On steering of entangled particles states in an intense laser field

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Plan

Laser modeled by a monochromatic plane wave

- Laser beam as transverse, monochromatic, elliptically polarised plane wave with pulse-shape factor

$$A^\mu = a(\xi)\left(0\,,\ \varepsilon\cos(\omega_L\xi)\,,\ \sqrt{1-\varepsilon^2}\sin(\omega_L\xi)\,,\ 0\right)\,,$$

- $a(\xi)$ — smooth and slowly varying function, vanishing at $\xi := (t - z/c) \to \pm\infty\,.$
    - As example: $a(\xi) = a\,\mathrm{sech}\big(\alpha\xi\big)$, with constant α determining a pulse interval; if $\alpha \ll \omega_L$, pulse is long.
    - Infinitely long pulse , $a(\xi) = a$ — constant
- Constant $0 \leq \varepsilon \leq 1$ measures a laser beam polarisation:
    - $\varepsilon = 0\,, 1$ — linear polarisation;
    - $\varepsilon^2 = 1/2$ — circular polarisation;

Classical View on a Laser- Charge Interaction

Dipole Approximation and Beyond

Intensities regimes

Three regimes for a charged (-e), massive (m) particle interacting with a laser field A, determined by a laser field strength parameter:

$$\eta^2 = -2\,\frac{e^2}{m^2c^4}\,\langle\langle A_\mu\, A^\mu\rangle\rangle\,,$$

- $\eta << 1$ – low intensity regime;
- $\eta \sim 1$ – semi-relativistic intensity regime;
- $\eta >> 1$ – ultra relativistic regime.

$\langle\langle\ldots\rangle\rangle$ - denotes time overage over the laser oscillations.

The field strength parameter

The laser field strength is

$$\eta^2 = \frac{2}{\pi}\,\frac{e^2}{m^2 c^5}\,\lambda_L^2 I_L\,,$$

λ_L- laser wavelength,
$I_L := c\,\mathbf{E}_0{}^2/8\pi$- laser peak intensity,
$\mathbf{E}_0$ - the electric field amplitude.

$\surd$ Numerically, $\eta^2 = 3\times 10^{-11}\, I_L\lambda_L^2$, $[I_L] = \frac{W}{cm^2}$ and $[\lambda_L] = cm$.

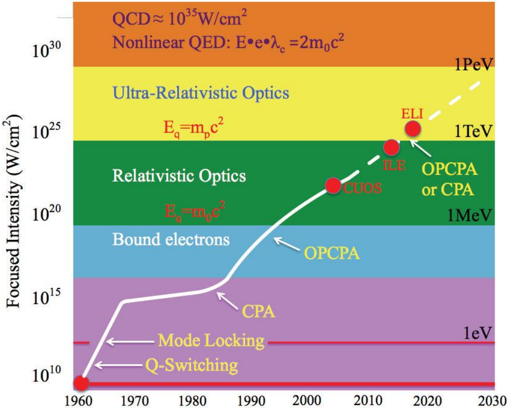
$\surd\surd$ For $\lambda_L = 10^{-4} cm$, an intensity of $I_L = 3\times 10^{18}\frac{W}{cm^2}$ is necessary to achieve $\eta^2 = 1$;

$\surd\surd\surd$ An ordinary light bulb corresponds to $\eta^2 \approx 10^{-18}$.

Strong Optical Laser Sources

Chirped Pulse Amplification (CPA) and Optical Parametric OPCPA

- THE NEW QUANTUM ERA
- Multipetawatt optical lasers
- Physics of intensive laser-matter interaction
  - √ From single photon to coherent photons;
  - √ Non-linear QED ;
  - √ Relativistic optics



Picture: G.A.Mourou et al. Optics Communications 285 (2012) 720-724

Charge + low intensity radiation

- The Newton equation with the Heaviside-Lorentz force

$$m\,\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} = e\,\mathbf{E}(t,\mathbf{x}(t)) + \frac{e}{c}\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times \mathbf{B}(t,\mathbf{x}(t))\,.$$

- Admits the linearization under assumptions:
  - $\surd$ dipole approximation — $\omega_L t - \mathbf{k}_L \cdot \mathbf{x} \approx \omega_L t$
  - $\surd$ magnetic force ignored — $\frac{v}{c}\,||\mathbf{B}_0|| \ll ||\mathbf{E}_0||$
- In a weak monochromatic wave background the Newton equation simplifies

$$m\,\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} = e\,\mathbf{E}_0 \cos\omega_L t\,.$$

- As result the electron executes harmonic motion at the same laser frequency ω_L:

$$\mathbf{x} = -\frac{e}{m\omega_L^2}\,\mathbf{E}_0\,\cos\omega_L t\,.$$

When does the approximation work?

- The magnetic field can be neglected only for non-relativistic velocities;
- Since $v_{max} = e||\mathbf{E}_0||/m\omega_L$ the condition $v \ll c$ holds true as long as

$$\eta^2 = \frac{e^2}{\omega_L^2 m^2 c^2}\,\mathbf{E}_0^2 = \frac{v_{max}^2}{c^2} \ll 1\,,$$

- The intensity scale is set by the dimensionless parameter η^2

Classical trajectories beyond the dipole approximation

Tackle the problem:

▷ Solve the Newton equations of motion for a charged spinless particle travelling in the plane electromagnetic wave:

$$A^{\mu} := \big(0\,, \mathbf{A}(ct - \mathbf{n}\cdot\mathbf{x})\big)\,, \qquad \mathbf{n}\cdot\mathbf{A} = 0\,, \quad \mathbf{n}^2 = 1\,.$$

- Can that problem be solved analytically

✓ not using the dipole approximation,

✓ taking into account the magnetic part of the Lorentz force ?

