Quantum Cosmology with Scalar Fields

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Introduction

Quantum cosmology

- **Quantum cosmology** is a tentative to apply the quantum concepts to the universe as a whole.
- Since there is no complete **Quantum Gravity Theory**, Quantum cosmology tries a less restrict approach, beginning from the General Relativity Theory in order to obtain a quantum description for certain gravitational phenomena.
- It does not touch the underlined structure of the space-time: it begins by the given description of the space-time, and from them obtained a Hamiltonian structure from which the quantisation follows by canonical methods.

Introduction

General Relativity

- General Relativity, our present theory of gravitation, is based on the Einstein-Hilbert Lagrangian,

$$\mathcal{L} = \sqrt{-g}R + \mathcal{L}_m, \tag{1}$$

where $\mathcal{L}_m$ is the Lagrangian describing the matter field.

- The structure of the space-time is given by the metric, defining the infinitesimal interval between two events in the space-time:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dt^2 + 2g_{0i}dt\,dx^i + g_{ij}dx^i\,dx^j. \tag{2}$$

Introduction

Cosmology

- For cosmology, the metric can be simplified considerably, since we believe that the universe is homogenous and isotropic at large scales:

$$ds^2 = dt^2 - a^2(dx^2 + dy^2 + dz^2). \quad (3)$$

where a is the scale factor.

- The gravitational equations can be solved, revealing an expanding universe.
- At the same time, it appears an initial singularity, called *big bang*.

Introduction

Cosmology

- In fact, General Relativity is a theory plagued with singularities in many situations.
- Can these singularities be cured by quantising gravity?
- In order to quantise gravity, we need a Hamiltonian.

Introduction

ADM decomposition

- The Hamiltonian formalism of General Relativity begins by selecting a special decomposition of the space-time metric.
- We consider that the four-dimensional space-time can be decomposed into space-like hyper- surfaces labeled by the time coordinate.
- This is known as the *ADM* decomposition and it leads to the metric,

$$ds^2 = (N^2 - N_i N^i)dt^2 - 2N_i\, dt\, dx^i - h_{ij}dx^i dx^j . \qquad (4)$$

- In this expression, N is the lapse function, N_i is the shifted function and h_{ij} is the induced metric in the three-dimensional spatial hyper-surface.

Introduction

Hamiltonian formulation

- The decomposition described before leads allows the Einstein-Hilbert Lagrangian to be rewritten as,

$$S = \int_{\mathcal{M}} N\sqrt{^3g}\biggl\{^3R + TrK^2 - K^2\biggr\}d^4x \\ -2N\int_{\partial\mathcal{M}} K\sqrt{^3g}d^3x, \qquad (5)$$

where K denotes the extrinsic curvature.

Introduction

The Wheeler-De Witt equation

- The previous action lead to a functional equation called the *Wheeler-De Witt equation*.

$$G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}\Psi + {}^3R\Psi = 0. \tag{6}$$

Introduction

Some of the problems

- The construction of a quantum cosmology must face many problems. Some examples are
  1. Since it is a Hamiltonian constraint, we loose de notion of time ($\mathcal{H} = 0$).
  2. It is not clear how to introduce matter fields.
  3. The resulting equation, the Wheeler-De Witt equation, is too complicated to admit any solution.

A quantum model

The mini-superspace

- For the complexity of the Wheeler-De Witt equation we can consider not all possible metrics, with infinite degrees of freedom, but just some (even one!) degrees of freedom.

A quantum model

The mini-superspace

- Let us consider the metric,

$$ds^2 = N(t)^2 dt^2 - a(t)^2 \gamma_{ij} dx^i dx^j \tag{7}$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor and γ_{ij} is the induced metric of the homogeneous and isotropic spatial hypersurfaces with curvature $k = 0, \pm 1$. For simplicity, we will fix $k = 0$

- With this metric, the gravitational Lagrangian becomes,

$$\mathcal{L}_G = \frac{V_0 a^3}{N}\left\{ -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\dot{a}}{a}\frac{\dot{N}}{N}\right]\right\} \quad , \tag{8}$$

where V_0 is a constant and can be interpreted as the physical volume of the compact universe (in this case a three-torus) divided by a^3.

A quantum model

The mini-superspace

- The gravitational Lagrangian can be written as,

$$\mathcal{L}_G = \frac{1}{N}\left\{6a\dot{a}^2\right\}. \tag{9}$$

- The canonical momenta associated with the scale factor is:

$$p_a = 12\frac{a\dot{a}}{N} \tag{10}$$

- The total Hamiltonian is:

$$H = N\left\{\frac{1}{24}\frac{p_a^2}{a}\right\}. \tag{11}$$

- The resulting Schrödinger-like equation is,

$$\frac{\partial^2\Psi}{\partial a^2} + \frac{q}{a}\frac{\partial\Psi}{\partial a} = 0, \tag{12}$$

A quantum model

Introducing matter

- Schutz has proposed a formalism to give dynamical degrees of freedom to a fluid:

$$L_m = \sqrt{-g}\,p, \tag{13}$$

with

$$u_\nu = \frac{1}{\mu}(\epsilon_{,\nu} + \zeta\beta_{,\nu} + \theta S_{,\nu}). \tag{14}$$

A quantum model

Introducing matter

- After some canonical transformation and adding matter to the gravitational Hamiltonian, we find:

$$\mathcal{H} = -\frac{1}{24}\frac{p_a^2}{a} + \frac{p_T}{a^{3\omega}} \tag{15}$$

where ω characterize the equation of state ($p = \omega\rho$), and p_T is the conjugated momentum associated to the fluid.

