒ The Newton equation

$$\frac{d}{dt}\vec{u}_m = -\frac{1}{m}\nabla V(\vec{r})$$

Variation of Action

$$\delta I = 0 \Rightarrow \left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2\nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

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when $\nu = 0$ → The Newton equation

$$\frac{d}{dt}\vec{u}_m = -\frac{1}{m}\nabla V(\vec{r})$$

The dynamics of ρ is given by the FP equation.

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho = -\nabla\left(\rho\vec{u}_m\right)$$

Derivation of Schrödinger Equation

Introduction of phase $\nabla \vartheta = \vec{u}_m /(2\nu)$

Eq. of variation

$$\partial_t \vartheta + \nu (\nabla \vartheta)^2 - \nu \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) + \frac{1}{m} \nabla V = 0$$

Introduction of wave function $\varphi \equiv \sqrt{\rho} e^{i\vartheta}$

Yasue, JFA 41, 327 (`81)

$$i\partial_t \varphi = \left[-\nu\Delta + \frac{1}{2\nu m} V \right] \varphi \xrightarrow{\nu = \frac{\hbar}{2m}} i\hbar\partial_t \varphi = \left[-\frac{\hbar^2}{2m} \Delta + V \right] \varphi$$

The Schrödinger equation

before

after

GLASSES

NOISE

The Newton equation

The Schrödinger equation

Canonical Quantization and SVM

$L(r,\dot{r})$

optimization

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

LEGENDRE TRANSFORM

$H(r,p)$

$p = \partial L / \partial \dot{r}$

optimization

$$\dot{r} = \{r, H\}_{PB}$$

$$\dot{p} = \{p, H\}_{PB}$$

Canonical Quantization and SVM

$$L(r,\dot{r})$$

optimization

SVM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

LEGENDRE TRANSFORM

$$H(r,p)$$

$$p = \partial L / \partial \dot{r}$$

optimization

canonical quantization

$$\dot{r} = \{r, H\}_{PB}$$

$$\rightarrow \frac{1}{i\hbar}[\hat{r}, \hat{H}]$$

$$\dot{p} = \{p, H\}_{PB}$$

$$\rightarrow \frac{1}{i\hbar}[\hat{p}, \hat{H}]$$

Canonical Quantization and SVM

$L(r,\dot{r})$

optimization

SVM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}-\frac{\partial L}{\partial r}=0$$

$$\tilde{D}\frac{\partial L}{\partial (Dr)}+D\frac{\partial L}{\partial (\tilde{D}r)}-\frac{\partial L}{\partial r}=0$$