- Surprisingly, one cannot find the solution in textbooks !!!

How one can solve the equations of motion ?

- Start with the "non-relativistic" Lagrange function

$$\mathcal{L}\left(\mathbf{x}\,,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\,,t\right) = \frac{m}{2}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{e}{c}\frac{\mathrm{d}x^i}{\mathrm{d}t}A_i(\xi) - e\,\Phi(\xi)\,,$$

- Perform the Dirac trick:
  - i). Introduce an auxiliary parameter s and consider a time t as a new dynamical coordinate $t(s)$,
  - ii). Define the new Lagrangian $\mathcal{L}^*$ on the extended space $t(s), \mathbf{x}(s)$

$$\mathcal{L}^*\left(\mathbf{x}(s)\,,t(s)\,,\dot{\mathbf{x}}(s)\,,\dot{t}(s)\right) := \dot{t}(s)\,\mathcal{L}\left(\mathbf{x}(s)\,,\frac{\dot{\mathbf{x}}(s)}{\dot{t}(s)}\,,t(s)\right).$$

  - P.Jameson and A.Khvedelidze, (2008)

The Hamilton-Jacobi solution; generating function

- The generating function of classical evolution reads

$$\mathcal{F}^{\pm}_{Cl.} = \mp c(mc - \Pi_z)\,\xi \pm c \int_0^{\xi} \mathrm{d}u \sqrt{(mc - \Pi_z)^2 + W(u, \mathbf{\Pi}_\perp)}\,.$$

- where

$$W(u, \mathbf{\Pi}_\perp) := -\frac{e^2}{c^2}\,\mathbf{A}_\perp(u)^2 + 2\,\frac{e}{c}\,\mathbf{A}_\perp(u)\cdot\mathbf{\Pi}_\perp$$

$\mathbf{\Pi}_z\,, \mathbf{\Pi}_\perp$ — constant 3-vector.

Particle's orbits in an arbitrary plane wave

- The parametric solution to the Newton's equation

$$
\begin{aligned}
t(s) &= mc \int_0^s \mathrm{d}u \, \frac{1}{\sqrt{(\Pi_z - mc)^2 + W(u, \mathbf{\Pi}_\perp)}} \\
z(s) &= -cs + mc^2 \int_0^s \mathrm{d}u \, \frac{1}{\sqrt{(\Pi_z - mc)^2 + W(u, \mathbf{\Pi}_\perp)}}\,, \\
\mathbf{x}_\perp(s) &= c \int_0^s \mathrm{d}u \, \frac{\mathbf{\Pi}_\perp - \frac{e}{c}\mathbf{A}_\perp(u)}{\sqrt{(\Pi_z - mc)^2 + W(u, \mathbf{\Pi}_\perp)}}
\end{aligned}
$$

✓ s is an auxiliary time variable.

The monochromatic background radiation

How the explicit form of a trajectory as function of LAB frame time looks like ?

- For the monochromatic background radiation the LAB frame time — an elliptic integral:

$$t(s) = \frac{2}{\omega_L} \int_0^{\tan(\omega_L s/2)} \mathrm{d}x \, \frac{1}{\sqrt{4^{th} \ order \ polynomial}} ,$$

- Therefore an auxiliary time s is expressible via the Weierstrass doubly periodic function: $\wp\,(\omega_L t/2\,; g_2\,, g_3)$.

Constants of motion and reference frame fixing

In the formulae above the vanishing position $\mathbf{x}(0) = 0$, at LAB time $t = 0$, has been fixed, while the initial velocity is encoded in the integrals of motion $\beta_+ := 1 + \Pi_z/mc$, and $\boldsymbol{\beta}_\perp = \boldsymbol{\Pi}_\perp/mc$

$$
\begin{aligned}
\boldsymbol{v}_\perp(0) &= c(\boldsymbol{\beta}_\perp - \eta\boldsymbol{\epsilon}_\perp)\,, \\
v_z(0) &= c - c\sqrt{\beta_+^2 + \boldsymbol{\beta}_\perp^2 - (\boldsymbol{\beta}_\perp - \eta\boldsymbol{\epsilon}_\perp)^2}\,,
\end{aligned}
$$

where $\boldsymbol{\epsilon}_\perp = (\varepsilon, 0)$.

- The formulae can be simplified by fixing the reference frame:
  - $\surd$ Transverse motion average rest frame $\langle \boldsymbol{v}_\perp \rangle = \boldsymbol{\beta}_\perp = 0$,
  - $\surd\surd$ Longitudinal motion average rest frame $\langle v_z \rangle = 0$.

Auxiliary time s as function of LAB frame time

- LAB frame time with a vanishing $\beta_\perp = 0$ is

$$t(s) = \frac{1}{\omega_L(1-\beta_z)} \int_0^{\omega_L s} \mathrm{d}u \, \frac{1}{\sqrt{1-\mu^2 \sin^2 u}} \,,$$

where

$$\mu^2 := \frac{1-2\,\varepsilon^2}{(1-\beta_z)^2}\,\eta^2 \,.$$

and the laser field strength $\eta^2 = -2\,\frac{e^2}{m^2c^4}\,\langle A_\mu\, A^\mu\rangle = \left(\frac{ae}{mc^2}\right)^2$.