A quantum model

Introducing matter

- The quantisation leads to a Schrödinger-like equation:

$$-\frac{1}{24}\partial_a^2\Psi = i24a^{1-3\omega}\partial_T\Psi. \qquad (16)$$

with the solution

$$\Psi = \Psi_0\sqrt{a}J_\nu\left(\frac{\sqrt{96E}}{3(1-\omega)}a^{3\frac{1-\omega}{2}}\right), \qquad (17)$$

where

$$\nu = \frac{1}{3(1-\omega)}. \qquad (18)$$

- The Hamiltonian operator is self-adjoint.

A quantum model

The wave packet

- It is possible to superpose the solutions and to construct a wave-packet:

$$\Psi = \sqrt{a}\int_0^\infty r^{\nu+1}e^{-B}J_\nu\left(ra^{3\frac{1-\omega}{2}}\right)dr, \tag{19}$$

$$= a\frac{e^{-a^{3\frac{3(1-\omega)}{4B}}}}{(-2B)^{\frac{4-3\omega}{3(1-\omega)}}}, \tag{20}$$

with

$$B = \gamma + i\frac{3}{32}(1-\omega)^2 T, \quad r = \frac{\sqrt{96E}}{1-\omega}. \tag{21}$$

A quantum model

Predictions

- The expectation value for the scale factor gives,

$$< a >_T = a_0(1 + T^2)^{\frac{1}{3(1-\omega)}}. \quad (22)$$

- Asymptotically, we re-obtain the classical solutions.
- But, there is no initial singularity.

Scalar-tensor theories

The Brans-Dicke theory

- An example of scalar tensor-theory is the Brans-Dicke theory:

$$\mathcal{L} = \sqrt{-g}\left\{\phi R - \omega\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right\}. \tag{23}$$

- ϕ is a scalar field. In this case, it correspond to the inverse of a (variable) gravitational coupling.
- Some fundamental theories, like the string theory, predict the existence of such a field.
- Remark that, depending on the value of ω the kinetic term for the scalar field can be negative - this would corresponds to the *phantom* case.

Scalar-tensor theories

The Brans-Dicke theory

- Performing a conformal transformation on the metric ($g = \Omega^2\tilde{g}$) and redefining the scalar field $\sigma = \sqrt{|\omega|}\ln\phi$, we obtain the new Lagrangian:

$$\mathcal{L} = \sqrt{-g}\left\{R - \epsilon\sigma_{;\rho}\sigma^{;\rho}\right\} + \mathcal{L}_m. \tag{24}$$

- We have added in the total Lagrangian the matter Lagrangian that we will suppose to be radiation.
- It must be remembered that a radiation field is conformal invariant.

A scalar-tensor model

The Hamiltonian

- Following the same procedure as before, we obtain the following Hamiltonian:

$$H = N\left\{\frac{1}{24}\frac{p_a^2}{a} - \epsilon\frac{p_\sigma^2}{4a^3} - \frac{p_T}{a}\right\}. \tag{25}$$

In this expression $\epsilon = \pm 1$, designating the *normal* and the *phantom* scalar field.

- The resulting Schrödinger-like equation is,

$$-\frac{\partial^2\Psi}{\partial a^2} + \frac{\epsilon}{a^2}\frac{\partial^2\Psi}{\partial\sigma^2} = i\frac{\partial\Psi}{\partial T}, \tag{26}$$

A scalar-tensor model

Some features

- This Schrödinger-type equation has some special features.
  1. The kinetic term has a hyperbolic signature, unless the scalar field is phantom.
  2. There is a non-trivial coupling between the variables.
- Questions:
  1. Is this effective Hamiltonian self-adjoint?
  2. Doe it have a positive energy spectrum?

Cosmological Scenarios

Exemple

- In order to obtain we use a specific ordering factor:

$$-\frac{\partial^2\Psi}{\partial a^2} - \frac{1}{a}\partial_a\Psi + \frac{\epsilon}{a^2}\frac{\partial^2\Psi}{\partial\sigma^2} = i\frac{\partial\Psi}{\partial T}, \qquad (27)$$

- For $\epsilon = -1$, a condition that assures the positivity of energy, the equation admits a solution in terms of stationary states of energy E:

$$\Psi = AJ_\nu(\sqrt{E}a)e^{ik\sigma}e^{-iET}, \quad \nu = k \quad , \qquad (28)$$

with A being a normalization constant and k is a separation constant.