LEGENDRE TRANSFORM

$H(r,p)$

optimization

$p=\partial L/\partial \dot{r}$

canonical quantization

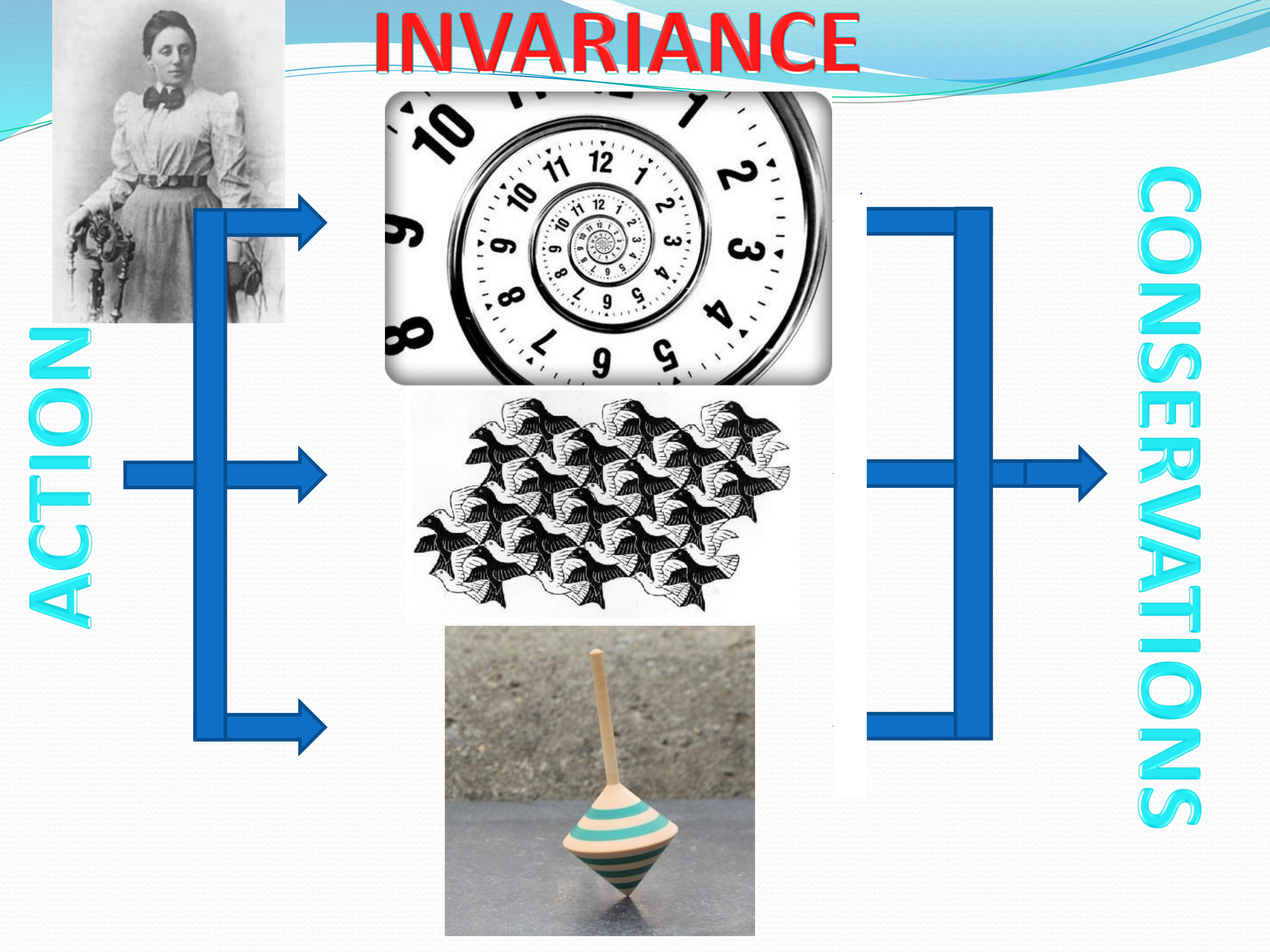
$$\dot{r}=\{r,H\}_{PB}$$

$$\rightarrow\frac{1}{i\hbar}[\hat{r},\hat{H}]$$

$$\dot{p}=\{p,H\}_{PB}$$

$$\rightarrow\frac{1}{i\hbar}[\hat{p},\hat{H}]$$

Noether Theorem


INVARIANCE
ACTION
CONSERVATIONS

Invariance for spatial translation

$$\vec{r}(t) \longrightarrow \vec{r}(t)+\vec{A}$$

The change
of the action

$$\Rightarrow \quad \delta I = \int_{t_i}^{t_f} dt E\left[L(\vec{r}+\vec{A}, D\vec{r}, \tilde{D}\vec{r})\right] - \int_{t_i}^{t_f} dt E\left[L(\vec{r}, D\vec{r}, \tilde{D}\vec{r})\right]$$

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} E\left[\frac{m}{2} D\vec{r} + \frac{m}{2} \tilde{D}\vec{r}\right] \cdot \vec{A}$$

Invariance for spatial translation

$$\vec{r}(t) \longrightarrow \vec{r}(t)+\vec{A}$$

The change of the action

$$\Rightarrow \quad \delta I = \int_{t_i}^{t_f} dt E\left[L(\vec{r}+\vec{A}, D\vec{r}, \tilde{D}\vec{r})\right] - \int_{t_i}^{t_f} dt E\left[L(\vec{r}, D\vec{r}, \tilde{D}\vec{r})\right]$$

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} E\left[\frac{m}{2} D\vec{r} + \frac{m}{2}\tilde{D}\vec{r}\right] \cdot \vec{A}$$

If the action is invariant for the spatial translation,

conserved

momentum operator!

$$\frac{m}{2} E\left[D\vec{r} + \tilde{D}\vec{r}\right] = \int d^3x\, \varphi^*(\vec{x},t)(-i\hbar\partial_x)\varphi(\vec{x},t)$$

As approximation method

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

classical variation ↓

N-body Newton's eq.

classical variation ↓

Ideal fluid eq.

Classical Many-Body Dynamics

coase-grainings →

Positions and velocities of all particles	Mass density and velocity of fluid
classical variation ↓ N-body Newton's eq.	classical variation ↓ Ideal fluid eq.
SVM ↓ N-body Scrödinger eq.	SVM ↓ ?