- Consider three allowed domains:

$$\text{(I)} \quad 0 < \mu^2 < 1\,, \quad \text{(II)} \quad \mu^2 > 1\,, \quad \text{(III)} \quad \mu^2 < 0\,.$$

- Special, degenerate cases $\mu^2 = 0\,, \quad \text{and} \quad \mu^2 = 1\,.$

"Fundamental solution"

- In the "fundamental domain" $(0 < \mu^2 < 1)$, LAB frame time $t(s)$ is recognised as inverse of the Jacobi amplitude function:

$$\boxed{\omega_L s = \mathrm{am}\left(\omega'_L t,\, \mu\right)\,,}$$

where μ stands for the modulus.

- Frequency is non-relativistically Doppler shifted:

$$\omega'_L := \omega_L\,(1 - \beta_z)\,.$$

- If $-\pi/2 \leq \omega_L s \leq \pi/2$ the amplitude am is single-valued function on the interval

$$-\,\mathbb{K}(\mu)\,\leq \omega'_L t\,\leq \mathbb{K}(\mu)\,,$$

$\mathbb{K}$ - "real" quarter period of Jacobi functions .

The explicit form of trajectories

- The trajectory as function of the LAB frame time

$$x(t) = -\frac{c}{\omega_L}\sqrt{\frac{\varepsilon^2}{1-2\varepsilon^2}}\,\arcsin\left[\sqrt{\mu^2}\,\mathrm{sn}(\omega_L' t\,,\mu)\right]\,,$$

$$y(t) = \frac{c}{\omega}\sqrt{\frac{1-\varepsilon^2}{1-2\varepsilon^2}}\,\ln\left[\frac{\sqrt{\mu^2}\,\mathrm{cn}(\omega_L' t\,,\mu)+\mathrm{dn}(\omega_L' t\,,\mu)}{1+\sqrt{\mu^2}}\right],$$

$$z(t) = ct - \frac{c}{\omega}\,\mathrm{am}(\omega_L' t\,,\mu)\,.$$

- Periodic motion in the plane orthogonal to the wave propagation

$$T_P := \frac{2\pi}{\omega_{\mathrm{P}}}\,.$$

- The fundamental frequency:

$$\boxed{\omega_{\mathrm{P}} := \frac{\pi}{2\mathbb{K}}\,\omega_L'}$$

Expansion over harmonics

$$x(t) = \frac{4c\varepsilon}{\omega_L\sqrt{1-2\,\epsilon^2}} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{(2n-1)(1+q^{2n-1})} \sin(2n-1)\omega_P\, t\,,$$

$$y(t) = \frac{8c\sqrt{1-\varepsilon^2}}{\omega_L} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{(2n-1)(1-q^{2n-1})} \sin^2\left(n-\frac{1}{2}\right)\omega_P\, t\,,$$

$$z(t) = \langle \upsilon_z\rangle\, t - \frac{c}{\omega_L} \sum_{n=1}^{\infty} \frac{2q^n}{n(1+q^{2n})} \sin 2n\,\omega_P\, t\,,$$

where q is the so-called *nome* parameter $q := \exp\left(-\pi\,\frac{\mathbb{K}'}{\mathbb{K}}\right)$.

The nome q for small intensities is approximately

$$q \approx \frac{1-2\,\varepsilon^2}{16(1-\beta_z)^2}\,\eta^2 + O(\eta^4)$$

Velocity and Average Rest Frame (ARF)

- A particle velocity reads

$$v_x(t) = -c\eta\varepsilon \, \mathrm{cn}\left(\omega'_L t,\, \mu\right) \, ,$$
$$v_y(t) = -c\eta\sqrt{1-\varepsilon^2} \, \mathrm{sn}\left(\omega'_L t,\, \mu\right) \, ,$$
$$v_z(t) = c - c(1-\beta_z) \, \mathrm{dn}\left(\omega'_L t,\, \mu\right) \, .$$

- The drift in the direction of propagation is a nonlinear function of laser beam intensity

$$\langle \beta_z \rangle = 1 - \frac{\pi}{2} \frac{(1-\beta_z)}{\mathbb{K}(\mu)} \, .$$

- The vanishing drift velocity $\langle v_z \rangle = 0$ for small intensity corresponds to fixation

$$\beta_z = -\frac{1}{4}(1 - 2\,\epsilon^2)\,\eta^2 + \dots \, ,$$

Low-intensity region

- Trajectory in the leading intensity order:

$$x = -\frac{c\varepsilon}{\omega'_L}\,\eta \sin\left(\omega'_L t\right) + o(\eta^3)\,,$$

$$y = -\frac{2c\sqrt{1-\varepsilon^2}}{\omega'_L}\,\eta \sin^2\left(\frac{\omega'_L t}{2}\right) + o(\eta^3)\,,$$

$$z = -\frac{c}{\omega}\,\frac{1-2\varepsilon^2}{8(1-\beta_z)^2}\,\eta^2 \sin\left(2\omega'_L t\right) + o(\eta^4)\,.$$

High-low intensities duality

- High-low intensity transformation:

$$\mu \dashrightarrow \mu' := \frac{1}{\mu}$$

- Duality: Solution in the *"fundamental domain"* determines all possible solutions. Any trajectory can be obtained from the *"fundamental solutions "* by modular transformation; combination of inversion $\mu \to 1/\mu$, and rotation to the imaginary axis $\mu \to \imath\mu$.