Cosmological Scenarios

Norm

- A particular wavepacket can be obtained by a convenient superposition of the constants E and k:

$$\Psi(a, \sigma, T) = C \frac{e^{-\frac{a^2}{4B(T)}}}{B(T)\, g_\alpha(a, B, \sigma)}, \tag{29}$$

where

$$B(T) = (\gamma + i\,T), \quad g_\alpha(a, \sigma, T) = -\alpha + \ln\left[\frac{a}{2B(T)}\right] \pm i\sigma, \tag{30}$$

γ and α being constants connected with the gaussian-type superposition, and C is a normalisation constant.

Cosmological Scenarios

Norm

- The norm of the wavefunction can be calculated explicitly:

$$N = \int_0^\infty \int_{-\infty}^{+\infty} \Psi^* \Psi \, da \, d\sigma = \frac{C^2}{(B\,B^*)^{1/2}} \pi I_1, \qquad (31)$$

where I_1 is the definite integral,

$$I_1 = \int_0^\infty \frac{e^{-\gamma u^2}}{\alpha + \ln\left(\frac{u}{2}\right)} du. \qquad (32)$$

- The norm is clearly time-dependent: the corresponding quantum model is not unitary.

Cosmological Scenarios

Norm

- Even though, the expectation value for the scalar field can be formally computed, leading to the expression,

$$< a > \propto (\gamma^2 + T^2)^{1/2}. \tag{33}$$

Self-adjointness

Inspecting the Hamiltonian

- Let us suppress the ordering factor introduced previously. The Schrödinger equation is,

$$-\partial_a^2\Psi + \frac{\epsilon}{a^2}\partial_\sigma^2\Psi = i\partial_T\Psi, \tag{34}$$

- Its solution is:

$$\Psi = \Psi_0\sqrt{a}J_\nu(\sqrt{E}a)e^{-i(k\sigma+ET)}. \tag{35}$$

Self-adjointness

Solution

- Constructing the wave-packet we obtain:

$$\Psi = \int_{-\infty}^{+\infty} A(k) e^{-ik\sigma} \frac{a^{\nu+1/2}}{(2B)^{\nu+1}} \exp\left(-\frac{a^2}{4B}\right) dk. \tag{36}$$

- The norm can be written as,

$$N = \frac{1}{2}\int_{-\infty}^{+\infty} |A(k)|^2 \Gamma\left(\frac{3+2\nu}{4}\right) dk. \tag{37}$$

- It is time-independent.

Unitarity

Recovering unitarity

- Let us consider the Hamiltonian with a give factoring order:

$$\hat{H} = -\partial_a^2 - \frac{q}{a}\partial_a + \frac{\epsilon}{a^2}\partial_\sigma^2. \tag{38}$$

- This Hamiltonian is symmetric under the inner product defined by,

$$(\phi, \Psi) = \int_0^\infty \int_{-\infty}^{+\infty} \phi^* \Psi \, a^q \, da \, d\sigma, \tag{39}$$

  if the functions ϕ and Ψ, as well as their first derivatives, are null in the extreme of the interval.

- A non-trivial measure is required.

Unitarity

Recovering unitarity

- A theory is unitary if the corresponding Hamiltonian is self-adjoint.
- To be self-adjoint, the Hamiltonian must be symmetric.
- But, moreover, the Hamiltonian and its adjoint must have the same domain.

Unitarity

Recovering unitarity

- In order to know if the operator is self-adjoint, or if admits a self-adjoint extension, we must evaluate the *deficiency indices*.
- They are given by the solutions of the following eigenvalue problem:

$$\hat{H}\Psi = \pm i\Psi. \quad (40)$$

Unitarity

Deficiency indices

- Let us call n_+ and n_- the number of square integrable solutions of the eigenvalue problem show previously. Then:
  1. If $n_+ = n_- = 0$, the operator $\hat{H}$ is already self-adjoint;
  2. if $n_+ = n_- \neq 0$, the operator is not self-adjoint but it admits self-adjoint extensions given by some restrictions in the wavefunctions;
  3. if $n_+ \neq n_-$ the operator is not self-adjoint and it does not admit any self-ajoint extension.

Unitarity

Results

- Performing this computation we find that:
  1. For $\epsilon = -1$ we have just divergent eigenfunctions. The Hamiltonian operator is self-adjoint.
  2. For $\epsilon = 1$, there are two possibilities.
     1. For an ordering factor such that $p > 1$, we find divergent eigenfunctions implying $n_+ = n_- = 0$ and the Hamiltonian operator is already self-adjoint.
     2. For an ordering factor such that $p \leq 1$, we have one divergent and one convergent function implying $n_+ = n_- = 1$. Hence, the Hamiltonian is not self-adjoint but admits a self-adjoint extension.

Conclusions

- Not all Hamiltonian that we find in quantum cosmology, in the mini-superspace, is self-adjoint.
- To restore the self-adjointness, when possible, we need to use the factoring oder, a specific measure, and, sometimes, appropriate boundary conditions.
- The final predictions seem to be insensitive to all these procedures.
- Hence, the meaning of a unitary evolution in quantum cosmology must be better understood.
- A word of caution: a specific method to recover a time variable has been used.