Loffredo&Morato, JPA40,8709(`07), Koide, arXiv:1412.6321

Classical variation of fluid

Action of (ideal) fluid

$$I(\rho_M, \vec{v}) = \int_{t_I}^{t_F} dt \int d^3x \left[\frac{\rho_M(\vec{x},t)}{2} \vec{v}^2(\vec{x},t) - \varepsilon(\rho_M) \right]$$

Mass density

Internal energy density

Classical variation

Euler equation

$$\left(\partial_t + \vec{v}\cdot\nabla\right)\vec{v} = -\frac{1}{\rho_M}\nabla P \qquad P = -\frac{d}{d\left(1/\rho_M\right)}\left(\frac{\varepsilon}{\rho_M}\right)$$

Pressure

Application of SVM

Applying SVM to the same action of (ideal) fluid,

Noise intensity

Koide&Kodama, JPA45, 255204 (`12)

$$i\partial_t \varphi = \left[-\nu\Delta + \frac{1}{2\nu}\frac{d\varepsilon}{d\rho_M} \right]\varphi$$

$$\nabla \vartheta = \frac{1}{2\nu}\vec{u}_m$$

$$\varphi \equiv \sqrt{\rho}e^{i\vartheta}$$

When we choose

$$\nu = \frac{\hbar}{2M}$$

$$\varepsilon(\rho_M) = V(\vec{x})\frac{\rho_M}{M} + \frac{1}{2}U_0\left(\frac{\rho_M}{M}\right)^2$$

quantum fluctuation

external force

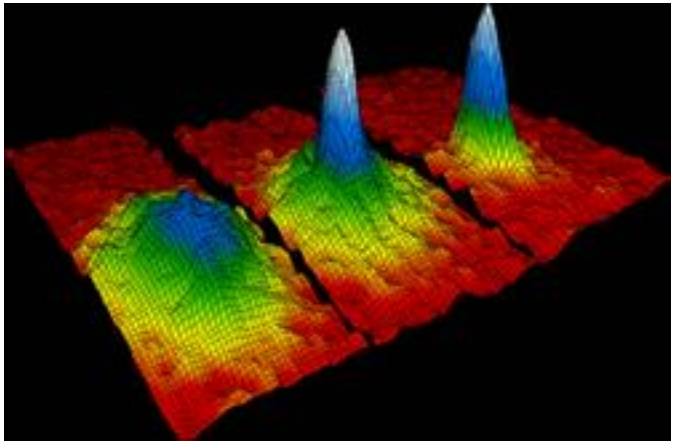
two-body interaction

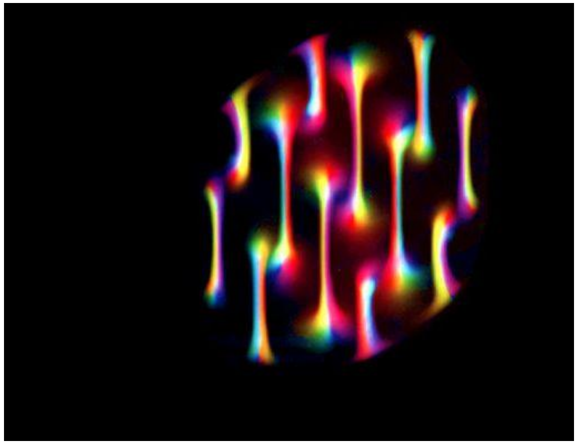
Application of SVM

Koide&Kodama, JPA45, 255204 (`12)

$$i\hbar\partial_t\varphi = \left[-\frac{\hbar^2}{2M}\Delta + V + U_0\left|\varphi\right|^2\right]\varphi$$

Gross-Pitaevskii equation





The Navier-Stokes-Fourier equation also can be formulated in SVM.

Concluding Remarks

- SVM is a useful method of for quantization of non-relativistic particles and bosonic fileds (Klein-Gordon, Abelian Gauge).
  T. Koide and T. Kodama, JPA45 255204 (2012) , arXiv:1306.6922, arXiv:1406.6295

- SVM is applicable as a method for coase-grainings of dynamics (Navier-Stokes, Gross-Pitaevskii).
  T. Koide and T. Kodama, JPA45 255204 (2012)

- The Noether theorem
  T. Misawa, JMP29 2178 (1988)

- The uncertainty relations
  T. Koide and T. Kodama, arXiv:1208.0258

- Classicalization of quantum variables
  T. Koide, arXiv:1412.6321

These successes are just accidental?

FERMION is a biggest open question!

See, also, Koide et al., arXiv:1412.5865.

ありがとう!

Köszönöm!

Gracias!

Merci!

OBRIGADO!

Grazie!

Danke!

Thank you!

謝謝!

Спасибо!