- The useful modular parameter is: $\tau := i\dfrac{\mathbb{K}'(\mu)}{\mathbb{K}(\mu)}$.

Modular transformations

- Under the modular transformation

$$\tau \to \tau' := \frac{a + b\tau}{c + d\tau}\,, \qquad a, b, c, d \in \mathbb{Z}\,,$$

elliptic functions are expressible through each other, e.g.:

- Jacobi imaginary transformation $\tau \to \tau' = -1/\tau$

$$\mathrm{sn}(iz\,,\mu) = i\frac{\mathrm{sn}(z\,,\mu')}{\mathrm{cn}(z,\mu')}\,, \qquad \mathrm{dn}(iz\,,\mu) = \frac{\mathrm{dn}(z,\mu')}{\mathrm{cn}(z,\mu')}\,,$$

- The shift transformation $\tau \to \tau' = \tau \pm 1$

$$\mathrm{sn}(\mu' z\,, \pm\frac{i\mu}{\mu'}) = \mu'\,\frac{\mathrm{sn}(z\,,\mu')}{\mathrm{dn}(z\,,\mu')}\,, \qquad \mathrm{dn}(\mu' z\,, \pm\frac{i\mu}{\mu'}) = \mu'\,\frac{\mathrm{dn}(z\,,\mu')}{\mathrm{cn}(z\,,\mu')}\,.$$

The $\pm$ signs of modulus correspond to $\mathrm{Re}(\tau) \lessgtr 0$ respectively.

A Laser- Charge Interaction

Quantum mechanical View

– The Dipole Approximation and Beyond

Quantum mechanics of a charged spinning particle

- *Charge & Spin quantum decomposition*

$$\Psi \subset L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

- The "composite" Hamilton operator

$$\hat{\mathrm{H}}(t) := \hat{\mathrm{H}}_{\mathrm{C}}(t) \otimes \mathrm{I}_{\mathrm{S}} + \mathrm{I}_{\mathrm{C}} \otimes \hat{\mathrm{H}}_{\mathrm{S}}(t)\,,$$

✓ with the charge-radiation Hamiltonian $\hat{\mathrm{H}}_{\mathrm{C}}$

$$\hat{\mathrm{H}}_{\mathrm{C}}(t) = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\,\mathbf{A}(t,\mathbf{x})\right)^2, \qquad \hat{\mathbf{p}} = -\imath\hbar\,\boldsymbol{\nabla}\,,$$

✓ and the spin-radiation Hamiltonian $\hat{\mathrm{H}}_{\mathrm{S}}$

$$\hat{\mathrm{H}}_{\mathrm{S}}(t) := -\frac{\hbar}{2}\,\kappa\,\mathbf{B}(t,\mathbf{x})\cdot\boldsymbol{\sigma}\,,$$

The WKB spin-charge decomposition

Charged spin-1/2 particle's pure state admits the semiclassical spin-charge degrees decomposition

$$|\Psi\rangle = a|\psi_+\rangle \otimes |\chi_0\rangle + b|\psi_+\rangle \otimes |\chi_1\rangle + c|\psi_-\rangle \otimes |\chi_0\rangle + d|\psi_-\rangle \otimes |\chi_1\rangle \,.$$

- ✓ $|\psi_\pm\rangle$ – two independent WKB solutions to the Schrödinger equation for spinless charge

$$i\hbar\frac{\partial}{\partial_t}|\psi\rangle = -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - \frac{\imath e}{\hbar c}\mathbf{A}(t,\mathbf{x})\right)^2|\psi\rangle \,.$$

- ✓ $|\chi_{0,1}\rangle$ – independent solutions to the spin precession equation.

The semi-classical solution for a spinless charged particle

The WKB solution to the Schrödinger equation for spinless particle

- As we will see in the leading semiclassical order

$$\langle \mathbf{x}, t|\psi_\pm\rangle = \frac{1}{\sqrt{|\partial_\xi \mathcal{F}^\pm_{Cl.}|}}\, e^{\pm\frac{i}{\hbar}\frac{\mathbf{\Pi}^2}{2m}t} e^{\pm\frac{i}{\hbar}\mathbf{\Pi}\cdot\mathbf{x}} \exp\frac{i}{\hbar}\mathcal{F}^\pm_{Cl.}(t-\frac{z}{c}, \mathbf{\Pi}),$$

✓ The phase $\mathcal{F}^\pm_{Cl.}$ is the Hamilton-Jacobi generating function written above.

✓ $\mathbf{\Pi}$ – constant 3-vector, classical integrals of motion.

The spin-precession equation

For a particle at REST the Spin Precession equation is

$$i\hbar\frac{\partial}{\partial t}|\chi\rangle = -g\mu_B\,\mathbf{B}_{\mathrm{REST}}(t)\cdot\mathbf{S}|\chi\rangle\,.$$

- ✓ While for a spin $\mathbf{S}$ moving in electromagnetic field $(\mathbf{E},\mathbf{B})$ with the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ the interaction Hamiltonian reads

$$\mathcal{H}' = -g\mu_B\,\underbrace{\boxed{\left(\mathbf{B}-\frac{1}{c}\,[\mathbf{v}\times\mathbf{E}]\right)\cdot\mathbf{S}}} - \underbrace{\boxed{\frac{1}{2c^2}\,[\mathbf{v}\times\mathbf{a}]\cdot\mathbf{S}}}$$

- ✓ **N.B.** Galilei boost & $\omega_{\mathbf{T}}\cdot\mathbf{S}$—Thomas precession

Three laser intensity induced effects

Spin-flip intensity resonance

Non-linearity of a charged particle's phase

Creation of entanglement between constituents spins

Spin evolution in dipole approximation

- In the non-relativistic limit

$$\mathbf{B}'^{\mathrm{Dipole}} = \frac{a\omega_L}{c}\left(\sqrt{1-\varepsilon^2}\cos(\omega_L t), \varepsilon\sin(\omega_L t), 0\right).$$

- The probability for a spin to flip in this field with the circular polarization, $(\varepsilon^2 = 1/2)$, is vanishing in the linear, $\eta \ll 1$, approximation:

$$\mathcal{P}^{\mathrm{Dipole}}_{\downarrow\uparrow} = \frac{1}{1+2/g^2\eta^2}\,\sin^2\frac{\omega_L}{2g}\,t \;\Rightarrow\; 0\,.$$

Beyond the dipole approximation , due to several factors: the retardation effect + magnetic part of Heaviside-Lorentz force distortion of a particle's classical orbit as well as the Thomas correction new effect appears.

Effective magnetic field for particle's spin

- Effective magnetic field = alternating field + "almost constant" magnitude field along the laser.
- In formulae:

$$
\begin{aligned}
\mathrm{B}'_x &= \frac{a\omega'_L}{gc}\sqrt{1-\varepsilon^2}[(g+1)\mathrm{dn}(u,\mu)-\gamma_z]\mathrm{cn}(u,\mu),\\
\mathrm{B}'_y &= \frac{a\omega'_L}{gc}\varepsilon\left[(g+1)\mathrm{dn}(u,\mu)-\gamma_z(1-\mu^2)\right]\mathrm{sn}(u,\mu)\\
\mathrm{B}'_z &= -\eta\frac{a\omega_L}{gc}\varepsilon\sqrt{1-\varepsilon^2}\left[g-\gamma_z\,\mathrm{dn}(u,\mu)\right].
\end{aligned}
$$

where

$$
\gamma_z^2\,\mu^2 = (1-2\,\varepsilon^2)\,\eta^2\,.
$$

Laser-spin interaction — Rabbi oscillation

Laser circularly polarized

$$\mathbf{B}_{\text{Circular}}(t) = \left(\mathcal{H}_0 \cos(\omega'_L t)\,, \quad \mathcal{H}_0 \sin(\omega'_L t)\,, \quad \mathcal{H}_z \right)$$

$$\mathcal{H}_0 := \frac{\eta \omega'_L g}{2\sqrt{2}}\,, \quad \mathcal{H}_z := \frac{\eta^2 \omega'_L (1-g)}{4}$$

The effective Laser-Spin interaction is famous NMR interaction via the rotated magnetic with field !

Spin flipping intensity resonance

The spin-flipping probability is

$$\mathrm{P}_{\downarrow\uparrow} = \frac{\kappa^2\eta^2}{\kappa^2\eta^2 + (\eta^2 - \eta_*^2)^2}\,\sin^2(\omega_S t)\,,$$

$$\omega_S := \frac{\omega_L|1-g|}{8}\sqrt{\kappa^2\eta^2 + (\eta^2-\eta_*^2)^2}\,, \qquad \kappa^2 := \frac{2g^2}{(1-g)^2}\,,$$

The spin flip resonance occurs at intensity

$$\eta_*^2 := \frac{4}{g-1}$$

Spin-flip in an elliptically polarised laser

- The evolution operator $U(t, t_0)$ in the factor form

$$U(t,0) = \exp\left(aS_+\right)\exp\left(bS_0\right)\exp\left(cS_-\right)\,,$$

where $S_\pm = 1/2\left(\sigma_1 \pm \imath\,\sigma_2\right)$ and $S_0 = 1/2\,\sigma_3\,.$

- Unknown $a(t), b(t)$ and $c(t)$ determine spin-1/2 state

$$|\chi(t)\rangle = U(t,0)|\chi(0)\rangle\,.$$

- Probabilities of transitions between states

$$\mathcal{P}_{\uparrow\uparrow} = \frac{1}{1+|a|^2}\,, \qquad \mathcal{P}_{\uparrow\downarrow} = \frac{|a|^2}{1+|a|^2}\,.$$

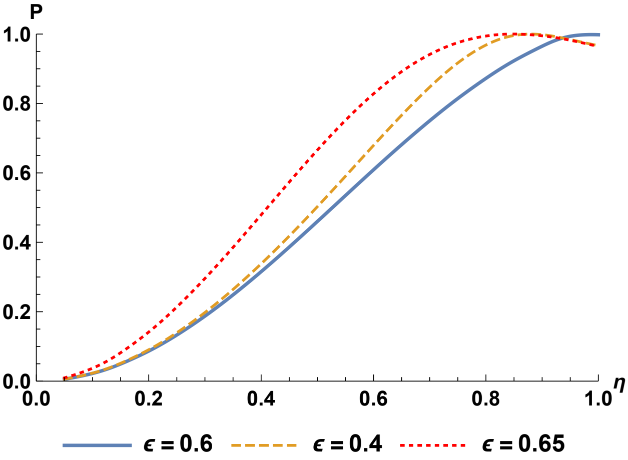
- Unknown $a(t)$ is subject to the Riccati equation:

$$\dot{a} = \imath(-\mathrm{B}_- + \mathrm{B}_0 a + \mathrm{B}_+ a^2)\,,$$

with functions $\mathrm{B}_\pm = 1/2\left(\mathrm{B}_x \pm \imath\,\mathrm{B}_y\right)$ and $\mathrm{B}_0 = 1/2\,\mathrm{B}_z\,.$

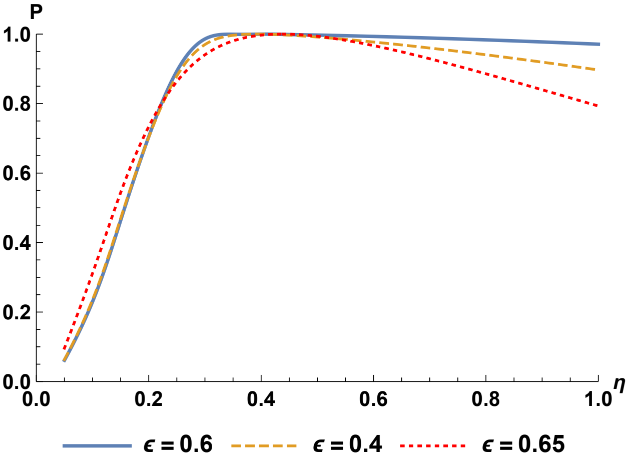
Figures: Spin-flip probability

Probability vs. Intensity; $g = 6$



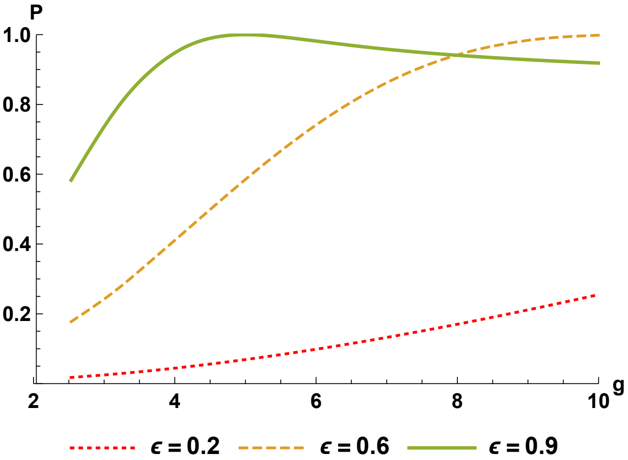
Figures: Spin-flip probability

Probability vs. Intensity; $g = 20$



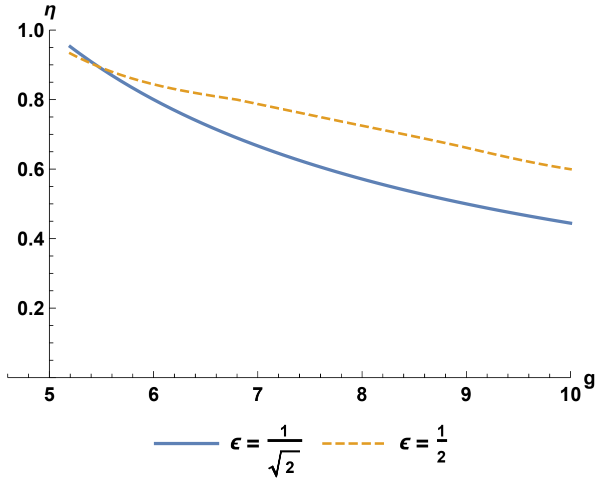
Figures: Spin-flip probability

Probability vs. Gyromagnetic ration



Figures: Spin-flip probability

Resonance curve: Intensity vs. Gyromagnetic ration



Quantum phase v.s. cyclic change of the environment

Laser linearly polarized

- Periodicity of radiation $t \to t + \frac{2\pi}{\omega_L}$
- Reply of particle's spin-1/2:

$$U\left(t + 2\pi/\omega_P\right) = U\left(t\right) M \,,$$

  Appearance of the monodromy matrix, M.
- The simplest diagonal monodromy matrix

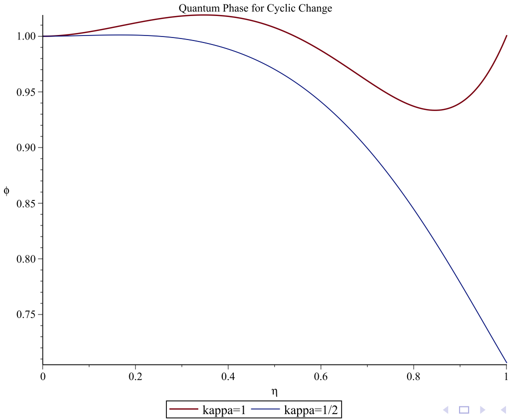
$$M_D = \begin{pmatrix} e^{i\pi\varphi} & 0 \\ 0 & e^{-i\pi\varphi} \end{pmatrix} .$$

Quantum phase depends nonlinearly on a laser intensity

$$\varphi = \frac{g\eta}{\sqrt{2}} \sqrt{1 + \frac{2}{g^2}\left(\frac{1}{\eta} + \frac{1-g}{2}\eta\right)^2} \,.$$

Quantum Phase & Laser Intensity

Deviation of quantum phase from its non-relativistic value $\phi = 1$



Model for a laser-bound state interaction

MODEL ASSUMPTIONS:

HILBERT SPACE: $\mathcal{H} = \mathcal{H}_{\rm CM} \otimes \mathcal{H}_{\rm RM} \otimes \mathcal{H}_{\rm SPIN}\,.$

Bound state: Mass – (M_B), Charge – $(-q_B)$;
Constituents: Masses – $(m^{(n)}, m^{(p)})$, Charges – $(e^{(n)}, e^{(p)})$,
Spins – $(1/2, 1/2)$, Gyromagnetic ratios – $(g^{(n)}, g^{(p)})$.

- Laser-charge interaction V_{CL} (Bound state charge is point-like piked at its center of mass $\mathbf{R}$):

$$V_{CL} := \frac{q_B}{c}\,\boldsymbol{\upsilon}_R \cdot \mathbf{A}(t, \mathbf{R})\,, \qquad \boldsymbol{\upsilon}_R = d\mathbf{R}/dt\,.$$

- Inter constituents interaction V_B:

$$V_B = V_0(r) + V_{SS}(r)\,, \qquad V_{SS}(r) := V_S(r)\,\mathbf{S} \otimes \mathbf{S}\,,$$

$V_0(r)$ and $V_S(r)$ – a scalar functions of $r = |\mathbf{r}_n - \mathbf{r}_p|$.

Model for a laser-bound state interaction (continuation)

- The spin-laser coupling V_{SL}:

$$V_{SL} := -\mathbf{\Omega}^{(n)}(t, \mathbf{r}_n) \cdot \mathbf{s}^{(n)} - \mathbf{\Omega}^{(p)}(t, \mathbf{r}_p) \cdot \mathbf{s}^{(p)} \,,$$

where the vector $\mathbf{\Omega}^{(i)}$ reads

$$\mathbf{\Omega}^{(i)} := \frac{e^{(i)}\, \mathrm{g}^{(i)}}{2\, m^{(i)} c} \left(\mathbf{B} - \frac{1}{c} \left[\mathbf{v}^{(i)} \times \mathbf{E} \right] \right) + \frac{1}{2\, c^2} \left[\mathbf{v}^{(i)} \times \mathbf{a}^{(i)} \right] .$$

E and **B** are the electric and magnetic component of a laser field evaluated along the *i*-th particle in the LAB frame.

Solving the problem

Step one

Evolution of center-mass motion of bound state

Step two

Evolution of spin degrees

Step three

Dynamics of spin degrees entanglement

Center-mass motion of bound state

- The semi-classical picture: The laser-spin interaction V_{SL} contribution to the phase of wave function is negligible small in the leading approximation.
- In the leading semi-classical order the density matrix admits *charge & spin decomposition*

$$\rho = \sum_{\alpha=\pm} c_\alpha |\psi_\alpha\rangle \otimes \varrho_\alpha \,,$$

  where ϱ_α is the constituents spin density, $|\psi_\pm\rangle$, – two linearly independent WKB solutions to the Schrödinger equation with the Hamiltonian $H_0 + V_{SS} + V_{CL}$.
- Separation of the relative and center-mass motion: Trajectory of the bound state's center of mass is similar to considered above a single charged particle

The evolution of spin degrees

- Spin density matrix ϱ satisfy the *spin evolution equation*

$$\dot{\varrho}(t) = -\frac{i}{\hbar}\left[H_S(t),\, \varrho(t)\right].$$

H_S is defined as projection of $V_{SS} + V_{SL}$:

$$H_S(t) = V_{SS} + V_{SL}\Big|_{\text{Constituents classical trajectory}}.$$

- Born-Oppenheimer approximation – "freeze" the relative motion of constituents $< r(t) >= r_0$ and neglect $o(v_r/c)$

An effective spin-laser Hamiltonian

- Effective laser-spin Hamiltonian reads:

$$H_S = -\mathfrak{B}^{(n)}(t)\cdot \mathbf{S}\otimes I - I\otimes \mathbf{S}\cdot \mathfrak{B}^{(p)}(t) + H_I\,,$$

$\mathfrak{B}^{(i)}(t)\,,\ i=(n,p)$ for $\widetilde{\mathrm{g}}^{(i)} = \left(e^{(i)}/m^{(i)}\right)\left(M_B/q_B\right)\mathrm{g}^{(i)}$,

$$\mathfrak{B}^{(i)}_x(t) = \eta\frac{\omega'_L}{2}\sqrt{1-\varepsilon^2}\left[(\widetilde{\mathrm{g}}^{(i)}+1)\mathrm{dn}(u,\,\mu)-\gamma_z\right]\mathrm{cn}(u\,,\mu)\,,$$

$$\mathfrak{B}^{(i)}_y(t) = \eta\frac{\omega'_L}{2}\,\varepsilon\left[(\widetilde{\mathrm{g}}^{(i)}+1)\mathrm{dn}(u\,\mu)-\gamma_z(1-\mu^2)\right]\mathrm{sn}(u\,,\mu)\,,$$

$$\mathfrak{B}^{(i)}_z(t) = -\eta^2\frac{\omega_L}{2}\,\varepsilon\sqrt{1-\varepsilon^2}\left[\widetilde{\mathrm{g}}^{(i)}-\gamma_z\mathrm{dn}(u\,,\mu)\right]\,.$$

- H_I – spin-spin interaction originates from V_{SS} under the same static approximation :

$$\hbar H_I = g\,\mathbf{S}\otimes\mathbf{S}\,. \qquad g := \hbar\, V_S(r_0)\,.$$

The evolution operator

- "Interaction picture" for the evolution operator:

$$U(t) = X(t)W(t)\,.$$

- $W(t) := U^{(n)}(t) \otimes U^{(p)}(t)$ describes the unitary evolution of non-interacting spins.

$$U^{(i)}(t) = \exp\left(\frac{i}{2}\,\vartheta^{(i)}(t)\,\sigma_1\right)$$

where the phase factor for linearly polarized laser reads

$$\vartheta^{(i)}(t) = \frac{\eta}{2}\left[\widetilde{\mathrm{g}}^{(i)} + 1\right]\,\mathrm{sn}(u,\mu) - \frac{1}{2}\arcsin\left[\mu\,\mathrm{sn}(u,\mu)\right]$$

For $\eta \ll 1$ it reduces to the non-relativistic precession:

$$2\vartheta_{NR} = \eta\,\widetilde{\mathrm{g}}^{(n)}\,\sin(\omega_L t)\,.$$

- The factor $X(t)$ affects the entanglement.

Composite systems

Bipartite system – system $A \otimes B$ composed from A and B subsystems

PRINCIPLE OF SUPERPOSITION

- The Hilbert space $\mathcal{H}_{A\otimes B}$ for bipartite system is the tensor product of the Hilbert spaces of its subsystems $\mathcal{H}_A^{d_A}$ and $\mathcal{H}_B^{d_B}$:

$$\mathcal{H}_{A\otimes B} \sim \mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B} \,,$$

  where $d_A = \dim\mathcal{H}_A^{d_A}$ and $d_B = \dim\mathcal{H}_B^{d_B}$.
- The joint system's density matrix ϱ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$

Separable & Entangled density matrices

CLASSICAL VS QUANTUM CORRELATIONS

ABSENCE OF CORRELATION:

$$\varrho_{A\otimes B} = \varrho_A \otimes \varrho_B$$

CLASSICAL CORRELATIONS = SEPARABILITY:

- A bipartite system is in separable state ϱ_{SEP} if

$$\varrho_{\mathrm{SEP}} = \sum_k^r p_k \varrho_A^k \otimes \varrho_B^k, \qquad \sum_k^r p_k = 1\,, \quad p_k \in [0,1]\,,$$

where $\{\varrho_A^k\}$ and $\{\varrho_B^k\}$ are states on $\mathcal{H}_A$ and $\mathcal{H}_B$ for some r .

- Otherwise the state is entangled .

Example: 2 spins pure states

- One spin 1/2 state

$$|1\rangle = \alpha_{\uparrow}|\uparrow\rangle + \beta_{\downarrow}|\downarrow\rangle\,,$$

- Two spin 1/2 state

$$|2\rangle = c_{\uparrow\uparrow}|\uparrow\uparrow\rangle + c_{\uparrow\downarrow}|\uparrow\downarrow\rangle + c_{\downarrow\uparrow}|\downarrow\uparrow\rangle + c_{\downarrow\downarrow}|\downarrow\downarrow\rangle\,,$$

QUESTION: When $|2\rangle$ is separable, i.e., $|2\rangle = |1\rangle \otimes |1'\rangle$?
ANSWER: If and only if the CONCURRENCE is vanishing:

$$C := \det\begin{pmatrix} c_{\uparrow\uparrow} & c_{\uparrow\downarrow} \\ c_{\downarrow\uparrow} & c_{\downarrow\downarrow} \end{pmatrix} = 0\,.$$

Example: 2 spins mixed states

- One spin 1/2 state: $\varrho_{1/2} = \frac{1}{2}[1 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}]\,.$
- Two spin 1/2 state:

$$\varrho_{1/2,1/2} = \frac{1}{4} R_{\mu\nu}\, \sigma_\mu \otimes \sigma_\nu\,, \qquad \mu, \nu = 0, 1, 2, 3\,.$$

$$\sigma_\mu = (I, \boldsymbol{\sigma})\,.$$

Concurrence :

$$C(R) := \max\left(0, \frac{1}{2}(-s_0 + s_1 + s_2 - s_3)\right)\,,$$

$s_0, \ldots s_3$-Lorentz singular values of $R = L_1 \mathrm{diag}(s_0, s_1, s_2, s_3) L_2$.

Dynamics of entanglement

- Initial Werner state

$$\varrho_W := \frac{1}{4}\left(I + p\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\right) ,$$

that for $\frac{1}{3} < p \leq 1$ describes the mixed entangled state.

- In the leading order in laser intensity the concurrence is stable under the influence of laser background:

$$C(\varrho_W) = max\left(0, \frac{3p-1}{2}\right)$$

Dynamics of entanglement

- Initially uncorrelated spins

$$\varrho_0 = \frac{1}{4}\left(\mathrm{I} + \alpha\,\frac{1}{2}\left(\sigma_{03} + \sigma_{30}\right) + \beta\,\frac{1}{2}\left(\sigma_{03} - \sigma_{30}\right)\right)$$

- The concurrence averaged over the period of laser field oscillations $2\pi/\omega_L$, in the leading order in laser intensity η

$$\langle C(\varrho_0)\rangle = max\left(0, \eta\frac{2\beta\omega_L g\Delta}{\pi(\omega_L^2 - 16g^2)}\left[1 - \sin\left(\frac{2\pi g}{\omega_L}\right)\right] - \sqrt{1-\alpha^2}\right).$$

where $\Delta = \widetilde{\mathrm{g}}^{(p)} - \widetilde{\mathrm{g}}^{(n)}$.

Thank you for attention !